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Frobenius flocks and algebraicity of matroids

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Guus Pieter Bollen

E-mail: guuspieterb@gmail.com

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PROEFSCHRIFT

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Guus Bollen
Eindhoven, October 2018

Contents

CHAPTER 1

Introduction

1. Matroids

Matroids are combinatorial objects that appear in many branches of mathematics. They consist of a set of elements, of which certain subsets are 'dependent', and others are 'independent'. The notion of (in)dependence in a matroid is abstracted from (in)dependence of a number of other structures. To name just two examples: linear independence of vectors in a vector space and algebraic dependence of rational expressions are both special cases of matroidal independence. Figure 1 depicts several different occurrences of the same matroid.

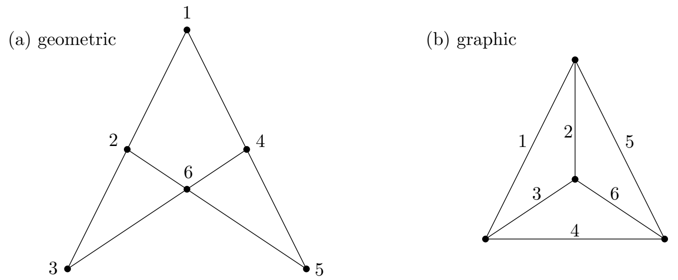
These are just some ways to obtain a matroid. The unifying property of all matroids is a property relating the *bases* of a matroid, the maximal independent sets of *elements*. The set of bases $\mathcal{B}$ of a matroid satisfies the following properties.

(B1) $\mathcal{B}$ is nonempty;

(B2) For all $B, B' \in \mathcal{B}$, and for all $e \in B \setminus B'$, there exists $f \in B' \setminus B$ such that both
$$B \cup \{f\} \setminus \{e\}$$
and
$$B' \cup \{e\} \setminus \{f\}$$
are elements of $\mathcal{B}$.

Definition 1.1. *Let E be a finite set, and let $\mathcal{B}$ be a set of subsets of E. If $\mathcal{B}$ satisfies (B1) and (B2), then the pair $(E, \mathcal{B})$ is a* matroid.

Whitney [**52**] was the first to define the notion of matroid, inspired by linear independence of vectors and by graphs. A *linear matroid* is a pair $(E, \mathcal{B})$, where E is a finite set of vectors, and $\mathcal{B}$ consists of the subsets of E that form a basis of the vector space $\langle E \rangle$. The vector space $\langle E \rangle$ is then called a linear representation of the matroid $(E, \mathcal{B})$. A *graphic matroid* is a pair $(E, \mathcal{B})$, where



(c) algebraic

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$
$$\left(x,\quad y,\quad \frac{x+y}{1+xy},\quad z,\quad \frac{x+z}{1+xz},\quad \frac{x+y+z+xyz}{1+xz+xy+yz} \right)$$

(d) linear

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

FIGURE 1. A matroid described in several different ways: (a) the collinear triples of *points* are dependent in the matroid; (b) the triples of *edges* that form a cycle are dependent in the matroid; (c) the algebraically dependent triples of *rational expressions* are dependent in the matroid; (d) the linearly dependent triples of *vectors* are dependent in the matroid.

E is the set of edges of a graph G, and $\mathcal{B}$ is the set of spanning forests of G. It should be noted that all graphic matroids are linear.

Already in this paper, he noticed that there are more matroids than those coming from the columns of a matrix in characteristic 0. The Fano matroid (Figure 2), which is a projective geometry, is the example mentioned in his paper.

2. Algebraic matroids

Algebraic matroids arise from the notion of algebraic dependence.

Definition 1.2. *Consider a field K which is contained in a field L. If $a_1, \dots, a_n \in L$, then $a_1, \dots, a_n$ are said to be* algebraically dependent *over K if there exists a nonzero polynomial $f(X_1, \dots, X_n)$ with coefficients in K such that*

$$f(a_1, \dots, a_n) = 0.$$

If we take any finite set of elements in L, then the maximal independent subsets are the bases of a matroid [**43**]. This type of matroid is called an *algebraic matroid*, and the set of elements in L is called an *algebraic representation* of this matroid.

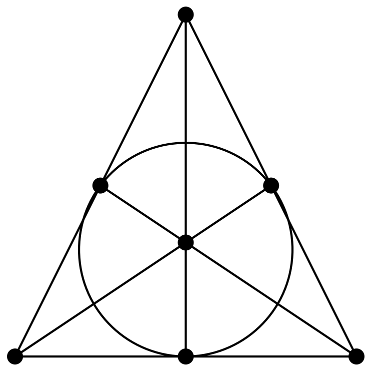


FIGURE 2. The Fano matroid.

While this is the most common definition of algebraic matroids, there are several equivalent ways to define them, which each have their own benefit. A second definition that proves to be rather useful in this thesis is the following, in the language of algebraic geometry. For basic terminology in algebraic geometry, see [**11**].

Definition 1.3. *Let K be an algebraically closed field, let E be a finite set and let $X \subseteq K^E$ be an irreducible algebraic variety. We declare a subset $I \subseteq E$* independent *if the projection of X on K^I is dominant, that is, if the closure of $\{(x_i)_{i \in I} : x \in X\}$ in the Zariski topology equals K^I.*

These two definitions define equivalent notions of algebraic (in)dependence, as will be discussed in Chapter 5. A finite set of elements in L in the first definition corresponds to the set of coordinates of the variety in the second definition. For more details about the correspondence between both definitions, see [**26**].

Algebraic dependence was studied by Van der Waerden [**51**], and by MacLane [**36**], who related the concept to Whitney's matroids. Only in the 1970's did the theory of algebraic matroids begin to gain traction. Ingleton discovered that all algebraic matroids over a field of characteristic 0 are linear [**24**]. Ingleton and Main [**25**] first found a matroid that is not algebraic over any field: the Vámos matroid. Their main argument was later generalised by Dress and Lovász [**13**]. The common point is that algebraic matroids satisfy a certain extension property within the class of algebraic matroids that is not satisfied by matroids in general. Thus if a matroid does not have this extension property within the class of matroids, it cannot be algebraic.

A huge contribution to the theory of algebraic matroids comes from Bernt Lindström, who wrote numerous papers on the subject in the 1980's and early 1990's, many of which will be referred to in this thesis. He characterised algebraic representations via p-polynomials [**34**], and found several algebraic matroids [**33, 29**] that are not linear over any field, but instead over a skew field, such as the Non-Pappus matroid [**30**]. Evans and Hrushovski later generalised Lindström's result on p-polynomials to algebraic representations coming from a connected one-dimensional algebraic group [**16**]. Lindström showed that algebraic matroids are closed under adding harmonic conjugates [**32**]. He showed that an infinite class of matroids, the Lazarson matroids, are only algebraic in a single characteristic [**31**]. Gordon published a similar result for Reid geometries [**18**]. The list goes on: for a comprehensive summary of Lindström's work, I refer to [**3**].

One of the pressing open questions is whether the class of algebraic matroids is closed under duality. Alfter and Hochstättler [**1**] found a matroid of rank 5 on 9 elements, the Tic-Tac-Toe matroid, that is closed under the extension properties from Dress and Lovász [**13**], but whose dual is non-algebraic. It is still unknown whether or not the Tic-Tac-Toe matroid is algebraic over any field.

Kromberg, and later Királyi, Rosen and Theran, made algebraic matroids accessible by methods in commutative algebra and algebraic geometry [**27, 26, 45**].

3. Valuated matroids

Dress and Wenzel introduced the concept of matroid valuations of a matroid M [**15**].

Definition 1.4. *Let M be a matroid with basis set $\mathcal{B}$. A map $\nu : \mathcal{B} \to \mathbb{R}$ is a* valuation *of M if for each pair $B, B' \in \mathcal{B}$ and for each $e \in B \setminus B'$, there exists $f \in B' \setminus B$ such that $B \cup \{f\} \setminus \{e\}, B' \cup \{e\} \setminus \{f\} \in \mathcal{B}$ and the following submodularity condition holds:*

$$\nu(B) + \nu(B') \geq \nu(B \cup \{f\} \setminus \{e\}) + \nu(B' \cup \{e\} \setminus \{f\}).$$

Example 1.5. *Let M be the uniform matroid of rank 2 on 4 elements, shown in Figure 3. The valuation $\nu = 0$ is clearly a valuation of M. A more interesting*

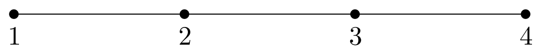


FIGURE 3. The uniform matroid of rank 2 on 4 elements $U_{2,4}$. Each pair of elements is a basis.

valuation is the following:

$$\begin{aligned} \nu(\{1,2\}) &= 0; & \nu(\{2,3\}) &= 0; \\ \nu(\{1,3\}) &= 0; & \nu(\{2,4\}) &= 0; \\ \nu(\{1,4\}) &= 0; & \nu(\{3,4\}) &= 1. \end{aligned}$$

We check the submodularity condition for $B = \{1,3\}$ and $B' = \{2,4\}$. For $e = 1$ we can choose $f = 2$ (but not $f = 4$), and for $e = 3$ we can choose $f = 4$ (but not $f = 2$), so that the inequality holds. ♙

Valuations of matroids give rise to numerous rich structures. The *matroid polytope* of M is a polytope in $\mathbb{R}^E$, where E is the ground set of M, given as the convex hull of the points $(\sum_{i\in B} e_i)_{B\in\mathcal{B}}$, where e_i is the i'th standard basis vector. Matroid valuations correspond to tropical Plücker vectors in $\mathbb{R}^{\mathcal{B}}$, and give rise to a regular subdivision of the matroid polytope of M into smaller matroid polytopes [**49**]. Moreover, a matroid valuation of M gives rise to a matroid for each $\alpha \in \mathbb{R}^E$ by taking as basis set the maximizers among $\mathcal{B}$ of the linear function $\sum_{i\in B} \alpha_i - \nu(B)$. For each α, the corresponding matroid is the matroid of one of the smaller matroid polytopes in the regular subdivision given by ν. The set of valuations of a matroid is also called the *Dressian*, and carries a natural fan structure as a subfan of the secondary fan of the matroid polytope [**20**].

4. This thesis

In this thesis, I bring together the theory of valuated matroids and the theory of algebraic matroids. The goal is to add a new tool to the toolbox for determining algebraic representability of matroids. This tool, along with known methods, is then used to determine algebraicity of as many matroids up to 9 elements as possible.

4.1. Algebraic representations and flocks. If X is an algebraic representation of a matroid M in characteristic 0, then Ingleton showed that there also exists a linear representation of M [**24**]. If X is regarded as an algebraic variety, the linear representation is obtained by taking a sufficiently general smooth point $x \in X$, and taking the tangent space of X at the point x. This tangent space is then a linear representation of M. In characteristic $p > 0$, this method of obtaining a linear representation fails in general, and the reason for that is that the derivative of x^p is 0. Hence algebraic matroids in characteristic p need no longer be linear; a simple example is the Non-Fano matroid (Figure 2 without the circle), which is algebraic in characteristic 2, but not linear.

One can still obtain information on algebraicity of a matroid in characteristic p using Ingleton's 'trick' of taking the tangent space. Lindström noticed that replacing a variable x_e in X by x_e^p leaves the algebraic matroid unchanged, but not Ingleton's tangent space. This allowed him to prove that algebraicity of certain matroids in characteristic p implies linearity, by choosing the right powers of the variables [**31**]. This argument works for instance for the Lazarson matroids, which will be discussed in Chapter 6, of which the Fano matroid is an example.

We apply Lindström's idea on an industrial scale. We assume the field K such that $X \subseteq K^E$ is algebraically closed. Then for each $\alpha \in \mathbb{Z}^E$, we consider the algebraic representation obtained from X by replacing the variable x_e with $x_e^{p^{\alpha_e}}$ for each $e \in E$. This leaves us with infinitely many similar algebraic representations of the same matroid, one for each $\alpha \in \mathbb{Z}^E$. For each of these algebraic representations αX, regarded as algebraic varieties, we may take the tangent space at a general point. This gives a collection of linear spaces, one for each $\alpha \in \mathbb{Z}^E$, which each represent a matroid $\mathcal{M}_\alpha$.

We find that there are simple relations between $\mathcal{M}_\alpha$ and $\mathcal{M}_{\alpha'}$ for any α and α' that differ by a unit vector or by the all-one vector in $\mathbb{Z}^E$. We then define any collection of matroids $(\mathcal{M}'_\alpha)_{\alpha \in \mathbb{Z}^E}$ with the local structure imposed by these relations to be a *matroid flock*. Hence every algebraic representation X gives rise to a matroid flock $\mathcal{M}(X)$. On the other hand, using Murota's discrete duality theory [**39**], we find that any matroid flock corresponds to a matroid valuation ν such that

$$\arg\max_{B \in \mathcal{B}} \{\sum_{i \in B} \alpha_i - \nu(B)\}$$

is the set of bases of $\mathcal{M}_\alpha$ for each $\alpha \in \mathbb{Z}^E$. If the matroid flock is $\mathcal{M}(X)$, this valuation is called the *Lindström valuation* of X. The structure of matroid flocks and their connection with matroid valuations is the subject of **Chapter 3**.

We return to the algebraic representations αX for $\alpha \in \mathbb{Z}^E$. Using the structure of matroid flocks, it turns out that there exists a general point x of X that, when twisted with the powers of p dictated by α, is a general point of each αX. With such a 'very general' x fixed, we again consider the tangent spaces of αX. These tangent spaces then display a local structure very similar to the local structure of matroid flocks. We define any collection of vector spaces $(\mathcal{V}_\alpha)_{\alpha \in \mathbb{Z}^E}$ with this local structure to be a *Frobenius flock*. **Chapter 4** deals with the structure of Frobenius flocks, or more generally, *linear flocks*. An important result from that chapter is that the entire Frobenius flock is determined by a finite number of vector spaces $\mathcal{V}_\alpha$, dictated by the underlying matroid flock. In

Chapter 5 the relation between algebraic matroid representations, matroid flocks and Frobenius flocks is explained in detail.

The fact that an algebraic representation of a matroid gives rise to a Frobenius flock can be used as a necessary condition for algebraicity of matroids. If no Frobenius flock exists for a matroid, then it can certainly not be algebraic. This is the new tool that the work in this thesis brings to the table.

4.2. Computations and algebraicity of small matroids. In the remainder of the thesis, we take the old tools from the matroid algebraicity toolbox and combine them with the new Frobenius-flock tool. We then use these tools to give an as accurate as possible account of the status quo regarding algebraicity of matroids on at most 9 elements in characteristic 2. The methods at our disposal to show that a matroid is algebraic over an algebraically closed field K of characteristic $p > 0$ are the following:

(1) linearity over K implies algebraicity over K;
(2) linearity over the endomorphism ring of a connected one-dimensional algebraic group defined over K implies algebraicity over K;
(3) finding an algebraic representation over K.

Conversely, the methods available to us to show that a matroid is non-algebraic over K are the following:

(1) algebraic matroids should satisfy the extension property from Dress and Lovász and from Ingleton and Main;
(2) harmonic points in algebraic matroids should have a unique harmonic conjugate;
(3) algebraic matroids over K should be *Frobenius-flock representable* over K.

Not all of these criteria are necessarily decidable. An example of an endomorphism ring of a connected one-dimensional algebraic group is an order in $\mathbb{Q}$, and it is not known to be decidable whether a matroid is linearly representable over $\mathbb{Q}$ [**50**]. Similarly, it is not known whether checking Frobenius-flock representability is decidable. However, for many matroids we can find a definite answer to these generally undecidable questions.

In **Chapter 6**, we investigate some classes of matroids, such as linear matroids and matroids from one-dimensional algebraic groups. We also apply our Frobenius-flock methods to the Lazarson matroids and Reid geometries, revisiting the results of Lindström and Gordon. Finally, we consider the interesting class of rank 3 Dowling matroids.

The algorithms that I use to test Frobenius-flock representability take up most of **Chapter 7**. The final part of the chapter contains algorithms that compute the Lindström valuation and the Frobenius flock of an algebraic representation, supporting the results from Chapter 5.

In **Chapter 8**, the computational results on all matroids on at most 9 elements are expounded.

CHAPTER 2

Preliminaries

This chapter introduces some mathematical notions that appear frequently throughout the thesis. We describe the definition and basic properties of matroids, matroid representations, matroid valuations and polyhedral complexes.

1. Matroids

A *matroid* is a pair $(E, \mathcal{B})$, where E is a finite set of elements and $\mathcal{B}$ is a set of subsets of E called the *bases*. The set of bases $\mathcal{B}$ of a matroid satisfies the following properties.

(B1) $\mathcal{B}$ is nonempty;
(B2) Suppose $B, B' \in \mathcal{B}$, and suppose $e \in B \setminus B'$. Then there exists $f \in B' \setminus B$ such that both $B + f - e$ and $B' + e - f$ are elements of $\mathcal{B}$.

Here and later we use the notation $B + e$ for $B \cup \{e\}$ and $B - e$ for $B \setminus \{e\}$. This is one of the many definitions of a matroid. We choose this definition because it bears the most similarity to matroid valuations. For a complete introduction of matroid theory, including proofs of the statements in this section, we refer to Oxley [**43**]. Here, we highlight some concepts related to matroids that will be important in this thesis.

All bases of a matroid necessarily have the same cardinality. The common cardinality of the bases is called the *rank* of the matroid. This induces the *rank function* r of a matroid M, which is defined by

$$r(I) = \max_{B \in \mathcal{B}} |I \cap B|$$

for $I \subseteq E$.

A *loop* of a matroid M is an element of the ground set that is not in any basis of M. A *coloop* of a matroid M is an element of the ground set that is in every basis of M. Since each matroid has at least one basis, an element cannot be both a loop and a coloop.

For a non-coloop $i \in E$, the *deletion* $M \backslash i$ is the matroid $(E - i, \{B \in \mathcal{B} : i \notin B\})$. For a non-loop i, the *contraction* M/i is the matroid $(E - i, \{B - i \in$

$\mathcal{B} : B \in \mathcal{B}, i \in B\})$. If i is a coloop, then $M\backslash i := M/i$. Similarly, if i is a loop, then $M/i := M\backslash i$. For subsets $I \subseteq E$ with $|I| > 1$, we define deletion and contraction recursively: $M\backslash I := (M\backslash I - i)\backslash i$ for some i in I, and similarly $M/I := (M/I - i)/i$ for some i in I, where it should be noted that the choice of i is irrelevant in both cases, since deletion and contraction of e commute with both deletion and contraction of f if e and f are distinct elements of E. When M' can be obtained from M by a series of deletions and contractions, then we call M' a *minor* of M.

In the context of the ground set E, we write $\overline{I} := E \setminus I$ when $I \subseteq E$.

The *dual* of a matroid M is the matroid $M^* := (E, \{\overline{B} : B \in \mathcal{B}\})$. From this definition, it is straightforward to see that $(M^*)^* = M$. Moreover, deletion and contraction are dual operations in the sense that $(M\backslash i)^* = M^*/i$.

The *connectivity function* λ of M is given by $\lambda(I) = r(I) + r(\overline{I}) - r(E)$. A matroid is *connected* if for all $\emptyset \neq I \subsetneq E$, we have $\lambda(I) > 0$.

Disconnected matroids occur frequently in this thesis. If $M := (E, \mathcal{B})$ and $M' := (E', \mathcal{B}')$ are matroids such that E and E' are disjoint, then the *direct sum* $M \oplus M'$ is the matroid $(E \cup E', \{B \cup B' : B \in \mathcal{B}, B' \in \mathcal{B}'\})$. For a matroid M on E with connectivity function λ, and for $I \subseteq E$ we have $\lambda(I) = 0$ if and only if $M = M\backslash I \oplus M\backslash \overline{I}$.

Lemma 2.1. *Let M be a matroid on E with connectivity function λ, and let $I \subseteq E$. Then $\lambda(I) = 0$ if and only if for all $J \subseteq I$: $M/J\backslash(I \setminus J) = M\backslash I$.*

Lemma 2.2. *Let M be a matroid on E with connectivity function λ, and let $I \subseteq E$. Then $\lambda(I) = 0$ if and only if for all bases B, B' of M, we have $|B \cap I| = |B' \cap I|$.*

For a set E and an integer r we denote by $\binom{E}{r}$ the set of cardinality r subsets of E.

Definition 2.3. *A matroid $(E, \mathcal{B})$ of rank r is* uniform *if $\mathcal{B} = \binom{E}{r}$.*

We use the notation $U_{r,n}$ for a uniform matroid of rank r on n elements.

Definition 2.4. *Two matroids $M = (E, \mathcal{B})$ and $M' = (E', \mathcal{B}')$ are* isomorphic*, notation $M \cong M'$, if there is a bijection $\varphi : E \to E'$ such that $\varphi(\mathcal{B}) = \mathcal{B}'$.*

Whenever matroids are counted, we always do so up to isomorphism.

2. Linear representations

Let K be a field, d a nonnegative integer and let E be a finite set. Let A be a $d \times E$ matrix over K of rank d. For $B \subseteq E$, denote by A_B the restriction of A to the columns in B. Then

$$\mathcal{B}_A := \{B \subseteq E : A_B \text{ is an invertible } d \times d \text{ matrix}\}$$

is the set of bases of a matroid $(E, \mathcal{B}_A)$. If Q is an invertible $d \times d$ matrix, then $\mathcal{B}_{QA} = \mathcal{B}_A$. So $\mathcal{B}_A$ only depends on the row space of A.

Definition 2.5. *Let K be a field, d a nonnegative integer, E a finite set and $V \in \mathrm{Gr}_d(K^E)$. Let A be a matrix such that V is the row space of A. Then $M(V) := (E, \mathcal{B}_A)$ is the* matroid of V. *Moreover, we call V a* linear representation *of $(E, \mathcal{B}_A)$.*

Here $\mathrm{Gr}_d(K^E)$ denotes the set of d-dimensional subspaces of K^E, the Grassmannian. When we call a matrix A a linear representation of M, we mean that its row space is a linear representation of M.

If D is an invertible $E \times E$ diagonal matrix, then $\mathcal{B}_{AD} = \mathcal{B}_A$. The row space of AD may be different from the row space of A, and in that case these row spaces are different (but equivalent) linear representations of the same matroid.

Definition 2.6. *We call two vector spaces V, W* linearly equivalent *if $W = VD$ for some invertible diagonal matrix D.*

We state a lemma similar to Lemma 2.1 for linear representations of a matroid. We first introduce notation and define deletion and contraction for vector spaces. When K is a field, E is a finite set, $I \subseteq E$, and $v \in K^E$, then we write

$$v_I := (v_i)_{i \in I}.$$

Now we define deletion and contraction. If $V \in \mathrm{Gr}_d(K^E)$ and $I \subseteq E$, then we define

$$V \backslash I := \{v_{\overline{I}} : v \in V\};$$
$$V / I := \{v_{\overline{I}} : v \in V, v_I = 0\}.$$

Lemma 2.7. *Let $V \in \mathrm{Gr}_d(K^E)$ and let $I \subseteq E$. Let $M = M(V)$ and let λ be the connectivity function of M. Then $\lambda(I) = 0$ if and only if for all $J \subseteq I$: $V/J \backslash (I \setminus J) = V \backslash I$.*

Informally, if a vector space V is a direct sum of its I and $\overline{I}$ parts, then the $\overline{I}$ part is unaffected by the choice of deletion or contraction for each element of I.

3. Algebraic matroids

One of the main objects of study in this thesis are algebraic matroids. Let K be a field and L an extension field of K. Elements $a_1, \ldots, a_n \in L$ are called *algebraically independent* over K if there exists no nonzero polynomial $f \in K[x_1, \ldots, x_n]$ such that $f(a_1, \ldots, a_n) = 0$. Algebraic independence satisfies the matroid independence axioms [**43**], and this leads to the following notions.

Definition 2.8. *Let M be a matroid on a finite set E, and let K be a field. An* algebraic representation *of M over K is a pair (L, ϕ) consisting of a field extension L of K and a map $\phi : E \to L$ such that any $I \subseteq E$ is independent in M if and only if the multiset $\phi(I)$ is algebraically independent over K.*

Definition 2.9. *If a matroid M admits an algebraic representation, M is called an* algebraic matroid.

Algebraic independence and algebraic representations can also be defined in a different manner as follows. For now, we call this type of representation 'algebro-geometric' rather than algebraic. In Section 2 of Chapter 5 we argue that these notions of algebraic representation are equivalent, which is also one of the results of [**26**]. First we define the notion of algebraic variety that we use throughout the thesis.

Definition 2.10. *Let K be an algebraically closed field. An* algebraic variety *over K is a set V such that there exist polynomials $f_1, \ldots, f_s$ in $K[x_1, \ldots, x_n]$ such that*

$$V = \{(a_1, \ldots, a_n) \in K^n : f_i(a_1, \ldots, a_n) = 0 \text{ for all } i \in \{1, \ldots, s\}\}.$$

We proceed to define algebro-geometric representations of matroids. Let K be a field, E a finite set and $I \subseteq E$. Then $\pi_I : K^E \to K^I$ is the projection from K^E on K^I given by $v \mapsto v_I$.

The vector space K^E is equipped with the Zariski topology, in which the closed sets are those defined by polynomial equations; we will use the term *variety* or *closed subvariety* for such a set. In particular, let $X \subseteq K^E$ be the closed subvariety defined as the $\{a \in K^E : \forall f \in P : f(a) = 0\}$. Since P is prime, X is an irreducible closed subvariety, and by Hilbert's Nullstellensatz, P is exactly the set of all polynomials that vanish everywhere on X.

Definition 2.11. *An* algebro-geometric representation *of a matroid M on the ground set E over the algebraically closed field K is an irreducible, closed subvariety Y of K^E such that $I \subseteq E$ is independent in M if and only if the Zariski-closure of $\pi_I(Y)$ equals K^I.*

We denote the matroid M represented by Y as $M(Y)$.

4. Octahedra, pure quadrangles and the Tutte group

Let M be a rank r matroid on a ground set E with basis set $\mathcal{B}$. Consider $S \in \binom{E}{r-2}$ and let $\{a, b, c, d\} \in \binom{E \setminus S}{4}$. Define an ordering on E and assume

$a < b < c < d$. Then we define

$$\begin{array}{rcl} B_{11} & := & S + a + b; \\ B_{12} & := & S + c + d; \\ B_{21} & := & S + a + c; \\ B_{22} & := & S + b + d; \\ B_{31} & := & S + a + d; \\ B_{32} & := & S + b + c. \end{array}$$

We define an *octahedron* of M to be an ordered tuple $(B_{11}, \ldots, B_{32})$ as above, such that all six sets $B_{11}, \ldots, B_{32}$ are in $\mathcal{B}$. An octahedron gives rise to a $U_{2,4}$-minor of M: $(M/S)|_{\{a,b,c,d\}} \cong U_{2,4}$.

If two pairs B_{i1}, B_{i2} and B_{j1}, B_{j2} are in $\mathcal{B}$, then we call the set $Q = \{B_{i1}, B_{i2}, B_{j1}, B_{j2}\}$ a *pure quadrangle* of M. Then if not both B_{k1} and B_{k2} are in $\mathcal{B}$, Q is called *degenerate*.

Degenerate pure quadrangles are used in the definition of the Tutte group. Non-degenerate pure quadrangles are part of an octahedron.

In order to define the Tutte group of M, we use Definition 1.2 from [**14**], which we will give next. Denote by $\mathbb{F}_M$ be the free abelian group generated by ε and brackets $[b_1, \ldots, b_r]$ for $\{b_1, \ldots, b_r\} \in \mathcal{B}$. Denote by $\mathbb{K}_M$ the subgroup of $\mathbb{F}_M$ generated by

(T1) ε^2;

(T2) $\varepsilon[b_1, \ldots, b_r][b_{\sigma(1)}, \ldots, b_{\sigma(r)}]^{-1}$ for all $\{b_1, \ldots, b_r\} \in \mathcal{B}$ and odd permutations $\sigma \in \mathrm{Sym}(r)$;

(T3)
$$\frac{[b_1, \ldots, b_{r-2}, a, b][b_1, \ldots, b_{r-2}, c, d]}{[b_1, \ldots, b_{r-2}, a, d][b_1, \ldots, b_{r-2}, c, b]}$$
for all degenerate pure quadrangles
$$\{S + a + b, S + c + d, S + a + d, S + b + c\},$$
where $S = \{b_1, \ldots, b_{r-2}\}$.

Definition 2.12. *The group*

$$\mathbb{T}_M := \mathbb{F}_M / \mathbb{K}_M$$

is called the Tutte group of M.

Let A be an $r \times E$ matrix over K linearly representing M. Consider the homomorphism $\varphi_A : \mathbb{F}_M \to K^*$ given by $\varepsilon \mapsto -1$ and $[b_1, \ldots, b_r] \mapsto \det(A_{b_1} \cdots A_{b_r})$ for $B \in \mathcal{B}$, where A_{b_i} is column b_i of A. The following lemma follows from [**14**, Prop. 3.1].

Lemma 2.13.
$$\mathbb{K}_M \subseteq \ker \varphi_A.$$

Proof. We show that each generator of $\mathbb{K}_M$ lies in the kernel of φ_A. We have $\varphi_A(\varepsilon^2) = (-1)^2 = 1$, so $\varepsilon^2 \in \ker \varphi_A$. Since the determinant function is alternating in the columns of the matrix, we also find that the generators of type (T2) lie in $\ker \varphi_A$. For generators of type (T3), let $\{S+a+b, S+c+d, S+a+d, S+b+c\}$ be a degenerate pure quadrangle. Then we have either $S+a+c \notin \mathcal{B}$ or $S+b+d \notin \mathcal{B}$. Without loss of generality, suppose $S+a+c \notin \mathcal{B}$. We may assume that A has an identity submatrix in the columns $(b_1, \ldots, b_{r-2}, a, b)$. Then

$$A = \begin{array}{c} \begin{array}{ccccc} S & a & b & c & d \end{array} \\ \begin{pmatrix} I_{r-2} & 0 & 0 & * & * \\ 0 & 1 & 0 & x & z \\ 0 & 0 & 1 & 0 & y \end{pmatrix} \end{array},$$

where $x, y \neq 0$. Now

$$\varphi_A\left(\frac{[b_1, \ldots, b_{r-2}, a, b][b_1, \ldots, b_{r-2}, c, d]}{[b_1, \ldots, b_{r-2}, a, d][b_1, \ldots, b_{r-2}, c, b]}\right) = \frac{1 \cdot xy}{y \cdot x} = 1,$$

and hence generators of type (T3) lie in $\ker \varphi_A$. Since all generators of $\mathbb{K}_M$ lie in $\ker \varphi_A$, we conclude that $\mathbb{K}_M \subseteq \ker \varphi_A$. □

So φ_A induces a homomorphism $\tilde{\varphi}_A : \mathbb{T}_M \to K^*$. This homomorphism can be used to relate $\mathbb{T}_M$ to linear representations of M over K. The following theorem is part of the statement of [**14**, Prop. 3.2].

Theorem 2.14. *Let M be a matroid on E that is linear over K. Let $S = \{b_1, \ldots, b_{r-2}\}$ be given. Suppose $Q = \{S+a+b, S+c+d, S+a+d, S+b+c\}$ is a pure quadrangle of M and suppose $S+a+c \in \mathcal{B}$. The following are equivalent:*

(1) *for all $r \times E$ matrices A over K linearly representing M we have*
$$\tilde{\varphi}_A\left(\frac{[b_1, \ldots, b_{r-2}, a, b][b_1, \ldots, b_{r-2}, c, d]}{[b_1, \ldots, b_{r-2}, a, d][b_1, \ldots, b_{r-2}, c, b]}\right) \neq 1;$$

(2)
$$\frac{[b_1, \ldots, b_{r-2}, a, b][b_1, \ldots, b_{r-2}, c, d]}{[b_1, \ldots, b_{r-2}, a, d][b_1, \ldots, b_{r-2}, c, b]} \neq 1 \text{ in } \mathbb{T}_M;$$

(3) *Q is non-degenerate.*

5. Matroid valuations

Matroid valuations were introduced by Dress and Wenzel [**15**]. All results in this section are due to them. Let E be a set, and $0 \leq r \leq |E|$ an integer. A *matroid valuation* is a map $\nu : \binom{E}{r} \to \mathbb{R}_\infty$ such that

(V0) there exists B such that $\nu(B) < \infty$; and

(V1) for all $B, B' \in \binom{E}{r}$ and $i \in B \setminus B'$ there exists $j \in B' \setminus B$ such that

$$\nu(B) + \nu(B') \geq \nu(B - i + j) + \nu(B' + i - j).$$

Here and later we use the notation $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$.

This definition bears a strong resemblance with the definition of a matroid. The B for which $\nu(B)$ is finite form the set of bases of a matroid. We use the notation $\mathcal{B}^\nu$ for the bases and M^ν for the matroid. If M is a matroid with basis set $\mathcal{B}$, then a *valuation of* M is a map $\nu : \mathcal{B} \to \mathbb{R}$ such that the map $\tau : \binom{E}{r} \to \mathbb{R}_\infty$ given by

$$\tau(B) = \begin{cases} \nu(B), & B \in \mathcal{B}; \\ \infty, & \text{else.} \end{cases}$$

is a matroid valuation. Conversely, if ν is a matroid valuation, then the restriction $\nu|_{\mathcal{B}^\nu}$ is a valuation of M^ν.

For $I \subseteq E$, we write $e_I = \sum_{i \in I} e_i$, where e_i is the i'th standard basis vector. A valuation ν of M is a *trivial valuation* if for every B, $\nu(B) = e_B \cdot w$ for some $w \in \mathbb{R}^E$. The following property of matroid valuations follows from the fact that the inequalities in (V1) are 'balanced' in the sense that each $e \in E$ appears the same number of times on the left-hand side of the inequality as on the right-hand side.

Lemma 2.15. *Let M be a matroid. Let τ be a trivial valuation of M. Then ν is a valuation of M if and only if $\nu + \tau$ is a valuation of M.*

The bases B for which $\nu(B)$ is minimal form the basis set of a matroid.

Lemma 2.16. *Let M be a matroid with basis set $\mathcal{B}$ and let $\nu \in \mathbb{R}^{\mathcal{B}}_{\geq 0}$ be a valuation of M. Let $\mathcal{B}' := \{B \in \mathcal{B} : \nu(B) = 0\}$. If $\mathcal{B}' \neq \emptyset$, then $\mathcal{B}'$ is the basis set of a matroid.*

Proof. We show the symmetric basis exchange axiom. Let $B, B' \in \mathcal{B}'$ and $e \in B \setminus B'$ be given. By (V1), there exists $f \in B' \setminus B$ such that

$$0 = \nu(B) + \nu(B') \geq \nu(B - e + f) + \nu(B + e - f).$$

Then since $\nu \geq 0$, equality holds and $\nu(B - e + f) = \nu(B + e - f) = 0$, as required. $\square$

The following characterisation of matroid valuations is implicit in [**15**].

Lemma 2.17. *A map* $\nu : \binom{E}{r} \to \mathbb{R}_\infty$ *is a matroid valuation if and only if*

(1) $\mathcal{B}^\nu$ *satisfies the base exchange axiom (B); and*
(2) *for all* $S \in \binom{E}{r-2}$ *and* $\{a, b, c, d\} \in \binom{E \setminus S}{4}$, *the minimum of*

$$\nu(S+a+b)+\nu(S+c+d), \quad \nu(S+a+c)+\nu(S+b+d), \quad \nu(S+a+d)+\nu(S+b+c)$$

is attained at least twice.

Proof. Necessity is straightforward. We prove sufficiency. Let ν, B, B', e be a counterexample with $|B \setminus B'|$ as small as possible. So conditions (1) and (2) hold for ν, and we have

$$\nu(B) + \nu(B') < \nu(B - e + f) + \nu(B' + e - f)$$

for all $f \in B' \setminus B$. It follows that $|B \setminus B'| > 1$. If $|B \setminus B'| = 2$, then

$$\nu(B)+\nu(B') < \min\{\nu(B-e+f)+\nu(B'+e-f), \nu(B-e+f')+\nu(B'+e-f')\}$$

where $B' \setminus B = \{f, f'\}$. Taking $S = B \cap B'$, $\{a, b, c, d\} = B \cup B' - S$, this violates assumption (2). So $|B \setminus B'| > 2$. We may assume without loss of generality that $\nu(B) + \nu(B') = 0$, using the trivial valuation τ from Lemma 2.15 with weight function $w = -(\nu(B) + \nu(B'))e_e$. While preserving that $\nu(B) + \nu(B') = 0$, we can also make sure that

$$0 < \min\{\nu(B - e + f), \nu(B' + e - f)\}$$

for each $f \in B' \setminus B$, by using weight functions $w = \lambda_f(e_f - e_e)$ for some $\lambda_f \in \mathbb{R}$.

Let X attain the minimum of

$$\min\{\nu(X) : \ X = B - e' + f', \ e' \in B \setminus B' - e, \ f' \in B' \setminus B\}$$

Since $\nu(B) + \nu(B') = 0$, we have $B, B' \in \mathcal{B}^\nu$, and by assumption (1) the base exchange axiom (B) holds for $\mathcal{B}^\nu$. Hence for each $e' \in B \setminus B' - e$ there exists an $f' \in B' \setminus B$ so that $B - e' + f' \in \mathcal{B}^\nu$, i.e. so that $\nu(B - e' + f') < \infty$. It follows that $\nu(X) < \infty$. As $|X \setminus B'| < |B \setminus B'|$, we have

$$\nu(X) + \nu(B') \geq \nu(X - e + f) + \nu(B' + e - f) > \nu(X - e + f)$$

for some $f \in B' \setminus X \subseteq B' \setminus B$. Put $Y := X - e + f$. Then $|B \setminus Y| = 2 < |B \setminus B'|$, hence

$$\nu(B) + \nu(Y) \geq \nu(B - e + g) + \nu(Y + e - g) > \nu(Y + e - g)$$

for some $g \in Y \setminus B \subseteq B' \setminus B$. Hence for $X' = Y + e - g$, we have

$$\nu(X) = \nu(X) + \nu(B) + \nu(B') > \nu(B) + \nu(Y) > \nu(X'),$$

which contradicts the choice of X. $\square$

Valuated matroids can be modelled in several different ways. For instance, if M is a matroid on E, then consider the *matroid polytope* of M, which is the convex hull of the points $e_B \in \mathbb{R}^E$ for all bases B of M. A map $\nu : \mathcal{B} \to \mathbb{R}$ is a valuation if and only if the regular subdivision of the matroid polytope corresponding to ν is a subdivision into smaller matroid polytopes [**49**].

5.1. Deletion and contraction. Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a matroid valuation. Let $i \in E$ be a non-coloop of M^ν. Then define $\nu\backslash i : \binom{E-i}{d} \to \mathbb{R}_\infty$ by $\nu\backslash i(B) = \nu(B)$. Similarly, let $j \in E$ be a non-loop of M^ν. Then define $\nu/j : \binom{E-j}{d-1} \to \mathbb{R}_\infty$ by $\nu/j(B) = \nu(B+j)$. For i a coloop, set $\nu\backslash i = \nu/i$, and if j is a loop, set $\nu/j = \nu\backslash j$. Note that an element cannot be both a loop and a coloop.

The following lemmas are from [**15**].

Lemma 2.18. *Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a matroid valuation, and let $i \in E$. Then the maps $\nu\backslash i$ and ν/i are matroid valuations.*

Define the relation of strong equivalence '$\cong$' on matroid valuations as follows: $\nu \cong \tau$ if and only if there exists $c \in \mathbb{R}$ such that $\nu = \tau + c$.

Lemma 2.19. *Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a matroid valuation, and let $i, j \in E$ be distinct elements. The following commutation properties are satisfied:*

(1) $\nu\backslash i\backslash j \cong \nu\backslash j\backslash i$;
(2) $\nu/i/j \cong \nu/j/i$;
(3) $\nu/i\backslash j \cong \nu\backslash j/i$,

For a valuation ν, we denote its equivalence class under '$\cong$' by $\nu + \mathbb{R}$. Let $I \subset E$. We recursively define $\nu\backslash I := \nu + \mathbb{R}$ if $I = \emptyset$, and $\nu\backslash I := (\nu\backslash I - i)\backslash i + \mathbb{R}$ otherwise. Analogously, define $\nu/I := \nu + \mathbb{R}$ if $I = \emptyset$, and $\nu/I := (\nu/I - i)/i + \mathbb{R}$ otherwise. By Lemma 2.19, the choices of i are irrelevant.

5.2. Duality. Just like deletion and contraction, there exists a notion of duality for matroid valuations. Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a matroid valuation. Define $\nu^* : \binom{E}{|E|-d} \to \mathbb{R}_\infty$ by $\nu^*(B) = \nu(\overline{B})$. The following lemma is due to Dress and Wenzel [**15**].

Lemma 2.20. *Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a matroid valuation. Then ν^* is a matroid valuation.*

Proof. (V0) is satisfied due to (V0) for ν. For (V1), consider $B, B' \in \binom{E}{|E|-d}$ and $i \in B \setminus B' = \overline{B'} \setminus \overline{B}$. By definition of ν^*, we have $\nu^*(B) + \nu^*(B') = \nu(\overline{B}) + \nu(\overline{B'})$. By (V1) for ν, there exists $j \in \overline{B} \setminus \overline{B'} = B' \setminus B$ such that

$\nu(\overline{B}) + \nu(\overline{B'}) \geq \nu(\overline{B} + i - j) + \nu(\overline{B'} - i + j) = \nu^*(B - i + j) + \nu^*(B' + i - j)$, proving (V1). Thus ν^* is a valuation. □

5.3. Circuit-hyperplane relaxation. If E is a set, and M is a matroid on E, then a *circuit-hyperplane* of M is a subset of E that is both a circuit and a hyperplane. If H is a circuit-hyperplane of M, then adding H to the list of bases of M (relaxing H) yields a matroid. Let M^H denote the matroid obtained from M by relaxing H. The following lemma about circuit-hyperplane relaxation in valuated matroids is due to Dress and Wenzel [**15**].

Lemma 2.21. *Let M be a matroid of rank d with basis set $\mathcal{B}$, and let H be a circuit-hyperplane of M. If ν is a valuation of M, then there exists $k_0 \in \mathbb{Z} \cup \{-\infty\}$ so that the map $\nu_k^H : \binom{E}{d} \to \mathbb{R}_\infty$ given by*

$$B \mapsto \begin{cases} \nu(B), & B \in \mathcal{B}; \\ k, & B = H; \\ \infty, & \text{else.} \end{cases}$$

is a valuation matroid valuation with $M^{\nu_k^H} = M^H$ for all $k > k_0$.

Proof. Let a, b be distinct elements of H. Let c, d be distinct elements of $\overline{H}$. Let $S = H \setminus \{a, b\}$. Then by Lemma 2.17, the minimum of

$$\nu(H) + \nu(S + c + d), \quad \nu(S + a + c) + \nu(S + b + d), \quad \nu(S + a + d) + \nu(S + b + c)$$

is attained at least twice. Since H is a circuit-hyperplane,

$$\{S + a + c, S + b + d, S + a + d, S + b + c\} \subseteq \mathcal{B}^\nu$$

and $\nu(H) = \infty$, the minimum is attained by both $\nu(S + a + c) + \nu(S + b + d)$ and $\nu(S + a + d) + \nu(S + b + c)$. So put

$$k_0 = \sup_{a,b,c,d} \{\nu(S + a + d) + \nu(S + b + c) - \nu(S + c + d)\}.$$

Hence for $k \geq k_0$ the minimum of

$$\nu^H(H) + \nu^H(S + c + d), \quad \nu^H(S + a + c) + \nu^H(S + b + d),$$

$$\nu^H(S + a + d) + \nu^H(S + b + c)$$

is attained at least twice. It follows from Lemma 2.17 that ν^H is a valuation of M^H. □

6. Polyhedral complexes

We define basic notions related to polyhedral complexes that are used in this paper, following Ziegler [**53**].

Definition 2.22. *A set $P \subseteq \mathbb{R}^E$ is a* polyhedron *if*

$$P = \{x \in \mathbb{R}^E : Ax \leq b\},$$

where $A \in \mathbb{R}^{n \times E}$ and $b \in \mathbb{R}^n$ for some nonnegative integer n.

Definition 2.23. *If P is a polyhedron, then F is a* face *of P if*

$$F = P \cap \{x \in \mathbb{R}^E : a \cdot x = b_0\},$$

where $a \in \mathbb{R}^E$ and $b_0 \in \mathbb{R}$ are such that $a \cdot x \leq b_0$ holds for all $x \in P$.

It follows that a nonempty cell F is a face of $P = \{x \in \mathbb{R}^E : Ax \leq b\}$ if $F = P \cap \{x \in \mathbb{R}^E : A'x = b'\}$, where A' is obtained from A by deleting some rows, and b' is obtained from b by deleting the same rows. Moreover, the empty polyhedron is a face of any other polyhedron.

Definition 2.24. *Let $P \subseteq \mathbb{R}^E$ be a polyhedron . Then the* lineality space *of P is defined by*

$$\Lambda(P) := \{y \in \mathbb{R}^E : x + ty \in P \textit{ for all } x \in P, t \in \mathbb{R}\}.$$

Clearly $\Lambda(P)$ is a linear subspace of $\mathbb{R}^E$.

Definition 2.25. *A* polyhedral complex *$\mathcal{D}$ is a finite collection of polyhedra in $\mathbb{R}^E$ such that*

1. *the empty polyhedron is in $\mathcal{D}$;*
2. *if $C \in \mathcal{D}$, then all the faces of C are also in $\mathcal{D}$;*
3. *the intersection $C \cap D$ of two polyhedra $C, D \in \mathcal{D}$ is a face of both C and D.*

We extend the notion of lineality space to polyhedral complexes.

Definition 2.26. *Let $\mathcal{D}$ be a polyhedral complex. Then the* lineality space *of $\mathcal{D}$ is*

$$\Lambda(\mathcal{D}) := \bigcap_{C \in \mathcal{D}} \Lambda(C).$$

We denote by $|\mathcal{D}| := \bigcup \mathcal{D}$ the *support* of $\mathcal{D}$. If $\mathcal{C}, \mathcal{D}$ are two polyhedral complexes, then their intersection is defined by

$$\mathcal{C} \wedge \mathcal{D} := \{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\},$$

which is a polyhedral complex.

When $\mathcal{D}$ is a polyhedral complex, we call an element of $\mathcal{D}$ a *cell.* Cells of a polyhedral complex have a natural notion of dimension.

Definition 2.27. *Let D be a polyhedral complex in $\mathbb{R}^E$. Let C be a nonempty cell of $\mathcal{D}$, and let $\alpha \in C$. Then we call* $\dim\langle \beta - \alpha : \beta \in C\rangle$ *the dimension of C. Furthermore,* $\dim \emptyset := -1$.

In the remainder of the thesis, we only consider polyhedral complexes with rational polyhedra. That is, with polyhedra defined by a rational matrix A and a rational vector b. The lowest and second-lowest dimensional cells play an important role in this thesis.

Definition 2.28. *Let $\mathcal{D}$ be a polyhedral complex and let $k \in \mathbb{N}$. Then we call*

$$\mathcal{S}^k(\mathcal{D}) := \{C \in \mathcal{D} : \dim C \leq k\}$$

the k-skeleton of $\mathcal{D}$. When $n = \dim \Lambda(\mathcal{D})$, we sometimes use the notation $\mathcal{S}_{k-n}(\mathcal{D})$ for the k-skeleton of $\mathcal{D}$.

The k-skeleton of a polyhedral complex is itself a polyhedral complex.

7. The Dressian

Let M be a matroid with basis set $\mathcal{B}$. For each octahedron

$$O = (B_{11}, \dots B_{32})$$

of M, we define the polyhedra

$$P_O^0 := \{\nu \in \mathbb{R}^{\mathcal{B}} : \nu(B_{11}) + \nu(B_{12}) = \nu(B_{21}) + \nu(B_{22}) = \nu(B_{31}) + \nu(B_{32})\},$$

and

$$P_O^i := \{\nu \in \mathbb{R}^{\mathcal{B}} : \nu(B_{i1}) + \nu(B_{i2}) \geq \nu(B_{j1}) + \nu(B_{j2}) = \nu(B_{k1}) + \nu(B_{k2})\},$$

where i, j, k are distinct members of $\{1, 2, 3\}$. Now

$$\mathcal{P}_O := \{\emptyset, P_O^0, P_O^1, P_O^2, P_O^3\}$$

is a polyhedral complex.

Next, for each degenerate pure quadrangle

$$Q = \{S + a + b, S + c + d, S + a + d, S + b + c\},$$

we define the polyhedron

$$P_Q := \{\nu \in \mathbb{R}^{\mathcal{B}} : \nu(S + a + b) + \nu(S + c + d) = \nu(S + a + d) + \nu(S + b + c)\}$$

and the polyhedral complex $\mathcal{P}_Q := \{\emptyset, P_Q\}$.

Definition 2.29. *Let M be a matroid. The* Dressian *of M is defined by*

$$\mathfrak{D}(M) := \bigwedge_O \mathcal{P}_O \wedge \bigwedge_Q \mathcal{P}_Q,$$

where O ranges over the octahedra of M, and Q ranges over the degenerate pure quadrangles of M.

Theorem 2.30. *Let M be a matroid on E of rank r with bases $\mathcal{B}$. The following are equivalent:*

(1) $\nu : \mathcal{B} \to \mathbb{R}_\infty$ *is a valuation of M;*
(2) $\nu \in |\mathfrak{D}(M)|$.

Proof. Suppose ν is a valuation of M, and let O be an octahedron of M. Due to Lemma 2.17, ν must lie in one of the polyhedra P_O^i for $i \in \{0, 1, 2, 3\}$, and hence in $|\mathcal{P}_O|$.

Next, suppose $Q = \{S+a+b, S+c+d, S+a+d, S+b+c\}$ is a degenerate pure quadrangle of M. Since either $S+a+c$ or $S+b+d$ is not in $\mathcal{B}$, we have $\nu(S+a+c) + \nu(S+b+d) = \infty$. Therefore due to Lemma 2.17, $\nu_{\mathcal{B}}$ must lie in P_Q. It follows that $\nu \in |\mathfrak{D}(M)|$.

Conversely, suppose $\nu \in |\mathfrak{D}(M)|$. Now let $S \in \binom{E}{r-2}$ and $\{a, b, c, d\} \in \binom{E \setminus S}{4}$, and consider the corresponding sets $\mathcal{S} = \{B_{11}, \ldots, B_{32}\}$ as in Section 4. We distinguish three cases. If $\mathcal{S}$ contains no pure quadrangle, then Lemma 2.17(2) holds trivially for S, a, b, c, d. If $\mathcal{S}$ contains a degenerate pure quadrangle Q, Lemma 2.17(2) is satisfied for S, a, b, c, d due to the fact that $\nu_{\mathcal{B}} \in P_Q$. Finally, if $\mathcal{S}$ contains a non-degenerate pure quadrangle, then $O = (B_{11}, \ldots, B_{32})$ is an octahedron of M. Since ν lies in one of the polyhedra P_O^i for $i \in \{0, 1, 2, 3\}$, Lemma 2.17(2) is satisfied for S, a, b, c, d. We conclude that Lemma 2.17(2) is satisfied for ν. Since $\mathcal{B}^\nu = \mathcal{B}$, ν is a valuation of M due to Lemma 2.17. □

Consider the trivial valuation $\tau_w : \mathcal{B} \to \mathbb{R}, B \mapsto e_B \cdot w$ for some $w \in \mathbb{R}^E$. We strengthen Lemma 2.15.

Lemma 2.31. *Let M be a matroid on E. Let $w \in \mathbb{R}^E$ be given. Then $\tau_w \in \Lambda(\mathfrak{D}(M))$.*

Proof. Each cell C of $\mathfrak{D}(M)$ is the intersection of a number of polyhedra of the form P_Q or P_O^i, where Q is a degenerate pure quadrangle, O is an octahedron of M and $i \in \{0, 1, 2, 3\}$. Clearly $\tau_w \in \Lambda(P_Q)$ for each Q, since for each $e \in E$ there are as many bases containing e on the left-hand side of the defining equation as on the right-hand side. Similarly, $\tau_w \in \Lambda(P_O^i)$ for each O and i. We conclude that $\nu + \tau_w \in C$ for each $\nu \in C$ and for each $C \in \mathfrak{D}(M)$, as required. □

We define the *space of trivial valuations* of M:

$$T(M) := \langle \tau_w : w \in \mathbb{R}^E \rangle.$$

By the previous lemma, $T(M) \subseteq \Lambda(\mathfrak{D}(M))$.

Definition 2.32. *Let M be a matroid with bases $\mathcal{B}$. If ν is a valuation of M, then the* combinatorial type *of ν is the inclusionwise minimal cell $C \in \mathfrak{D}(M)$ such that $\nu \in C$.*

Let $O(M)$ be the set of octahedra of M. Then the combinatorial type of ν is characterised by the map $\kappa_\nu : O(M) \to \{0,1,2,3\}$, mapping O to the (unique) i such that ν lies in the interior of P_O^i. Then we have

$$\nu \in \bigcap_{O \in O(M)} P_O^{\kappa_\nu(O)} \cap \bigcap_Q P_Q \in \mathfrak{D}(M).$$

From Lemma 2.31, we may conclude that the combinatorial type of ν does not change by adding a trivial valuation. The same is true for scaling by a positive scalar.

Lemma 2.33. *Let M be a matroid and $\nu \in |\mathfrak{D}(M)|$. Let $\lambda \in \mathbb{R}_{>0}$ be given. Then $\lambda\nu$ is a valuation of M with $\kappa_\nu = \kappa_{\lambda\nu}$.*

Proof. The defining inequalities of the polyhedra P_Q and P_O^i are all homogeneous in ν. Therefore they are satisfied by ν if and only if they are satisfied by $\lambda\nu$, since $\lambda > 0$. Hence $\lambda\nu$ lies in the interior of the same cell of $\mathfrak{D}(M)$ as ν. By Theorem 2.30, $\lambda\nu$ is a valuation. $\square$

8. Murota's discrete duality theory

We briefly review the definitions and results we use from [**39**]. For an $x \in \mathbb{Z}^n$, let

$$\operatorname{supp}(x) := \{i : x_i \neq 0\},\ \operatorname{supp}^+(x) := \{i : x_i > 0\},$$

$$\operatorname{supp}^-(x) := \{i : x_i < 0\}.$$

For any function $f : \mathbb{Z}^n \to \mathbb{R}_\infty$, we write $\operatorname{dom}(f) := \{x \in \mathbb{Z}^n : f(x) \in \mathbb{R}\}$.

Definition 2.34. *A function $f : \mathbb{Z}^n \to \mathbb{Z}_\infty$ is called* M-convex *if:*

(1) *$dom(f) \neq \emptyset$; and*
(2) *for all $x, y \in dom(f)$ and $i \in supp^+(x-y)$, there exists $j \in supp^-(x-y)$ so that $f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$.*

Let $x, y \in \mathbb{Z}^n$. We write $x \vee y := (\max\{x_i, y_i\})_i$ and $x \wedge y := (\min\{x_i, y_i\})_i$, and let $\mathbf{1}$ denote the all-one vector.

Definition 2.35. *A function $g : \mathbb{Z}^n \to \mathbb{Z}_\infty$ is* L-convex *if*

(1) *$dom(g) \neq \emptyset$;*
(2) *$g(x) + g(y) \geq g(x \vee y) + g(x \wedge y)$ for all $x, y \in \mathbb{Z}^n$; and*
(3) *there exists an $r \in \mathbb{Z}$ so that $g(x + \mathbf{1}) = g(x) + r$ for all $x \in \mathbb{Z}^n$.*

For any $h : \mathbb{Z}^n \to \mathbb{Z}_\infty$ with nonempty domain, the *Legendre-Fenchel dual* is the function $h^\bullet : \mathbb{Z}^n \to \mathbb{Z}_\infty$ defined by

$$h^\bullet(x) = \sup\{x \cdot y - h(y) : y \in \mathbb{Z}^n\}.$$

The following key theorem describes the duality of M-convex and L-convex functions [**39**].

Theorem 2.36. *Let $f, g : \mathbb{Z}^n \to \mathbb{Z}_\infty$. The following are equivalent.*

(1) *f is M-convex, and $g = f^\bullet$; and*
(2) *g is L-convex, and $f = g^\bullet$.*

Finally, Murota provides the following local optimality criterion for L-convex functions.

Lemma 2.37. *(Murota [**39**], 7.14) Let G be an L-convex function on $\mathbb{Z}^n$, and let $x \in \mathbb{Z}^n$. Then*

$$\forall y \in \mathbb{Z}^n : G(x) \le G(y) \iff \begin{cases} \forall I \subset \{1, \ldots, n\} : G(x) \le G(x + e_I), \\ G(x) = G(x + \mathbf{1}). \end{cases}$$

CHAPTER 3

Matroid flocks and cell complexes

This chapter is based on joint work with Jan Draisma and Rudi Pendavingh [**7, 6**].

1. Introduction

In this chapter, I introduce matroid flocks and their associated cell complexes.

Definition 3.1. *A* matroid flock of rank d on E *is a map $\mathcal{M}$ which assigns a matroid $\mathcal{M}_\alpha$ on E of rank d to each $\alpha \in \mathbb{Z}^E$, satisfying the following two axioms:*

(MF1) $\mathcal{M}_\alpha/i = \mathcal{M}_{\alpha+e_i}\backslash i$ *for all* $\alpha \in \mathbb{Z}^E$ *and* $i \in E$*;*
(MF2) $\mathcal{M}_\alpha = \mathcal{M}_{\alpha+\mathbf{1}}$ *for all* $\alpha \in \mathbb{Z}^E$*.*

Matroid flocks started out as a 'side product' of vector space flocks, which are the subject of chapter 4. However, after we had discovered the relation between matroid flocks and integer-valued matroid valuations, they turned into an object of greater interest. In the second section of this chapter we make this relation precise.

There is also a bijective relation between matroid flocks and certain polyhedral complexes, which will be studied in the third section of this chapter. As a consequence we have three different interpretations of matroid flocks that are all equivalent. This allows us to gain a firm understanding of the structure of matroid flocks, as will be the focus of the fifth section of this chapter.

2. Matroid flock characterisation

We begin with an example of a matroid flock as defined in Definition 3.1.

Example 3.2. *Let $E = \{1, 2\}$ and let $\mathcal{M}_0$ be the matroid on E with bases $\{1\}$ and $\{2\}$. We claim that this extends in a unique manner to a matroid flock $\mathcal{M}$ on E of rank* 1*. Indeed, by (MF1) the rank-zero matroid $\mathcal{M}_0/1$ equals $\mathcal{M}_{e_1}\backslash 1$, so that $\{1\}$ is the only basis in $\mathcal{M}_{e_1}$. Repeating this argument, we find $\mathcal{M}_{ke_1} = \mathcal{M}_{e_1}$ for all $k > 0$. Similarly, $\mathcal{M}_{ke_2}$ is the matroid with only one* *basis* $\{2\}$ *for all* $k > 0$. *Using (MF2) this determines* $\mathcal{M}_\alpha$ *for all* $\alpha = (k, l)$*: it equals* $\mathcal{M}_0$ *if* $k = l$, $\mathcal{M}_{e_1}$ *if* $k > l$, *and* $\mathcal{M}_{e_2}$ *if* $k < l$. *The phenomenon that most* $\mathcal{M}_\alpha$ *follow from a small number of them, like migrating birds follow a small number of leaders, inspired our term* matroid flock. ♙

We proceed with a characterisation of matroid flocks. For a matroid valuation $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ and an $\alpha \in \mathbb{R}^E$, let

$$\mathcal{B}^\nu_\alpha := \arg\max_{B \in \binom{E}{d}} \{e_B \cdot \alpha - \nu(B)\}$$

and put $\mathcal{M}^\nu_\alpha := (E, \mathcal{B}^\nu_\alpha)$. Then $\mathcal{M}^\nu_\alpha$ is a matroid. Moreover, let

$$g^\nu(\alpha) := \max_{B \in \binom{E}{d}} \{e_B \cdot \alpha - \nu(B)\}$$

be the maximal value obtained by the elements of $\mathcal{B}^\nu_\alpha$.

The main result of this section will be the following characterization of matroid flocks.

Theorem 3.3. *Let* E *be a finite set, let* $d \in \mathbb{N}$, *and let* $\mathcal{M}_\alpha$ *be a matroid on* E *of rank* d *for each* $\alpha \in \mathbb{Z}^E$. *The following are equivalent:*

1. $\mathcal{M} : \alpha \mapsto \mathcal{M}_\alpha$ *is a matroid flock; and*
2. *there is a matroid valuation* $\nu : \binom{E}{d} \to \mathbb{Z}_\infty$ *so that* $\mathcal{M}_\alpha = \mathcal{M}^\nu_\alpha$ *for all* $\alpha \in \mathbb{Z}^E$.

The implication (2)⇒(1) of Theorem 3.3 is relatively straightforward.

Lemma 3.4. *Let* $\nu : \binom{E}{d} \to \mathbb{Z}_\infty$ *be a valuation. Then*

1. $\mathcal{M}^\nu_\alpha / i = \mathcal{M}^\nu_{\alpha + e_i} \backslash i$ *for all* $\alpha \in \mathbb{Z}^E$ *and* $i \in E$*; and*
2. $\mathcal{M}^\nu_\alpha = \mathcal{M}^\nu_{\alpha + \mathbf{1}}$ *for all* $\alpha \in \mathbb{Z}^E$.

Proof. We prove (1). Consider $\alpha \in \mathbb{Z}^E$ and $i \in E$. If i is a loop of $\mathcal{M}^\nu_\alpha$, then $g^\nu(\alpha + e_i) = g^\nu(\alpha)$. Then B is a basis of $\mathcal{M}^\nu_\alpha / i$ if and only if

$$e_B \cdot (\alpha + e_i) - \nu(B) = e_B \cdot \alpha - \nu(B) = g^\nu(\alpha) = g^\nu(\alpha + e_i)$$

if and only if B is a basis of $\mathcal{M}^\nu_{\alpha+e_i} \backslash i$. On the other hand, if i is not a loop of $\mathcal{M}^\nu_\alpha$, then $g^\nu(\alpha + e_i) = g^\nu(\alpha) + 1$, and then B' is a basis of $\mathcal{M}^\nu_\alpha / i$ if and only if $B = B' + i$ is a basis of $\mathcal{M}^\nu_\alpha$, if and only if

$$e_B \cdot (\alpha + e_i) - \nu(B) = e_B \cdot \alpha - \nu(B) + 1 = g(\alpha) + 1 = g(\alpha + e_i)$$

if and only if B' is a basis of $\mathcal{M}^\nu_{\alpha+e_i} \backslash i$.

To see (2), note that $g^\nu(\alpha + \mathbf{1}) = g^\nu(\alpha) + d$. Then B is a basis of $\mathcal{M}^\nu_\alpha$ if and only if

$$e_B \cdot (\alpha + \mathbf{1}) - \nu(B) = e_B \cdot \alpha - \nu(B) + d = g^\nu(\alpha) + d = g^\nu(\alpha + \mathbf{1})$$

if and only if B is a basis of $\mathcal{M}^{\nu}_{\alpha+\mathbf{1}}$. □

We show the implication (1)⇒(2) of Theorem 3.3. Our proof makes essential use of the discrete duality theory of Murota [**39**]. Specifically, we will first construct an L-convex function $g : \mathbb{Z}^E \to \mathbb{Z}$ from the matroid flock $\mathcal{M}$. The Fenchel dual f of g is then an M-convex function, from which we derive the required valuation ν. Before we can prove the existence of a suitable function g, we need a few technical lemmas.

In the context of a matroid flock $\mathcal{M}$, we will write r_α for the rank function of the matroid $\mathcal{M}_\alpha$. We first extend (MF1).

Lemma 3.5. *Let $\mathcal{M}$ be a matroid flock on E, let $\alpha \in \mathbb{Z}^E$ and let $I \subseteq E$. Then $\mathcal{M}_\alpha/I = \mathcal{M}_{\alpha+e_I}\backslash I$.*

Proof. By induction on I. Clearly the lemma holds if $I = \emptyset$. If $I \neq \emptyset$, pick any $i \in I$. Using the induction hypothesis for α and $I - i$, followed by (MF1) for $\alpha + e_{I-i}$ and i, we have

$$\begin{aligned}
\mathcal{M}_\alpha/I = \mathcal{M}_\alpha/(I-i)/i &= \mathcal{M}_{\alpha+e_{I-i}}\backslash(I-i)/i \\
&= \mathcal{M}_{\alpha+e_{I-i}}/i\backslash(I-i) \\
&= \mathcal{M}_{\alpha+e_I}\backslash i\backslash(I-i) \\
&= \mathcal{M}_{\alpha+e_I}\backslash I,
\end{aligned}$$

as required. □

The above lemma allows us to extend (MF1) as follows.

(MF1') $\mathcal{M}_\alpha/I = \mathcal{M}_{\alpha+e_I}\backslash I$ for all $\alpha \in \mathbb{Z}^E$ and $I \subseteq E$.

Lemma 3.6. *Let $\mathcal{M}$ be a matroid flock on E, let $\alpha \in \mathbb{Z}^E$ and let $I \subseteq J \subseteq E$. Then*

$$r_\alpha(J) = r_\alpha(I) + r_{\alpha+e_I}(J\backslash I).$$

Proof. Using Lemma 3.5, we have $r_{\alpha+e_I}(J \backslash I) = r(\mathcal{M}_{\alpha+e_I} \backslash I \backslash \overline{J}) = r(\mathcal{M}_\alpha/I\backslash\overline{J}) = r_\alpha(J) - r_\alpha(I)$, as required. □

If $\alpha, \beta \in \mathbb{R}^E$, we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i \in E$.

Lemma 3.7. *Let $\mathcal{M}$ be a matroid flock on E, let $\alpha, \beta \in \mathbb{Z}^E$ and let $I \subseteq E$. If $I \cap supp(\beta - \alpha) = \emptyset$ and $\alpha \leq \beta$, then $r_\alpha(I) \geq r_\beta(I)$.*

Proof. We use induction on $\max_i(\beta_i - \alpha_i)$. Let $J := \mathrm{supp}(\beta - \alpha)$. Then

$$\begin{aligned}
r_\alpha(I) &= r(\mathcal{M}_\alpha\backslash J\backslash\overline{I \cup J}) \geq r(\mathcal{M}_\alpha/J\backslash\overline{I \cup J}) \\
&= r(\mathcal{M}_{\alpha+e_J}\backslash J\backslash\overline{I \cup J}) = r_{\alpha+e_J}(I).
\end{aligned}$$

Taking $\alpha' := \alpha + e_J$, we have $\alpha' \leq \beta$, $\max_i(\beta_i - \alpha'_i) < \max_i(\beta_i - \alpha_i)$ and $\operatorname{supp}(\beta - \alpha') \subseteq \operatorname{supp}(\beta - \alpha)$. Hence

$$r_\alpha(I) \geq r_{\alpha+e_J}(I) \geq r_\beta(I),$$

by using the induction hypothesis for α', β. □

We are now ready to show the existence of the function g alluded to above.

Lemma 3.8. *Let $\mathcal{M}$ be a matroid flock on E. There is a unique function $g : \mathbb{Z}^E \to \mathbb{Z}$ so that*

(1) *$g(0) = 0$, and*
(2) *$g(\alpha + e_I) = g(\alpha) + r_\alpha(I)$ for all $\alpha \in \mathbb{Z}^E$ and $I \subseteq E$.*

Proof. Let $D = \big(\mathbb{Z}^E, A\big)$ be the infinite directed graph with arcs

$$A := \{(\alpha, \alpha + e_I) : \ \alpha \in \mathbb{Z}^E, \ \emptyset \neq I \subseteq E\}.$$

Let $l : A \to \mathbb{Z}$ be a length function on the arcs determined by

$$l(\alpha, \alpha + e_I) = r_\alpha(I).$$

This length function extends to the undirected walks $W = (\alpha^0, \dots, \alpha^k)$ of D in the usual way, by setting

$$l(W) := \begin{cases} l(\alpha^0, \dots, \alpha^{k-1}) + l(\alpha^{k-1}, \alpha^k) & \text{if } (\alpha^{k-1}, \alpha^k) \in A \\ l(\alpha^0, \dots, \alpha^{k-1}) - l(\alpha^k, \alpha^{k-1}) & \text{if } (\alpha^k, \alpha^{k-1}) \in A \end{cases}$$

if $k > 0$, and $l(W) = 0$ otherwise. A walk $(\alpha^0, \dots, \alpha^k)$ is *closed* if it starts and ends in the same vertex, i.e. if $\alpha^0 = \alpha^k$.

If we assume that $l(W) = 0$ for each closed walk W, then we can construct a function g satisfying (1) and (2) as follows. For each $\alpha \in \mathbb{Z}^E$, let W^α be an arbitrary walk from 0 to α, and put $g(\alpha) = l(W^\alpha)$. Then $g(0) = l(W^0) = 0$ by our assumption, since W^0 is a walk from 0 to 0. Also, if $\alpha \in \mathbb{Z}^E$ and $I \subseteq E$, then writing $\beta := \alpha + e_I$ we have

$$l(W^\alpha) + l(\alpha, \beta) - l(W^\beta) = l(\alpha^0, \dots, \alpha^k, \beta^m, \dots, \beta^0) = 0,$$

by our assumption, where $W^\alpha = (\alpha^0, \dots, \alpha^k)$ and $W^\beta = (\beta^0, \dots, \beta^m)$. It follows that

$$g(\alpha + e_I) = l(W^\beta) = l(W^\alpha) + l(\alpha, \beta) = g(\alpha) + r_\alpha(I),$$

as required. So to prove the lemma, it will suffice to show that $l(W) = 0$ for each closed walk W.

Suppose for a contradiction that $W = (\alpha^0, \ldots, \alpha^k)$ is a closed walk with $l(W) \neq 0$. Fix any $i \in E$ so that $\alpha_i^0 \neq \alpha_i^1$. If $J \subseteq E$ is such that $i \in J$, then for any $\alpha \in \mathbb{Z}^E$ we have

$$(1) \quad l(\alpha, \alpha + e_J) = r_\alpha(J) = r_\alpha(i) + r_{\alpha+e_i}(J - i) = l(\alpha, \alpha + e_i, \alpha + e_J)$$

by applying Lemma 3.6 with $I = \{i\}$. Hence, if we replace each subsequence $(\alpha^{t-1}, \alpha^t) = (\alpha, \alpha + e_J)$ of W with $i \in J$ by $(\alpha, \alpha + e_i, \alpha + e_J)$, and each subsequence $(\alpha^{t-1}, \alpha^t) = (\alpha + e_J, \alpha)$ with $i \in J$ by $(\alpha + e_J, \alpha + e_i, \alpha)$, then we obtain a closed walk $U = (\beta^0, \ldots, \beta^m)$ with $l(U) = l(W) \neq 0$, such that if $\beta^t - \beta^{t-1} = \pm e_J$, then $i \notin J$ or $J = \{i\}$, and moreover such that $\beta^t - \beta^{t-1} = \pm e_i$ for some t. Pick such U, i with m as small as possible, and minimizing $|U|_i := \sum\{t \in \{1, \ldots, m\} : \beta_i^t \neq \beta_i^{t-1}\}$.

We claim that there is no $t > 0$ so that $\beta_i^{t-1} = \beta_i^t \neq \beta_i^{t+1}$. Consider that by applying Lemma 3.6 with $I = J - i$, we have

$$(2) \quad l(\alpha, \alpha + e_J) = r_\alpha(J) = r_\alpha(J - i) + r_{\alpha+e_{J-i}}(i) = l(\alpha, \alpha + e_{J-i}, \alpha + e_J),$$

so that using (1) we obtain $l(\alpha, \alpha + e_{J-i}, \alpha + e_J, \alpha + e_i, \alpha) = 0$. Hence, $l(\alpha, \alpha + e_{J-i}, \alpha + e_J) = l(\alpha, \alpha + e_i, \alpha + e_J)$ and $l(\alpha + e_{J-i}, \alpha, \alpha + e_i) = l(\alpha + e_{J-i}, \alpha + e_J, \alpha + e_i)$. It follows that any subsequence $(\beta^{t-1}, \beta^t, \beta^{t+1})$ of U with $\beta_i^{t-1} = \beta_i^t \neq \beta_i^{t+1}$ can be rerouted to $(\beta^{t-1}, \beta', \beta^{t+1})$ with $\beta_i^{t-1} \neq \beta'_i = \beta_i^{t+1}$, which would result in a closed walk U' with $|U'|_i < |U|_i$, a contradiction.

So there exists an $m' \in \{1, \ldots, m\}$ such that $\beta^t - \beta^{t-1} = \pm e_i$ if and only if $t \leq m'$. Then $\beta^0 = \beta^{m'} = \beta^m$, and $l(\beta^0, \ldots, \beta^{m'}) = 0$. Hence

$$l(\beta^{m'}, \ldots, \beta^m) = l(\beta^0, \ldots, \beta^{m'}) + l(\beta^{m'}, \ldots, \beta^m) = l(U) \neq 0,$$

which contradicts the minimality of m. $\square$

For any matroid flock $\mathcal{M}$, let $g^{\mathcal{M}}$ denote the unique function g from Lemma 3.8.

Theorem 3.9. *Let $\mathcal{M}$ be a matroid flock of rank d on E, and let $g = g^{\mathcal{M}}$. Then*

(1) $g(\alpha) + g(\beta) \geq g(\alpha \vee \beta) + g(\alpha \wedge \beta)$ *for all* $\alpha, \beta \in \mathbb{Z}^E$*; and*
(2) $g(\alpha + \mathbf{1}) = g(\alpha) + d$ *for all* $\alpha \in \mathbb{Z}^E$.

Proof. We first show (1). Let $\alpha, \beta \in \mathbb{Z}^E$. Since $(\beta - \alpha) \vee 0 \geq 0$, there are $I_1 \subseteq \ldots \subseteq I_k \subseteq E$ so that $(\beta - \alpha) \vee 0 = \sum_{j=1}^k e_{I_j}$. Let $\gamma(t) := \alpha \wedge \beta + \sum_{j=1}^t e_{I_j}$. Then $\gamma(0) = \alpha \wedge \beta$, $\gamma(k) = \alpha \wedge \beta + (\beta - \alpha) \vee 0 = \beta$, and $\gamma(t) = \gamma(t-1) + e_{I_t}$, so that

$$g(\beta) - g(\alpha \wedge \beta) = \sum_{t=1}^k g(\gamma(t)) - g(\gamma(t-1)) = \sum_{t=1}^k r_{\gamma(t-1)}(I_t).$$

Let $\delta := (\alpha - \beta) \vee 0$. Then $\gamma(0) + \delta = \alpha$ and $\gamma(k) + \delta = \alpha \vee \beta$, and we also have

$$g(\alpha \vee \beta) - g(\alpha) = \sum_{t=1}^{k} g(\gamma(t) + \delta) - g(\gamma(t-1) + \delta) = \sum_{t=1}^{k} r_{\gamma(t-1)+\delta}(I_t).$$

For each t we have $I_t \cap \operatorname{supp}(\delta) \subseteq \operatorname{supp}((\beta - \alpha) \vee 0) \cap \operatorname{supp}((\alpha - \beta) \vee 0) = \emptyset$, and $\delta \geq 0$. By Lemma 3.7, it follows that $r_{\gamma(t-1)}(I_t) \geq r_{\gamma(t-1)+\delta}(I_t)$ for each t, and hence

$$g(\beta) - g(\alpha \wedge \beta) = \sum_{t=1}^{k} r_{\gamma(t)}(I_t) \geq \sum_{t=1}^{k} r_{\gamma(t)+\delta}(I_t) = g(\alpha \vee \beta) - g(\alpha),$$

which implies (1).

To see (2), note that $g(\alpha + \mathbf{1}) = g(\alpha) + r_\alpha(E) = g(\alpha) + d$. □

It follows that for any matroid flock $\mathcal{M}$, the function $g^{\mathcal{M}}$ is L-convex in the sense of Murota.

Lemma 3.10. *Let $\mathcal{M}$ be a matroid flock on E, let $g = g^{\mathcal{M}}$ and $f := g^\bullet$, and let $\alpha, \omega \in \mathbb{Z}^E$. The following are equivalent.*

(1) *$\omega \cdot \alpha = f(\omega) + g(\alpha)$; and*
(2) *$\omega = e_B$ for some basis B of $\mathcal{M}_\alpha$.*

Proof. We first show that (1) implies (2). So assume that $\omega \cdot \alpha = f(\omega) + g(\alpha)$. Then $f(\omega)$ is finite, as $g(\alpha)$ and $\omega \cdot \alpha$ are both finite. Since $f = g^\bullet$, we have

$$\omega \cdot \alpha - g(\alpha) = f(\omega) = \sup\{\omega \cdot \beta - g(\beta) : \beta \in \mathbb{Z}^E\},$$

and hence α minimizes the function $G : \beta \mapsto g(\beta) - \omega \cdot \beta$ over all $\beta \in \mathbb{Z}^E$. Since

$$0 \leq G(\alpha - e_i) - G(\alpha) = g(\alpha - e_i) - \omega \cdot (\alpha - e_i) - g(\alpha) + \omega \cdot \alpha = -r_{\alpha - e_i}(i) + \omega_i$$

for each $i \in E$, it follows that $\omega \geq 0$. Since

$$0 \leq G(\alpha + e_i) - G(\alpha) = g(\alpha + e_i) - \omega \cdot (\alpha + e_i) - g(\alpha) + \omega \cdot \alpha = r_\alpha(i) - \omega_i,$$

we have $\omega \leq \mathbf{1}$. Hence $\omega = e_B$ for some $B \subseteq E$. Then

$$0 \leq G(\alpha + e_B) - G(\alpha) = g(\alpha + e_B) - \omega \cdot (\alpha + e_B) - g(\alpha) + \omega \cdot \alpha = r_\alpha(B) - |B|,$$

so that $r_\alpha(B) = |B|$. Moreover,

$$0 \leq G(\alpha - e_{\overline{B}}) - G(\alpha) = g(\alpha - e_{\overline{B}}) - \omega \cdot (\alpha - e_{\overline{B}}) - g(\alpha) + \omega \cdot \alpha = -r_{\alpha - e_{\overline{B}}}(\overline{B}),$$

so that $r_{\alpha - e_{\overline{B}}}(\overline{B}) = 0$. It follows by Lemma 3.5 that

$$|B| = r_\alpha(B) = r(\mathcal{M}_\alpha \backslash \overline{B}) = r(\mathcal{M}_{\alpha - e_{\overline{B}}} / \overline{B}) = d - r_{\alpha - e_{\overline{B}}}(\overline{B}) = d,$$

and hence that B is a basis of $\mathcal{M}_\alpha$.

We now show that (2) implies (1). Suppose $\omega = e_B$ for some basis B of $\mathcal{M}_\alpha$. Consider again the function $G : \alpha \mapsto g(\alpha) - \omega \cdot \alpha$ over $\mathbb{Z}^E$. As g is L-convex, G is L-convex. We show that α minimizes G over $\mathbb{Z}^E$, using the optimality condition for L-convex functions given in Lemma 2.37. First, note that as $g(\alpha + \mathbf{1}) = g(\alpha) + d$, we have

$$G(\alpha + \mathbf{1}) = g(\alpha + \mathbf{1}) - \omega \cdot (\alpha + \mathbf{1}) = g(\alpha) + d - \omega \cdot \alpha - |B| = G(\alpha).$$

Let $I \subseteq E$. As B is a basis of $\mathcal{M}_\alpha$, we have $|B \cap I| \leq r_\alpha(I)$, and hence

$$G(\alpha + e_I) - G(\alpha) = g(\alpha + e_I) - \omega \cdot (\alpha + e_I) - g(\alpha) - \omega \cdot \alpha = r_\alpha(I) - |B \cap I| \geq 0.$$

Thus α minimizes G over $\mathbb{Z}^E$, hence $f(\omega) = \sup\{-G(\alpha) : \alpha \in \mathbb{Z}^E\} = \omega \cdot \alpha - g(\alpha)$, as required. ☐

Let $\mathcal{M}$ be a matroid flock on E of rank d. We define the function $\nu^{\mathcal{M}} : \binom{E}{d} \to \mathbb{Z}_\infty$ by setting $\nu^{\mathcal{M}}(B) := f(e_B)$ for each $B \in \binom{E}{r}$, where $f = g^\bullet$ is the Lagrange-Fenchel dual of $g = g^{\mathcal{M}}$.

Lemma 3.11. *Let $\mathcal{M}$ be a matroid flock, and let $\nu = \nu^{\mathcal{M}}$. Then ν is a valuation, and $\mathcal{M}^\nu_\alpha = \mathcal{M}_\alpha$ for all $\alpha \in \mathbb{Z}^E$.*

Proof. Suppose $\mathcal{M}$ is a matroid flock. Then $g = g^{\mathcal{M}}$ is L-convex by Theorem 3.9, and $f := g^\bullet$ is M-convex by Theorem 2.36. That $\nu : B \mapsto f(e_B)$ is a matroid valuation is straightforward from the fact that f is M-convex. We show that $\mathcal{M}^\nu_\alpha = \mathcal{M}_\alpha$ for all $\alpha \in \mathbb{Z}^E$. By Theorem 2.36, we have $g = f^\bullet$. By Lemma 3.10, we have

$$g(\alpha) = f^\bullet(\alpha) = \sup\{\omega \cdot \alpha - f(\omega) : \omega \in \mathbb{Z}^E\}$$

$$= \sup\left\{e_{B'} \cdot \alpha - \nu(B') : B' \in \binom{E}{r}\right\},$$

as the first supremum is attained by ω only if $\omega = e_{B'}$ for some $B' \in \binom{E}{r}$. Again by Lemma 3.10, B is a basis of $\mathcal{M}_\alpha$ if and only if $g(\alpha) = e_B \cdot \alpha - \nu(B)$, i.e. if B is a basis of $\mathcal{M}^\nu_\alpha$. ☐

This proves the implication (1)⇒(2) of Theorem 3.3. Finally, we note:

Lemma 3.12. *Let $\mathcal{M}$ be a matroid flock, and let $\nu = \nu^{\mathcal{M}}$. Then $g^{\mathcal{M}} = g^\nu$.*

Proof. Let $f : \mathbb{Z}^E \to \mathbb{Z}_\infty$ be defined by $f(e_B) = \nu(B)$ for all $B \in \binom{E}{d}$, and $= \infty$ otherwise. Then $g^{\mathcal{M}} = f^\bullet = g^\nu$, as required. ☐

3. Polyhedral complexes associated to matroid flocks

3.1. The support matroid and the cells of a matroid flock. If $\mathcal{M}$ is a matroid flock, then the *support matroid of* $\mathcal{M}$, denoted by $M(\mathcal{M})$, is just the support matroid M^ν of the associated valuation $\nu = \nu^{\mathcal{M}}$.

Lemma 3.13. *Suppose* $\mathcal{M} : \alpha \mapsto \mathcal{M}_\alpha = (E, \mathcal{B}_\alpha)$ *is a matroid flock with* $M(\mathcal{M}) = (E, \mathcal{B})$*. Then* $\mathcal{B} = \bigcup_{\alpha \in \mathbb{Z}^E} \mathcal{B}_\alpha$*.*

Proof. Let $\nu = \nu^{\mathcal{M}}$, and put $g = g^\nu$. By Lemma 3.11, we have

$$\mathcal{B}_\alpha = \mathcal{B}^\nu_\alpha = \left\{ B \in \binom{E}{d} : e_B \cdot \alpha - \nu(B) = g(\alpha) \right\}$$

for all $\alpha \in \mathbb{Z}^E$, and $\mathcal{B} = \{B \in \binom{E}{d} : \nu(B) < \infty\}$. Since $\nu(B') < \infty$ for some $B' \in \binom{E}{d}$ by (V0), we have $g(\alpha) > -\infty$ for all $\alpha \in \mathbb{Z}^E$. Consider a $B \in \binom{E}{d}$.

Suppose first that $B \in \mathcal{B}$, i.e. $\nu(B) < \infty$. Consider the difference $h(\alpha) := g(\alpha) - e_B \cdot \alpha + \nu(B)$. Then $h(\alpha)$ is nonnegative and finite for all $\alpha \in \mathbb{Z}^E$, and $B \in \mathcal{B}_\alpha$ if and only if $h(\alpha) = 0$. Moreover, if B is not a basis of $\mathcal{M}^\nu_\alpha$, then $g(\alpha + e_B) \leq g(\alpha) + |B| - 1$ and $e_B \cdot (\alpha + e_B) = e_B \cdot \alpha + |B|$, so that $h(\alpha + e_B) \leq h(\alpha) - 1$. It follows that for any fixed α and any sufficiently large $k \in \mathbb{Z}$, we have $h(\alpha + ke_B) = 0$, and then $B \in \mathcal{B}_{\alpha + ke_B}$. Then $B \in \bigcup_{\alpha \in \mathbb{Z}^E} \mathcal{B}_\alpha$.

If on the other hand $B \notin \mathcal{B}^\nu$, i.e. $\nu(B) = \infty$, then $e_B \cdot \alpha - \nu(B) = -\infty < g(\alpha)$ for all $\alpha \in \mathbb{Z}^E$, so that $B \notin \mathcal{B}_\alpha$ for any $\alpha \in \mathbb{Z}^E$. Then $B \notin \bigcup_{\alpha \in \mathbb{Z}^E} \mathcal{B}_\alpha$. □

The geometry of valuations is quite intricate, and is studied in much greater detail in tropical geometry [**49, 19**]. We mention only those results we need in this thesis. For any matroid valuation $\nu : \binom{E}{d} \to \mathbb{R}_\infty$, put $C^\nu_\beta := \{\alpha \in \mathbb{R}^E : \mathcal{B}^\nu_\alpha \supseteq \mathcal{B}^\nu_\beta\}$.

Lemma 3.14. *Let* $\nu : \binom{E}{d} \to \mathbb{Z}_\infty$ *be a matroid valuation, and let* $\beta \in \mathbb{R}^E$*. Then*

$$\begin{aligned} C^\nu_\beta = \quad & \{\alpha \in \mathbb{R}^E : \alpha_i - \alpha_j \geq \nu(B) - \nu(B') \\ & \textit{for all } B \in \mathcal{B}^\nu_\beta,\ B' \in \mathcal{B}^\nu \textit{ s.t. } B' = B - i + j\}. \end{aligned}$$

Proof. Let C denote the right-hand side polyhedron in the statement of the lemma. Directly from the definition of $\mathcal{B}^\nu_\alpha$, it follows that $\mathcal{B}^\nu_\alpha \supseteq \mathcal{B}^\nu_\beta$ if and only if

$$e_B \cdot \alpha - \nu(B) \geq e_{B'} \cdot \alpha - \nu(B')$$

for all $B \in \mathcal{B}^\nu_\beta$ and $B' \in \mathcal{B}^\nu$. In particular, $C^\nu_\beta \subseteq C$.

To see that $C^\nu_\beta \supseteq C$, suppose that $\alpha \notin C^\nu_\beta$, that is, $\mathcal{B}^\nu_\alpha \not\supseteq \mathcal{B}^\nu_\beta$, so that

$$e_B \cdot \alpha - \nu(B) < e_{B'} \cdot \alpha - \nu(B')$$

for some $B \in \mathcal{B}^\nu_\beta$ and $B' \in \mathcal{B}^\nu$. Consider the valuation $\nu' : B \mapsto \nu(B) - e_B \cdot \alpha$. Pick $B \in \mathcal{B}^\nu_\beta, B' \in \mathcal{B}^\nu$ such that $\nu'(B) > \nu'(B')$ with $B \backslash B'$ as small as possible. If $|B\backslash B'| > 1$, then by minimality of $|B\backslash B'|$ we have

$$\nu'(B)+\nu'(B) > \nu'(B)+\nu'(B') \geq \nu'(B-i+j)+\nu'(B'+i-j) \geq \nu'(B)+\nu'(B),$$

for some $i \in B\backslash B'$ and $j \in B'\backslash B$, since ν' is a valuation. This is a contradiction, so $|B\backslash B'| = 1$ and $B' = B - i + j$, and hence

$$\alpha_i - \alpha_j = (e_B - e_{B'}) \cdot \alpha < \nu(B) - \nu(B'),$$

so that $\alpha \notin C$. □

Thus the cells C^ν_β are 'alcoved polytopes' (see [**28**]). The relative interior of such cells is connected also in a discrete sense.

Lemma 3.15. *Let $\mathcal{M} : \alpha \mapsto \mathcal{M}_\alpha$ be a matroid flock on E, and let $\alpha, \beta \in \mathbb{Z}^E$. If $\mathcal{M}_\alpha = \mathcal{M}_\beta$, then there is a walk $\gamma^0, \ldots, \gamma^k \in \mathbb{Z}^E$ from $\alpha = \gamma^0$ to $\beta = \gamma^k$ so that $\mathcal{M}_{\gamma^i} = \mathcal{M}_\alpha$ for $i = 0, \ldots, k$, and for each i there is a J_i so that $\gamma^i - \gamma^{i-1} = \pm e_{J_i}$*

Proof. By (MF2), there is such a walk from α to $\alpha + k\mathbf{1}$ for any $k \in \mathbb{Z}$, taking steps of the form $\pm\mathbf{1}$. Fixing any $i_0 \in E$, we may assume that $\alpha_{i_0} = 0$, and similarly that $\beta_{i_0} = 0$.

Let $\nu = \nu^{\mathcal{M}}$. Using Lemma 3.14, we have $\{\gamma \in \mathbb{R}^E : \mathcal{M}_\gamma = \mathcal{M}_\alpha\} = (C^\nu_\alpha)^\circ$, where $(C^\nu_\alpha)^\circ$ denotes the relative interior of C^ν_α. For each i, j let $c_{ij} := \min\{\alpha_i - \alpha_j, \beta_i - \beta_j\}$. Then by inspection of the system of inequalities which defines C^ν_α (Lemma 3.14), we have

$$\alpha, \beta \in C := \{\gamma \in \mathbb{R}^E : \gamma_i - \gamma_j \geq c_{ij} \text{ for all } i, j, \text{ and } \gamma_{i_0} = 0\}$$

$$\subseteq \{\gamma \in \mathbb{R}^E : \mathcal{M}_\gamma = \mathcal{M}_\alpha\}.$$

Then C is a bounded polyhedron defined by a totally unimodular system of inequalities with integer constant terms c_{ij}. It follows that C is an integral polytope. Moreover α, β are both vertices of C, and hence there is a walk from α to β over the 1-skeleton of C. Since C has integer vertices, and each edge of C is parallel to e_J for some $J \subseteq E$, the lemma follows. □

Lemma 3.16. *Let ν be a matroid valuation, and let $\alpha \in \mathbb{R}^E$ be given. Then*

$$\langle \beta - \alpha \mid \beta \in C^\nu_\alpha \rangle = \langle e_J \mid J \text{ component of } \mathcal{M}^\nu_\alpha \rangle.$$

Moreover, $\dim C^\nu_\alpha$ equals the number of components of $\mathcal{M}^\nu_\alpha$.

Proof. For the inclusion '$\subseteq$'. Suppose $\beta \in C^\nu_\alpha$. Then for all $B, B' \in \mathcal{B}^\nu_\alpha$, we have $e_B \cdot \alpha - \nu(B) = e_{B'} \cdot \alpha - \nu(B')$ and $e_B \cdot \beta - \nu(B) = e_{B'} \cdot \beta - \nu(B')$. When we consider the difference, we get that for all $B, B' \in \mathcal{B}^\nu_\alpha$, we have

$$e_B \cdot (\beta - \alpha) = e_{B'} \cdot (\beta - \alpha). \tag{3}$$

Let J be a component of $\mathcal{M}^\nu_\alpha$. We show that $\beta - \alpha$ is constant on J by induction. Suppose $\beta - \alpha$ is constant on I such that $\emptyset \neq I \subsetneq J$, which is clearly true when I is a singleton. Then for each $I \subsetneq J$, due to Lemma 2.2 there exists a pair of bases $B, B' \in \mathcal{B}^\nu_\alpha$ such that $|B \cap I| \neq |B' \cap I|$. As the base exchange graph of any matroid is connected, we may assume $B' = B - e + f$ for some $e \in I, f \in J \setminus I$. Hence $(\beta - \alpha)_e = (\beta - \alpha)_f$ due to (3), and so $\beta - \alpha$ is constant on $I + f$. Thus by induction $\beta - \alpha$ is constant on J, proving the inclusion '$\subseteq$'.

For the inclusion '$\supseteq$', let J be a component of M^ν_α. By Lemma 2.2, for all $B, B' \in \mathcal{B}^\nu_\alpha$, $|B \cap J| = |B' \cap J|$. Hence for all $\varepsilon > 0$ and $B, B' \in \mathcal{B}^\nu_\alpha$, we have $e_B \cdot \alpha - \nu(B) = e_{B'} \cdot \alpha - \nu(B')$ if and only if $e_B \cdot (\alpha + \varepsilon e_J) - \nu(B) = e_{B'} \cdot (\alpha + \varepsilon e_J) - \nu(B')$. Denote $h^\nu(\alpha) := \sup_{B \in \mathcal{B}^\nu \setminus \mathcal{B}^\nu_\alpha} \{e_B \cdot \alpha - \nu(B)\}$, the second highest value taken by $e_B \cdot \alpha - \nu(B)$ on $\mathcal{B}^\nu$. Picking $\varepsilon \leq \frac{g^\nu(\alpha) - h^\nu(\alpha)}{r^\nu(J)}$ thus ensures that $\mathcal{B}_\alpha \subseteq \mathcal{B}_{\alpha + \varepsilon e_J}$. Hence $\alpha + \varepsilon e_J \in C^\nu_\alpha$, as required.

Finally, taking the dimension on both sides we get that $\dim C^\nu_\alpha$ equals the number of components of $\mathcal{M}^\nu_\alpha$. $\square$

As a consequence, the cells $C^\nu_B := \{\alpha \in \mathbb{R}^E \mid B \in \mathcal{B}^\nu_\alpha\}$ for $B \in \mathcal{B}^\nu$ are the $|E|$-dimensional cells, since in its interior, where $\mathcal{B}^\nu_\alpha = \{B\}$, all elements of M^ν_α are loops or coloops.

Let $\mathcal{D}^\nu := \{C^\nu_\alpha \mid \alpha \in \mathbb{R}^E\} \cup \{\emptyset\}$.

Theorem 3.17. *Let ν be a matroid valuation. Then $\mathcal{D}^\nu$ is a polyhedral complex.*

Proof. First note that $\mathcal{D}^\nu$ is finite, since each cell C^ν_α uniquely determined by $\mathcal{B}^\nu_\alpha$, and $\mathcal{B}^\nu$ only has finitely many subsets. By Lemma 3.14, each $C \in \mathcal{D}^\nu$ is a polyhedron.

Now we show that for any $C, C' \in \mathcal{D}^\nu$, $C \cap C'$ is a face of both C and C'. Consider $\alpha, \beta \in \mathbb{Z}^E$ such that $C = C^\nu_\alpha$ and $C' = C^\nu_\beta$. We claim

$$C \cap C' = C \cap A,$$

where

$$\begin{aligned} A := \quad & \{\gamma \in \mathbb{R}^E : \gamma_i - \gamma_j = \nu(B) - \nu(B') \\ & \text{for all } B \in \mathcal{B}^\nu_\alpha, B' \in \mathcal{B}^\nu_\beta \text{ s.t. } B' = B - i + j\}. \end{aligned}$$

The inclusion '$\subseteq$' is straightforward from Lemma 3.14. For the inclusion '$\supseteq$', let γ be an element of the right-hand side. Clearly $\gamma \in C$. To see $\gamma \in C'$, we check that all inequalities for C^ν_β in Lemma 3.14 are satisfied. Thus let $B \in \mathcal{B}^\nu_\beta, B' \in \mathcal{B}^\nu$ be given such that $B' = B - i + j$. We need to show that

$$\gamma_i - \gamma_j \geq \nu(B) - \nu(B').$$

Pick $B'' \in \mathcal{B}^\nu_\alpha$. Since $\gamma \in C$, we have

$$(e_{B''} - e_{B'}) \cdot \gamma \geq \nu(B'') - \nu(B').$$

Furthermore as $\gamma \in A$, we have

$$(e_{B''} - e_B) \cdot \gamma = \nu(B'') - \nu(B).$$

Subtracting these equations we obtain the desired inequality. Hence $C \cap C'$ is a face of C. By symmetry, $C \cap C'$ is a face of both C and C'.

Finally we show that for each $C \in \mathcal{D}^\nu$, each face F of C is in D^ν. Let α be such that $C = C^\nu_\alpha$. As F is a face of C, there exist $\mathcal{S} \subseteq \mathcal{B}^\nu_\alpha$ and $\mathcal{T} \subseteq \mathcal{B}^\nu \setminus \mathcal{B}^\nu_\alpha$ such that each $B \in \mathcal{T}$ has a neighbor in S, and

$$\begin{aligned} F = C \cap \quad & \{\gamma \in \mathbb{R}^E : \gamma_i - \gamma_j = \nu(B) - \nu(B') \\ & \text{for all } B \in \mathcal{S}, B' \in \mathcal{T} \text{ s.t. } B' = B - i + j\}, \end{aligned}$$

due to Lemma 3.14. We want to show

$$\begin{aligned} F = \quad & \{\gamma \in \mathbb{R}^E : \gamma_i - \gamma_j \geq \nu(B) - \nu(B') \\ & \text{for all } B \in \mathcal{B}^\nu_\alpha \cup \mathcal{T}, B' \in \mathcal{B}^\nu \text{ s.t. } B' = B - i + j\}. \end{aligned} \tag{4}$$

The inclusion '$\supseteq$' holds since the set of inequalities for F is a subset of the inequalities on the right-hand side. We now prove the inclusion '$\subseteq$'. Let $\gamma \in F$ be given. Now suppose $B \in \mathcal{B}^\nu_\alpha \cup \mathcal{T}$ and $B' \in \mathcal{B}^\nu$ such that $B' = B - i + j$. So we must show that $\gamma_i - \gamma_j \geq \nu(B) - \nu(B')$. As $\gamma \in C$, this clearly holds if $B \in \mathcal{B}^\alpha$. So suppose $B \in \mathcal{T}$. Then there exists $B'' \in \mathcal{S}$ neighboring to B, when by assumption

$$(e_{B''} - e_B) \cdot \gamma = \nu(B'') - \nu(B).$$

As $\gamma \in C$ and $B'' \in \mathcal{B}^\nu_\alpha$, we also have

$$(e_{B''} - e_{B'}) \cdot \gamma \geq \nu(B'') - \nu(B').$$

Hence indeed, by subtracting these equations we obtain $\gamma_i - \gamma_j \geq \nu(B) - \nu(B')$. So (4) holds, and hence

$$F = \bigcap_{B \in \mathcal{B}^\nu_\alpha \cup \mathcal{T}} C^\nu_B,$$

which is a cell in $\mathcal{D}^\nu$, as required.

We conclude that $\mathcal{D}^\nu$ is a polyhedral complex. □

We remark that a matroid valuation ν is uniquely determined (up to tropical scalar multiplication) by its polyhedral complex $\mathcal{D}^\nu$. Hence there is a three-way relation between matroid flocks, matroid valuations and polyhedral complexes. If $\mathcal{M}$ is a matroid flock such that $\mathcal{M} = \mathcal{M}^\nu$, then we call $\mathcal{D}^\nu$ the polyhedral complex of $\mathcal{M}$. In this case, we will use the notation $\mathcal{S}^k(\mathcal{M}) := \mathcal{S}^k(\mathcal{D}^\nu)$ for the k-skeleton of the polyhedral complex of $\mathcal{M}$. Similarly, we use the notation $\Lambda(\mathcal{M}) := \Lambda(\mathcal{D}^\nu)$ for the lineality space.

In the following lemma we characterize the set of components of $\mathcal{M}_\alpha$ for $\alpha \in \mathbb{Z}^E$.

Lemma 3.18. *Let $\mathcal{M}$ be a matroid flock. Let $\alpha \in \mathbb{Z}^E$ be given. Then the set of components of $\mathcal{M}_\alpha$ is a refinement of the set of components of $M(\mathcal{M})$.*

PROOF. Let J be a component of $M(\mathcal{M})$. Note that $r_\alpha(S) \leq r(S)$ for any $S \subseteq E$, since each basis of $\mathcal{M}_\alpha$ is a basis of $M(\mathcal{M})$. So comparing connectivity functions, we get

$$\lambda_\alpha(J) = r_\alpha(J) + r_\alpha(\overline{J}) - r_\alpha(E) \leq r(J) + r(\overline{J}) - r(E) = \lambda(J) = 0,$$

so $\lambda_\alpha(J) = 0$. Hence J is a union of components of $\mathcal{M}_\alpha$, as required. $\square$

Next, we characterize the lineality space of the polyhedral complex of $\mathcal{M}$.

Lemma 3.19. *Let $\mathcal{M}$ be a matroid flock. Then $\Lambda(\mathcal{M}) = \langle e_{J_1}, \ldots, e_{J_n} \rangle$, where $J_1, \ldots, J_n$ are the components of $M(\mathcal{M})$.*

PROOF. Let ν be the valuation such that $\mathcal{M} = \mathcal{M}^\nu$ by Theorem 3.3. Suppose $y \in \Lambda(\mathcal{M})$ is not constant on the components of $M := M(\mathcal{M})$. Let $i, j \in J$ such that $y_i < y_j$ be given for some component J of M. As i, j are in the same component of M, there exist bases B, B' of M such that $B' = B - i + j$. Consider the cell $C_B^\nu \in \mathcal{D}^\nu$ and let $\alpha \in C_B^\nu$ be given. By Lemma 3.14, we have $\alpha_i - \alpha_j \geq \nu(B) - \nu(B')$. But then since $y_i - y_j < 0$, we have that for t large enough, $\alpha + ty \notin C_B^\nu$. Hence $y \notin \Lambda(C_B^\nu)$; contradiction.

Conversely, let J be a component of M. Then by Lemma 2.2 each basis of M intersects J in $r(J)$ elements, and hence the same is true for each $\mathcal{M}_\alpha^\nu$, whose basis set is contained in that of M. Therefore $B \in \mathcal{B}_\alpha^\nu$ if and only if $B \in \mathcal{B}_{\alpha + te_J}^\nu$ for each $\alpha \in \mathbb{R}^E$ and $t \in R$. Hence $e_J \in \Lambda(\mathcal{M})$, as required. $\square$

Next we show that the boundary of any cell C_α of the polyhedral complex of $\mathcal{M}$ is connected in a special way.

Lemma 3.20. *Let $\mathcal{M}$ be a matroid flock. Let $\alpha \in \mathbb{R}^E$. Then for each pair $\beta, \beta' \in C_\alpha \cap |\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E$, there is a sequence $\beta = \gamma_0, \ldots, \gamma_m = \beta'$ such that for each $0 \leq i < m$:*

(1) *$\gamma_i \in |\mathcal{S}_0(\mathcal{M})|$;*

(2) *there exist* $k_i \in \mathbb{Z}$ *and* $I \subseteq E$ *such that* $\gamma_{i+1} = \gamma_i + k_i e_I$;
(3) *the line segment in* $\mathbb{R}^E$ $[\gamma_i, \gamma_{i+1}] \subseteq C_\alpha \cap |\mathcal{S}_1(\mathcal{M})|$.

Proof. Since the cell C_α is a polyhedron, its boundary is connected in the sense that for each pair $\beta, \beta' \in C_\alpha \cap |\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E$ there is a sequence of cells $\beta \in D_0, D_1, \dots, D_{2k} \ni \beta'$ where $D_{2i} \in \mathcal{S}_0(\mathcal{M})$ for all $0 \le i \le k$, and $D_{2i}, D_{2i+2} \subset D_{2i+1} \in \mathcal{S}_1(\mathcal{M}) \setminus \mathcal{S}_0(\mathcal{M})$ for all $0 \le i < k$. We will now construct a sequence $\beta = \gamma_0, \dots, \gamma_m = \beta'$, where $\gamma_i \in D_{2i}$ for all $0 \le i \le k$ and $\gamma_j \in D_{2k}$ for all $j \ge k$.

For a cell D, denote $M(D) := \mathcal{M}_\beta$ where β is such that $D = C_\beta$. Due to Lemma 3.18, for all $D \in \mathcal{S}_0(\mathcal{M})$, the set of components of $M(D)$ is equal to the set of components of $M(\mathcal{M})$. Let n be the size of this set of components. Let $0 \le i < k$ be given, and suppose $\gamma_i \in D_{2i}$. Then D_{2i+1} is $(n+1)$-dimensional and D_{2i} is n-dimensional, so due to Lemma 3.16 there exists a component I of $M(D_{2i+1})$ that is not a component of $M(\mathcal{M}) = M(D_{2i}) = M(D_{2i+2})$. Hence D_{2i+1} is bounded in both directions e_I and $-e_I$, and D_{2i} and D_{2i+2} are the cells at the boundaries. Thus since D_{2i+1} is an integral polyhedron, there exists $k_i \in \mathbb{Z}$ such that $\gamma_{i+1} := \gamma_i + k_i e_I \in D_{2i+2}$. By convexity of D_{2i+1}, the line segment $[\gamma_i, \gamma_{i+1}]$ then lies completely inside D_{2i+1}.

Now γ_k and β' both lie in D_{2k}. Thus by Lemma 3.16, $\beta' - \gamma_k = \sum_{i=1}^n k_i e_{J_i}$ where $J_1, \dots, J_n$ are the components of $M(D_{2k})$. So we may put $\gamma_{k+t} := \sum_{i=1}^t k_i e_{J_i}$ for all $1 \le t \le n$, when $\gamma_{k+t+1} = \gamma_{k+t} + k_t e_{J_t}$ and $\gamma_{k+n} = \gamma_m = \beta'$. ☐

We call the elements of $|\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E$ the *central points* of $\mathcal{M}$.

4. Matroid flock translation

If ν is a valuation of M, then adding a trivial valuation τ of M to ν yields a valuation due to Lemma 2.15. The effect of adding a trivial valuation τ on the matroid flock $\mathcal{M}^\nu$ is translation by τ.

Lemma 3.21. *Let M be a matroid on E. Let ν be a valuation of M. Let $\gamma \in \mathbb{R}^E$ and let τ be the trivial valuation of M defined by $B \mapsto e_B \cdot \gamma$. Then $\mathcal{M}^\nu_\alpha = \mathcal{M}^{\nu+\tau}_{\alpha+\gamma}$ for all $\alpha \in \mathbb{R}^E$.*

Proof. We have

$$\mathcal{B}^\nu_\alpha = \operatorname*{arg\,max}_{B \in \mathcal{B}(M)} \{e_B \cdot \alpha - \nu(B)\} = \operatorname*{arg\,max}_{B \in \mathcal{B}(M)} \{e_B \cdot (\alpha + \gamma) - (\nu(B) + e_B \cdot \gamma)\} = \mathcal{B}^{\nu+\tau}_{\alpha+\gamma},$$

which is well-defined as $\nu + \tau$ is a valuation of M due to Lemma 2.15. ☐

We next characterize when two valuations determine the same matroid flock.

Theorem 3.22. *Let M be a matroid on E. Let ν, ν' be valuations of M. Then the following are equivalent:*

(1) $\mathcal{M}^{\nu} = \mathcal{M}^{\nu'}$;
(2) $\nu \cong \nu'$.

Proof. We first show the implication (2) $\Rightarrow$ (1). As $\nu \cong \nu'$, there exists $k \in \mathbb{R}$ such that $\nu' = \nu + k$. If d is the rank of M, let $\gamma := \frac{k}{d}\mathbf{1}$. Then for all bases B of M we have $e_B \cdot \gamma = k$, and hence $\nu'(B) - \nu(B) = e_B \cdot \gamma$. Thus for each $\alpha \in \mathbb{R}^E$ we have $\mathcal{B}^{\nu}_{\alpha} = \mathcal{B}^{\nu}_{\alpha-\gamma}$. By Lemma 3.21, for each $\alpha \in \mathbb{R}^E$ we now have $\mathcal{M}^{\nu}_{\alpha} = \mathcal{M}^{\nu}_{\alpha-\gamma} = \mathcal{M}^{\nu'}_{\alpha}$, proving the implication (2) $\Rightarrow$ (1).

Conversely, let B, B' be neighboring bases. That is, $B' = B - i + j$ for some $i, j \in E$. Then there exists $\alpha \in \mathbb{R}^E$ such that $\{B, B'\} \subseteq \mathcal{B}^{\nu}_{\alpha} = \mathcal{B}^{\nu'}_{\alpha}$. By Lemma 3.14, $\alpha_i - \alpha_j = \nu(B) - \nu(B') = \nu'(B) - \nu'(B')$. Hence since the basis exchange graph of a matroid is connected, for all bases B, B' we have

$$\nu'(B) - \nu(B) = \nu'(B') - \nu(B'),$$

proving the implication (1) $\Rightarrow$ (2). $\square$

Due to the above theorem, we may define $\mathcal{M}^{\nu+\mathbb{R}} := \mathcal{M}^{\nu}$ for any matroid valuation ν.

5. Matroid flock structure

In this section we will explore structural properties of matroid flocks.

5.1. Deletion and contraction. We define deletion and contraction for matroid flocks.

Definition 3.23. *(Matroid flock deletion) Let $\mathcal{M}$ be a matroid flock, and let $\nu : \binom{E}{d} \to \mathbb{Z}_{\infty}$ be a matroid valuation such that $\mathcal{M} = \mathcal{M}^{\nu}$. Let $I \subset E$. Then we define $\mathcal{M}\backslash I := \mathcal{M}^{\nu\backslash I}$.*

Definition 3.24. *(Matroid flock contraction) Let $\mathcal{M}$ be a matroid flock, and let $\nu : \binom{E}{d} \to \mathbb{Z}_{\infty}$ be a matroid valuation such that $\mathcal{M} = \mathcal{M}^{\nu}$. Let $I \subset E$. Then we define $\mathcal{M}/I := \mathcal{M}^{\nu/I}$.*

Each matroid flock is of the form $\mathcal{M}^{\nu}$ for some matroid valuation ν due to Theorem 3.3. Thus the following is a direct consequence of Lemma 2.19.

Lemma 3.25. *Let $\mathcal{M}$ be a matroid flock on E, and let $I, J \subset E$ be disjoint. The following commutation properties are satisfied:*

(1) $\mathcal{M}\backslash I\backslash J = \mathcal{M}\backslash J\backslash I$;
(2) $\mathcal{M}/I/J = \mathcal{M}/J/I$;
(3) $\mathcal{M}/I\backslash J = \mathcal{M}\backslash J/I$.

Lemma 3.26. *Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a valuation. Let $i \in E$ and $\alpha \in \mathbb{Z}^{E-i}$ be given. Then there exists $k_0 \in \mathbb{Z}$ such that for all $\alpha' \in \mathbb{Z}^E$ such that $\alpha'_{E-i} = \alpha$ and $\alpha'_i < k_0$: $(\mathcal{M}^\nu \backslash i)_\alpha = \mathcal{M}^\nu_{\alpha'} \backslash i$.*

Proof. If i is a coloop of M^ν, then i is a coloop in $\mathcal{M}^\nu_{\alpha'}$ for each α'. Now

$$\mathcal{B}^{\nu\backslash i}_\alpha = \mathcal{B}^{\nu/i}_\alpha = \underset{B \in \binom{E-i}{d-1}}{\arg\max}\{e_B \cdot \alpha - \nu/i(B)\} = \underset{B \in \binom{E-i}{d-1}}{\arg\max}\{e_{B+i} \cdot \alpha' - \nu(B+i)\}.$$

But this is just the basis set of $\mathcal{M}^\nu_{\alpha'}/i = \mathcal{M}^\nu_{\alpha'} \backslash i$ for any k, since i is in every basis of M^ν.

If, on the other hand, i is not a coloop, then we reason similarly:

$$\mathcal{B}^{\nu\backslash i}_\alpha = \underset{B \in \binom{E-i}{d}}{\arg\max}\{e_B \cdot \alpha - \nu\backslash i(B)\} = \underset{B \in \binom{E-i}{d}}{\arg\max}\{e_B \cdot \alpha - \nu(B)\}.$$

Now if $k = \alpha'_i$ is small enough, then for all $B \in \binom{E}{d}$ with $i \in B$ we have $e_B \cdot \alpha' - \nu(B) < \sup_{B' \in \binom{E}{d}}\{e_{B'} \cdot \alpha' - \nu(B')\}$, so that $B \notin \mathcal{B}^\nu_{\alpha'}$. On the other hand, if $i \notin B$, then $B \in \mathcal{B}^\nu_{\alpha'}$ if and only if $B \in \mathcal{B}^{\nu\backslash i}_\alpha$. Hence there exists $k_0 \in \mathbb{Z}$ so that for all $k < k_0$ we have $\mathcal{B}^{\nu\backslash i}_\alpha = \mathcal{B}^\nu_{\alpha'}$, and thus $(\mathcal{M}^\nu \backslash i)_\alpha = \mathcal{M}^\nu_{\alpha'} \backslash i$. □

We generalise the lemma. Let S be a set and $I \subseteq S$. We say $\alpha' \in \mathbb{Z}^S$ is a $(k, I, <)$-*expansion* of $\alpha \in \mathbb{Z}^{S-I}$ if $\alpha'_{S-I} = \alpha$ and $\alpha'_i < k$ for all $i \in I$.

Lemma 3.27. *Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a valuation. Let $I \subset E$ and $\alpha \in \mathbb{Z}^{E-I}$ be given. Then there exists k such that $(\mathcal{M}^\nu \backslash I)_\alpha = \mathcal{M}^\nu_{\alpha'} \backslash I$, for all $(k, I, <)$-expansions α' of α.*

Proof. By induction on $|I|$. The statement is trivial when $I = \emptyset$. Now suppose I is nonempty. Let $i \in I$ be given. Then $(\mathcal{M}^\nu \backslash I)_\alpha = \mathcal{M}^{\nu\backslash I}_\alpha = \mathcal{M}^{\nu\backslash I-i\backslash i}_\alpha$ by definition. By Lemma 3.26, for some $k'' \in \mathbb{Z}$ this equals $\mathcal{M}^{\nu\backslash I-i}_{\alpha''} \backslash i$ for all $(k'', i, <)$-expansions α'' of α. Then by the induction hypothesis, there exists $k' \in \mathbb{Z}$ so that this is equal to $\mathcal{M}^\nu_{\alpha'} \backslash I-i \backslash i$ for all $(k', I-i, <)$-expansions α' of α''. Choosing $k = \min\{k', k''\}$, we get that for all α' that are $(k, I, <)$-expansions of α, we have that $(\mathcal{M}^\nu \backslash I)_\alpha$ equals $\mathcal{M}^\nu_{\alpha'} \backslash I$, as required. □

A similar fact holds for contraction. Let S be a set and $I \subseteq S$. We say $\alpha' \in \mathbb{Z}^S$ is a $(k, I, >)$-*expansion* of $\alpha \in \mathbb{Z}^{S-I}$ if $\alpha'_{S-I} = \alpha$ and $\alpha'_i > k$ for all $i \in I$.

Lemma 3.28. *Let $\nu : \binom{E}{d} \to \mathbb{R}_\infty$ be a valuation. Let $I \subset E$ and $\alpha \in \mathbb{Z}^{E-I}$ be given. Then there exists k such that $(\mathcal{M}^\nu / I)_\alpha = \mathcal{M}^\nu_{\alpha'} / I$, for all $(k, I, >)$-expansions α' of α.*

We omit the proof as it is analogous to the proof of Lemma 3.27. Alternatively, the lemma follows from Lemma 3.27 using the notion of duality from the following subsection.

5.2. Duality. We define duality for matroid flocks.

Definition 3.29. *(Matroid flock duality) Let $\mathcal{M}$ be a matroid flock, and let ν be a valuation so that $\mathcal{M} = \mathcal{M}^{\nu}$. Then we define the dual matroid flock $\mathcal{M}^* := \mathcal{M}^{\nu^*}$.*

Lemma 3.30. *Let $\mathcal{M}$ be a matroid flock. Then $\mathcal{M}^*_\alpha = (\mathcal{M}_{-\alpha})^*$ for all $\alpha \in \mathbb{Z}^E$.*

Proof. Using Theorem 3.3, let ν be a valuation such that $\mathcal{M} = \mathcal{M}^{\nu}$. Rewriting the definition of $\mathcal{B}^{\nu^*}_\alpha$, we obtain

$$\begin{aligned}
\mathcal{B}^{\nu^*}_\alpha &= \operatorname*{arg\,max}_{B \in \binom{E}{|E|-d}} \{e_B \cdot \alpha - \nu^*(B)\} \\
&= \operatorname*{arg\,max}_{B \in \binom{E}{|E|-d}} \{(\mathbf{1} - e_{\overline{B}}) \cdot \alpha - \nu(\overline{B})\} \\
&= \operatorname*{arg\,max}_{B \in \binom{E}{|E|-d}} \{e_{\overline{B}} \cdot (-\alpha) - \nu(\overline{B})\}.
\end{aligned}$$

This is the set of cobases of $\mathcal{M}^{\nu}_{-\alpha} = \mathcal{M}_{-\alpha}$. The statement follows. □

5.3. Lines. In this subsection we investigate matroid flocks along lines $\alpha + \mathbb{Z}e_I$ for $\alpha \in \mathbb{Z}^E$ and $I \subseteq E$. The following lemma is a straightforward consequence of (MF1') and (MF2).

Lemma 3.31. *Let $\mathcal{M}$ be a matroid flock on E. Let $\alpha \in \mathbb{Z}^E$ and $I \subseteq E$ be given. Suppose $S \subseteq I \subseteq S'$ and $T \subseteq \overline{I} \subseteq T'$. Then:*

(1) $r_\alpha(S) \leq r_{\alpha+e_I}(S)$;
(2) $r_\alpha(S') \leq r_{\alpha+e_I}(S')$;
(3) $r_\alpha(T) \geq r_{\alpha+e_I}(T)$;
(4) $r_\alpha(T') \geq r_{\alpha+e_I}(T')$;

Lemma 3.32. *Let $\mathcal{M}$ be a matroid flock on E, let $I \subseteq E$ and let $\alpha \in \mathbb{Z}^E$. Then $\lambda_\alpha(I) = r_\alpha(I) - r_{\alpha-e_I}(I)$.*

Proof. By (MF1'), we have $\mathcal{M}_{\alpha-e_I}/I = \mathcal{M}_\alpha \backslash I$. The rank of the left-hand matroid is $d - r_{\alpha-e_I}(I)$, while the rank of the right-hand matroid is $r_\alpha(\overline{I})$. Thus we have $\lambda_\alpha(I) = r_\alpha(I) + r_\alpha(\overline{I}) - d = r_\alpha(I) - r_{\alpha-e_I}(I)$, as required. □

When M, M' are two matroids, we write $M \geq M'$ when for all $I \subseteq E$: $r^M(I) \geq r^{M'}(I)$. M' is then also said to be a weak image of M.

Lemma 3.33. *Let $\mathcal{M}$ be a matroid flock on E, let $\alpha \in \mathbb{Z}^E$ and let $I \subseteq E$. The following are equivalent.*

(1) $\mathcal{M}_\alpha \geq \mathcal{M}_{\alpha+e_I}$;
(2) $r_\alpha(I) = r_{\alpha+e_I}(I)$;
(3) $\lambda_{\alpha+e_I}(I) = 0$;
(4) $\mathcal{M}_{\alpha+e_I} = \mathcal{M}_\alpha/I \oplus \mathcal{M}_\alpha\backslash\overline{I}$.

Proof. (1) $\Rightarrow$ (2) is trivial using Lemma 3.31(1), and the implication (2) $\Rightarrow$ (3) is immediate from Lemma 3.32.

For the implication (3) $\Rightarrow$ (4), note that by (MF1') and (MF2), $\mathcal{M}_{\alpha+e_I}/\overline{I} = \mathcal{M}_{\alpha+\mathbf{1}}\backslash\overline{I} = \mathcal{M}_\alpha\backslash\overline{I}$. Similarly, by (MF1'), $\mathcal{M}_{\alpha+e_I}\backslash I = \mathcal{M}_\alpha/I$. By (3) we have $\mathcal{M}_{\alpha+e_I}/\overline{I} = \mathcal{M}_{\alpha+e_I}\backslash\overline{I}$ so that $\mathcal{M}_{\alpha+e_I}$ is indeed a direct sum as in (4).

For the implication (4) $\Rightarrow$ (1), let $J \subseteq E$ be given. Then $r_\alpha(J \cap I) = r_{\alpha+e_I}(J \cap I)$ and $r_\alpha(J) = r_\alpha(J \cap I) + r_\alpha(J \cap \overline{I})$ by (4). On the other hand, by Lemma 3.31, $r_\alpha(J \cap \overline{I}) \geq r_{\alpha+e_I}(J \cap \overline{I})$. It follows that $r_\alpha(J) \geq r_{\alpha+e_I}(J)$, proving (1). ☐

Theorem 3.34. *Let $\mathcal{M}$ be a matroid flock on E, let $I \subseteq E$, and let $\alpha \in \mathbb{Z}^E$. Then the set $S = \{k \in \mathbb{Z} : \lambda_{\alpha+ke_I}(I) > 0\}$ is finite. Moreover, $\lambda(I) = \sum_{k\in S} \lambda_{\alpha+ke_I}(I)$.*

Proof. By Lemma 3.27 and 3.28 respectively there exist integers k_0 and k_1 such that $r_{\alpha+k_0e_I}(I) = r^{\mathcal{M}\backslash\overline{I}}(I) = r(I)$ and $r_{\alpha+k_1e_I}(I) = r^{\mathcal{M}/\overline{I}}(I) = d - r(\overline{I})$ and that moreover, the difference $r_{\alpha+ke_I}(I) - r_{\alpha+(k-1)e_I}(I)$ can only be nonzero when k lies in the interval $(k_1, k_0]$. Hence S is finite. Thus

$$\begin{aligned} \lambda(I) &= r_{\alpha+k_0e_I}(I) - r_{\alpha+k_1e_I}(I) \\ &= \sum_{k=k_1+1}^{k_0} r_{\alpha+ke_I}(I) - r_{\alpha+(k-1)e_I}(I) = \sum_{k\in S} \lambda_{\alpha+ke_I}(I) \end{aligned}$$

using Lemma 3.32 for each k to obtain the last equation. ☐

Corollary 3.35. *Let $\mathcal{M}$ be a matroid flock on E. Let $I \subseteq E$ and let $\alpha \in \mathbb{Z}^E$. Then $\mathcal{M}_{\alpha+ke_I} = \mathcal{M}_{\alpha+(k+1)e_I}$ for all but finitely many $k \in \mathbb{Z}$.*

Proof. By Theorem 3.34, the set $S = \{k \in \mathbb{Z} : \lambda_{\alpha+ke_I}(I) > 0\}$ is finite. Hence the set T of consecutive pairs $k, k+1$ such that not both $\lambda_{\alpha+ke_I}$ and $\lambda_{\alpha+(k+1)e_I}$ are zero is finite. Only these pairs in T have $\mathcal{M}_{\alpha+ke_I} \neq \mathcal{M}_{\alpha+(k+1)e_I}$ due to Lemma 3.33. ☐

Remark 3.36. *Figure 1 illustrates a line in a matroid flock. A point α lies within a gray rectangle if $\lambda_\alpha(I) = 0$. Two points α and $\alpha + e_I$ are connected*

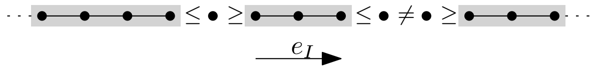


FIGURE 1. A line $\alpha + \mathbb{Z}e_I$ in a matroid flock.

by a line segment if $\mathcal{M}_\alpha = \mathcal{M}_{\alpha+e_I}$*. The relations inferred from Lemma 3.33 between other neighboring points are indicated on the line. By Corollary 3.35, the matroid will be equal for all points far enough to the right on the line. By Lemma 3.28, for such a point* β *we will have* $\mathcal{M}_\beta/I = (\mathcal{M}/I)_{\beta_{\overline{I}}}$*. Similarly, if* β *is far enough to the left, then* $\mathcal{M}_\beta\backslash I = (\mathcal{M}\backslash I)_{\beta_{\overline{I}}}$*.*

5.4. Direct sums. We define direct sums of matroid flocks.

Definition 3.37. *Let* $\mathcal{M}, \mathcal{M}'$ *be two matroid flocks on disjoint sets* E *and* E' *respectively. The* direct sum *of* $\mathcal{M}$ *and* $\mathcal{M}'$ *is the map* $\mathcal{M} \oplus \mathcal{M}' : \mathbb{Z}^{E\cup E'} \to \mathcal{M}_{E\cup E'}$ *given by* $\alpha \mapsto \mathcal{M}_{\alpha_E} \oplus \mathcal{M}_{\alpha_{E'}}$*.*

Theorem 3.38. *Let* $\mathcal{M}, \mathcal{M}'$ *be two matroid flocks on disjoint sets* E *and* E' *respectively. Then* $\mathcal{M} \oplus \mathcal{M}'$ *is a matroid flock.*

PROOF. Let $\alpha \in \mathbb{Z}^{E\cup E'}$ be given, and let $i \in E \cup E'$. Without loss of generality we may assume $i \in E$. By (MF1) for $\mathcal{M}$, $\mathcal{M}_{\alpha_E}/i = \mathcal{M}_{\alpha_E+e_i}\backslash i$. Since $\lambda_\alpha^{\mathcal{M}\oplus\mathcal{M}'}(E) = 0$, we get

$$(\mathcal{M} \oplus \mathcal{M}')_\alpha/i = \mathcal{M}_{\alpha_E}/i \oplus \mathcal{M}'_{\alpha'_E} = \mathcal{M}_{\alpha_E+e_i}\backslash i \oplus \mathcal{M}'_{\alpha'_E} = (\mathcal{M} \oplus \mathcal{M}')_{\alpha+e_i}\backslash i.$$

This proves (MF1) for $\mathcal{M} \oplus \mathcal{M}'$.

By (MF2) for $\mathcal{M}$ and $\mathcal{M}'$, we have

$$(\mathcal{M} \oplus \mathcal{M}')_\alpha = \mathcal{M}_{\alpha_E} \oplus \mathcal{M}_{\alpha_{E'}} = \mathcal{M}_{\mathbf{1}+\alpha_E} \oplus \mathcal{M}_{\mathbf{1}+\alpha_{E'}} = (\mathcal{M} \oplus \mathcal{M}')_{\mathbf{1}+\alpha},$$

proving (MF2). Hence $\mathcal{M} \oplus \mathcal{M}'$ is a matroid flock. □

Theorem 3.39. *Let* $\mathcal{M}$ *be a matroid flock on* E*, and let* $I \subseteq E$*. Let* M *be the matroid of* $\mathcal{M}$*. If* $\lambda^M(I) = 0$*, then* $\mathcal{M} = \mathcal{M}\backslash I \oplus \mathcal{M}\backslash\overline{I}$*.*

PROOF. Let $\alpha \in \mathbb{Z}^E$ be given. Then $\lambda_\alpha(I) = 0$. Hence for each $J \subseteq I$, $r_\alpha^{\mathcal{M}}(J) = r_{\alpha_I}^{\mathcal{M}\backslash\overline{I}}(J)$. Similarly for $J' \subseteq \overline{I}$, $r_\alpha^{\mathcal{M}}(J') = r_{\alpha_{\overline{I}}}^{\mathcal{M}\backslash I}(J')$. Thus $\mathcal{M}_\alpha = (\mathcal{M}\backslash I)_{\alpha_{\overline{I}}} \oplus (\mathcal{M}\backslash\overline{I})_{\alpha_I}$, as required. □

5.5. The $\boxdot$-operator. Let M be a matroid on E and $I \subseteq E$. Then define the left-associative operator $\boxdot$ by $M \boxdot I := M/I \oplus M\backslash\overline{I}$. The following are some basic properties of this operator.

Lemma 3.40. *Let* M *be a matroid on* E*, and let* $I \subseteq E$*. Then* $M \boxdot I \boxdot I = M \boxdot I$*.*

Lemma 3.41. *Let M be a matroid on E, and let $J \subseteq I \subseteq E$ be given. Then $M \boxdot I \boxdot J = M \boxdot J \boxdot I$.*

PROOF.

$$\begin{aligned} M \boxdot I \boxdot J &= (M/I \oplus M\backslash\overline{I}) \boxdot J \\ &= M/I \cup J \oplus M\backslash\overline{I}/J \oplus M/I\backslash\overline{J} \oplus M\backslash\overline{I} \cup \overline{J} \end{aligned}$$

Note that $I \cup \overline{J} = E$, so that the factor $M/I\backslash\overline{J}$ vanishes. Moreover, since J and $\overline{I}$ are disjoint, we have $M\backslash\overline{I}/J = M/J\backslash\overline{I}$, and thus

$$\begin{aligned} \ldots &= M/J \cup I \oplus M/J\backslash\overline{I} \oplus M\backslash\overline{J} \cup \overline{I} \\ &= M \boxdot J \boxdot I. \end{aligned}$$

□

Lemma 3.42. *Let M be a matroid on E, and let $I, J \subseteq E$ be disjoint. Suppose the connectivity of I in $M \boxdot I \cup J$ is zero. Then $M \boxdot I \boxdot J = M \boxdot I \cup J$.*

PROOF. Since I and $\overline{I \cup J}$ are disjoint, we have $M/I\backslash\overline{J} = M/I\backslash\overline{I \cup J} = M\backslash\overline{I \cup J}/I$. Now as the connectivity of I is zero in $M \boxdot I \cup J$, and hence in $M\backslash\overline{I \cup J}$, by Lemma 2.1 we get $M\backslash\overline{I \cup J}/I = M\backslash\overline{I \cup J}\backslash I = M\backslash\overline{J}$. Also note that $M\backslash\overline{I}/J = M\backslash\overline{I}$, since $J \subseteq \overline{I}$.

Thus we compute

$$\begin{aligned} M \boxdot I \boxdot J &= M/I \cup J \oplus M\backslash\overline{I}/J \oplus M/I\backslash\overline{J} \\ &= M/I \cup J \oplus M\backslash\overline{I} \oplus M\backslash\overline{J} \\ &= M/I \cup J \oplus M\backslash\overline{I \cup J} \\ &= M \boxdot I \cup J, \end{aligned}$$

where in the third equality we use that the connectivity of I is zero in $M \boxdot I \cup J$. □

For $\alpha \in \mathbb{Z}^E$, we define the left-associative $\boxdot$ operator inductively as follows.

$$M \boxdot \alpha := \begin{cases} M & \text{if } \alpha = k\mathbf{1} \\ M \boxdot (\alpha - e_I) \boxdot I & \text{otherwise, where } I = \arg\max_{i \in E}\{\alpha_i\} \end{cases}$$

We may derive a direct formula for $M \boxdot \alpha$.

Lemma 3.43. *Let M be a matroid on E, and let $\alpha \in \mathbb{Z}^E$. Write $\alpha = \sum_{i=1}^n c_i e_{I_i}$, where $\emptyset = I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_n = E$ and $c_i > 0$ for $i < n$. Then $M \boxdot \alpha = \bigoplus_{i=1}^n M\backslash\overline{I_i}/I_{i-1}$.*

Proof. By definition we have

$$M \boxdot \alpha = M(\boxdot I_{n-1})^{c_{n-1}} \boxdot \dots (\boxdot I_1)^{c_1}.$$

By Lemma 3.40, we may simplify this expression to

$$M \boxdot I_{n-1} \boxdot \dots \boxdot I_1.$$

Let $1 \leq i \leq n$ be given and define $I := I_i \setminus I_{i-1}$. Then $I \subseteq I_j$ for all $j \geq i$ and $I \subseteq \overline{I_l}$ for all $l < i$. Thus, restricting $M \boxdot \alpha$ to I, we get

$$M \backslash \overline{I_{n-1}} \backslash \dots \backslash \overline{I_i} / I_{i-1} / \dots / I_1.$$

This is exactly $M \backslash \overline{I_i} / I_{i-1}$, as required.

To see that $M \boxdot \alpha$ is a direct sum of these matroids, note that $\lambda^{M \boxdot \alpha}(I_j) = 0$ for all j, and hence also $\lambda^{M \boxdot \alpha}(I_i \setminus I_{i-1}) = 0$. $\square$

Lemma 3.44. *Let M be a matroid. Then $\mathcal{M} : \alpha \mapsto M \boxdot \alpha$ is a matroid flock.*

Proof. Denote by ν the zero valuation of M. We will show that $\mathcal{M}^\nu_\alpha = M \boxdot \alpha$. We have $\mathcal{B}^\nu_\alpha = \arg\max_{B \in \mathcal{B}(M)} \{e_B \cdot \alpha\}$. Hence if $\alpha = \sum_{i=1}^n c_i e_{I_i}$ with $\emptyset = I_0 \subsetneq I_1 \subsetneq \dots \subsetneq I_n = E$ and $c_i > 0$ for $i < n$, then $B \in \mathcal{B}^\nu_\alpha$ if and only if for all $i \leq n$, $|B \cap (I_i \setminus I_{i-1})| = r(I_i) - r(I_{i-1})$. Indeed, when the index i is smaller, $\alpha_{I_i \setminus I_{i-1}}$ is larger. But then $\mathcal{M}^\nu_\alpha|_{I_i \setminus I_{i-1}} = M \backslash \overline{I_i} / I_{i-1}$ and by Lemma 2.2, $\lambda^{M^\nu_\alpha}(I_i \setminus I_{i-1}) = 0$. So in total, $\mathcal{M}^\nu_\alpha = \bigoplus_{i=1}^n M \backslash \overline{I_i} / I_{i-1}$, which equals $M \boxdot \alpha$ due to Lemma 3.43. $\square$

Lemma 3.45. *Let $\mathcal{M}$ be a matroid flock. Suppose $\beta \in C_\alpha$. Write $\alpha - \beta = c\mathbf{1} + \sum_{i=1}^n e_{I_i}$, where $I_1 \subseteq \dots \subseteq I_n \subsetneq E$ and $c \in \mathbb{Z}^E$. Then*

$$\mathcal{M}_\beta \geq \mathcal{M}_{\beta + e_{I_1}} \geq \mathcal{M}_{\beta + e_{I_1 + I_2}} \geq \dots \geq \mathcal{M}_{\beta + \sum_{i=1}^n e_{I_i}} = \mathcal{M}_\alpha.$$

Proof. By (MF2), we have $\mathcal{M}_\beta = \mathcal{M}_{\beta + s\mathbf{1}}$ so that we may assume without loss of generality that $c = 0$. Let $\gamma(t) := \beta + \sum_{i=1}^t e_{I_i}$. As $\beta \in C_\alpha$, we have $\mathcal{M}_\beta \geq \mathcal{M}_\alpha$. Then by Lemma 3.7 we have for each t

$$r_\alpha(I_t) \geq r_{\gamma(t-1)}(I_t) \geq r_{\gamma(t)}(I_t) \geq r_\beta(I_t).$$

Since also $r_\beta(I_t) \geq r_\alpha(I_t)$ by choice of β, we have equality throughout, so that $r_{\gamma(t)}(I_t) = r_{\gamma(t-1)}(I_t)$ for each t, and by Lemma 3.33 that implies that $\mathcal{M}_{\gamma(t-1)} \geq \mathcal{M}_{\gamma(t)}$, as required. $\square$

Lemma 3.46. *Let $\mathcal{M}$ be a matroid flock. Let $\alpha \in \mathbb{Z}^E$ and $\beta \in C_\alpha$. Then $\mathcal{M}_\alpha = \mathcal{M}_\beta \boxdot (\alpha - \beta)$.*

Proof. Write $\alpha - \beta$ as in Lemma 3.45. By (MF2) we may assume $c = 0$. Let $\gamma(t) := \beta + \sum_{i=1}^{t} e_{I_i}$. For each $t < n$, $\mathcal{M}_{\gamma(t)} \geq \mathcal{M}_{\gamma(t+1)}$ due to Lemma 3.45. By Lemma 3.33(4), $\mathcal{M}_{\gamma(t+1)} = \mathcal{M}_{\gamma(t)} \boxdot I_{t+1}$. Hence by induction on t, $\mathcal{M}_\alpha = \mathcal{M}_{\gamma(n)} = \mathcal{M}_\beta \boxdot I_1 \boxdot \ldots \boxdot I_n$. By Lemma 3.41 this equals $\mathcal{M}_\beta \boxdot I_n \boxdot \ldots \boxdot I_1 = \mathcal{M}_\beta \boxdot (\alpha - \beta)$, as required. □

5.6. Circuit-hyperplanes. We state some results about matroid flocks related to circuit-hyperplanes.

Lemma 3.47. *Let M be a matroid, and let H be a circuit-hyperplane of M. Let ν be a valuation of M. Then there exists a valuation ν^H of M^H such that*

1. *if $\alpha \in |\mathcal{S}_0(\mathcal{M}^\nu)|$, then $\mathcal{M}^{\nu^H}_\alpha = \mathcal{M}^\nu_\alpha$;*
2. *if $H \notin \mathcal{B}^{\nu^H}_\alpha$, then $\mathcal{M}^{\nu^H}_\alpha = \mathcal{M}^\nu_\alpha$.*

Proof. Let C be the set of central points of $\mathcal{M}^\nu$. Pick

$$k > \sup\{(e_H - e_B) \cdot \alpha + \nu(B) : \alpha \in C, B \in \mathcal{B}\}.$$

Note that this supremum is finite, since $(e_H - e_B) \cdot \mathbf{1} = 0$ for each B, and there are only finitely many classes of central points. Put $\nu^H = \nu^H_k$ as in Lemma 2.21. Then ν^H is a valuation of M^H. Next, let α be a central point of $\mathcal{M}^\nu$. Then

$$e_H \cdot \alpha - \nu^H(H) < e_H \cdot \alpha - ((e_H - e_B) \cdot \alpha + \nu^H(B)) = e_B \cdot \alpha - \nu^H(B)$$

for all $B \in \mathcal{B}$. Thus $H \notin \mathcal{B}^{\nu^H}_\alpha$ and $\mathcal{B}^{\nu^H}_\alpha = \mathcal{B}^\nu_\alpha$, proving property (1).

For property (2), note that for any α such that $H \notin \mathcal{B}^{\nu^H}_\alpha$, we have

$$\mathcal{B}^{\nu^H}_\alpha = \operatorname*{arg\,max}_{B \in \mathcal{B}^{\nu^H}} \{e_B \cdot \alpha - \nu^H(B)\} = \operatorname*{arg\,max}_{B \in \mathcal{B}^{\nu^H} \setminus H} \{e_B \cdot \alpha - \nu^H(B)\}.$$

Since $\nu^H(B) = \nu(B)$ for all $B \neq H$, the latter equals $\mathcal{B}^\nu_\alpha$, as required. □

Lemma 3.48. *Let $\mathcal{M}$ be a matroid flock. Let H be a circuit-hyperplane of $M(\mathcal{M})$. Then there is a unique 2-dimensional cell $C \in \mathcal{S}^2(\mathcal{M})$ such that for any $\alpha \in C$, H is a circuit-hyperplane of $\mathcal{M}_\alpha$.*

Proof. Let $M = M(\mathcal{M})$. Since H is a circuit, $M \backslash \overline{H}$ is a cycle matroid. Thus let $\alpha \in \mathbb{Z}^H \cap |\mathcal{S}^1(\mathcal{M} \backslash \overline{H})|$. Note that α is unique up to adding multiples of e_H, since any matroid $N < M \backslash \overline{H}$ is disconnected. Then $(\mathcal{M} \backslash \overline{H})_\alpha = M \backslash \overline{H}$. Now there exists $k \in \mathbb{Z}$ such that

$$(\mathcal{M} \backslash \overline{H})_\alpha = \mathcal{M}_\beta \backslash \overline{H}$$

for all $(k, \overline{H}, <)$-expansions β of α by Lemma 3.27.

Similarly, since H is a hyperplane, M/H is a uniform matroid of rank 1. Thus let $\alpha' \in \mathbb{Z}^{\overline{H}} \cap |\mathcal{S}^1(\mathcal{M}/H)|$. Note that α' is unique up to adding multiples of $e_{\overline{H}}$, since any matroid $N < M/H$ is disconnected. Then $(\mathcal{M}/H)_{\alpha'} = M/H$. Again there exists $k' \in \mathbb{Z}$ such that

$$(\mathcal{M}/H)_{\alpha'} = \mathcal{M}_{\beta'}/H$$

for all $(k', H, >)$-expansions β' of α' by Lemma 3.28.

Let $\hat{\alpha} \in \mathbb{Z}^E$ be such that $\hat{\alpha}_H = \alpha$ and $\hat{\alpha}_{\overline{H}} = 0$ and let $\hat{\alpha}' \in \mathbb{Z}^E$ be such that $\hat{\alpha}'_H = 0$ and $\hat{\alpha}'_{\overline{H}} = \alpha'$. Now pick

$$l > \max\{k - \min_{i \in H} \alpha_i, k' + \max_{i \in \overline{H}} \alpha'_i\}$$

and let $\gamma := \hat{\alpha} + \hat{\alpha}' + l e_H$. Then γ is a $(k', H, >)$-expansion of α'. Hence $\mathcal{M}_\gamma/H$ is a uniform matroid of rank 1, so that H is a hyperplane in $\mathcal{M}_\gamma$. Similarly, $\gamma - l\mathbf{1} = \hat{\alpha} + \hat{\alpha}' - l e_{\overline{H}}$ is a $(k, \overline{H}, <)$-expansion of α. Thus $\mathcal{M}_{\gamma - l\mathbf{1}} \backslash \overline{H}$ is a cycle matroid, so that H is a circuit in $\mathcal{M}_{\gamma - l\mathbf{1}}$. By (MF2), $\mathcal{M}_{\gamma - l\mathbf{1}} = \mathcal{M}_\gamma$, and thus H is a circuit-hyperplane of $\mathcal{M}_\gamma$.

In particular, $\mathcal{M}_\gamma$ has at most 2 components, so that $\dim C_\gamma \leq 2$ due to Lemma 3.16. Note that $\gamma + c e_H \in C_\gamma$ for all $c > 0$ since H is not a basis of $M(\mathcal{M})$. So $C_{\gamma + e_H} \in \mathcal{S}^2(\mathcal{M}) \setminus \mathcal{S}^1(\mathcal{M})$, and hence $C_{\gamma + e_H}$ is as required.

By uniqueness of α and α', only points in $\mathbb{Z}^E$ which are, up to addition of multiples of e_H and $e_{\overline{H}}$, both a $(k, \overline{H}, <)$-expansion of α and a $(k', H, >)$-expansion of α' can have H as a circuit-hyperplane. Hence $C_{\gamma + e_H}$ is the unique maximal cell satisfying the requirements of the lemma. $\square$

If $M(\mathcal{M})$ is connected, then the unique 2-dimensional cell from the above lemma contains a unique cell $C' \in \mathcal{S}^1(\mathcal{M})$.

CHAPTER 4

Linear flocks

Throughout this chapter, let E be a finite set, let K be a field equipped with an automorphism f and let $d \leq |E|$ be a positive integer. Whenever important, we denote this field by (K, f). This chapter is largely based on [**6**].

1. Introduction

Define an action of $\mathbb{Z}^E$ on subsets X of K^E by

$$\alpha X = \{(f^{-\alpha_i}(x_i))_{i \in E} : x \in X\}.$$

The automorphism f of K is applied $-\alpha_i$ times to the i'th entry of all elements of X. Note that if $V \in \mathrm{Gr}_d(K^E)$, then $\mathbf{1}V \in \mathrm{Gr}_d(K^E)$. However, not necessarily $\alpha V \in \mathrm{Gr}_d(K^E)$ for all $\alpha \in \mathbb{Z}^E$.

Definition 4.1. *A* linear flock *over (K, f) is a map $\mathcal{V} : \mathbb{Z}^E \to \mathrm{Gr}_d(K^E)$, mapping α to $\mathcal{V}_\alpha$, satisfying the following axioms:*

(LF1) $\mathcal{V}_\alpha / i = \mathcal{V}_{\alpha+e_i} \backslash i$ *for all $\alpha \in \mathbb{Z}^E$ and $i \in E$; and*
(LF2) $\mathcal{V}_{\alpha+\mathbf{1}} = \mathbf{1}\mathcal{V}_\alpha$ *for all $\alpha \in \mathbb{Z}^E$.*

When f is the Frobenius automorphism of K, then we call a linear flock over (K, f) a *Frobenius flock*. Frobenius flocks play a special role as algebraic matroid representations give rise to such flocks. In Chapter 5 we will elaborate on this relation.

This chapter is concerned with understanding many features of linear flocks. The map $\mathcal{M}(\mathcal{V}) : \alpha \mapsto M(\mathcal{V}_\alpha)$ is a matroid flock. One main goal we will be working towards is the following.

Theorem 4.2. *Let $\mathcal{V}, \mathcal{V}'$ be linear flocks.Then $\mathcal{V} = \mathcal{V}'$ if and only if:*

(1) $\mathcal{M}(\mathcal{V}) = \mathcal{M}(\mathcal{V}')$*;*
(2) *for each $C \in \mathcal{S}_0(\mathcal{M}(\mathcal{V}))$ there exists $\alpha^C \in \mathbb{Z}^E \cap C$ such that $\mathcal{V}_{\alpha^C} = \mathcal{V}'_{\alpha^C}$.*

Not each combination of matroid flock and vector spaces as in the above theorem yields a linear flock. Our second main goal is to characterize exactly when these data yield a linear flock.

Using the powerful linear flock characterisation theorem (Theorem 4.30) we then proceed by investigating the class of linear-flock representable matroids. If $\mathcal{M} = \mathcal{M}(\mathcal{V})$, then we say $\mathcal{V}$ is a linear-flock representation of both $\mathcal{M}$ and $M(\mathcal{M})$. We show that the class of linear-flock representable matroids is closed under circuit-hyperplane relaxation. We use this fact to conclude that the Vámos matroid, among others, is linear-flock representable. Moreover, we are able to bound the number of linear-flock representable matroids from below.

Theorem 4.3. *Let E be a set with $|E| = 2n$. The number of pairwise non-isomorphic matroids on E that are linear-flock representable over some field is at least*

$$\frac{2^{2^{n-1}}}{(2n)!}.$$

2. Linear flocks and matroid flocks

Linear flocks and matroid flocks are similar objects. The following lemma shows that each linear flock has an underlying matroid flock.

Lemma 4.4. *Let $\mathcal{V}$ be a linear flock of rank d on E over (K, f). Then $\mathcal{M} : \mathbb{Z}^E \to \mathbb{M}_E$ given by $\alpha \mapsto M(\mathcal{V}_\alpha)$ is a matroid flock.*

Proof. As f is an automorphism of K, a set of vectors $v_1, \ldots, v_k \in K^d$ is linearly dependent if and only if $f(v_1), \ldots, f(v_k)$ are linearly dependent. By (LF2), for all $\alpha \in \mathbb{Z}^E$, $\mathcal{M}_\alpha$ and $\mathcal{M}_{\alpha+\mathbf{1}}$ are thus equal, proving (MF2) for $\mathcal{M}$.

Next, let $V \in \mathrm{Gr}_d(K^E)$, and let $M = M(V)$. For any $i \in E$, $M(V/i) = M/i$ and $M(V\backslash i) = M\backslash i$. So let $\alpha \in \mathbb{Z}^E$ and $i \in E$ be given. By (LF1), $\mathcal{M}_\alpha/i = \mathcal{M}_{\alpha+e_i}\backslash i$, proving (MF1) for $\mathcal{M}$.

It follows that $\mathcal{M}$ is a matroid flock. $\square$

Definition 4.5. *Let $\mathcal{M}$ be a matroid flock. We call a linear flock $\mathcal{V} : \mathbb{Z}^E \to \mathrm{Gr}_d(K^E)$ a* linear-flock representation *of $\mathcal{M}$ if $M(\mathcal{V}_\alpha) = \mathcal{M}_\alpha$ for all $\alpha \in \mathbb{Z}^E$. Conversely, if $\mathcal{V}$ is a linear flock, then $\mathcal{M}(\mathcal{V})$ denotes the matroid flock $\alpha \mapsto M(\mathcal{V}_\alpha)$.*

3. Linear flock structure

In this section, we investigate the structure of linear flocks. We consider lines, deletion, contraction, duality, direct sums and the boxdot operator. Many of the results for linear flocks are very similar to their matroid flock counterparts.

3.1. Lines. The following generalisation of (LF1) holds in analogy with (MF1'):

(LF1') $\mathcal{V}_\alpha/I = \mathcal{V}_{\alpha+e_I}\backslash I$ for all $\alpha \in \mathbb{Z}^E$ and $I \subseteq E$.

Using (LF1') we see that on the line $\alpha + \mathbb{Z}e_I$, the following chain of inclusions hold:

$$\cdots \subseteq \mathcal{V}_\alpha/I = \mathcal{V}_{\alpha+e_I}\backslash I \subseteq \mathcal{V}_{\alpha+e_I}/I \subseteq \mathcal{V}_{\alpha+2e_I}/I \subseteq \cdots .$$

The same holds when we replace I by $\overline{I}$, and thus after applying (LF2) we obtain

$$\cdots \supseteq \mathcal{V}_\alpha/\overline{I} \supseteq (-\mathbf{1})\mathcal{V}_{\alpha+e_I}/\overline{I} \supseteq (-2\mathbf{1})\mathcal{V}_{\alpha+2e_I}/\overline{I} \supseteq \cdots .$$

Lemma 4.6. *Let $\mathcal{V}$ be a linear flock and let $\mathcal{M} = \mathcal{M}(\mathcal{V})$. Let $\alpha \in \mathbb{Z}^E$ and $I \subseteq E$. Then $\mathcal{M}_\alpha \geq \mathcal{M}_{\alpha+e_I}$ if and only if $\mathcal{V}_{\alpha+e_I} = V_\alpha/I \oplus e_I V_\alpha\backslash\overline{I}$.*

Proof. Note that due to Lemma 3.33, $\mathcal{M}_\alpha \geq \mathcal{M}_{\alpha+e_I}$ is equivalent to $\mathcal{M}_{\alpha+e_I} = M_\alpha/I \oplus M_\alpha\backslash\overline{I}$. So suppose $\mathcal{V}_{\alpha+e_I} = V_\alpha/I \oplus e_I V_\alpha\backslash\overline{I}$. Then taking the matroid on both sides, we obtain $\mathcal{M}_{\alpha+e_I} = M_\alpha/I \oplus M_\alpha\backslash\overline{I}$, as required.

Conversely, by (LF1'), $\mathcal{V}_\alpha/I = \mathcal{V}_{\alpha+e_I}\backslash I$ and $\mathcal{V}_{\alpha+e_I}/\overline{I} = \mathcal{V}_{\alpha+\mathbf{1}}\backslash\overline{I}$. Due to (LF2), the latter equals $e_I V_\alpha\backslash\overline{I}$. Due to the assumption $\mathcal{M}_{\alpha+e_I} = M_\alpha/I \oplus M_\alpha\backslash\overline{I}$, $\mathcal{V}_{\alpha+e_I}$ now splits in the same way as $V_\alpha/I \oplus e_I V_\alpha\backslash\overline{I}$, as required. □

Lemma 4.7. *Let $\mathcal{V}$ be a linear flock. Let $k > 0$ be given and suppose $\lambda_{\alpha+le_I}(I) = 0$ for all $0 < l < k$. Then*

(1) *$\mathcal{V}_\alpha/I = \mathcal{V}_{\alpha+ke_I}\backslash I$; and*
(2) *$\mathcal{V}_\alpha\backslash\overline{I} = -k\mathbf{1}\mathcal{V}_{\alpha+ke_I}/\overline{I}$.*

Proof. By repeated application of (LF1') and Lemma 2.7, we have

$$\mathcal{V}_\alpha/I = \mathcal{V}_{\alpha+e_I}\backslash I = \mathcal{V}_{\alpha+e_I}/I = \ldots = \mathcal{V}_{\alpha+(k-1)e_I}/I = \mathcal{V}_{\alpha+ke_I}\backslash I.$$

And similarly we have

$$\mathcal{V}_\alpha\backslash\overline{I} = \mathcal{V}_{\alpha-ke_{\overline{I}}}/\overline{I} = -k\mathbf{1}\mathcal{V}_{\alpha+ke_I}/\overline{I},$$

using (LF2). □

3.2. Deletion and contraction. Before we can define deletion and contraction for linear flocks, we need the following technical lemmas.

Lemma 4.8. *Let $\mathcal{V}$ be a linear flock on E, $I \subset E$ and $\alpha \in \mathbb{Z}^{\overline{I}}$. Then there exist $k \in \mathbb{Z}$ and V^I_α so that $V^I_\alpha = \mathcal{V}_{\alpha'}\backslash I$ for all $(k, I, <)$-expansions α' of α.*

Proof. It suffices to show that $\mathcal{V}_{\alpha'}\backslash I = \mathcal{V}_{\alpha'+e_i}\backslash I$ for all α' and $i \in I$ so that both α' and $\alpha' + e_i$ are $(k, I, <)$-expansions of α, since each pair of $(k, I, <)$-expansions of α is connected by a path of $(k, I, <)$-expansions in which each pair of consecutive points differs by e_i for some $i \in I$.

Let $\mathcal{M} = \mathcal{M}(\mathcal{V})$. By Lemma 3.27, there exists $k \in \mathbb{Z}$ such that $(\mathcal{M}\backslash I)_\alpha = \mathcal{M}_{\alpha'}\backslash I$ for all $(k, I, <)$-expansions α' of α.

Let α' be a $(k-1, I, <)$-expansion of α. Then both α' and $\alpha' + e_I$ are $(k, I, <)$-expansions of α. Thus we know $\mathcal{M}_{\alpha'}\backslash I = (\mathcal{M}\backslash I)_\alpha = \mathcal{M}_{\alpha'+e_I}\backslash I$, implying $r^{\mathcal{M}}_{\alpha'}(\overline{I}) = r^{\mathcal{M}}_{\alpha'+e_I}(\overline{I}) = r^{\mathcal{M}}_{\alpha'-e_{\overline{I}}}(\overline{I})$. Lemma 3.33 then yields $\lambda^{\mathcal{M}}_{\alpha'}(I) = 0$.

So let $i \in I$ be given, and suppose α' and $\alpha' + e_i$ are both $(k, I, <)$-expansions of α. Then $\lambda^{\mathcal{M}}_{\alpha'}(I) = \lambda^{\mathcal{M}}_{\alpha'+e_i}(I) = 0$, and we compute

$$\mathcal{V}_{\alpha'}\backslash I = \mathcal{V}_{\alpha'}/i\backslash I - i = \mathcal{V}_{\alpha'+e_i}\backslash i\backslash I - i = \mathcal{V}_{\alpha'+e_i}\backslash I,$$

using Lemma 2.7 in the first equation and (LF1) in the middle equation. □

A similar statement holds for contraction. We omit the proof as it is analogous to the proof of the previous lemma.

Lemma 4.9. *Let $\mathcal{V}$ be a linear flock on E, $I \subset E$ and $\alpha \in \mathbb{Z}^{\overline{I}}$. Then there exist $k \in \mathbb{Z}$ and W^I_α so that $W^I_\alpha = \mathcal{V}_{\alpha'}/I$ for all $(k, I, >)$-expansions α' of α.*

We are now ready to define deletion and contraction for linear flocks.

Definition 4.10. *(linear flock deletion) Let $\mathcal{V}$ be a linear flock on E, and let $I \subset E$. For $\alpha \in \mathbb{Z}^{\overline{I}}$, let V^I_α be as in Lemma 4.8. Then we set $(\mathcal{V}\backslash I)_\alpha := V^I_\alpha$.*

Theorem 4.11. *Let $\mathcal{V}$ be a linear flock on E, and let $I \subset E$. Then $\mathcal{V}\backslash I$ is a linear flock on $\overline{I}$.*

Proof. Let $\alpha \in \mathbb{Z}^{\overline{I}}$ and $j \in \overline{I}$ be given. Using Lemma 4.8, let k be so that $(\mathcal{V}\backslash I)_\alpha = \mathcal{V}_{\alpha'}\backslash I$ and $(\mathcal{V}\backslash I)_{\alpha+e_j} = \mathcal{V}_{\alpha'+e_j}\backslash I$ for all $(k, I, <)$-expansions α' of α. Then $(\mathcal{V}\backslash I)_\alpha/j = \mathcal{V}_{\alpha'}/j\backslash I = \mathcal{V}_{\alpha'+e_j}\backslash j\backslash I = (\mathcal{V}\backslash I)_{\alpha+e_j}\backslash j$, proving (LF1) for $\mathcal{V}\backslash I$.

For (LF2), compute $\mathbf{1}^{\overline{I}}(\mathcal{V}\backslash I)_\alpha = \mathbf{1}^E\mathcal{V}_{\alpha'}\backslash I = \mathcal{V}_{\alpha'+\mathbf{1}^E}\backslash I = (\mathcal{V}\backslash I)_{\alpha+\mathbf{1}^{\overline{I}}}$, where the exponent of $\mathbf{1}$ indicates its dimensions. □

Definition 4.12. *(linear flock contraction) Let $\mathcal{V}$ be a linear flock on E, and let $I \subset E$. For $\alpha \in \mathbb{Z}^{\overline{I}}$, let W^I_α be as in Lemma 4.9. Then we set $(\mathcal{V}/I)_\alpha := W^I_\alpha$.*

Theorem 4.13. *Let $\mathcal{V}$ be a linear flock on E, and let $I \subset E$. Then $\mathcal{V}/I$ is a linear flock on $\overline{I}$.*

Lemma 4.14. *Let $\mathcal{V}$ be a linear flock on E, and let $I, J \subset E$ be disjoint. The following commutation properties are satisfied:*

(1) $\mathcal{V}\backslash I\backslash J = \mathcal{V}\backslash J\backslash I$;
(2) $\mathcal{V}/I/J = \mathcal{V}/J/I$;
(3) $\mathcal{V}/I\backslash J = \mathcal{V}\backslash J/I$.

3.3. Duality. Just like for matroid flocks, there is a notion of duality for linear flocks.

Definition 4.15. *(linear flock duality) Let $\mathcal{V}$ be a linear flock. The dual $\mathcal{V}^* : \mathbb{Z}^E \to \mathrm{Gr}_{d-|E|} K^E$ is defined by $\mathcal{V}^*_\alpha = \mathcal{V}^\perp_{-\alpha}$.*

Theorem 4.16. *Let $\mathcal{V}$ be a linear flock on E over (K, f) of dimension d. Then $\mathcal{V}^*$ is a linear flock on E over (K, f^{-1}) of dimension $|E| - d$.*

Proof. For each $\alpha \in \mathbb{Z}^E$, $\mathcal{V}^\perp_{-\alpha}$ is an $(|E| - d)$-dimensional vector space. To see that (LF1) holds for $\mathcal{V}^*$, we calculate for given $\alpha \in \mathbb{Z}^E$ and $i \in E$:

$$\mathcal{V}^*_{\alpha+e_i}/i = \mathcal{V}^\perp_{-\alpha-e_i}/i = (\mathcal{V}_{-\alpha-e_i}\backslash i)^\perp = (\mathcal{V}_{-\alpha}/i)^\perp = \mathcal{V}^\perp_{-\alpha}\backslash i = \mathcal{V}^*_\alpha\backslash i.$$

To see (LF2) for $\mathcal{V}^*$, we evaluate $\mathcal{V}^*_{\alpha+\mathbf{1}} = \mathcal{V}^\perp_{-\alpha-\mathbf{1}} = f[\mathcal{V}_{-\alpha}]^\perp = f^{-1}[\mathcal{V}^\perp_{-\alpha}] = \mathbf{1}\mathcal{V}^*_\alpha$. □

3.4. Direct sums. The definition of direct sums for linear flocks is analogous to the matroid flock case.

Definition 4.17. *Let $\mathcal{V}$ be a linear flocks on E of rank d, and let $\mathcal{V}'$ be a linear flock on E' of rank d', where E and E' are disjoint. The* direct sum *of $\mathcal{V}$ and $\mathcal{V}'$ is the map $\mathcal{V} \oplus \mathcal{V}' : \mathbb{Z}^{E \cup E'} \to \mathrm{Gr}_{d+d'}(K^{E\cup E'})$ given by $\alpha \mapsto \mathcal{V}_{\alpha_E} \oplus \mathcal{V}_{\alpha_{E'}}$.*

Theorem 4.18. *Let $\mathcal{V}, \mathcal{V}'$ be two linear flocks on disjoint sets E and E' respectively. Then $\mathcal{V} \oplus \mathcal{V}'$ is a linear flock.*

Theorem 4.19. *Let $\mathcal{V}$ be a linear flock on E, and let $I \subseteq E$. Let $M = M(\mathcal{V})$. If $\lambda^M(I) = 0$, then $\mathcal{V} = \mathcal{V}\backslash I \oplus \mathcal{V}\backslash \overline{I}$.*

The proofs are analogous to the matroid flock case, and therefore omitted.

3.5. The $\boxdot$-operator. Let $V \in \mathrm{Gr}_d(K^E)$ and $I \subseteq E$. Then we may define $\boxdot$ for vector spaces similarly to the matroid version as follows. Define the left-associative operator $\boxdot$ by $V \boxdot I := V/I \oplus e_I V\backslash\overline{I}$. The following basic properties are similar to their matroid versions. The proofs are analogous, and therefore omitted.

Lemma 4.20. *Let $V \in \mathrm{Gr}_d(K^E)$, and let $I \subseteq E$. Then $V \boxdot I \boxdot I = e_I V \boxdot I$.*

Lemma 4.21. *Let $V \in \mathrm{Gr}_d(K^E)$, and let $J \subseteq I \subseteq E$ be given. Then $V \boxdot I \boxdot J = V \boxdot J \boxdot I$.*

Lemma 4.22. *Let $V \in \mathrm{Gr}_d(K^E)$, and let $I, J \subseteq E$ be disjoint. Suppose the connectivity of I in $M(V) \boxdot I \cup J$ is zero. Then $V \boxdot I \boxdot J = V \boxdot I \cup J$.*

We again extend the $\boxdot$-operator to take its second argument in $\mathbb{Z}^E$. Let $\alpha \in \mathbb{Z}^E$.

$$V \boxdot \alpha := \begin{cases} k\mathbf{1}V & \text{if } \alpha = k\mathbf{1} \\ V \boxdot (\alpha - e_I) \boxdot I & \text{otherwise, where } I = \arg\max_{i \in E}\{\alpha_i\} \end{cases}$$

We may give a direct formula for $V \boxdot \alpha$.

Lemma 4.23. *Let $V \in \mathrm{Gr}_d(K^E)$, and let $\alpha \in \mathbb{Z}^E$. Write $\alpha = \sum_{i=1}^n c_i e_{I_i}$, where $\emptyset = I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_n = E$ and $c_i > 0$ for $i < n$. Then $V \boxdot \alpha = \alpha \bigoplus_{i=1}^n V \backslash \overline{I_i} / I_{i-1}$.*

Lemma 4.24. *Let $V \in \mathrm{Gr}_d(K^E)$, $\alpha \in \mathbb{Z}^E$ and $i \in E$. Then*

(1) $V \boxdot \alpha / i = V \boxdot (\alpha + e_i) \backslash i$;
(2) $\mathbf{1}V \boxdot \alpha = V \boxdot (\alpha + \mathbf{1})$.

Proof. Write $\alpha = c\mathbf{1} + \sum_{i=0}^n e_{I_i}$, where $\emptyset = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n \subsetneq E$ and $c \in \mathbb{Z}^E$. Let j be the largest index such that $i \notin I_j$. By Lemma 4.21,

$$V \boxdot (\alpha + e_i) = V \boxdot I_0 \boxdot \ldots \boxdot (I_j + i) \boxdot \ldots \boxdot I_n = V \boxdot I_0 \boxdot \ldots \boxdot I_n \boxdot (I_j + i).$$

If $\lambda^{M(V) \boxdot (\alpha + e_i)}(i) = 0$, then by Lemma 4.22,

$$V \boxdot (\alpha + e_i) = V \boxdot I_0 \boxdot \ldots \boxdot I_n \boxdot I_j \boxdot i,$$

which equals $V \boxdot \alpha \boxdot i$ by Lemma 4.21. Then

$$V \boxdot (\alpha + e_i) \backslash i = V \boxdot (\alpha + e_i) / i = (V \boxdot \alpha \boxdot i)/i = V \boxdot \alpha / i,$$

as required.

Similarly, if $\lambda^{M(V) \boxdot \alpha}(i) = 0$, then

$$V \boxdot \alpha = -\mathbf{1}V \boxdot (\alpha + \mathbf{1}) = -\mathbf{1}V \boxdot (\alpha + e_i) \boxdot \bar{i}.$$

Thus

$$V \boxdot \alpha / i = V \boxdot \alpha \backslash i = -e_{\bar{i}}(V \boxdot (\alpha + e_i) \boxdot \bar{i}) \backslash i = V \boxdot (\alpha + e_i) \backslash i,$$

as required.

The case when both $\lambda^{M(V) \boxdot (\alpha + e_i)}(i) > 0$ and $\lambda^{M(V) \boxdot \alpha}(i) > 0$ remains. But $\mathcal{M} : \alpha \mapsto M(V \boxdot \alpha)$ is a matroid flock by Lemma 3.44. Thus by Theorem 3.34

$$\lambda^{\mathcal{M}_\alpha}(i) + \lambda^{\mathcal{M}_{\alpha + e_i}}(i) \leq \lambda^{M(V)}(i) \leq 1,$$

contradicting the assumption that both connectivities are positive. $\square$

Theorem 4.25. *Let $V \in \mathrm{Gr}_d(K^E)$. Then the map $\mathcal{V} : \alpha \mapsto V \boxdot \alpha$ is a linear flock.*

Proof. Let $\alpha \in \mathbb{Z}^E$ and $i \in E$ be given. By Lemma 4.24, we have $\mathcal{V}_\alpha/i = \mathcal{V}_{\alpha+e_i}$, proving (LF1), and moreover, $\mathbf{1}\mathcal{V}_\alpha = \mathcal{V}_{\alpha+\mathbf{1}}$, proving (LF2). Finally, $\dim \mathcal{V}_\alpha = \dim V \boxdot \alpha = \dim V$, which is in particular independent of α. $\square$

4. Flock characterisation

In this section we prove Theorem 4.2 and the linear flock characterization theorem.

Lemma 4.26. *Let $\mathcal{V}$ be a linear flock. Let $\alpha \in \mathbb{Z}^E$ and $\beta \in C_\alpha$. Then $\mathcal{V}_\alpha = \mathcal{V}_\beta \boxdot (\alpha - \beta)$*

The proof is similar to the proof of Lemma 3.46.

Proof. Write $\alpha - \beta$ as in Lemma 3.45. We may assume $c = 0$, since by (LF2) $\mathcal{V}_{\alpha - c\mathbf{1}} = -c\mathbf{1}\mathcal{V}_\alpha$ and by the definition of $\boxdot$, $\mathcal{V}_\beta \boxdot (\alpha - \beta - c\mathbf{1}) = -c\mathbf{1}\mathcal{V}_\beta \boxdot (\alpha - \beta)$.

Let $\gamma(t) := \beta + \sum_{i=1}^{t} e_{I_i}$. For each $t < n$, $\mathcal{M}_{\gamma(t)} \geq \mathcal{M}_{\gamma(t+1)}$ due to Lemma 3.45. By Lemma 4.6, $\mathcal{V}_{\gamma(t+1)} = \mathcal{V}_{\gamma(t)} \boxdot I_{t+1}$. Hence by induction on t,

$$\mathcal{V}_\alpha = \mathcal{V}_{\gamma(n)} = \mathcal{V}_\beta \boxdot I_1 \boxdot \ldots \boxdot I_n = \mathcal{V}_\beta \boxdot I_n \boxdot \ldots \boxdot I_1 = \mathcal{V}_\beta \boxdot (\alpha - \beta),$$

as required. $\square$

The following technical lemma is essential for the linear flock characterization theorem.

Lemma 4.27. *Let $V, V' \in \mathrm{Gr}_d(K^E)$, $\alpha \in \mathbb{Z}^E$, $k \in \mathbb{Z}$ and $I \subseteq E$, such that*

1. *$V'/I = V \backslash I$;*
2. *$V' \backslash \overline{I} = ke_I V/\overline{I}$;*
3. *$M(V) \boxdot \alpha = M(V') \boxdot (\alpha + ke_I)$;*
4. *$\lambda^{M(V)\boxdot\alpha}(I) = 0$.*

Then $V \boxdot \alpha = V' \boxdot (\alpha + ke_I)$.

Proof. Denote $M := M(V) \boxdot \alpha = M(V') \boxdot (\alpha + ke_I)$. Write

$$\alpha = \sum_{i=1}^{n} c_i e_{S_i} \text{ where } \emptyset = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_n = E$$

and $c_i > 0$ for $i < n$. Similarly, write

$$\alpha + ke_I = \sum_{i=1}^{m} d_i e_{T_i} \text{ where } \emptyset = T_0 \subsetneq T_1 \subsetneq \ldots \subsetneq T_m = E$$

and $d_i > 0$ for $i < m$. Then by Lemma 3.43,

$$M = \bigoplus_{i=1}^{n} M(V)\backslash\overline{S_i}/S_{i-1} = \bigoplus_{i=1}^{m} M(V')\backslash\overline{T_i}/T_{i-1}.$$

Similarly by Lemma 4.23,

$$V \boxdot \alpha = \alpha \bigoplus_{i=1}^{n} V\backslash\overline{S_i}/S_{i-1}$$

and

$$V' \boxdot (\alpha + ke_I) = (\alpha + ke_I) \bigoplus_{i=1}^{m} V'\backslash\overline{T_i}/T_{i-1}.$$

Let J be a component of M. Then $J \subseteq S_a \setminus S_{a-1} =: X$ for some a, and $J \subseteq T_b \setminus T_{b-1} =: Y$ for some b. Since by assumption $\lambda^M(I) = 0$, we have either $J \subseteq I$ or $J \subseteq \overline{I}$. Suppose $J \subseteq \overline{I}$.

Write $W_J := V \boxdot \alpha\backslash\overline{J}$ and $W'_J := V' \boxdot (\alpha + ke_I)\backslash\overline{J}$. We will show that $W_J = W'_J$. Since $M(W_J) = M(W'_J)$, we know $\dim W_J = \dim W'_J = r^M(J)$. So it suffices to show $W_J \subseteq W'_J$.

Let $w \in W_J \times \{0\}^{\overline{J}} \subseteq V \boxdot \alpha$ be given. Then since

$$W_J = \alpha_J V\backslash\overline{S_a}/S_{a-1}/(X \setminus J),$$

there exists $v \in V$ such that $\alpha_J v_J = w_J$ and $\mathrm{supp}(v) \subseteq J \cup \overline{S_a}$.

Now by assumption (1) there exists $v' \in V'$ such that $v'_{\overline{I}} = v_{\overline{I}}$ and $\mathrm{supp}(v') \subseteq \overline{I}$. Then $\mathrm{supp}(v') = \mathrm{supp}(v) \setminus I = J \cup \overline{S_a} \setminus I$.

We claim $J \cup \overline{S_a} \setminus I \subseteq J \cup \overline{T_b}$. As J is disjoint from both $\overline{S_a}$ and $\overline{T_b}$, we may show instead that $\overline{S_a} \setminus I \subseteq \overline{T_b}$. Let l be such that $\alpha_J = le_J$. Let $j \in \overline{S_a} \setminus I$, when necessarily $\alpha_j < l$. Since $J \subseteq \overline{I}$, $(\alpha + ke_I)_J = le_J$ and since $j \in \overline{I}$, $(\alpha + ke_I)_j = \alpha_j$. Hence $(\alpha + ke_I)_j < l$ and thus $j \in \overline{T_b}$, proving the claim. So we may conclude that $\mathrm{supp}(v') \subseteq J \cup \overline{T_b}$.

Thus since

$$W'_J = \alpha_J V'\backslash\overline{T_b}/T_{b-1}/(Y \setminus J),$$

there exists $w' \in W'_J \times \{0\}^{\overline{J}} \subseteq V' \boxdot (\alpha + ke_I)$ such that $w'_J = \alpha_J v'_J$. But then we have $w_J = \alpha_J v_J = \alpha_J v'_J = w'_J \in W'_J$, as required.

Note that assumption (2) was not used in the above. Finally we claim that the case $J \subseteq I$ follows from the former case by setting $\tilde{V}' = k\mathbf{1}V$, $\tilde{V} = V'$, $\tilde{\alpha} = \alpha + ke_I$ and $\tilde{I} = \overline{I}$. By assumption (2), we get $\tilde{V}/\tilde{I} = \tilde{V}'\backslash\tilde{I}$. Moreover by (3),

$$M(\tilde{V}) \boxdot \tilde{\alpha} = M(V') \boxdot (\alpha + ke_I) = M(V) \boxdot \alpha = M(\tilde{V}) \boxdot (\tilde{\alpha} + ke_{\overline{I}}).$$

Similarly $\lambda^{M(\tilde{V})\boxdot\tilde{\alpha}}(\tilde{I}) = 0$ by (4). Finally

$$\tilde{V} \boxdot \tilde{\alpha} = V' \boxdot (\alpha + ke_I)$$

and

$$V \boxdot \alpha = (-k\mathbf{1}\tilde{V}') \boxdot (\tilde{\alpha} + ke_{\tilde{I}} - k\mathbf{1}) = \tilde{V}' \boxdot (\tilde{\alpha} + ke_{\tilde{I}}),$$

proving the claim. $\square$

Next we define skeleton representations of a matroid flock. Recall that $\mathcal{S}^k(\mathcal{M}) = \mathcal{S}_{k-n}(\mathcal{M})$ is the k-skeleton of a matroid flock, where n is the dimension of the lineality space of $\mathcal{M}$. Due to Lemma 3.16 n is also the number of components of $M(\mathcal{M})$.

Definition 4.28. *Let $\mathcal{M}$ be a matroid flock. Then we call a map $\mathcal{V} : |\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E \to \mathrm{Gr}_d(K^E)$ a* skeleton representation *of $\mathcal{M}$ if and only if*

1. *for all $\alpha \in |\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E$, $M(\mathcal{V}_\alpha) = \mathcal{M}_\alpha$;*
2. *for all $\alpha \in |\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E$ and all components J of $M(\mathcal{M})$ we have*
$$e_J \mathcal{V}_\alpha = \mathcal{V}_{\alpha+e_J};$$
3. *for all $\alpha \in \mathbb{Z}^E$, $k \in \mathbb{Z}$ and $I \subseteq E$ such that $\alpha, \alpha + ke_I \in |\mathcal{S}_0(\mathcal{M})| \cap C$ for some $C \in \mathcal{S}_1(\mathcal{M})$, we have*
$$\mathcal{V}_\alpha / I = \mathcal{V}_{\alpha+ke_I} \backslash I.$$

Skeleton representations are intuitive in the sense that any linear flock restricted to its 0-skeleton is a skeleton representation of its underlying matroid flock.

Lemma 4.29. *Let $\mathcal{V}$ be a linear flock. Let $\mathcal{M} = \mathcal{M}(\mathcal{V})$. Then $\mathcal{V}|_{|\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E}$ is a skeleton representation of $\mathcal{M}$.*

Proof. We verify the points of Definition 4.28. Property (1) holds by definition of $\mathcal{M}$.

By Theorem 4.19, $\mathcal{V}$ is a direct sum of linear flocks on its components. Property (2) holds due to (LF2) on each component.

For property (3), consider $\alpha, \alpha + ke_I \in |\mathcal{S}_0(\mathcal{M})| \cap C$ for some $C \in \mathcal{S}_1(\mathcal{M})$. Then due to Lemma 3.16, $\lambda(I) = 0$ in the interior of C, so in particular $\lambda_{\alpha+le_I}(I) = 0$ for all $0 < l < k$. So by Lemma 4.7, property (3) holds. $\square$

We are now ready to prove the linear flock characterization theorem, which is the following.

Theorem 4.30. *Let $\mathcal{M}$ be a matroid flock. Then any skeleton representation of $\mathcal{M}$ extends uniquely to a linear-flock representation of $\mathcal{M}$.*

Proof. Suppose $\mathcal{V}'$ is a skeleton representation of $\mathcal{M}$. Now for each $\alpha \in \mathbb{Z}^E \setminus |\mathcal{S}_0(\mathcal{M})|$ we want to choose $\mathcal{V}_\alpha \in \mathrm{Gr}_d(K^E)$ in such a way that $M(\mathcal{V}_\alpha) = \mathcal{M}_\alpha$ and that $\mathcal{V} : \mathbb{Z}^E \to \mathrm{Gr}_d(K^E)$ is a linear flock. For $\alpha \in |\mathcal{S}_0(\mathcal{M})|$, pick $\mathcal{V}_\alpha := \mathcal{V}'_\alpha$. We will first show that for each $\beta \in C_\alpha \cap |\mathcal{S}_0(\mathcal{M})|$, $\mathcal{V}'_\beta \boxdot (\alpha - \beta)$ is identical. This will allow us to put $\mathcal{V}_\alpha := \mathcal{V}'_\beta \boxdot (\alpha - \beta)$ for any $\alpha \in \mathbb{Z}^E \setminus |\mathcal{S}_0(\mathcal{M})|$ and any choice of $\beta \in C_\alpha \cap |\mathcal{S}_0(\mathcal{M})|$. Then finally we will verify that $\mathcal{V} : \mathbb{Z}^E \to \mathrm{Gr}_d(K^E)$ is a linear flock.

Let $\alpha \in \mathbb{Z}^E$ be given. We now show that for each $\beta \in C_\alpha \cap |\mathcal{S}_0(\mathcal{M})|$, $\mathcal{V}'_\beta \boxdot (\alpha - \beta)$ is identical. Consider $\beta, \beta' \in C_\alpha \cap |\mathcal{S}_0(\mathcal{M})|$. As $C_\alpha \cap |\mathcal{S}_1(\mathcal{M})|$ is connected in the sense of Lemma 3.20, we may assume $\beta' = \beta - ke_I$ for some $k \in \mathbb{Z}$ and $I \subseteq E$.

We now want to apply Lemma 4.27. As $\mathcal{V}'$ is a skeleton representation of $\mathcal{M}$, we have $\mathcal{V}'_{\beta'}/I = \mathcal{V}'_\beta \backslash I$ and $\mathcal{V}'_{\beta'} \backslash \overline{I} = ke_I \mathcal{V}'_\beta / \overline{I}$. As $\mathcal{M}$ is a matroid flock, by Lemma 3.46 we have

$$\mathcal{M}_\alpha = \mathcal{M}_\beta \boxdot (\alpha - \beta) = \mathcal{M}_{\beta'} \boxdot (\alpha - \beta') = \mathcal{M}_{\beta'} \boxdot (\alpha - \beta + ke_I).$$

Finally since e_I is parallel to C_α, by Lemma 3.16 we have $\lambda^{\mathcal{M}_\beta \boxdot (\alpha - \beta)}(I) = 0$. Thus by Lemma 4.27 we have

$$\mathcal{V}'_\beta \boxdot (\alpha - \beta) = \mathcal{V}'_{\beta'} \boxdot (\alpha - \beta').$$

So we may put $\mathcal{V}_\alpha := \mathcal{V}'_\beta \boxdot (\alpha - \beta)$ for any $\beta \in C_\alpha \cap |\mathcal{S}_0(\mathcal{M})|$. Then $M(\mathcal{V}_\alpha) = \mathcal{M}_\alpha$ due to Lemma 3.46.

Next, we show $\mathcal{V} : \mathbb{Z}^E \to \mathrm{Gr}_d(K^E)$ is a linear flock. Let $\alpha \in \mathbb{Z}^E$ and $i \in E$ be given. Then either $C_\alpha \subseteq C_{\alpha+e_i}$ or $C_{\alpha+e_i} \subseteq C_\alpha$. So there exists $\beta \in |\mathcal{S}_0(\mathcal{M})| \cap (C_\alpha \cup C_{\alpha+e_i})$, for which $\mathcal{V}_\alpha = \mathcal{V}_\beta \boxdot (\alpha - \beta)$ and $\mathcal{V}_{\alpha+e_i} = \mathcal{V}_\beta \boxdot (\alpha + e_i - \beta)$. By Lemma 4.24, we obtain $\mathcal{V}_\alpha / i = \mathcal{V}_{\alpha+e_i} \backslash i$, proving (LF1). Furthermore, $\alpha + \mathbf{1} \in C_\alpha$, so that (LF2) holds similarly by Lemma 4.24. Hence indeed, $\mathcal{V}$ is a linear flock.

To see that the extension is unique, we note that due to Lemma 4.26, none of the $\mathcal{V}_\alpha$ could have been chosen differently. □

If $\mathcal{V}$ is a skeleton representation of a matroid flock $\mathcal{M}$, then $\mathcal{V}$ is determined by a finite number of vector spaces, as will be clear from the following lemma.

Lemma 4.31. *Let $\mathcal{M}$ be a matroid flock, and let $\mathcal{V}$ be a skeleton representation of $\mathcal{M}$. Let a map $\zeta : \mathcal{S}_0(\mathcal{M}) \to \mathbb{Z}^E$ be given with the property that $\zeta(p) \in p \cap \mathbb{Z}^E$. Let the map $W : \mathcal{S}_0(\mathcal{M}) \to \mathrm{Gr}_d(K^E)$ be given by $p \mapsto \mathcal{V}_{\zeta(p)}$. Then for each central point α of $\mathcal{M}$, $\mathcal{V}_\alpha = (\alpha - \zeta(C_\alpha))W(C_\alpha)$.*

Proof. Since α and $\zeta(C_\alpha)$ lie in the same cell of $\mathcal{S}_0(\mathcal{M})$, their difference $\alpha - \zeta(C_\alpha) = \sum_{i=1}^n k_i e_{J_i}$, where $J_1, \ldots, J_n$ are the components of $M(\mathcal{M})$ and

$k_i \in \mathbb{Z}$ for all i. By repeated application of (2) of definition 4.28, we get

$$\mathcal{V}_\alpha = \mathcal{V}_{\zeta(C_\alpha)+(\alpha-\zeta(C_\alpha))} = (\alpha - \zeta(C_\alpha))\mathcal{V}_{\zeta(C_\alpha)} = (\alpha - \zeta(C_\alpha))W(C_\alpha).$$

□

Thus the map W together with a description of the polyhedral complex of $\mathcal{M}$ completely determine the central points of $\mathcal{V}$. Since the polyhedral complex of $\mathcal{M}$ is completely determined by the valuation ν such that $\mathcal{M}^\nu = \mathcal{M}(\mathcal{V})$, and since W determines a skeleton representation of $\mathcal{M}$ due to Lemma 4.31, which in turn uniquely determines $\mathcal{V}$ due to Theorem 4.30, we obtain the following result.

Theorem 4.2. *Let $\mathcal{V}, \mathcal{V}'$ be linear flocks. Then $\mathcal{V} = \mathcal{V}'$ if and only if:*

(1) *$\mathcal{M}(\mathcal{V}) = \mathcal{M}(\mathcal{V}')$;*
(2) *for each $C \in \mathcal{S}_0(\mathcal{M}(\mathcal{V}))$ there exists $\alpha^C \in \mathbb{Z}^E \cap C$ such that $\mathcal{V}_{\alpha^C} = \mathcal{V}'_{\alpha^C}$.*

We may conclude that a matroid valuation ν such that $\mathcal{M}^\nu = \mathcal{M}(\mathcal{V})$ and a vector space for each n-dimensional cell of $\mathcal{M}(\mathcal{V})$ completely determine $\mathcal{V}$. In particular, if the vector spaces can be finitely described, then so can $\mathcal{V}$.

4.1. The skeleton graph. To the aid of the next section, we introduce the skeleton graph of a matroid flock. Let $\mathcal{M}$ be a matroid flock. The set

$$P(\mathcal{M}) := \{C/\Lambda(\mathcal{M}) \mid C \in \mathcal{S}_0(\mathcal{M})\}$$

is then a set of points in $\mathbb{R}^E/\Lambda(\mathcal{M})$. Similarly,

$$L(\mathcal{M}) := \{C/\Lambda(\mathcal{M}) \mid C \in \mathcal{S}_1(\mathcal{M}) \setminus \mathcal{S}_0(\mathcal{M})\}$$

is a set of line segments in $\mathbb{R}^E/\Lambda(\mathcal{M})$.

Definition 4.32. *Let $\mathcal{M}$ be a matroid flock. Then*

$$G_\mathcal{M} := (P(\mathcal{M}), L(\mathcal{M}), \mathit{ends})$$

is the skeleton graph *of $\mathcal{M}$, where the map*

$$\mathit{ends}: L(\mathcal{M}) \to 2^{P(\mathcal{M})}, l \mapsto \{p \in P(\mathcal{M}) \mid p \subset l\}$$

assigns endpoints to the edges.

Remark 4.33. *If $\mathcal{M} = \mathcal{M}^1 \oplus \ldots \oplus \mathcal{M}^n$, then $G_\mathcal{M}$ is the Cartesian product of the graphs $G_{\mathcal{M}^1}, \ldots, G_{\mathcal{M}^n}$. Thus skeleton graphs of disconnected matroid flocks can be reconstructed from the skeleton graphs of their components. It is often more practical to work with skeleton graphs of connected matroid flocks.*

5. The class of linear-flock representable matroids

Since each linear flock corresponds to a matroid, we may regard a linear flock as a representation of this matroid. This section is concerned with the question which matroids are *linear-flock representable* over a fixed field (K, f). That is, for a matroid M on E of rank d we wonder if there exists a map $\mathcal{V} : \mathbb{Z}^E \to \mathrm{Gr}_d(K^E)$ with $M(\mathcal{V}) = M$. Each valuated matroid corresponds to a matroid flock by Theorem 3.3, but not necessarily to a linear flock for any K.

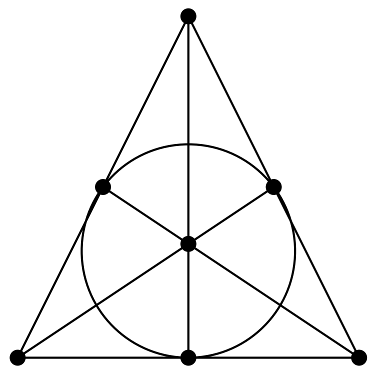


FIGURE 1. The Fano matroid.

Example 4.34. *Not every matroid is linear-flock representable. Let M be the Fano matroid, depicted in Figure 1, and let K be a field of characteristic > 2. It can be shown that any valuation of M is trivial, and hence in any matroid flock $\mathcal{M}$ of M, there exists α such that $\mathcal{M}_\alpha = M$. However, since M is only linear in characteristic 2, there cannot be a linear flock $\mathcal{V}$ such that $M(\mathcal{V}) = M$, as $\mathcal{V}_\alpha$ would then be a linear representation of M over K.*

The Fano matroid is in the class of Lazarson matroids, which will be discussed in Chapter 6 Section 4. ♙.

Lemma 4.35. *The class of linear-flock representable matroids over (K, f) is minor-closed.*

PROOF. Suppose $\mathcal{V}$ is a linear flock, and let $M = M(\mathcal{V})$. It suffices to show that, for each i, there exist linear flocks $\mathcal{V}'$ and $\mathcal{V}''$ such that $M(\mathcal{V}') = M/i$ and $M(\mathcal{V}'') = M\backslash i$. By Theorems 4.13 and 4.11 respectively, $\mathcal{V}' = \mathcal{V}/i$ and $\mathcal{V}'' = \mathcal{V}\backslash i$ have these properties. $\square$

Lemma 4.36. *If $f = f^{-1}$, then the class of linear-flock representable matroids over (K, f) is closed under duality.*

PROOF. Suppose $\mathcal{V}$ is a linear flock over (K, f), and $M = M(\mathcal{V})$. By Theorem 4.16, the dual $\mathcal{V}^*$ is a linear flock over $(K, f) = (K, f^{-1})$. The matroid of $\mathcal{V}^*$ is M^*. □

5.1. Circuit hyperplane relaxation. Recall that if H is a circuit-hyperplane of M, M^H denotes the matroid obtained from M by relaxing H. The following theorem states that the class of linear-flock representable matroids over (K, f) is closed under circuit-hyperplane relaxation.

Theorem 4.37. *Let $\mathcal{V}$ be a linear flock over (K, f), and let $M = M(\mathcal{V})$. Suppose M is connected and H is a circuit-hyperplane of M. Then there exists a linear flock of M^H over (K, f).*

PROOF. Let ν be such that $\mathcal{M}^\nu = \mathcal{M}(\mathcal{V})$. Using Lemma 3.47, pick a valuation ν' of M^H so that for all central points α of $\mathcal{M}^\nu$, we have $\mathcal{M}^{\nu'}_\alpha = \mathcal{M}^\nu_\alpha$ and for all α such that $H \notin \mathcal{B}^{\nu'}_\alpha$, we have $\mathcal{M}^{\nu'}_\alpha = \mathcal{M}^\nu_\alpha$.

Let C be the unique 2-dimensional cell from Lemma 3.48 such that for all $\alpha \in C$, H is a circuit-hyperplane of $\mathcal{M}^\nu_\alpha$. Then we claim

$$C = \bigcap_{e \in H, f \in \overline{H}} C^\nu_{H-e+f}.$$

Indeed, all bases $H - e + f$ for $e \in H$ and $f \in \overline{H}$ must be present in order for H to be a circuit-hyperplane, proving the inclusion '$\subseteq$'. To see equality, note that the right-hand side has at most 2 components, and therefore due to Lemma 3.16 has dimension at most 2. Since $\nu(H) = \infty$, it is closed under addition of positive integer multiples of e_H, and hence has dimension exactly 2. But then due to uniqueness of C, we have equality, proving the claim.

Next let $\alpha \in C \cap |\mathcal{S}^1(\mathcal{M}^\nu)|$ be given. Now there exists a minimal $k \in \mathbb{Z}_{\geq 0}$ so that $H \in \mathcal{B}^{\nu'}_{\alpha+ke_H}$. Since for all central points α of $\mathcal{M}^\nu$, we have $\mathcal{M}^{\nu'}_\alpha = \mathcal{M}^\nu_\alpha$, we have $k > 0$ and thus the cell $C^{\nu'}_H$ only intersects the basis cells of neighboring bases to H of $M^{\nu'}$. Then

$$\mathcal{B}^{\nu'}_{\alpha+ke_H} = \{H - e + f \mid e \in H, f \in \overline{H}\} \cup \{H\},$$

which is the basis set of a connected matroid, and hence $\alpha + ke_H$ is a central point of $\mathcal{M}^{\nu'}$. Since $C^{\nu'}_H$ only intersects the basis cells of neighboring bases to H of $M^{\nu'}$, there cannot be any other central point in $C^{\nu'}_H$. Thus the skeleton graph $G_{\mathcal{M}^{\nu'}}$ only differs from $G_{\mathcal{M}^\nu}$ in the extra vertex $p := \alpha + ke_H + A(\mathcal{M})$ and an edge between this vertex and $q := \alpha + A(\mathcal{M})$.

We now construct a skeleton representation $\mathcal{V}'$ of $\mathcal{M}^{\nu'}$. For each central point β of $\mathcal{M}^\nu$, put $\mathcal{V}'_\beta = \mathcal{V}_\beta$. Let v be a spanning vector in $\mathcal{V}_\alpha/H \times \{0\}^H$. Pick nonzero $s \in K$, and pick $h \in H$. Put $\mathcal{V}'_{\alpha+ke_H} = \text{span}(v + se_h) + ke_H\mathcal{V}_\alpha \backslash \overline{H}$.

Furthermore, put $\mathcal{V}'_{\alpha+ke_H+\gamma} = \gamma\mathcal{V}'_{\alpha+ke_H}$ for all $\gamma \in q$, so that property (2) of Definition 4.28 is satisfied.

Now as $\mathcal{V}$ is a linear flock, property (3) of Definition 4.28 is satisfied between all neighboring pairs of vertices of $G_{\mathcal{M}^{\nu'}}$ that are also in $G_{\mathcal{M}^{\nu}}$ due to Lemma 4.7. It remains to check property (3) of Definition 4.28 between p and q. Indeed, $\mathcal{V}'_{\alpha+ke_H}\backslash H = \mathrm{span}(v\backslash H) = \mathcal{V}'_{\alpha}/H$. Moreover, $\mathcal{V}'_{\alpha+ke_H}/\overline{H} = ke_H\mathcal{V}'_{\alpha}\backslash\overline{H}$ as required. Finally, it is straightforward to verify that $M(\mathcal{V}'_{\alpha+ke_H}) = \mathcal{M}^{\nu'}_{\alpha+ke_H}$.

Hence $\mathcal{V}'$ is a skeleton representation of $\mathcal{M}^{\nu'}$. By Theorem 4.30, $\mathcal{V}'$ can now be extended to a linear-flock representation of M^H. □

We may generalise Theorem 4.37 to allow several circuit-hyperplanes to be relaxed at once.

Theorem 4.38. *Let (K, f) be a field. Let $\mathcal{V}$ be a linear flock over (K, f). Let $M = M(\mathcal{V})$. Let $\mathcal{H}$ be a set of circuit-hyperplanes of M. Let $M^{\mathcal{H}}$ be the matroid obtained from M by relaxing all circuit-hyperplanes in $\mathcal{H}$. Then there exists a linear flock of $M^{\mathcal{H}}$ over (K, f).*

Proof. Suppose $\mathcal{R}$ is a subset of $\mathcal{H}$ of maximal cardinality so that there exists a linear flock of $M^{\mathcal{R}}$. Suppose $H \in \mathcal{H}\backslash\mathcal{R}$. Then H is a circuit-hyperplane of $M^{\mathcal{R}}$. By Theorem 4.37, $(M^{\mathcal{R}})^H = M^{\mathcal{R}\cup\{H\}}$ is linear-flock representable. But $\mathcal{R}$ was maximal, so $\mathcal{R} = \mathcal{H}$. □

The case where f is the Frobenius automorphism is the most important case if we want to say something about algebraic matroids. As will become apparent in Chapter 5, each algebraic representation of a matroid gives rise to a Frobenius flock. One of the implications of Theorem 4.37 is that the class of Frobenius-flock representable matroids is strictly larger than the class of algebraic matroids.

Corollary 4.39. *The Vámos matroid is Frobenius-flock representable.*

Let $E = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4\}$. The Vámos matroid is the matroid on E where all four-element subsets of E are bases, except

$$\{a_1, b_1, a_2, b_2\}, \{a_1, b_1, a_3, b_3\}, \{a_1, b_1, a_4, b_4\}, \{a_2, b_2, a_3, b_3\}, \{a_2, b_2, a_4, b_4\}.$$

See figure 2. The Vámos matroid bears significance as one of the smallest matroids that is not algebraically representable [**25**].

If the set $\{a_3, b_3, a_4, b_4\}$ is added to the list of non-bases, we obtain the Non-Vámos matroid, which has $\{a_3, b_3, a_4, b_4\}$ as a circuit-hyperplane. The Non-Vámos matroid is linear, and hence linear-flock representable over (K, f), where K is any field over which the Non-Vámos matroid is linearly representable, and f is any automorphism of K, due to Theorem 4.25. Thus relaxing $\{a_3, b_3, a_4, b_4\}$

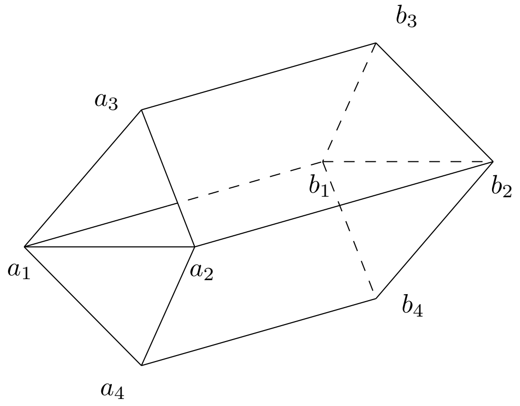


FIGURE 2. The Vámos matroid

in a linear flock of the Non-Vámos matroid yields a linear flock for the Vámos matroid. We conclude that the Vámos matroid is Frobenius-flock representable, but non-algebraic [**25**].

Corollary 4.40. *The Tic-Tac-Toe matroid is Frobenius-flock representable.*

Let $E = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$. The Tic-Tac-Toe matroid is the matroid on E where all five-element subsets of E are bases, except

$$\{a_1, a_2, a_3, b_1, c_1\}, \{a_1, a_2, a_3, b_2, c_2\}, \{a_1, a_2, a_3, b_3, c_3\},$$

$$\{b_1, b_2, b_3, a_1, c_1\}, \{b_1, b_2, b_3, a_3, c_3\},$$

$$\{c_1, c_2, c_3, a_1, b_1\}, \{c_1, c_2, c_3, a_2, b_2\}, \{c_1, c_2, c_3, a_3, b_3\}.$$

The non-bases can be seen as the L-shapes and T-shapes in the Tic-Tac-Toe-like grid

$$\begin{array}{c|c|c} a_1 & a_2 & a_3 \\ \hline b_1 & b_2 & b_3 \\ \hline c_1 & c_2 & c_3 \end{array}.$$

The dual of the Tic-Tac-Toe matroid is non-algebraic [**27**], while it is unknown whether the Tic-Tac-Toe matroid itself is algebraic. According to Hochstättler [**22**], this matroid is a good candidate to answer the question whether the class of algebraic matroids is closed under duality ([**43**] problem 6.7.15) negatively.

If the set $\{b_1, b_2, b_3, a_2, c_2\}$, the $+$-shape, is added to the list of non-bases, then once again the resulting matroid is linearly representable, and $\{b_1, b_2, b_3, a_2, c_2\}$ is a circuit-hyperplane.

5.2. The number of linear-flock representable matroids. The class of linear-flock representable matroids is large. This is follows from the fact that binary spike matroids are linear-flock representable and have many circuit-hyperplanes.

Definition 4.41. *Let M be a matroid on $\{a, e_1, f_1, e_2, f_2, \ldots, e_n, f_n\}$ so that $\{a, e_1, \ldots, e_n\}$ is a circuit and for each i, $\{a, e_i, f_i\}$ is a circuit. Then $M \backslash a$ is called a* spike matroid*, and its rank is n. The pairs $\{e_i, f_i\}$ are called the* legs *of the spike matroid. See Figure 3.*

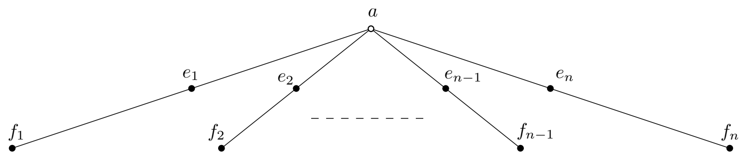


FIGURE 3. A spike matroid

For each $n \geq 3$, there is an unique spike matroid on n legs that is binary. Peter Nelson made us aware of the following [**42**].

Lemma 4.42. *Let M be the binary spike matroid with n legs. Then M has 2^{n-1} circuit-hyperplanes.*

PROOF. By [**21**] (Corollary 2.4(2)) applied to binary spikes, exactly half of the transversals of the legs of M are bases. Here a transversal of the legs is a set containing exactly one element from each leg. Moreover, if $\{e, f\}$ is a leg and $X \ni e$ is a non-basis transversal of the legs of M, then $X - e + f$ is a basis of M. Hence each non-basis transversal is a circuit, but also a hyperplane as it has size n. As there are 2^n transversals of the legs, the statement follows. $\square$

Using Theorem 4.38, Nelson proceeded to conclude the following.

Theorem 4.3. *Let E be a set with $|E| = 2n$. The number of pairwise non-isomorphic matroids on E that are linear-flock representable over some field is at least*

$$\frac{2^{2^{n-1}}}{(2n)!}.$$

PROOF. Let M be the rank n binary spike on $|E| = 2n$ points. Due to Theorem 4.25, M is linear-flock representable. By Lemma 4.42, M has 2^{n-1} circuit-hyperplanes $H_1, \ldots, H_{2^{n-1}}$. Let $S \subseteq \{1, \ldots, 2^{n-1}\}$ be given. Let M' be the matroid obtained by relaxing $(H_s)_{s \in S}$ in M. By Theorem 4.38, M' is linear-flock representable. Since each choice of S yields a linear-flock representable matroid in this way, the number of linear-flock representable matroids is at least $2^{2^{n-1}}$. We divide by $(2n)!$ to account for possible isomorphisms. $\square$

Note that $(2n)!$ is negligible compared to $2^{2^{n-1}}$ for large n. In contrast, the number of linear matroids over any field is at most $2^{n^3/4}$ if $n = |E| \geq 12$ [**41**].

CHAPTER 5

Algebraic matroid representations

The first four sections of this chapter are based on [**7**].

1. Introduction

In this chapter we discuss how algebraic matroid representations in positive characteristic give rise to a Frobenius flock.

Theorem 5.1. *Let K be an algebraically closed field of nonzero characteristic. Then every algebraic matroid over K is Frobenius-flock representable over K.*

Throughout this chapter, we use the language of algebraic geometry to describe matroid representations, and we work over an algebraically closed field K. An irreducible algebraic variety $X \subseteq K^E$ determines a matroid $M(X)$ with ground set E by declaring a set $I \subseteq E$ independent if the projection of X on K^I is dominant, that is, if the closure of $\{x_I : x \in X\}$ in the Zariski topology equals K^I. In the special case that X is a linear space, $M(X)$ is exactly the matroid represented by the columns of any matrix whose rows span X. We next translate results of Ingleton and Lindström to the language of algebraic geometry.

Ingleton [**24**] argued that if K has characteristic 0, then for any sufficiently general point $x \in X$, the tangent space T_xX of X at x will have the same dominant projections as X itself, so that then $M(X) = M(T_xX)$. Since such a sufficiently general point always exists — it suffices that x avoids finitely many hypersurfaces in X — a matroid which has an algebraic representation over K also has a linear representation over K, in the form of a tangent space T_xX.

Ingleton's argument does not generalize to fields K of positive characteristic p. Consider the variety $X = V(x_1 - x_2^p) = \{x \in K^2 : x_1 - x_2^p = 0\}$, which represents the matroid on $E = \{1, 2\}$ with independent sets $\emptyset, \{1\}, \{2\}$. The tangent space of X at any $x \in X$ is $T_xX = V(x_1)$, which represents a matroid in which $\{1\}$ is a dependent set. Thus, $M(X) \neq M(T_xX)$ for all $x \in X$.

Lindström demonstrated that for some varieties X this obstacle may be overcome by applying the Frobenius map $F : x \mapsto x^p$ to some of the coordinates, to derive varieties from X which represent the same matroid. In case of the counterexample $X = V(x_1 - x_2^p)$ above, we could define $X' = \{(x_1, F(x_2)) : x \in X\}$. Then $M(X') = M(X)$, and $X' = V(x_1 - x_2)$, so that $M(X') = M(T_{x'}X')$ for some (in fact, all) $x' \in X'$. In general for an $X \subseteq K^E$, if we fix a vector $\alpha \in \mathbb{Z}^E$, put

$$\alpha x := (F^{-\alpha_i} x_i)_{i \in E} \text{ and } \alpha X := \{\alpha x : x \in X\},$$

then it can be argued that $M(\alpha X) = M(X)$. This gives additional options for finding a suitable tangent space.

Lindström showed in [**31**] that if X is any algebraic representation of the Fano matroid, then there necessarily exists an α so that $M(X) = M(T_\xi \alpha X)$ for a sufficiently general $\xi \in \alpha X$. Thus any algebraic representation of the Fano matroid spawns a linear representation in the same characteristic. Since the Fano matroid is linear only in characteristic 2, it follows that the Fano matroid is non-algebraic over K if $\mathrm{char}(K) \neq 2$.

The choice of the matroid in Lindström's argument is not arbitrary. If we consider an algebraic representation X of the non-Fano matroid, then there does not necessarily exist an $\alpha \in \mathbb{Z}^E$ so that in a sufficiently general point $\xi \in \alpha X$ we have $M(X) = M(T_\xi \alpha X)$. Indeed, since the non-Fano matroid is algebraic in characteristic 2, but not linear.

In the present chapter, we consider the overall structure of the map $\alpha \mapsto M(T_{\xi_\alpha} \alpha X)$, where for each α, ξ_α is the generic point of αX. A central result of this chapter is that a sufficiently general $x \in X$ satisfies $M(T_{\alpha x} \alpha X) = M(T_{\xi_\alpha} \alpha X)$ for all $\alpha \in \mathbb{Z}^E$ (Theorem 5.17). This is nontrivial, as now x must *a priori* avoid countably many hypersurfaces, which one might think could cover all of X. Fixing such a general x, we show that the assignment $\mathcal{V} : \alpha \mapsto \mathcal{V}_\alpha := T_{\alpha x} \alpha X$ is Frobenius flock.

Due to Lemma 4.4, any Frobenius flock $\mathcal{V}$ gives rise to a matroid flock $\mathcal{M}$ by taking $\mathcal{M}_\alpha = M(V_\alpha)$. In particular, each algebraic matroid representation $X \subseteq K^E$ gives rise to a matroid flock $\mathcal{M} : \alpha \mapsto M(T_{\alpha x} \alpha X)$. This matroid flock does not depend on the choice of the general point $x \in X$, since $M(T_{\alpha x} \alpha X) = M(T_{\alpha x'} \alpha X)$ for any two such general points. Due to Theorem 3.3, there exists a matroid valuation ν such that $\mathcal{M} = \mathcal{M}^\nu$. Recognizing the seminal work of Bernt Lindström, we named ν the *Lindström valuation* of X.

Theorem 5.1 can be applied to show non-algebraicity of matroids over certain fields, much like Lindströms result. If a matroid admits no Frobenius flock over a given field, then this matroid is non-algebraic over that field. In particular, matroids that admit only trivial valuations, the so-called *rigid* matroids, are prone to this argument. For example, the Fano matroid only has trivial valuations. So each of the corresponding matroid flocks $\mathcal{M}$ has a point

α such that $\mathcal{M}_\alpha$ is the Fano matroid. Hence none of them admit a skeleton representation over fields of characteristic $\neq 2$. Thus by Lemma 4.29, the Fano matroid is not Frobenius-flock representable over such fields, and hence not algebraic.

It is an open problem to characterize the valuated matroids which may arise from algebraic representations in characteristic p.

In the final part of the chapter we define a notion of algebraic equivalence of algebraic representations. We show that all algebraic representations of the uniform matroid $U_{1,2}$ are algebraically equivalent, but that $U_{2,3}$ admits algebraically inequivalent algebraic representations. If two linear representations are linearly equivalent, it is not hard to see that they are also algebraically equivalent. Finally, we investigate how much more general algebraic equivalence of linear representations is compared linear equivalence. We show that algebraically equivalent linear representations must be componentwise field-equivalent.

2. Algebraic matroids in the algebro-geometric setting

Definition 2.8 is the most common definition of algebraic matroids. In this chapter it will be useful to take a more geometric viewpoint on algebraic matroids; a good general reference for the algebro-geometric terminology that we will use is [**11**]; and we refer to [**26, 45**] for details on the link to algebraic matroids.

First, we assume throughout this chapter that K is algebraically closed. This is no loss of generality in the following sense: take an algebraic closure L' of L and let K' be the algebraic closure of K in L'. Let E be a finite set and consider a map $\phi : E \to L$. Then for any subset $I \subseteq E$ the set $\phi(I) \subseteq L$ is algebraically independent over K if and only if $\phi(I)$ is algebraically independent over the algebraically closed field K'.

Second, there is clearly no harm in assuming that L is generated by $\phi(E)$. Then let P be the kernel of the K-algebra homomorphism from the polynomial ring $R := K[(x_i)_{i\in E}]$ into L that sends x_i to $\phi(i)$. Since L is a domain, P is a prime ideal, so the quotient R/P is a domain and L is isomorphic to the field of fractions of this domain.

By Hilbert's basis theorem, P is finitely generated, and one can store algebraic representations on a computer by means of a list of generators (of course, this requires that one can already compute with elements of K). In these terms, a subset $I \subseteq E$ is independent if and only if $P \cap K[x_i : i \in I] = \{0\}$. Given generators of P, this intersection can be computed using Gröbner bases [**11**, Chapter 3, §1, Theorem 2].

We have now seen how to go from an algebraic representation over K of a matroid on E to an irreducible subvariety of K^E. Conversely, every irreducible closed subvariety Y of K^E determines an algebraic representation of some matroid, as follows: let $Q \subseteq R$ be the prime ideal of polynomials vanishing on Y, let $K[Y] := R/Q$ be the integral domain of regular functions on Y, and set $L := K(Y)$, the fraction field of $K[Y]$. The map ϕ sending i to the class of x_i in L is a representation of the matroid M in which $I \subseteq E$ is independent if and only if $Q \cap K[x_i : i \in I] = \{0\}$. This latter condition can be reformulated as saying that the image of Y under the projection $\pi_I : K^E \to K^I$ is dense in the latter space, i.e., that Y projects *dominantly* into K^I. Our discussion is summarised in the following lemma.

Recall the definition of an algebro-geometric representation of a matroid (Definition 2.11). We have seen:

Lemma 5.2. *A matroid M admits an algebraic representation over the algebraically closed field K if and only if it admits an algebro-geometric representation over K.*

The rank function on M corresponds to dimension:

Lemma 5.3. *If Y is an algebro-geometric representation of M, then for each $I \subseteq E$ the rank of I in M is the dimension of the Zariski closure $\overline{\pi_I(Y)}$.*

Because of this equivalence between algebraic and algebro-geometric representations, we will continue to use the term algebraic representation for algebro-geometric representations.

3. Tangent spaces

Crucial to our construction of a flock from an algebraic matroid are tangent spaces, which were also used in [**31**] in the study of characteristic sets of algebraic matroids. In this section, $Y \subseteq K^E$ is an irreducible, closed subvariety with vanishing ideal $Q \subseteq R$, $K[Y] := R/Q$ is its coordinate ring, and $K(Y)$ its function field.

Definition 5.4. *Define the $K[Y]$-module*

$$J_Y := \left\{ \left(\frac{\partial f}{\partial x_j} + Q \right)_{j \in E} : f \in Q \right\} \subseteq K[Y]^E.$$

For any $v \in Y$, define the tangent space $T_vY := J_Y(v)^\perp \subseteq K^E$, where $J_Y(v) \subseteq K^E$ is the image of J_Y under evaluation at v. Let $\eta \in K(Y)^E$ be the generic point of Y, i.e., the point $(x_j + Q)_{j \in E}$, and define $T_\eta Y$ as $(K(Y) \otimes_{K[Y]} J_Y)^\perp \subseteq K(Y)^E$. The variety Y is called smooth *at v (and v a* smooth point *of Y) if $\dim_K T_vY = \dim_{K(Y)} T_\eta Y$.*

The $K[Y]$-module J_Y is generated by the rows of the Jacobi matrix $(\frac{\partial f_i}{\partial x_j} + Q)_{i,j}$ for any finite set of generators $f_1, \ldots, f_r$ of Q. The right-hand side in the smoothness condition also equals the transcendence degree of $K(Y)$ over K and the Krull dimension of Y. The smooth points in Y form an open and dense subset of Y.

We recall the following property of smooth points.

Lemma 5.5. *Assume that Y is smooth at $v \in Y$ and let S be the local ring of Y at v, i.e., the subring of $K(Y)$ consisting of all fractions f/g where $g(v) \neq 0$. Then $M := S \otimes_{K[Y]} J_Y \subseteq S^E$ is a free S-module of rank equal to $|E| - \dim Y$, which is saturated in the sense that $su \in M$ for $s \in S$ and $u \in S^E$ implies $u \in M$.*

Now we come to a fundamental difference between characteristic zero and positive characteristic.

Lemma 5.6. *Let $v \in Y$ be smooth, and let $I \subseteq E$. Then $\dim \pi_I(T_v Y) \leq \dim \overline{\pi_I(Y)}$. If, moreover, the characteristic of K is equal to zero, then the set of smooth points $v \in Y$ for which equality holds is an open and dense subset of Y.*

The inequality is fairly straightforward, and it shows that $M(T_v Y)$ is always a weak image of $M(Y)$ (of the same rank as the latter). For a proof of the statement in characteristic zero, see for instance [**47**, Chapter II, Section 6]. A direct consequence of this lemma is the following, well-known theorem [**24**].

Theorem 5.7 (Ingleton). *If $\operatorname{char} K = 0$, then every matroid that admits an algebraic representation over K also admits a linear representation over K.*

PROOF. If $Y \subseteq K^E$ represents M, then the set of smooth points $v \in Y$ such that $\dim \pi_I(T_v Y) = \dim \overline{\pi_I(M)}$ for all $I \subseteq E$ is a finite intersection of open, dense subsets, and hence open and dense. For any such point v, the linear space $T_v Y$ represents the same matroid as Y. $\square$

A fundamental example where this reasoning fails in positive characteristic was given in the introduction. As we will see next, in positive characteristic Frobenius flocks take the role of the linear representations in Theorem 5.7.

4. Positive characteristic

Assume that K is algebraically closed of characteristic $p > 0$. Then we have the action of $\mathbb{Z}^E$ on K^E by $\alpha w := (F^{-\alpha_i} w_i)_{i \in E}$.

Let $X \subseteq K^E$ be an irreducible closed subvariety. To study the orbit of X under $\mathbb{Z}^E$, we need the following lemma.

Lemma 5.8. *The action of $\mathbb{Z}^E$ on K^E is via homeomorphisms in the Zariski topology.*

These homeomorphisms are not polynomial automorphisms since $F^{-1} : K \to K, c \mapsto c^{1/p}$, while well-defined as a map, is not polynomial.

Proof. Let $\alpha \in \mathbb{Z}^E$ and let $Y \subseteq K^E$ be closed with vanishing ideal Q. Then

$$\alpha Y = \{v \in K^E : \forall_{f \in Q} f((-\alpha)v) = 0\}.$$

Now for $f \in Q$ the function $g : K \to K,\ v \mapsto f((-\alpha)v)$ is not necessarily polynomial if α has negative entries. But for every $e \in \mathbb{Z}$ the equation $g(v) = 0$ has the same solutions as the equation $g(v)^{p^e} = 0$; and by choosing e sufficiently large, $h(v) := g(v)^{p^e}$ *does* become a polynomial. Hence αY is Zariski-closed, and the map $K^E \to K^E$ defined by α is continuous. The same applies to $-\alpha$, so α is a homeomorphism. $\square$

As a consequence of the lemma, αX is an irreducible subvariety of K^E for each $\alpha \in \mathbb{Z}^E$, and has the same Krull dimension as X—indeed, both of these terms have purely topological characterisations. The ideal of αX can be obtained explicitly from that of X by writing $\alpha = c\mathbf{1} - \beta$ with $c \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^E_{\geq 0}$ and applying the following two lemmas.

Lemma 5.9. *Let $Y \subseteq K^E$ be closed with vanishing ideal Q, and $\beta \in \mathbb{Z}^E_{\geq 0}$. Then the ideal of $(-\beta)Y$ equals*

$$\left\{ f((x_i)_{i \in E}) : f\left((x_i^{(p^{\beta_i})})_{i \in E} \right) \in Q \right\}.$$

Proof. The variety $(-\beta)Y$ is the image of Y under a polynomial map, and by elimination theory [**11**, Chapter 3, §3, Theorem 1] its ideal is obtained from the intersection $Q \cap K\left[\left(x_i^{(p^{\beta_i})}\right)_{i \in E}\right]$ by replacing $x^{(p^{\beta_i})}$ by x_i. $\square$

Note that the ideal in the lemma can be computed from Q by means of Gröbner basis calculations, again using [**11**, Chapter 3,§1, Theorem 2].

Lemma 5.10. *Let $Y \subseteq K^E$ be closed with vanishing ideal generated by $f_1, \ldots, f_k$. Then for each $c \in \mathbb{Z}$ the ideal of $(c\mathbf{1})Y$ is generated by $g_1 := F^{-c}(f_1), \ldots, g_k := F^{-c}(f_k)$, where F^{-c} acts on the coefficients of these polynomials only.*

Proof. A point $a \in K^E$ lies in $(c\mathbf{1})Y$ if and only if $(-c\mathbf{1})a \in Y$, i.e., if and only if $f_i(F^c(a)) = 0$ for all i, which is equivalent to $g_i(a) = 0$ for all i. So by Hilbert's Nullstellensatz the vanishing ideal of αY is the radical of the ideal generated by the g_i. But the g_i are the images of the f_i under a ring automorphism $R \to R$, and hence generate a radical ideal since the f_i do. □

Lemma 5.11. *For each $\alpha \in \mathbb{Z}^E$, the variety αX represents the same matroid as X.*

PROOF. If $I \subseteq E$ is independent in the matroid represented by X, then the map $\pi_I : X \to K^I$ has a dense image. But the image of the projection $\alpha X \to K^I$ equals $(\alpha|_I)\operatorname{im}\pi_I$, and is hence also dense in K^I. So all sets independent for X are independent for αX, and the same argument with $-\alpha$ yields the converse. □

Lemma 5.12. *Let $Y \subseteq K^E$ be an irreducible, closed subvariety with generic point η and let $j \in E$. Then $e_j \in T_\eta Y$ if and only if the vanishing ideal of Y is generated by polynomials in x_j^p and the x_i with $i \neq j$.*

PROOF. If the ideal Q of Y is generated by polynomials in which all exponents of x_j are multiples of p, then Q is stable under the derivation $\frac{\partial}{\partial x_j}$. This means that the projection of $J_Y \subseteq K[Y]^E$ onto the j-th coordinate is identically zero, so that $e_j \perp (K(Y) \otimes_{K[Y]} J_Y)$. This proves the "if" direction.

For "only if" suppose that $e_j \in T_\eta Y$, let G be a reduced Gröbner basis of Q relative to any monomial order, and let $g \in G$. Then $f := \frac{\partial g}{\partial x_j}$ is zero in $K[Y]$, i.e., $f \in Q$. Assume that f is a nonzero polynomial. Then the leading monomial u of f is divisible by the leading monomial u' of some element of $G \setminus \{g\}$. But u equals v/x_j for some monomial v appearing in g, and hence v is divisible by u'; this contradicts the fact that G is reduced. Hence $\frac{\partial g}{\partial x_j} = 0$, and therefore all exponents of x_j in elements of G are multiples of p. □

For any closed, irreducible subvariety $X \subseteq K^E$, let $M(X, \alpha) := M(T_\xi \alpha X)$, where ξ is the generic point of αX.

Theorem 5.13. *Let K be algebraically closed of characteristic $p > 0$, $X \subseteq K^E$ a closed, irreducible subvariety. Let $v \in X$ be such that for each $\alpha \in \mathbb{Z}^E$, we have*

$$M(T_{\alpha v} \alpha X) = M(X, \alpha) \tag{*}$$

Then the assignment $\mathcal{V} : \alpha \mapsto \mathcal{V}_\alpha := T_{\alpha v}\alpha X$ is a Frobenius flock that satisfies $M(\mathcal{V}) = M(X)$.

Definition 5.14. *This Frobenius flock is called the* Frobenius flock associated *to the pair (X, v).*

Proof of Theorem 5.13. For each $\alpha \in \mathbb{Z}^E$ we have $M(T_{\alpha v}\alpha X) = M(T_\xi \alpha X)$, and this implies that $\dim_K \mathcal{V}_\alpha = \dim X =: d$.

Next, for $j \in E$ the action by $(-e_j) \in \mathbb{Z}^E$ sends $Y := (\alpha + e_j)X$ into $(-e_j)Y = \alpha X$ by raising the j-th coordinate to the power p. Hence the derivative of this map at $y := (\alpha + e_j)v$, which is the projection onto $e_j^\perp$ along e_j, maps $\mathcal{V}_{\alpha+e_j} = T_y Y$ into $\mathcal{V}_\alpha = T_{(-e_j)y}(-e_j)Y$, and therefore $\mathcal{V}_{\alpha+e_j} \backslash j \subseteq \mathcal{V}_\alpha / j$. If the left-hand side has dimension d, then equality holds, and (LF1) follows.

If not, then $e_j \in T_y Y$, i.e., j is a co-loop in $M(T_y Y)$. Then by (*) j is also a co-loop in $M(T_\eta Y)$, where η is the generic point of Y. By Lemma 5.12 the ideal of Y is generated by polynomials $f_1, \ldots, f_r$ in which all exponents of x_j are multiples of p. By Lemma 5.9, replacing x_j^p by x_j in these generators yields generators $g_1, \ldots, g_r$ of the ideal of $e_j Y$. Now the Jacobi matrix of $g_1, \ldots, g_r$ at $(-e_j)y$ equals that of $f_1, \ldots, f_r$ at y except that the j-th column may have become nonzero. But this means that $\mathcal{V}_\alpha / j$ has dimension equal to that of $\mathcal{V}_{\alpha+e_j} \setminus j$, namely, $d - 1$. Hence (LF1) holds in this case, as well.

For (LF2), let $Z := \alpha Y$ and $z := \alpha y$, pick any generating set $f_1, \ldots, f_r$ of I_Z, raise all f_i to the $(1/p)$-th power, and replace each x_j in the result by x_j^p. By Lemma 5.10, the resulting polynomials $g_1, \ldots, g_r$ generate $I_{\mathbf{1}Z}$. The Jacobi matrix of $g_1, \ldots, g_r$ at $\mathbf{1}z$ equals that of $f_1, \ldots, f_r$ at z except with F^{-1} applied to each entry. Hence $\mathcal{V}_{\alpha+\mathbf{1}} = \mathbf{1}\mathcal{V}_\alpha$ as claimed. This proves that $\mathcal{V}$ is a Frobenius flock.

We next verify that $M(\mathcal{V}) = M(X)$; by Lemma 5.11 the right-hand side is also $M(\alpha X)$ for each $\alpha \in \mathbb{Z}^E$. Assume that I is independent in $M(\mathcal{V}_\alpha)$ for some α. This means that the projection $T_{\alpha v}\alpha X \to K^I$ is surjective and since αv is a smooth point of αX by (*) we find that the projection $\alpha X \to K^I$ is dominant, i.e., I is independent in $M(X)$.

Conversely, assume that I is a basis for $M(X)$, so that $|I| = d$. Then $K(X)$ is an algebraic field extension of $K(x_i : i \in I) =: K'$. If this is a separable extension, then by [**8**, AG.17.3] the projection $T_u X \to K^I$ is surjective (i.e., a linear isomorphism) for general $u \in X$, hence also for $u = v$ by (*). If not, then for each $j \in \bar{I}$ let $\alpha_j \in \mathbb{Z}_{\geq 0}$ be minimal such that $x_j^{(p^{\alpha_j})}$ *is* separable over K', and set $\alpha_i = 0$ for $i \in I$. Then the extension $K' \subseteq K((-\alpha)X)$ *is* separable, and hence the projection $T_{(-\alpha)v} : (-\alpha X) \to K^I$ surjective. □

For *fixed* $\alpha \in \mathbb{Z}^E$, the condition (*) holds for v in some open dense subset of X, i.e., for *general* v in the language of algebraic geometry—indeed, it says that certain subdeterminants of the Jacobi matrix that do not vanish at the generic point of αX do not vanish at the point αv either. However, we require that (*) holds for *all* $\alpha \in \mathbb{Z}^E$. This means that v must lie outside a countable union of proper Zariski-closed subsets of X, i.e., it must be *very general* in the language of algebraic geometry. *A priori*, that K is algebraically closed does not imply the existence of such a very general point v. This can be remedied by enlarging K. For instance, if we change the base field to $K(X)$, then the generic point of X satisfies (*); alternatively, if K is taken uncountable (in addition to algebraically closed), then a very general v also exists. But in fact, as we will see in Theorem 5.17, certain general finiteness properties of flocks imply that, after all, it *does* suffice that K is algebraically closed.

Corollary 5.15. *Let K be algebraically closed of characteristic $p > 0$, and let $X \subseteq K^E$ be a closed, irreducible subvariety. Then $\mathcal{M} : \alpha \mapsto M(X, \alpha)$ is a matroid flock with support matroid $M(X)$.*

PROOF. Let v be the generic point of X over the enlarged base field $K(X)$. Then the Frobenius flock $\mathcal{V}$ associated with (X, v) satisfies $M(\mathcal{V}_\alpha) = M(X, \alpha)$. As $\mathcal{V}$ is a Frobenius flock, the assignment $\alpha \mapsto M(\mathcal{V}_\alpha)$, and hence $\mathcal{M}$, is a matroid flock. By Theorem 5.13, the support matroid of $\mathcal{V}$, and hence of its matroid flock $\mathcal{M}$, is $M(X)$. $\square$

Due to Theorem 3.3, there is a valuation associated to $\mathcal{M}$, which we call the Lindström valuation. Cartwright found a direct construction of this valuation from an algebraic representation [**9**]. In the setting of Definition 2.8, the value assigned to a basis B is the p-logarithm of the inseparable degree of the extension L of the set of elements of $K(\phi(B))$ that are separable over L:

$$\nu(B) = \log_p[L : K(\phi(B))^{\mathrm{sep}(L)}].$$

We now prove that under certain technical assumptions, if v satisfies (*) at some $\alpha \in \mathbb{Z}^E$, then it also satisfies (*) at $\alpha - e_I$. This will reduce the number of conditions on v from countable to finite, whence ensuring that a general v satisfies them.

Lemma 5.16. *Let $v \in X$ satisfy (*) for some $\alpha \in \mathbb{Z}^E$, and let $I \subseteq E$ be such that $M(X, \alpha) = M(X, \alpha - e_I)$. Then v satisfies (*) for $\alpha - e_I$.*

PROOF. Set $Y := \alpha X$ and set $W := K(Y) \otimes_{K[Y]} J_Y \subseteq K(Y)^E$. By Lemma 3.33 applied to the matroid flock $\alpha \mapsto M(X, \alpha)$, we find that the connectivity of I in $M(W) = M(X, \alpha)$ is zero, and hence that $W = (W/I) \times (W/\bar{I})$.

We claim that the same decomposition happens over the local ring S of Y at αv. Let $M := S \otimes_{K[Y]} J_X \subseteq S^E$. Let $m \in M$ and write $m = m_1 + m_2$ where m_1, m_2 have nonzero entries only in $I, \bar{I}$, respectively. By the decomposition of W, we have $m_1, m_2 \in W$, and by clearing denominators it follows that

$s_1 m_1, s_2 m_2 \in J_Y \subseteq M$ for suitable $s_1, s_2 \in K[Y] \subseteq S$. Then by Lemma 5.5, m_1, m_2 themselves already lie in M. Thus $M = (M/I) \times (M/\bar{I})$, as claimed.

This means that there exist generators $f_1, \dots, f_r, g_1, \dots, g_s$ of the maximal ideal of S with $r+s = |E| - \dim Y$ and such that $\frac{\partial f_i}{\partial x_j} = 0 \in S$ for all $i = 1, \dots, r$ and $j \in \bar{I}$ and $\frac{\partial g_i}{\partial x_j} = 0 \in S$ for all $i = 1, \dots, s$ and $j \in I$. Thus the Jacobi matrix of $f_1, \dots, f_r, g_1, \dots, g_t$ looks as follows:

$$A = \begin{array}{c} \\ f_1 \\ \vdots \\ f_r \\ g_1 \\ \vdots \\ g_s \end{array} \begin{array}{c} \begin{array}{cc} I & \bar{I} \end{array} \\ \left[\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right] \end{array}$$

By (*) the K-row space of $A(y)$ defines the same matroid as the $K(Y)$-row space of A itself.

It follows that the exponents of x_j with $j \in I$ in the g_i are multiples of p, and so are the exponents of the x_j with $j \in \bar{I}$ in the f_i. Let g_i' be the polynomial obtained from g_i by replacing x_j^p with x_j for $j \in I$, and let f_i' be the polynomial obtained from f_i^p by replacing each x_j^p with x_j for $j \in I$. Then the f_i' and g_i' lie in the maximal ideal of $(-e_I)Y$ at $(-e_I)y$, and their Jacobi matrix looks like this:

$$A = \begin{array}{c} \\ f_1' \\ \vdots \\ f_r' \\ g_1' \\ \vdots \\ g_s' \end{array} \begin{array}{c} \begin{array}{cc} I & \bar{I} \end{array} \\ \left[\begin{array}{c|c} FA_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}$$

Here FA_{11} is the matrix over S whose entries are obtained by applying the Frobenius automorphism to all coefficients. In particular, the evaluation $A'((-e_I)y)$ has full K-rank $r+s$, $(-e_I)Y$ is smooth at $(-e_I)y$, and the f_i' and the g_i' generate the maximal ideal of the local ring of $(-e_I)Y$ at $(-e_I)y$. Moreover, if a subset I' of the columns of A is independent, then the columns labelled by I' are also independent in $A'((-e_I)y)$. This means that $M(T_{(-e_I)y}(-e_I)Y)$ has at least as many bases as $M(T_\xi Y) = M(T_\eta e_I Y)$, but it cannot have more, so that v satisfies (*) at $\alpha - e_I$. $\square$

Theorem 5.17. *Let K be algebraically closed of characteristic $p > 0$ and let $X \subseteq K^E$ be an irreducible closed subvariety of dimension d. Then for a general* *point $v \in X$ the map $\mathcal{V} : \alpha \mapsto T_{\alpha v}\alpha X$ is a Frobenius flock of rank d over K such that $M(X, \alpha) = M(\mathcal{V}_\alpha)$ for each $\alpha \in \mathbb{Z}^E$.*

Proof. By Theorem 5.13 it suffices to prove that there exists a $v \in X$ satisfying (*) at every $\alpha \in \mathbb{Z}^E$. For some field extension $K' \supseteq K$ there does exist a K'-valued point $v' \in X(K')$ that satisfies (*) at every α, and we may form the Frobenius flock $\mathcal{V}'$ associated to (X, v') over K'.

For each matroid M' on E the points $\alpha \in \mathbb{Z}^E$ with $M(\mathcal{V}'_\alpha) = M'$ are connected to each other by means of moves of the form $\alpha \to \alpha + \mathbf{1}$ or $\alpha \to \alpha - e_I$ for some subset $I \subseteq E$ of connectivity 0 in M'; see Lemma 3.15. Hence by Lemma 5.16, for a $v \in X(K)$ to satisfy (*) for all $\alpha \in \mathbb{Z}^E$ it suffices that v satisfies this condition for one representative α for each matroid M'. Since there are only finitely many matroids M' to consider, we find that, after all, a general $v \in X(K)$ suffices these conditions. $\square$

Observe the somewhat subtle structure of this proof: apart from commutative algebra, it also requires the entire combinatorial machinery of flocks. Theorem 5.1 is a direct consequence of this theorem.

Example 5.18. *Let $E = \{1, 2, 3, 4\}$ and consider the polynomial map $\phi : K^2 \to K^4$ defined by $\phi(s, t) = (s, t, s + t, s + t^{(p^g)})$ where $p = \operatorname{char} K$ and $g > 1$. This is a morphism of (additive) algebraic groups, hence $X := \operatorname{im} \phi$ is closed. The polynomials in the parameterisation ϕ are pairwise algebraically independent, so that $M(X)$ is the uniform matroid on E of rank 2.*

One can verify that the point 0 is general in the sense of Theorem 5.17, and

$$T_0 X = \operatorname{im} d_0\phi = \textit{the row space of } \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Note that in $M(T_0X)$ the elements 1 and 4 are parallel. Compute

$$\begin{aligned} (-e_2 - e_3)X &= \{(s, t^p, s^p + t^p, s + t^{(p^g)}) : s, t \in K\} \\ &= \{(s, t, s^p + t, s + t^{(p^{g-1})}) : s, t \in K\}; \end{aligned}$$

so

$$T_0(-e_2 - e_3)X = \textit{the row space of } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Here, not only 1 and 4 are parallel, but also 2 and 3. We see the same matroid for $(-ke_2 - ke_3)X$ with $k = 2, \ldots, g - 1$. But $(-ge_2 - ge_3)X = \{(s, t, s^{p^g} + t, s + t) : s, t \in K\}$, so

$$T_0(-ge_2 - ge_3)X = \textit{the row space of } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

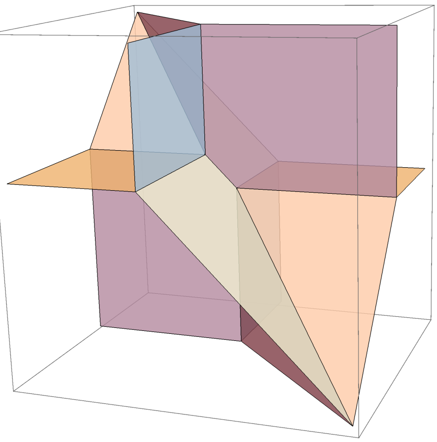


FIGURE 1. The cell decomposition for a matroid flock of $U_{2,4}$; the two zero-dimensional cells are 0 and $-ge_2 - ge_3$.

Here only 2 *and* 3 *are parallel. The cell decomposition of the underlying matroid flock, intersected with the hyperplane where one of the coordinates is zero, is depicted in Figure 1.* ♙

5. Rigid matroids

Following Dress and Wenzel, we call a matroid M *rigid* if all valuations of M are trivial. For rigid matroids, Ingleton's equivalence between algebraicity and linearity not only holds over fields of characteristic 0, but over any algebraically closed field, as is shown in the following theorem.

Theorem 5.19. *Let M be a matroid, and let K be an algebraically closed field of positive characteristic. If M is rigid, then M is algebraic over K if and only if M is linear over K.*

PROOF. If M is linear over K, then clearly M is also algebraic over K. We prove the converse. Let $X \subseteq K^E$ be an algebraic representation of N, and let $\nu = \nu^X$ be the Lindström valuation of X. Since we assumed that M is rigid, and ν is a valuation with support matroid $M(X) = M$, it follows that ν is trivial. Hence $\nu(B) = e_B \cdot \alpha$ for some $\alpha \in \mathbb{Z}^E$. Then $M(X, \alpha) = \mathcal{M}^\nu_\alpha = M$. For a general $x \in \alpha X$, we have $M(T_x \alpha X) = M(X, \alpha)$, and then $T_x\alpha X$ is a linear representation of M over K. □

Using a consideration about the Tutte group, Dress and Wenzel showed [**15**, Thm 5.11]:

Theorem 5.20. *If the inner Tutte group of a matroid M is a torsion group, then M is rigid. In particular:*

(1) *binary matroids are rigid; and*
(2) *if $r \geq 3$ and q is a prime power, then the finite projective space $PG(r-1,q)$ is rigid.*

Next, we discuss rigidity of matroids with parallel elements. Recall that $T(M)$ is the space of trivial valuations of a matroid M.

Lemma 5.21. *Suppose M is a matroid, and $\{i,j\}$ is a circuit of M. If ν,ν' are valuations of M such that $\nu\backslash j = \nu'\backslash j$, then $\nu' \in \nu + T(M)$.*

Proof. Let ν be any valuation of M, and B, B' be such that $i \in B \cap B'$ and $B' = B - k + l$, where $j \neq k, l$. By Lemma 2.17, we have $\nu(B) + \nu(B' - i + j) = \nu(B - i + j) + \nu(B')$, since $B - k + j = B' - l + j$ is not a basis as it contains the dependent set $\{i,j\}$. Hence $\nu(B) - \nu(B - i + j) = \nu(B') - \nu(B' - i + j)$ for any adjacent bases B, B' both containing i. Since any two bases of $M\backslash j$ are connected by a walk along adjacent bases, it follows that there is a constant c so that $\nu(B) - \nu(B - i + j) = c$ for any basis B of $M\backslash j$ with $i \in B$. If ν' is any other valuation of M, then by the same reasoning there is a c' so that $\nu'(B) - \nu'(B - i + j) = c'$ for any basis B of $M\backslash j$ with $i \in B$. If $\nu'\backslash j = \nu\backslash j$, then $\nu(B) + e_B \cdot (ce_j) = \nu'(B) + e_B \cdot (c'e_j)$ for all bases B of M, and it follows that $\nu' \in \nu + T(M)$, as required. □

If M is a matroid then $si(M)$, the *simplification* of M, is a matroid whose elements are the parallel classes of M, and which is isomorphic to any matroid which arises from M by restricting to one element from each parallel class. Directly from the previous lemma, we obtain:

Lemma 5.22. *Suppose M is a matroid. If $si(M)$ is rigid, then M is rigid.*

Similarly, matroids of rank or corank 1 are rigid.

Lemma 5.23. *Suppose $M \cong U_{1,n}$ or $M \cong U_{n-1,n}$. Then M is rigid.*

Proof. If $M \cong U_{1,n}$, then $si(M) \cong U_{1,1}$ which is rigid. Hence by Lemma 5.22, M is rigid. If $M = U_{n-1,n}$, then $M^* = U_{1,n}$ is rigid, and hence M is rigid. □

Another way to see this is by considering the Dressian. Matroids of rank or corank 1 contain no octahedra, and hence the Dressian contains a single nonempty cell.

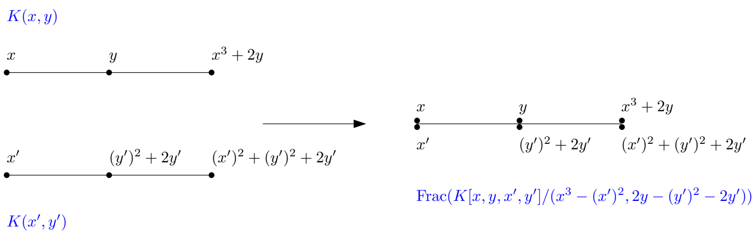


FIGURE 2. Two algebraic representations of $U_{2,3}$ that are algebraically equivalent, witnessed by an algebraic representation of $U^2_{2,3}$

6. Algebraic equivalence of algebraic representations

It would be beneficial for our understanding of algebraic matroids to be able to identify algebraic representations with similar properties. In an attempt to achieve that, we introduce an equivalence relation on algebraic representations.

Definition 5.24. *Let M be a matroid on E and let k be a positive integer. Define the matroid M^k on $E \times \{0, \ldots, k-1\}$ as an extension of M by relabeling $e \in E$ to $(e, 0)$, and adding parallel copies (e, i) of $(e, 0)$ for each $e \in E$ and $i \in \{0, \ldots, k-1\}$.*

In this section we will use the notation $S_i := S \times \{i\}$ for $S \subseteq E$ and $i \in \{0, \ldots, k-1\}$, and similarly $e^i := (e, i)$ for $e \in E$, $i \in \{0, \ldots, k-1\}$. Note that $M^k|_{E_i}$, the restriction of M^k to E_i, is isomorphic to M for $i \in \{0, \ldots, k-1\}$. We proceed to define a notion of algebraic equivalence of algebraic representations of a matroid.

Definition 5.25. *Let M be a matroid on E. Let X, X' be two algebraic representations of M. Then X, X' are said to be* algebraically equivalent *if there exists an algebraic representation Y of M^2 such that $Y|_{E_0} = X$ and $Y|_{E_1} = X'$.*

Example 5.26. *Figure 2 is an example of two representations of $U_{2,3}$ over K that are equivalent, witnessed by the right-hand representation. Equivalently, the prime ideal giving the algebraic equivalence is*

$$\langle x_1^3 + 2x_2 - x_3, (x_1')^2 + x_2' - x_3', x_1^3 - (x_1')^2, 2x_2 - x_2', x_3 - x_3' \rangle.$$

Algebraic equivalence is, as the name suggests, an equivalence relation. In order to show this, we will use the geometric interpretation of algebraic representations.

Lemma 5.27. *Let the algebraic representation Y of M^2 witness algebraic equivalence of X and X'. Then* $\dim Y = \dim X = \dim X'$. *Moreover, if S is an independent set of M, then the image $Z := \overline{\pi_{S_0,S_1}(Y)}$ is equal to a product $\prod_{s \in S} C_s$ with $C_s \subseteq K^{s^0,s^1}$ an irreducible curve projecting dominantly on both coordinate axes of s^0 and s^1.*

PROOF.

$$\dim(X) = \dim(X') = \mathrm{rank}(M) = \mathrm{rank}(N) = \dim(Y).$$

For the second statement, since S is independent, the map $Z \to K^{S_0}$ is dominant, hence in particular Z projects dominantly to K^{s^0} for each $s \in S$. By assumption, then, the image of Z in K^{s^0,s^1} is a curve C_s projecting dominantly to both coordinate axes. Now Z is contained in the product of the C_s, and for dimension reasons equal to that product. □

We call the defining polynomials of each curve C_s the *equivalence polynomials* of Y.

Theorem 5.28. *Algebraic equivalence of algebraic K-representations is an equivalence relation.*

PROOF. Let X, X', X'' be algebraic representations of M. Let $Y \subseteq K^{E_0} \times K^{E_1}$ be an algebraic equivalence of X, X' and let $Y' \subseteq K^{E_1} \times K^{E_2}$ be an algebraic equivalence of X', X''. We want to construct a suitable algebraic representation $Y'' \subseteq K^{E_0} \times K^{E_2}$. For this, let

$$Q := \{(y, y') \in Y \times Y' \mid \pi_{E_1}(y) = \pi_{E_1}(y')\}_{\mathrm{red}}$$

be the reduced fibre product of Y and Y' over X'. Note that in the matroid of Q, the elements E_0 are parallel to elements of E_1 by virtue of Y. Similarly, the elements E_2 are parallel to elements of E_1 by virtue of Y'. Hence $\dim Q = \dim Y = \dim Y' = \dim X = \dim X'$. Moreover, the map $Q \to X'$ is dominant, as both maps $Y \to X'$ and $Y' \to X'$ are dominant. Take any irreducible component P of Q that dominates X', and let Y'' be the closure of the image of P in $K^{E_0} \times K^{E_2}$. For each non-loop $e^0 \in E_0$, the image of P in K^{e^0,e^1,e^2} is an irreducible curve that projects dominantly on K^{e^1}, hence (by properties of Y) also on K^{e^0} and (by properties of Y') also on K^{e^2}. Hence the image of Y'' in K^{e^0,e^2} is a curve that projects dominantly on both coordinate axes. For each loop e^0, the images of Y, Y' in K^{e^0,e^1} and K^{e^1,e^2} have dimension zero, hence so do the images of P and Y'' in K^{e^0,e^2}.

It remains to check that the map $P \to X$, and hence also the map $Y'' \to X$, is dominant (the corresponding statement about $P \to X''$ follows similarly).

But this follows since, on P, the coordinates x_{e^1} and x_{e^2} (with e^0 a loop or not) are algebraic over x_{e^0}. Thus the fibre of the map $P \to X$ over the generic point has dimension 0, and since $\dim P = \dim X'$ the map must be dominant.

We conclude that the relation of algebraic equivalence is transitive. That it is symmetric and reflexive is straightforward. □

While it is easy to construct many equivalent algebraic representations from a given representation, we do not know of a general method to determine whether two algebraic representations are equivalent, even for representations of matroids as small as $U_{2,3}$. Given two algebraic representations of a matroid, is it decidable whether these representations are algebraically equivalent?

For a very small matroid we can show that all algebraic representations are equivalent.

Theorem 5.29. *Let K be an algebraically closed field. All algebraic representations of $U_{1,2}$ over K are algebraically equivalent.*

Proof. Let the algebraic representations be given by $V(\langle f\rangle)$ and $V(\langle g\rangle)$, for irreducible bivariate polynomials $f \in K[x, y]$ and $g \in K[x', y']$. Pick $P(x, x') = x - x'$ as the first equivalence polynomial. Then the resultant $\mathrm{res}_x(f(x, y), P(x, x')) = f(x', y)$.

Now set $Q(y, y') := \mathrm{res}_{x'}(f(x', y), g(x', y'))$. As f and g are irreducible with full support, Q is nonzero, irreducible [**35**], and has support $\{y, y'\}$. It follows that any irreducible component of $V(\langle f, g, P, Q\rangle)$ witnesses algebraic equivalence of the two algebraic representations. □

We show an invariant of equivalence classes of algebraic representations.

Theorem 5.30. *Algebraically equivalent representations have the same Lindström valuation up to translation.*

Proof. Let X and X' be algebraically equivalent representations of a matroid M witnessed by an algebraic representation Y of M^2, with an associated matroid flock $\mathcal{M}$. Let ν be the Lindström valuation of Y. Then by submodularity of ν, for each parallel pair (e^0, e^1), the difference $\nu(S + e^0) - \nu(S + e^1)$ is the same for each $S \subset E_0 \cup E_1$ not containing e^0 and e^1 such that $S + e^0$ is a basis of M^2. By translation, we may assume this difference to be 0. Then for all bases of B of M, we have $\nu(B_0) = \nu(B_1)$. Hence the valuations $\nu\backslash E_0$ and $\nu\backslash E_1$ are the same up to translation. □

6.1. Inequivalent algebraic representations. In some special cases we can solve the problem of determining whether two algebraic representations are algebraically equivalent. We will now give an example of a case where two algebraic representations are not algebraically equivalent using the technique of derivations [**24**].

Theorem 5.31. *Let $M = U_{2,3}$. Suppose K is a field of characteristic 0, and let X, X' be algebraic representations given by the ideals $(x+y+z)$ and $(x'+y'z')$ respectively. Then X and X' are not algebraically equivalent.*

PROOF. Suppose $X \subseteq K^{x,y,z}$ and $X' \subseteq K^{x',y',z'}$ are algebraically equivalent with parallel pairs (x,x'), (y,y') and (z,z'). Let an algebraic equivalence be given by an algebraic representation Y of M^2. Let $P(x,x')$, $Q(y,y')$ and $R(z,z')$ be the equivalence polynomials. Let

$$I = \langle x+y+z, x'+y'z', P(x,x'), Q(y,y'), R(z,z')\rangle$$

be the ideal of Y. Now consider the Jacobian matrix of I over

$$L := \mathrm{Frac}(K[x,y,z,x',y',z']/I):$$

$$J(I) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z' & y' \\ \frac{\partial P}{\partial x} & 0 & 0 & \frac{\partial P}{\partial x'} & 0 & 0 \\ 0 & \frac{\partial Q}{\partial y} & 0 & 0 & \frac{\partial Q}{\partial y'} & 0 \\ 0 & 0 & \frac{\partial R}{\partial z} & 0 & 0 & \frac{\partial R}{\partial z'} \end{bmatrix}.$$

Since P, Q and R are irreducible and have full support, none of their partial derivatives in this matrix are zero. Then the zero pattern makes sure that this matrix represents $(M^2)^*$: indeed, any quadruple of columns that does not correspond to $\{x,x',y,y'\}, \{x,x',z,z'\}$ or $\{y,y',z,z'\}$ is independent. As the set of columns $\{2,3,5,6\}$ is dependent in $(M^2)^*$,

$$0 = \det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & z' & y' \\ \frac{\partial Q}{\partial y} & 0 & \frac{\partial Q}{\partial y'} & 0 \\ 0 & \frac{\partial R}{\partial z} & 0 & \frac{\partial R}{\partial z'} \end{bmatrix} = z'\frac{\partial R}{\partial z'}\frac{\partial Q}{\partial y} - y'\frac{\partial Q}{\partial y'}\frac{\partial R}{\partial z}.$$

Dividing by $\frac{\partial Q}{\partial y}\frac{\partial R}{\partial z}$ gives

$$z'\frac{\frac{\partial R}{\partial z'}}{\frac{\partial R}{\partial z}} - y'\frac{\frac{\partial Q}{\partial y'}}{\frac{\partial Q}{\partial y}} \in I \cap K(y,y',z,z').$$

Due to Lemma 5.27, $I \cap K(y,y',z,z') = \langle Q, R\rangle$. In particular, the term $y'\frac{\frac{\partial Q}{\partial y'}}{\frac{\partial Q}{\partial y}}$ is constant modulo $\langle Q\rangle$. So we get for some $s \in K$ and $f \in K[y,y']$:

$$y'\frac{\partial Q}{\partial y'} + s\frac{\partial Q}{\partial y} = fQ. \tag{5}$$

Since the degree of the left-hand side cannot exceed the degree of Q, we must have $f = c \in K$. We now try to find a polynomial $Q(y, y')$ satisfying this partial differential equation. Let

$$Q = \sum_{k=0}^{N} d_k \cdot (y')^k$$

be of degree N in y', where $d_k = d_k(y)$ is a polynomial in y for each k. Note that $N > 0$, since Q has full support. As the degree of Q is assumed to be N, we have $d_N \neq 0$. Then (5) gives

$$\begin{aligned} 0 &= y' \sum_{k=0}^{N} k d_k \cdot (y')^{k-1} + s \sum_{k=1}^{N} \frac{\partial d_k}{\partial y} \cdot (y')^k - c \sum_{k=0}^{N} d_k \cdot (y')^k \\ &= \sum_{k=0}^{N} \left((k-c) d_k + s \frac{\partial d_k}{\partial y} \right) \cdot (y')^k. \end{aligned}$$

Hence, all coefficients of $(y')^k$ in this sum must be zero. Now we distinguish between the cases $s = 0$ and $s \neq 0$. If $s = 0$, then since $d_N \neq 0$, we must have $c = N$. Then furthermore, we must have $d_k = 0$ for $k < N$. But then $Q = d_N \cdot (y')^N$, when either Q does not depend on y, or Q is reducible; contradiction.

If, on the other hand, $s \neq 0$, then d_N is non-constant. If now $c \neq k$, then d_k must be an exponential function, contradicting the assumption that Q is polynomial. But then d_k must be zero unless $c = k$. But then again $c = N$ and $Q = d_N \cdot (y')^N$, which is reducible; contradiction.

□

7. Algebraic equivalence of linear representations

There exists a natural notion of equivalence for linear matroid representations. Two linear representations of a matroid, viewed as row spaces of matrices A and B respectively, are linearly equivalent if B can be obtained from A by a column scaling and applying elementary row operations. In this section, we consider algebraic equivalence of linear representations. Clearly, linearly equivalent linear representations are also algebraically equivalent, by taking the equivalence polynomials to be $x - c_e x'$, where c_e is the column scalar of the column labeled by e in the linear equivalence. A natural question is how much larger algebraic equivalence classes of linear representations are compared to linear equivalence classes. In characteristic 0, the answer is: not at all.

Theorem 5.32. *Let M be a matroid and let K be a field of characteristic 0. Let V, W be two algebraically equivalent linear representations of M over K. Then V, W are linearly equivalent.*

Proof. Let Y be an algebraic representation of M^2 witnessing algebraic equivalence of V and W. Like in Theorem 5.7, for some smooth point $v \in Y$, the matroid represented by the tangent space T_vY is M^2. Since $Y|_{E_0} = V$ and $Y|_{E_1} = W$ are both linear, we also have $(T_vY)|_{E_0} = V$ and $(T_vY)|_{E_1} = W$. Since for each $e \in E$, e^0 is parallel to e^1 in T_vY, V and W must be linearly equivalent. □

In positive characteristic we use the theory of Frobenius flocks.

Lemma 5.33. *Let M be a matroid and let K be an algebraically closed field of positive characteristic. Let V be a linear representation of M over K, and let X an algebraic representation of M over K. Furthermore, suppose V and X are algebraically equivalent witnessed by an algebraic representation Y of M^2. Let $\mathcal{V} := \mathcal{V}(Y)$ be the Frobenius flock of Y and let $\mathcal{M} := \mathcal{M}(\mathcal{V})$ be the matroid flock of Y. Then there exists $\beta \in \mathbb{Z}^{E_V \cup E_X}$ such that $(\mathcal{V} \backslash E_X)_{\beta_{E_V}} = V$ and $\mathcal{M}_\beta = M^2$.*

Proof. As $Y|_{E_V}$ is linear over K, we have $(\mathcal{M}\backslash E_X)_0 = M(V) = M$, and $(\mathcal{V}\backslash E_X)_0 = V$. By Lemma 3.27, let $\alpha' \in \mathbb{Z}^{E_V \cup E_X}$ be such that $\mathcal{M}_{\alpha'} \backslash E_X = (\mathcal{M}\backslash E_X)_0$. Now consider a central point β of $\mathcal{M}$ such that $\beta \in C_{\alpha'}$. We may assume $\beta_{E_V} = 0$. Since β is central in $\mathcal{M}$, the non-loops of M are non-loops in $\mathcal{M}_\beta$. Then as $M^2 \geq \mathcal{M}_\beta$, each $e \in E_X$ must be parallel in $\mathcal{M}_\beta$ to its counterpart in E_V. Hence $M^2 = \mathcal{M}_\beta$, as required. □

If X' is also linear, then both representations V and X' appear as a minor of some, not necessarily the same, central point in the Frobenius flock of M^2. If these central points are indeed different, the Frobenius automorphism comes into play.

Definition 5.34. *Two algebraic representations X, X' of a matroid M over an algebraically closed field K of characteristic $p \neq 0$ are said to be* field-equivalent *if there exists $i \geq 0$ such that they are algebraically equivalent witnessed by an algebraic representation of M^2 with equivalence polynomials of the form $x - d_e(x')^{p^i}$ or $x' - d_e x^{p^i}$, where $d_e \in K \setminus \{0\}$.*

Theorem 5.35. *Let M be a connected matroid and let K be an algebraically closed field of positive characteristic. Let V, W be two algebraically equivalent linear representations of M over K. Then V and W are field-equivalent.*

Proof. Suppose the algebraic representation Y of M^2 witnesses the algebraic equivalence of V and W. By Theorem 5.17, let $\mathcal{V} := \mathcal{V}(Y)$ be the Frobenius flock of Y. Then by Lemma 5.33, there exist $\beta \in \{0\}^{E_V} \times \mathbb{Z}^{E_W}$ and $\beta' \in \mathbb{Z}^{E_V} \times \{0\}^{E_W}$ such that $\mathcal{M}_\beta = \mathcal{M}_{\beta'} = M^2$, and moreover $\mathcal{V}_\beta \backslash E_W = V$ and $\mathcal{V}_{\beta'} \backslash E_V = W$. As M is connected, so is M^2, and hence due to Lemma 3.16, $\beta = \beta' + k\mathbf{1}$ for some $k \in \mathbb{Z}$. Without loss of generality, $k \geq 0$. Then $\mathcal{V}_\beta \backslash E_W$ and $\mathcal{V}_\beta \backslash E_V$ are linearly equivalent representations of M, since each column of the matrix of $\mathcal{V}_\beta$ is a nonzero multiple of its parallel counterpart. Due to (LF2), $\mathcal{V}_\beta \backslash E_V = k\mathbf{1}\mathcal{V}_{\beta'} \backslash E_V$. Thus V is linearly equivalent to $k\mathbf{1}W$. As K is algebraically closed, V and W are thus field-equivalent. $\square$

Note that we assumed connectedness of the matroid in the last theorem. This is because the theorem is false for disconnected matroids. Instead, the linear representations are then componentwise field-equivalent.

Example 5.36. *Consider the following linear representations of $U_{2,4} \oplus U_{2,4}$ over $\overline{GF(2)(t)}$:*

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & t \end{pmatrix};$$

$$B = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & t^2 \end{pmatrix}.$$

These representations are algebraically equivalent, but not field-equivalent, because the equivalence polynomials in the first component must be linear, while the equivalence polynomials in the second component must be quadratic.

Corollary 5.37. *Two algebraically equivalent linear representations of a matroid are componentwise field-equivalent.*

Proof. Apply Theorem 5.35 to each component. $\square$

CHAPTER 6

Some classes of matroids

1. Introduction

This chapter concerns algebraicity questions for certain interesting classes of matroids. First of all we discuss linear matroids in the first section. Linear matroids over K are algebraic over K. But this is also true for linear matroids over $\mathbb{Q}$, or more generally, linear matroids over the endomorphism ring of a connected one-dimensional algebraic group, as will be discussed in Section 3.

In Sections 4 and 5, we discuss two classes of matroids with a small characteristic set, revisiting results from Lindström [**31**] and Gordon [**18**]. We obtain slightly stronger results using Frobenius-flock methods. None of the classes of matroids discussed here are rigid. Still, Frobenius-flock methods are effective in showing that these matroids are non-algebraic in certain characteristics.

In the final section of this chapter, we turn to Dowling geometries as a class of matroids that is worth investigating. We discuss a matroid that was investigated by Aner Ben-Efraim [**2**], which is single-element extension of a Dowling geometry of rank 3. He showed that this matroid is only algebraic in a single characteristic in a similar way to Lindström.

2. Linear matroids

Linear relations are a special case of algebraic relations, so linearly representable matroids are also algebraic. This observation remains true when we fix a characteristic p and consider (algebraic) representability over a field of characteristic p.

Definition 6.1. *For a matroid M, $\chi_L(M)$ denotes the* linear characteristic set *of M, the set of integers p such that there exists a field of characteristic p over which M is linearly representable. Similarly, $\chi_A(M)$ denotes the* algebraic characteristic set *of M, the set of characteristics in which M is algebraically representable.*

Hence we have $\chi_L(M) \subseteq \chi_A(M)$. Moreover, Ingleton showed that if $0 \in \chi_A(M)$ if and only if $0 \in \chi_L(M)$, so that over fields of characteristic 0, linear and algebraic matroids are the same. The following was observed by Lindström [**34**]. For $v \in \mathbb{Q}^d$, denote

$$x^v := x_1^{v_1} \cdot \ldots \cdot x_d^{v_d}.$$

Lemma 6.2. *Let K be any field. Let $a_1, \ldots, a_n \in \mathbb{Z}^d$. The vectors $a_1, \ldots, a_n$ are linearly dependent over $\mathbb{Q}$ if and only if $x^{a_1}, \ldots, x^{a_n} \in K(x_1, \ldots, x_d)$ are algebraically dependent over K.*

Proof. Suppose $a_1, \ldots, a_n$ are linearly dependent. Then there exist $c_1, \ldots, c_n \in \mathbb{Q}$ which are not all zero such that $\sum_{i=1}^n c_i a_i = 0$. Hence

$$\prod_{i=1}^n (x^{a_i})^{c_i} = x^{\sum_{i=1}^n c_i a_i} = 1.$$

Conversely, suppose $a_1, \ldots, a_n$ are linearly independent. By contradiction, let a minimal polynomial P of $x^{a_1}, \ldots, x^{a_n}$ over F be given. Consider a nonzero term of P, which equals

$$c(x^{a_1})^{c_1} \cdot \ldots \cdot (x^{a_n})^{c_n} = cx^{\sum_{i=1}^n c_i a_i}$$

for some $c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{Z} \setminus \{0\}$. This term must vanish, so there is a different term

$$c'(x^{a_1})^{c'_1} \cdot \ldots \cdot (x^{a_n})^{c'_n} = c'x^{\sum_{i=1}^n c'_i a_i}$$

for some $c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^n c'_i a_i = \sum_{i=1}^n c_i a_i$. But since $a_1, \ldots, a_n$ are linearly independent, $c_i = c'_i$ for all i, contradicting that the second term is different from the first. Hence $x^{a_1}, \ldots, x^{a_n}$ are algebraically independent, as required. □

The above lemma relates linear representability over $\mathbb{Q}$ to algebraic representability over any commutative field. Thus we obtain the following theorem.

Theorem 6.3. *If a matroid M is linearly representable over $\mathbb{Q}$, then M is algebraic in each characteristic.*

This theorem is a special case Theorem 6.4 in the next section.

3. Matroids from connected one-dimensional algebraic groups

This section is based on ongoing work with Jan Draisma and Dustin Cartwright [**5**]. The goal of this section is to show that matroids that are linear over certain skew fields are algebraic. We also describe how the Lindström valuation can be obtained from a linear representation over such a skew field.

An algebraic group is an algebraic variety G endowed with a group structure, where the multiplication and inversion operations of G are morphisms of varieties [**23**]. The dimension of an algebraic group is its dimension as an algebraic variety. We say G is connected if it is irreducible as a variety. We list a classification of connected one-dimensional algebraic groups. Let K be an algebraically closed field of characteristic $p > 0$, and let L be an algebraically closed extension field of K. Then a connected one-dimensional algebraic group G in L is one of the following [**16**, Section 3.1]:

(1) the multiplicative group $\mathbb{G}_m$ of L;
(2) the additive group $\mathbb{G}_a$ of L;
(3) an elliptic curve defined over K.

In all of these cases, G is abelian, in which case $\mathbb{E} := \mathrm{End}(G)$ is the ring of endomorphisms of G. so that the set $\mathbb{E} := \mathrm{End}(G)$ of endomorphisms of G as an algebraic group carries a natural (in general, noncommutative) ring structure, with multiplication defined by $(\phi \cdot \psi)(g) = \phi(\psi(g))$ and addition defined by $(\phi + \psi)(g) = \phi(g) + \psi(g)$. We proceed to list the possible endomorphism rings of connected one-dimensional algebraic groups G in nonzero characteristic [**16**, Section 3.1]:

(1) if $G = \mathbb{G}_m$, then $\mathrm{End}(G) \cong \mathbb{Z}$;
(2) if $G = \mathbb{G}_a$, then $\mathrm{End}(G) \cong K[F]$, a skew polynomial ring in which for all $a \in K$, $Fa = a^p F$;
(3) if G is an elliptic curve, then $\mathrm{End}(G)$ is one of the following [**48**, Theorem V.3.1]:
  - an order in $\mathbb{Q}(\sqrt{D})$ for some $D < 0$;
  - an order in a definite quaternion algebra over $\mathbb{Q}$.

In each of these cases, $\mathrm{End}(G)$ embeds into a skew field S: $\mathrm{End}(\mathbb{G}_m)$ embeds into $\mathbb{Q}$, and $\mathrm{End}(\mathbb{G}_a)$ embeds into $K(F)$, the field of fractions of $K[F]$ [**10**, Cor. 1.3.3]. When G is an elliptic curve, $\mathrm{End}(G)$ embeds into $\mathbb{Q}(\sqrt{D})$ if G is ordinary, or into a definite quaternion algebra over $\mathbb{Q}$ if G is supersingular.

Now let E be a finite set, and let G be a connected one-dimensional algebraic group. Let $X \subseteq G^E$ be a closed, connected d-dimensional subgroup. Then X gives rise to an algebraic matroid $M(X)$ over G in the usual sense: a subset $I \subseteq E$ is independent in the matroid precisely when the projection of X on G^I is dominant. Now choose a non-constant rational function $h : G \dashrightarrow K$ defined in an open neighborhood of the neutral element of G. Then we obtain a rational map $h^E : G^E \dashrightarrow K^E$. The variety $\overline{h^E(X)} \subseteq K^E$ is then an algebraic representation of $M(X)$ over K.

We associate to X a right submodule N of $\mathbb{E}^E$ as follows:

$$N(X) := \{\psi : \psi, \text{ seen as a map } G \to G^E, \text{ has its image in } X\}.$$

The algebraic matroid $M(X)$ is also the linear matroid defined by $N(X)S$.

Theorem 6.4. *Let K be an algebraically closed field of nonzero characteristic and let M be a matroid. If M is linearly representable over*

(1) $\mathbb{Q}$*; or*
(2) $K(F)$*; or*
(3) $\mathbb{Q}(\sqrt{D})$*, where $D < 0$ and there exists an elliptic curve over K with discriminant D; or*
(4) *a definite quaternion algebra over $\mathbb{Q}$, an order in which is an endomorphism ring of some elliptic curve over K.*

then M is algebraic over K.

Proof. The listed skew fields are the skew fields of fractions of $\mathbb{E}$ for some connected one-dimensional algebraic group G defined over K. Let S be one of the given skew fields. Without loss of generality, M is right-representable over S. Let $V \subseteq S^E$ be a right subspace representing M. Consider the saturated right submodule $N = V \cap \mathbb{E}^E$ of $\mathbb{E}^E$. One can show that as N is a saturated right submodule of $\mathbb{E}^E$, there exists a closed, connected subgroup $X \subseteq G^E$ such that $N = N(X)$ [**5**]. Then we have $M(X) = M(NS) = M(V) = M$. Hence M is algebraic over K. □

For an elliptic curve G defined over the finite field $GF(q)$, the Frobenius endomorphism $\phi : x \mapsto x^q$ satisfies $\phi^2 - a\phi + q = 0$, where $a = q + 1 - \#G(GF(q))$ is the *trace of Frobenius* [**48**, Theorem V.2.3.1]. Therefore one of the roots of the polynomial $x^2 - ax + q$, namely

$$\frac{a \pm \sqrt{a^2 - 4q}}{2},$$

must be in $\mathbb{E}$. The number of points on G can be computed in polynomial time [**46**]. Hence the number field $\mathbb{Q}(\sqrt{D})$ in which $\mathbb{E}$ is an order can be computed from G in polynomial time. This will be relevant in Chapter 8, where we focus on small matroids in characteristic 2.

We proceed to construct the Lindström valuation of $X \subseteq G^E$. In order to do that, we need a valuation on $\mathbb{E}$ and a notion of determinant for matrices over S.

We first construct a valuation on $\mathbb{E}$ given by $v : \mathbb{E} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$. If φ is a nonzero element of $\mathbb{E}$, then φ gives rise to an injective homomorphism

$\varphi^* : K(G) \to K(G)$. Suppose K has characteristic p. Define $v(0) := \infty$ and

$$v(\varphi) := \log_p[K(G) : \varphi^* K(G)]_i,$$

the base-p logarithm of the inseparable degree of the given field extension. Then v is a valuation on $\mathbb{E}$, and v extends uniquely to S. Denote by S^* the multiplicative group of S. Since commutators in S have valuation 0, v also induces a map $v : S^*/[S^*, S^*] \to \mathbb{Z}$.

The notion of determinant we use is the *Dieudonné determinant* for matrices over S. This is the unique group homomorphism $\mathrm{Ddet} : GL_r(S) \to S^*/[S^*, S^*]$ sending the diagonal matrix $\mathrm{diag}(a, 1, \ldots, 1)$ to a, and sending $I_r + bE_r^{i,j}$ to 1 for any $b \in S^*$ and $i \neq j$, where $E_r^{i,j}$ is the $r \times r$ matrix with 1 at position (i, j) and 0 everywhere else. We now state a result from [**5**].

Theorem 6.5. *Let G be a connected one-dimensional algebraic group with endomorphism ring $\mathbb{E}$. Let S be the skew field of fractions of $\mathbb{E}$. Let $X \subseteq G^E$ be a closed, connected subgroup. Let A be a matrix with column space $N(X)S$. Then the Lindström valuation of X is given by:*

$$\nu : B \mapsto v(\mathrm{Ddet}(A_B))$$

This theorem shows how to obtain the Lindström valuation of an algebraic matroid from Theorem 6.4 directly from its linear representation over S.

Example 6.6. *Let K be an algebraically closed field of characteristic $p > 0$. Let $G = \mathbb{G}_m$. Then $\mathbb{E} \cong \mathbb{Z}$, $S = \mathbb{Q}$, v is the p-adic valuation on $\mathbb{Q}$, and* Ddet *is the regular determinant on $\mathbb{Q}$.* ♙

4. Lazarson matroids

In this section, we revisit Lindström's theorem on the class of Lazarson matroids [**31**]. We strengthen his result by proving that each Lazarson matroid is only linear-flock representable in a single characteristic.

Let p be prime. We denote by M_p the matroid which is linearly represented over $GF(p)$ by the matrix

$$\begin{array}{c} \begin{array}{cccccccccc} x_0 & x_1 & \cdots & x_p & z & y_0 & y_1 & \cdots & y_p \end{array} \\ \begin{pmatrix} 1 & & & & 1 & 0 & 1 & & 1 \\ & 1 & & & 1 & 1 & 0 & & 1 \\ & & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & 1 & 1 & 1 & 1 & & 0 \end{pmatrix}. \end{array} \tag{6}$$

By M_p^- we denote the matroid which is represented by the same matrix over $\mathbb{Q}$. The Lazarson matroids are the matroids

$$\{M_p : p \text{ prime}\}.$$

It is easy to verify that the following sets are the circuits of M_p:

- $\{x_0, \dots, x_p, z\}$;
- $\{x_0, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_p\}$ for $i = 0, \dots, p$;
- $\{x_i, z, y_i\}$ for $i = 0, \dots, p$;
- $\{y_0, \dots, y_p\}$.

Moreover, M_p^- has the same circuits, except $\{y_0, \dots, y_p\}$.

The following is known about M_p.

Theorem 6.7. *(Lindström [**31**]) If M_p has an algebraic representation over a field K, then the characteristic of K is p.*

We prove a stronger result, which implies this theorem.

Theorem 6.8. *Let K be a field and f an automorphism of K. If char$(K) = p$, then for any linear flock $\mathcal{V}$ of M_p over (K, f) there exists $\alpha \in \mathbb{Z}^E$ such that $M(\mathcal{V}_\alpha) = M_p$. If char$(K) \neq p$, then no linear flock of M_p over (K, f) exists.*

PROOF. Let K be a field and let $\mathcal{V}$ be a flock over (K, f) with $M(\mathcal{V}) = M_p$. Let $\mathcal{M} := \mathcal{M}(\mathcal{V})$.

Note that M_p restricted to $C = \{x_0, \dots, x_p, z\}$ is isomorphic to $U_{p+1,p+2}$. By Lemma 5.23, $M_p \backslash \overline{C}$ is rigid. Thus using Lemma 3.27, let $\alpha' \in \mathbb{Z}^E$ be given so that $\{x_0, \dots, x_p, z\}$ is a circuit of $\mathcal{M}_{\alpha'}$. Now consider $\alpha \in C_{\alpha'} \cap |\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E$. Since C is a spanning circuit, $\mathcal{M}_\alpha$ also contains C as a spanning circuit. Moreover, since α is a central point, $\mathcal{M}_\alpha$ contains no loops.

We now argue that $\mathcal{V}_\alpha$ equals the row space of (6) regarded as a matrix over K (up to column scaling). As $\{x_0, \dots, x_p, z\}$ is a circuit, we may row-reduce so that the columns corresponding to this circuit are as in the matrix. Now let i be given. As $\{x_0, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_p\}$ is a circuit, the i'th entry of column y_i is 0. Since $\{x_i, z, y_i\}$ is a circuit, all entries $j \neq i$ of the column y_i are equal, say b_i. As $\mathcal{M}_\alpha$ is loop-free, we must have $b_i \neq 0$. So by column scaling, we may choose $b_i = 1$. Hence indeed, $\mathcal{V}_\alpha$ equals the row space of (6) up to column scaling.

Finally, $\{y_0, \dots, y_p\}$ is a circuit. Hence the determinant corresponding to the columns $y_0, \dots, y_p$ must be zero. This determinant equals $(-1)^{p+1}p$. If char$(K) \neq p$, then we have a contradiction, and there exists no flock of M_p over (K, f). If on the other hand char$(K) = p$, then $M(\mathcal{V}_\alpha) = M_p$. $\square$

The argumentation in the above proof also yields a similar result about M_p^-.

Theorem 6.9. *Let K be a field with char$(K) \neq p$ and f an automorphism of K. Then for any linear flock $\mathcal{V}$ of M_p^- over (K, f) there exists $\alpha \in \mathbb{Z}^E$ such that $M(\mathcal{V}_\alpha) = M_p^-$.*

Proof. Let K be a field with $\mathrm{char}(K) \neq p$ and let $\mathcal{V}$ be a linear flock over (K, f) with $M(\mathcal{V}) = M_p^-$. Using the circuits of M_p^-, we may construct the matrix of $\mathcal{V}_\alpha$ for a central point α in the same way as in the proof of Theorem 6.8. Finally, $\{y_0, \ldots, y_p\}$ is not a circuit in M_p^- as opposed to M_p. As $\mathrm{char}(K) \neq p$, the determinant corresponding to the columns $y_0, \ldots, y_p$ is nonzero. So $M(\mathcal{V}_\alpha) = M_p^-$ as required. □

Theorem 6.7 follows by noting that any algebraic representation of M_p gives rise to a linear flock due to Theorem 5.13. Theorems 6.8 and 6.9 show that M_p and M_p^- have a property that is satisfied by rigid matroids. However, neither class of matroid is rigid: for large enough p there exist nontrivial valuations of M_p and M_p^-.

5. Reid geometries

We now move on to a result from Gordon on Reid geometries [**18**]. Again, we show that each Reid geometry is only linear-flock representable in a single characteristic, strengthening Gordon's result.

Let p be prime. Let N_p be the matroid which is linearly represented over $GF(p)$ by the matrix

$$\tag{7} \begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & a_0 & b_0 & a_1 & b_1 & & a_{p-1} \quad b_{p-1} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & \cdots & p-1 & p-1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & & 1 & 1 \end{pmatrix} \end{array}.$$

Let N_p^- be the matroid which is represented by the same matrix over $\mathbb{Q}$. Note that the circuits of N_p^- are exactly the circuits of N_p, except $\{a_{p-1}, b_0, x_3\}$. The Reid geometries are the matroids

$$\{N_p : p \text{ prime}\}.$$

Theorem 6.10. *(Gordon [**18**]) If N_p has an algebraic representation over a field K, then the characteristic of K is p.*

We prove a stronger result, which implies this theorem.

Theorem 6.11. *Let K be a field and f an automorphism of K. If $\mathrm{char}(K) = p$, then for any linear flock $\mathcal{V}$ of N_p over (K, f) there exists $\alpha \in \mathbb{Z}^E$ such that $M(\mathcal{V}_\alpha) = N_p$. If $\mathrm{char}(K) \neq p$, then no linear flock of N_p over (K, f) exists.*

Proof. Let K be a field and let $\mathcal{V}$ be a linear flock over (K, f) with $M(\mathcal{V}) = N_p$. Denote $\mathcal{M} := \mathcal{M}(\mathcal{V})$.

Note that $\mathcal{V} \backslash \overline{\{x_1, x_2, x_3, a_0, a_1, b_0, b_1\}}$ is a linear flock of a matroid isomorphic to M_2 if $\mathrm{char}(K) = 2$ or to M_2^- if $\mathrm{char}(K) \neq 2$. In both cases, due to

Theorems 6.8 and 6.9 there is β such that $(\mathcal{V}\backslash\overline{\{x_1,x_2,x_3,a_0,a_1,b_0,b_1\}})_\beta$ equals the row space of

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & a_0 & b_0 & a_1 & b_1 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

up to column scaling.

Thus using Lemma 4.8, let $\alpha' \in \mathbb{Z}^E$ be given so that

$$\mathcal{V}_{\alpha'}\backslash\overline{\{x_1,x_2,x_3,a_0,a_1,b_0,b_1\}} = (\mathcal{V}\backslash\overline{\{x_1,x_2,x_3,a_0,a_1,b_0,b_1\}})_\beta.$$

Now consider $\alpha \in C_{\alpha'} \cap |\mathcal{S}_0(\mathcal{M})| \cap \mathbb{Z}^E$. Then

$$\mathcal{V}_{\alpha}\backslash\overline{\{x_1,x_2,x_3,a_0,a_1,b_0,b_1\}} = \mathcal{V}_{\alpha'}\backslash\overline{\{x_1,x_2,x_3,a_0,a_1,b_0,b_1\}}.$$

Moreover, since α is a central point, $\mathcal{M}_\alpha$ contains no loops.

In each of the subsequent steps, we use the following important fact. By definition of $\mathcal{M}$, we have that for all $I \subseteq E$, $r_\alpha(I) \leq r(I)$. Hence if I is a circuit of N_p of size 3, then I is dependent in $\mathcal{M}_\alpha$.

For $i = 2, \dots, p-1$ we consecutively find the columns b_i and a_i in the matrix of $\mathcal{V}_\alpha$ as follows. Consider the circuits $\{x_2, b_0, b_i\}$ and $\{x_3, a_{i-1}, b_i\}$. We find the column of b_i:

$$\begin{array}{ccccc} x_2 & b_0 & b_i & x_3 & a_{i-1} \end{array}$$
$$\begin{pmatrix} 0 & 1 & \mathbf{1} & 1 & 0 \\ 1 & 0 & \mathbf{i} & 1 & i-1 \\ 0 & 1 & \mathbf{1} & 0 & 1 \end{pmatrix}.$$

The column b_i must be nonzero and lie in the intersection of the subspace spanned by the columns x_2, b_0 and the subspace spanned by the columns x_3, a_{i-1}, fixing b_i up to scaling.

And then using similar reasoning, considering the circuits $\{x_2, a_0, a_i\}$ and $\{x_1, a_i, b_i\}$, we find the column of a_i:

$$\begin{array}{ccccc} x_2 & a_0 & a_i & x_1 & b_i \end{array}$$
$$\begin{pmatrix} 0 & 0 & \mathbf{0} & 1 & 1 \\ 1 & 0 & \mathbf{i} & 0 & i \\ 0 & 1 & \mathbf{1} & 0 & 1 \end{pmatrix}.$$

Finally we distinguish between the possible characteristics of K. As $\{a_{p-1}, b_0, x_3\}$ is a circuit, the corresponding columns must be dependent.

$$\det \begin{array}{c} \begin{array}{ccc} a_{p-1} & b_0 & x_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ p-1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array} = p = 0$$

if and only if the characteristic of K equals p. Hence if $\mathrm{char}(K) \neq p$, $\{a_{p-1}, b_0, x_3\}$ would be independent, contradicting the assumption that $\mathcal{V}$ is a linear flock of N_p. If, on the other hand, $\mathrm{char}(K) = p$, then $\mathcal{M}_\alpha = N_p$ as required. □

Theorem 6.12. *Let K be a field with $char(K) \neq p$ and f an automorphism of K. Then for any linear flock $\mathcal{V}$ of N_p^- over (K, f) there exists $\alpha \in \mathbb{Z}^E$ such that $M(\mathcal{V}_\alpha) = N_p^-$.*

The proof is similar to the proof of Theorem 6.9. Similarly to the Lazarson matroids, the matroid classes N_p and N_p^- are not rigid either for sufficiently large p.

6. Dowling geometries

In this section, we consider Dowling group geometries of rank 3.

Let G be a finite group with identity e. We define a matroid from G as follows. Let $J := \{j_{AB}, j_{BC}, j_{AC}\}$ be a set of three points called 'joints'. Furthermore, let a point g_x be given for each $g \in G$ and $x \in \{A, B, C\}$ and define $G_x := \{g_x : g \in G\}$. Let

$$E := J \cup G_A \cup G_B \cup G_C.$$

Then we define $D_3(G)$, the rank 3 Dowling geometry of G, to be the matroid on E whose lines are

- $\{j_{AB}, j_{AC}\} \cup G_A$;
- $\{j_{AB}, j_{BC}\} \cup G_B$;
- $\{j_{AC}, j_{BC}\} \cup G_C$;
- $\{f_A, g_B, h_C\}$ for all $f, g, h \in G$ such that $fgh = e$.

In Figure 1 the rank 3 Dowling geometry of the trivial group is shown. This particular matroid is $M(K_4)$, the cycle matroid of the complete graph on 4 vertices, and it is a matroid for which Evans and Hrushovski showed that its algebraic representations must necessarily come from a connected one-dimensional algebraic group [**16**].

On the other hand, Dowling [**12**, Thms. 9,10] showed that $D_3(G)$ is linear over a field K if and only if G is a subgroup of the multiplicative group K^* of K. In particular, $D_3(G)$ is only linear if G is cyclic. A natural question is:

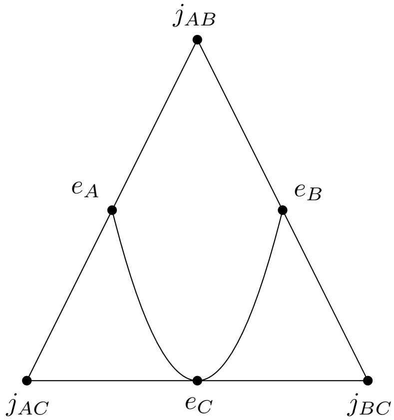


FIGURE 1. The rank 3 Dowling geometry of the trivial group.

what can be said about algebraic representations or representations over skew fields of rank 3 Dowling geometries?

The presence of many $M(K_4)$-minors in rank 3 Dowling geometries suggests the following conjecture.

Conjecture 6.13. *Suppose M is a rank 3 Dowling geometry. Then any algebraic representation of M comes from a connected one-dimensional algebraic group.*

Aner Ben-Efraim constructed a matroid which is only algebraic in characteristic 2 [**2**]. Let $Q_8 = \{e, -e, i, -i, j, -j, k, -k\}$ be the quaternion group, where e is the identity element. As Q_8 is not abelian, $D_3(Q_8)$ is not linear over any field. We extend the matroid $D_3(Q_8)$ by an element O such that the circuits containing O are $\{j_{AB}, O, (-e)_C\}$, $\{j_{AC}, O, (-e)_B\}$ and $\{j_{BC}, O, (-e)_A\}$. We call this extension $D_3^O(Q_8)$. It is straightforward to check that the restriction of $D_3^O(Q_8)$ to the elements $\{j_{AB}, j_{AC}, j_{BC}, O, (-e)_A, (-e)_B, (-e)_C\}$ is isomorphic to the Non-Fano matroid. Now Ben-Efraim shows the following.

Theorem 6.14. *Suppose $D_3^O(Q_8)$ is algebraic over K. Then K has characteristic 2.*

We sketch the proof in terms of flocks. Let X be an algebraic representation of $D_3^O(Q_8)$ over K. Since $D_3^O(Q_8)$ is nonlinear, the Lindström valuation of X is not a trivial valuation. Let $\mathcal{M} = \mathcal{M}(X)$. Then Ben-Efraim shows that there must be a central point $\alpha \in \mathcal{S}_0(\mathcal{M}) \cap \mathbb{Z}^E$ such that $\mathcal{M}_\alpha$ contains the Fano matroid as a minor. Hence $\mathcal{M}_\alpha$ can only be linear over K if K has characteristic 2. Thus if $\mathrm{char}(K) \neq 2$, then $\mathcal{M}$ does not admit a linear-flock representation over K. Due to Theorem 5.1, $D_3^O(Q_8)$ is then not algebraic over K.

CHAPTER 7

Computational problems

1. Introduction

This chapter is concerned with the computational aspects of Frobenius flocks. There are two main goals of this chapter: determining Frobenius-flock representability of matroids, and computing the Frobenius flock of an algebraic matroid representation. Both of these are computational tasks. All algorithms mentioned in this chapter are supplemented with an implementation in Sage. The code is available on
www.github.com/gpbollen/Algebraicity-of-Matroids-and-Frobenius-Flocks.

1.1. Determining Frobenius-flock representability. The first goal is to determine whether a given matroid is Frobenius-flock representable in a given characteristic. For that we will use Theorem 4.30. However, the theorem applies to matroid flocks, and not to matroids. So we have to consider all possible matroid flocks of a given matroid. Due to Theorem 3.3, matroid flocks correspond to integer-valued valuations. The set of integer-valued valuations of a matroid is infinite, as for instance scaling a valuation by a positive scalar yields another valuation.

For many matroids, there is no need to check every valuation to know whether or not the matroid is Frobenius-flock representable. This is obvious if the answer is yes — one just needs to find a valuation ν such that $\mathcal{M}^\nu$ is Frobenius-flock representable. If the answer is no, it often suffices to check a finite number of valuations: one for each combinatorial type. Valuations of the same combinatorial type share certain properties. One of these properties (Theorem 7.13) is that the image of the matroid flock $\mathcal{M}^\nu$ is equal to the image of $\mathcal{M}^\tau$ if ν and τ are valuations of the same combinatorial type. If one of the matroids in the image is nonlinear in the given characteristic, then none of the valuations of this combinatorial type will give rise to a matroid flock which is Frobenius-flock representable. Then only one valuation of the combinatorial type needs to be checked in order to exclude all valuations of the same combinatorial type.

Section 2 is devoted to computing one valuation for each cell of the Dressian of a matroid. An algorithm is described that computes one valuation for each occurring combinatorial type. In Section 3, we compute the relevant properties of a valuation so that the conditions of Theorem 4.30 can be checked. In particular, the central points of the matroid flock $\mathcal{M}^\nu$, and their relative positions in the matroid flock are considered. Then in Section 4 an algorithm is described that checks whether a matroid flock satisfies the conditions of Theorem 4.30. Finally in Section 5, we use the algorithms to say something about Frobenius-flock representability of a matroids. There are three possible outcomes:

(1) a Frobenius-flock representation has been found;
(2) each matroid flock contains a non-linear matroid and thus the matroid is not Frobenius-flock representable;
(3) neither (1) nor (2) hold.

Rigid nonlinear matroids will always fall into the second case. If a matroid M is rigid, then each valuation ν of M is trivial, and M has only one combinatorial type. Hence there exists $\alpha \in \mathbb{Z}^E$ such that $\mathcal{M}^\nu_\alpha = M$. If M is nonlinear, then the matroid flock $\mathcal{M}^\nu$ is not Frobenius-flock representable. Similarly, as was shown in the previous chapter, Lazarson matroids and Reid geometries are not Frobenius-flock representable due to the second case.

In the third case, we generally do not know whether the matroid is Frobenius-flock representable. However, in some cases we can still argue that no Frobenius-flock representation exists. We illustrate this on the *quaternary butterfly matroid* (Figure 1).

1.2. Computing Frobenius flocks. The second goal of this chapter is to compute a Frobenius flock of an algebraic representation of a matroid. That is, given an algebraic representation X, we want to compute a general $v \in X$, and then compute the Frobenius flock associated to (X, v) according to Theorem 5.13. In order to find a $v \in X$ satisfying (*) from Theorem 5.13, which exists due to Theorem 5.17, it suffices to check (*) for a specific set of $\alpha \in \mathbb{Z}^E$. Due to Theorem 4.30, these α are exactly the central points of $\mathcal{M}(X)$. So it makes sense to first compute the underlying matroid flock of X, which is derived from the Lindström valuation of X. In order to compute the Lindström valuation, I use the construction by Cartwright [**9**], which is described in Section 6. The required properties of the corresponding matroid flock are computed in Section 3.

Finally, in Section 7, we compute the tangent space $T_{\alpha v}\alpha X$ for a given $v \in X$. This gives us the tools we need to check whether some v satisfies (*) from Theorem 5.13 at all central points α of $\mathcal{M}(X)$. Once such a general v is found, the algorithm that computes $T_{\alpha v}\alpha X$ is, as a function of α, the Frobenius flock associated to (X, v).

2. Computing the cells of the Dressian

The aim of this section is to provide an algorithm that yields one integral valuation in the interior of each cell of the Dressian of a matroid M.

A naive approach to this would be to enumerate all cells of $\mathfrak{D}(M)$ by computing the intersections of all possible combinations of polyhedra from the polyhedral complexes $\mathcal{P}_Q$ and $\mathcal{P}_O$ for degenerate pure quadrangles Q and octahedra O, and then computing a valuation in the interior of each of the cells. However, this number of cells is very large. As for each O, $\mathcal{P}_O$ contains 4 nonempty cells, this approach requires enumeration and inspection of $4^{|O(M)|}$ open cells. Most of these cells are empty, and as $O(M)$ is generally quite large, this approach is extremely inefficient.

Instead, we use a type of backtracking approach, which is made precise in Algorithm 1.

Lemma 7.1. *Let a natural number n be given. On input*

(1) *a polyhedron $D \subseteq \mathbb{R}^n$;*
(2) *a finite set U;*
(3) *a finite set A_O of disjoint polyhedra in $\mathbb{R}^n$ for each $O \in U$,*

Algorithm 1 returns the set

$$\mathcal{D}(D, U, A) := \left\{ P : P = D \cap \bigcap_{O \in U} P_O \neq \emptyset, \text{ where } P_O \in A_O \text{ for each } O \right\}.$$

Proof. We use induction on $|U|$. If $U = \emptyset$, the algorithm returns either $\emptyset$ (if $D = \emptyset$) or $\{D\}$, as required.

Now suppose $|U| > 0$. If after constructing A'_O for each $O \in U$, there exists O such that $A'_O = \emptyset$, then $D \cap P = \emptyset$ for all $P \in A_O$, and it follows that $\mathcal{D}(D, U, A) = \emptyset$. Otherwise, for each $O \in U$ such that $|A'_O| = 1$, each element of $\mathcal{D}$ must be contained in the unique set P'_O in A'_O. Let D' and U' be as constructed in the algorithm. Then $\mathcal{D}(D, U, A)$ equals $\mathcal{D}(D', U', A')$. Thus if D' is empty, so is $\mathcal{D}(D, U, A)$. If D' is nonempty and U' is empty, then $|A'_O| = 1$ for each $O \in U$, and $\mathcal{D}(D, U, A)$ equals the singleton set $\{D'\}$. Now suppose D' and U' are both nonempty, and consider any $O \in U'$. Then any element of $\mathcal{D}(D, U, A)$ is contained in some $P \in A'_O$. Hence $C \in \mathcal{D}(D, U, A) = \mathcal{D}(D', U', A')$ if and only if $C \in \bigcup_{P \in A'_O} \mathcal{D}(D' \cap P, U' - O, (A'_T)_{T \in U' - O})$. By induction on $|U|$, this is exactly the set that is returned.

Algorithm 1 CombinatorialTypes(D, U, A)

input: A polyhedron D; a finite set U; a finite set A_O of disjoint polyhedra for each $O \in U$.
output: The set $\{P : P = D \cap \bigcap_{O\in U} P_O \neq \emptyset,$ where $P_O \in A_O$ for each $O \in U\}$.
for $O \in U$ **do**
    $A'_O \leftarrow A_O$
    for $P \in A_O$ **do**
        if $D \cap P = \emptyset$ **then**
            $A'_O \leftarrow A'_O - P$
        end if
    end for
end for
if $\exists O : A_O = \emptyset$ **then**
    return $\emptyset$
end if
$D' \leftarrow D$
$U' \leftarrow U$
for $O \in U : |A'_O| = 1$ **do**
    $P \leftarrow \bigcup A'_O$
    $D' \leftarrow D' \cap P$
    $U' \leftarrow U - O$
end for
if $D' = \emptyset$ **then**
    return $\emptyset$
end if
if $U' = \emptyset$ **then**
    return $\{D'\}$
end if
pick $O \in U'$
return $\bigcup_{P\in A'_O}$ CombinatorialTypes($D' \cap P, U' - O, (A'_T)_{T\in U'-O}$)

Checking whether a polyhedron is empty can be done in polynomial time by an LP-solver. The algorithm terminates because U and A are finite, and $|U|$ decreases in each iteration. $\square$

Let $D := \bigcap_{Q \in Q(M)} P_Q$ and let $U := O(M)$ be the set of octahedra of M. For $O \in O(M)$ with $O = (B_{11}, \ldots, B_{32})$, we set

$$\begin{aligned} \tilde{P}_O^0 &:= P_O^0; \\ \tilde{P}_O^i &:= \{\nu \in \mathbb{R}^{\mathcal{B}} : \nu(B_{i1}) + \nu(B_{i2}) - 1 \\ &\qquad \geq \nu(B_{j1}) + \nu(B_{j2}) = \nu(B_{k1}) + \nu(B_{k2})\} \subseteq \operatorname{int}(P_O^i), \end{aligned}$$

where i, j, k are distinct members of $\{1, 2, 3\}$ and int denotes the interior. Let

$$A_O := \{\tilde{P}_O^i : i \in \{0, 1, 2, 3\}\}$$

for each $O \in O(M)$. Let $\mathcal{D}$ be the set $\mathcal{D}(D, U, A)$ as in the previous lemma.

Lemma 7.2. *Let M be a matroid and let $\mathcal{D}$ be as above. Then there exists a bijection from $\mathfrak{D}(M)$ to $\mathcal{D}$ given by $T \mapsto C$ such that $C \subseteq T$.*

Proof. Let $\kappa : O(M) \to \{0, 1, 2, 3\}$ be given such that

$$T = \bigcap_{O \in O(M)} P_O^{\kappa(O)} \cap \bigcap_Q P_Q \neq \emptyset.$$

We show that there exists $C \in \mathcal{D}$ such that $C \subset T$.

If $\kappa = 0$, then this existence follows from Lemma 7.1. So suppose $\kappa \neq 0$. Define the slack of an inequality $a \geq b$ to be $a - b$. Suppose $\nu \in T$ and let ε be the smallest occurring slack of the defining inequality of $P_O^{\kappa(O)}$ for $O \in O(M)$ such that $\kappa(O) > 0$. Then $\varepsilon > 0$. Hence by Lemma 2.33, $\varepsilon^{-1}\nu \in T$, and the smallest occurring slack for $\varepsilon^{-1}\nu$ is 1. Hence as T is nonempty, neither is

$$C = \bigcap_{O \in O(M)} \tilde{P}_O^{\kappa(O)} \cap \bigcap_Q P_Q.$$

We have $C \subset T$ and, due to Lemma 7.1, $C \in \mathcal{D}$.

Conversely, each element of $\mathcal{D}$ is contained in a cell of $\mathfrak{D}(M)$, but no two in the same cell. This makes the map $T \mapsto C$ a bijection, as required. □

Now each element $C \in \mathcal{D}$ is a polyhedron. Using an LP-solver, we can find some $\nu^C \in C \cap \mathbb{Q}^{\mathcal{B}}$. Using Lemma 2.33, we may scale ν^C so that $\nu^C \in C \cap \mathbb{Z}^{\mathcal{B}}$. Due to Theorem 2.30, ν^C is a valuation of M. Due to the previous lemma, each ν^C has a distinct combinatorial type. Hence the set

$$\{\nu^C : C \in \mathcal{D}\}$$

contains an integral valuation for each combinatorial type of M. With that, we have achieved the goal of this section.

3. Computing the skeleton graph of a matroid flock

In this section we compute the points and lines of the skeleton graph $G_{\mathcal{M}}$ of a matroid flock $\mathcal{M}$. For each point of $G_{\mathcal{M}}$, we compute a representative $\alpha \in \mathbb{Z}^E$. Then for each pair of points, we check whether they are connected by a line in $G_{\mathcal{M}}$. First, we recall how to compute $\mathcal{M}$ from its valuation.

By Theorem 3.3, it is straightforward to compute a matroid flock from its valuation. Let $\mathcal{M} = \mathcal{M}^\nu$. Then for each $\alpha \in \mathbb{Z}^E$, we have $\mathcal{M}_\alpha = \mathcal{M}^\nu_\alpha$. Computing $\mathcal{M}^\nu_\alpha$ requires finding the maximizers of a linear function on $\mathcal{B}^\nu$:

$$\mathcal{B}^\nu_\alpha = \arg\max_{B \in \mathcal{B}^\nu} \{e_B \cdot \alpha - \nu(B)\}.$$

This allows us to compute $\mathcal{M}_\alpha$ for each α. However, we also want to locate the cells in $\mathcal{S}_1(\mathcal{M})$ (the points and lines of the skeleton graph) for the linear-flock representability algorithms. We show how we find them here.

Denote by $T = T(M)$ the linear subspace of $\mathbb{R}^{\mathcal{B}^\nu}$ of trivial valuations. By Lemma 2.31, $T \subseteq \Lambda(\mathfrak{D}(M^\nu))$. For any $\rho \in (\nu + T) \cap \mathbb{R}^{\mathcal{B}^\nu}_{\geq 0}$, denote

$$\mathcal{D}_\rho := \{B \in \mathcal{B} : \rho(B) = 0\}$$

Due to Lemma 2.16, $\mathcal{D}_\rho$ is the basis set of a matroid.

Lemma 7.3. *Let M be a matroid on E with basis set $\mathcal{B}$. Let ν be a valuation of M. Suppose $\alpha \in \mathbb{R}^E$ and $\rho \in \mathbb{R}^{\mathcal{B}}$ are such that $\rho = \nu - \sum_{e\in E} \alpha_e \tau_e$, and $\min_{B\in\mathcal{B}} \rho(B) = 0$. Then $\mathcal{M}^\nu_\alpha = (E, \mathcal{D}_\rho)$.*

Proof. The valuation $\sum_{e\in E} \alpha_e\tau_e \in T$ is the trivial valuation of M given by $B \mapsto \sum_{e\in B} \alpha_e$. Thus by Lemma 3.21, $\mathcal{M}^\nu_\alpha = \mathcal{M}^\rho_0$. But $\mathcal{B}^\rho_0$ is just the set of bases B for which $\rho(B)$ is minimal, and hence 0 by the assumptions on ρ. In other words, $\mathcal{B}^\rho_0 = \mathcal{D}_\rho$, as required. □

For a given valuation ν, the set $(\nu + T) \cap \mathbb{R}^{\mathcal{B}}_{\geq 0}$ is a polyhedron in $\mathbb{R}^{\mathcal{B}}$. The vertices of this polyhedron correspond to valuations ρ for which $\mathcal{D}_\rho$ is maximal. Hence each vertex gives rise to a central point α of $\mathcal{M}^\nu$ due to the above lemma. So in order to find the central points of $\mathcal{M}^\nu$, we compute the vertices (modulo lineality) of this polyhedron and retrieve one $\alpha^C \in C \cap \mathbb{Q}^E$ for each $C \in \mathcal{S}_0(\mathcal{M}^\nu)$.

Due to Lemma 3.16, we may choose a canonical representative β^C of C satisfying $\min_{e\in J} \beta_e = 0$ for each component J of M. That is, let $\zeta : \mathbb{Z}^E \to \mathbb{Z}^E$ be given by

$$\zeta(\alpha^C) = \alpha^C - \sum_J \min_{e\in J}(\alpha^C_e) e_J.$$

If ν is integral, then so is $\beta^C = \zeta(\alpha^C)$, and in that case ζ coincides with ζ from Lemma 4.31 on the central points of $\mathcal{M}$. Define

$$\mathcal{C}(\mathcal{M}^\nu) := \{\zeta(\alpha^C) : C \in \mathcal{S}_0(\mathcal{M}^\nu)\}$$

as the set of representatives of the cells in $\mathcal{S}_0(\mathcal{M}^\nu)$. This is the vertex set of the skeleton graph of $\mathcal{M}^\nu$.

Next, we characterise when two vertices are neighbors in the skeleton graph. So first, we find the pairs of points that are at distance ke_I from each other, modulo $\Lambda(\mathcal{M}^\nu)$. For simplicity, we restrict ourselves to the case of connected matroids, where the lineality space is just generated by $\mathbf{1}$ due to Lemma 3.19. This is no loss of generality, as the skeleton graph of a disconnected matroid flock can be reconstructed from the skeleton graphs of its components.

Lemma 7.4. *Let E be a nonempty set. Let $\alpha, \beta \in \mathbb{Z}^E$. Then there exists $I \subset E$ and $k, l \in \mathbb{Z}$ such that*

$$\beta - \alpha = ke_I + l\mathbf{1}$$

if and only if

$$|\{(\beta - \alpha)_e : e \in E\}| \leq 2.$$

PROOF. If $\beta - \alpha = ke_I + l\mathbf{1}$, then the values assumed by $(\beta - \alpha)_e$ for $e \in E$ are either k or $l + k$ (which may coincide). Conversely, if $\{(\beta - \alpha)_e : e \in E\} = \{a, b\}$, then we pick I such that $(\beta - \alpha)_e = a$. Pick $k = a$ and $l = b - a$. Then indeed, $\beta - \alpha = ke_I + l\mathbf{1}$. $\square$

Let

$$\begin{aligned} N(\mathcal{M}^\nu) := \quad & \{(\alpha, \beta, I) : \alpha, \beta \in \mathcal{C}(\mathcal{M}^\nu), \alpha \neq \beta, I \subseteq E, \\ & \text{and } \exists k, l \in \mathbb{Z} : \beta - \alpha = ke_I + l\mathbf{1}\}. \end{aligned}$$

Due to Lemma 7.4, $N(\mathcal{M}^\nu)$ is straightforward to compute. Note that being at distance ke_I of each other modulo the lineality space is not sufficient for points to be neighbors, as the points between them might not all lie in the same cell of $\mathcal{S}_1(\mathcal{M}^\nu)$. Define

$$A(\mathcal{M}^\nu) := \left\{(\alpha, \beta, I) \in N(\mathcal{M}^\nu) : \mathcal{M}^\nu_\alpha / I = \mathcal{M}^\nu_\beta \backslash I\right\}.$$

Then whenever (α, β, I) and $(\beta, \alpha, \overline{I})$ are both elements of $A(\mathcal{M}^\nu)$, $\{\alpha, \beta\}$ is an edge of the skeleton graph.

We conclude that points and lines of the skeleton graph $G_{\mathcal{M}}$ are given by $\mathcal{C}(\mathcal{M})$ and

$$\left\{\{\alpha, \beta\} \in \binom{\mathcal{C}(\mathcal{M})}{2} : \exists I \subseteq E : (\alpha, \beta, I), (\beta, \alpha, \overline{I}) \in A(\mathcal{M})\right\}$$

respectively.

4. Deciding Frobenius-flock representability of matroid flocks

Let K be an algebraically closed field of nonzero characteristic. A Frobenius-flock representation of a matroid flock $\mathcal{M}$ over K exists if and only if $\mathcal{M}$ admits a skeleton representation over K due to Lemma 4.29 and Theorem 4.30. In the previous section we have seen how to compute the set of representative central points $\mathcal{C}(\mathcal{M})$ of $\mathcal{M}$ and the pairs of $\mathcal{C}(\mathcal{M})$ that are neighboring in the skeleton graph of $\mathcal{M}$. Moreover, it was described how to compute the labeled arcs $A(\mathcal{M})$ that determine the edges of the skeleton graph. So in order to find a skeleton representation $\mathcal{V}$ of $\mathcal{M}$, we must find a set of vector spaces $\mathcal{V}_\alpha$ for $\alpha \in \mathcal{C}(\mathcal{M})$ satisfying two types of conditions.

The first condition, the *linearity condition*, is that for each central point $\alpha \in \mathcal{C}(\mathcal{M})$, $\mathcal{V}_\alpha$ linearly represents $\mathcal{M}_\alpha$. The second condition, the *neighbors condition*, is that the linear representations $\mathcal{V}_\alpha, \mathcal{V}_{\alpha+e_I+l\mathbf{1}}$ at neighboring central points should be related by

$$\mathcal{V}_\alpha/I = (-l)\mathbf{1}\mathcal{V}_{\alpha+ke_I+l\mathbf{1}}\backslash I; \tag{8}$$

$$(-(k+l)\mathbf{1})\mathcal{V}_{\alpha+ke_I+l\mathbf{1}}/\overline{I} = \mathcal{V}_\alpha\backslash\overline{I}. \tag{9}$$

The third property of a skeleton representation relates $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ in the same cell of $\mathcal{S}_0(\mathcal{M})$, and hence it does not yield a condition on the $\mathcal{V}_\alpha$ for $\alpha \in \mathcal{C}(\mathcal{M})$. Fixing $\mathcal{V}_\alpha$ for one central point per cell fixes $\mathcal{V}_\alpha$ for all central points in agreement with this third property due to Lemma 4.31. Then due to Theorem 4.30, if the $\mathcal{V}_\alpha$ for $\alpha \in \mathcal{C}(\mathcal{M})$ satisfy the linearity and neighbors conditions, $\mathcal{V}$ uniquely extends to a Frobenius flock.

The remainder of this section is devoted to finding a set of vector spaces $\mathcal{V}_\alpha$ for each $\alpha \in \mathcal{C}(\mathcal{M})$ that satisfies the linearity and neighbors conditions. We describe these conditions by a system of polynomial equations over K. Finally we check if all of these polynomial equations have a common solution. We take the ideal $\mathcal{I}$ generated by the polynomial equations. Due to Hilbert's Nullstellensatz, $1 \in \mathcal{I}$ if and only if there exists no common solution to the polynomial equations. We can check whether $1 \in \mathcal{I}$ by computing the Gröbner basis g of $\mathcal{I}$ and checking if $1 \in g$.

Note that a matroid flock $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2$ is Frobenius-flock representable if and only if both $\mathcal{M}^1$ and $\mathcal{M}^2$ are Frobenius-flock representable. So there is no harm in assuming $\mathcal{M}$ is connected.

4.1. The linearity condition.

Definition 7.5. *A matrix $r \times E$ $A(x_1, \ldots, x_N)$ is a* general matrix representation *of a matroid M over an algebraically closed field K if and only if for* *each linear representation $V \in \mathrm{Gr}_r(K^E)$ of M there exist $a_1, \ldots, a_N \in K$ such that $A(a_1, \ldots, a_N) = V$.*

Clearly, an $r \times E$ matrix with a variable at each entry is a general matrix representation of any rank r matroid on E. If M is a matroid, A is an $r \times E$ matrix, Q is an invertible $r \times r$ matrix and D is a nonsingular $E \times E$ diagonal matrix, then A is a general matrix representation of M if and only if QAD a general matrix representation of M. We use this fact to bring A into the simplest possible form.

Let M be a matroid over K of rank r on E. Let B be a basis of M. For $f \in \overline{B}$, let C_f^B denote the *fundamental circuit* of f with respect to B, which is the unique circuit in $B + f$. Consider the matrix $A^B(x_{e,f} : e \in B, f \in \overline{B})$ over K defined by $A_B^B = I_r$, the rank r identity matrix, and the column $A_e^B = \sum_{e \in C_f^B - f} x_{e,f}$.

Lemma 7.6. *Let M be a matroid over K of rank r on E. Let B be a basis of M. Then $A^B(x_{e,f} : e \in B, f \in \overline{B})$ is a general matrix representation of M over K.*

Proof. Let V be any general matrix representation of M. Row operations preserve the fact that V is a general matrix representation of M. Hence for any basis B we may row-reduce the matrix so that there is an identity matrix at B. The fundamental circuits $(C_f^B)_{f \in \overline{B}}$ with respect to B then fix the zero pattern of the matrix. It follows that A^B is a general matrix representation of M. □

There is further freedom of choosing the matrix in the form of column scaling. While column scaling with a nonzero scalar alters the row space of the matrix, it does not change the matroid it represents. Let B be the basis of M corresponding to the identity, and consider the bipartite graph $G(M, B)$ on E where (e, f) is an edge if and only if $e \in B, f \in \overline{B}$ and $e \in C_f^B - f$. Let T be a spanning forest of G. Define the matrix $A^{B,T}$ be the matrix obtained from A^B by setting $x_{e,f} = 1$ for all $(e, f) \in T$.

Lemma 7.7. *Let M be a matroid, let B be a basis of M, and let T be a spanning forest of $G(M, B)$. Then $A^{B,T}$ is a general matrix representation of M up to column scaling.*

Proof. Let $[I, A] = A^B$. By row and column scaling, we morph this matrix into $[I, A'] = A^{B,T}$.

We label row i of A by the element $e \in B$ such that the column corresponding to e is e_i, so that each edge of G corresponds to a nonzero entry of A. Now consider a spanning forest T of G. Without loss of generality, assume $G(M, B)$ is connected, so that T is a tree. Let the weight function w on T be given by

$w(e, f) = A_{e,f}$. Then scaling a column or row e of A amounts to scaling the weights of all edges adjacent to e. Note that scaling both endpoints of an edge by mutually inverse scalars keeps the edge weight the same. So pick any edge (e, f) of T for which $w(e, f) \neq 1$. We scale e by $w(e, f)^{-1}$, so that the new weight of (e, f) is 1. Next, for all g in the component of e of $T \backslash \{(e, f)\}$, we scale g by

$$w(e, f)^{(-1)^{1+d(e,g)}},$$

where $d(e, g)$ is the distance between e and g. The distance is indeed well defined, since T is a tree. This ensures that all other edge weights remain intact. Hence by induction, there are scalars λ_e for each vertex such that all weights of the edges of T are 1.

Finally, we return to the matrix $[I, A]$. Applying the row and column scalars as above yields a matrix $[D, A']$, where A' has 1 at the entries corresponding to edges of T, and D is a diagonal matrix. The identity matrix can be restored by scaling the columns $e \in B$ by the inverse of the diagonal elements of D. Since $[I, A']$ was obtained from $[I, A]$ by only row and column scaling, $A^{B,T}$ is a general matrix representation of M up to column scaling if and only if A^B is a general matrix representation of M. The latter is ensured by Lemma 7.6. □

The following lemma argues that linear representability of a matroid over a field can be expressed by a system of polynomial equations.

Lemma 7.8. *Let K be an algebraically closed field. Let M be a matroid of rank r on E. Suppose $A(x_1, \dots, x_N)$ is a general matrix representation of M over K. Then M is linear over K if and only if there exist $a_1, \dots, a_N \in K$ and $t_B \in K$ for each basis B of M such that:*

(1) *for each nonbasis S of M, $\det A(a_1, \dots, a_N)_S = 0$;*
(2) *for each basis B of M, $t_B \det A(a_1, \dots, a_N)_B = 1$.*

Proof. We have $\det A(a_1, \dots, a_N)_B = 0$ if and only if $A(a_1, \dots, a_N)_B$ is not invertible. On the other hand, there exists $t_B \in K$ such that

$$t_B \det A(a_1, \dots, a_N)_B = 1$$

if and only if

$$\det A(a_1, \dots, a_N)_B \neq 0.$$

□

For $\alpha \in \mathcal{C}(\mathcal{M})$ and a general matrix representation $A(x_1, \dots, x_N)$ of $\mathcal{M}_\alpha$, let the ideal in $K[x_1, \dots, x_N, (t_B)_{B \in \mathcal{B}(\mathcal{M})}]$

$$\mathcal{I}_\alpha(A)$$

be generated by

$$\det A(x_1, \ldots, x_N)_S = 0$$

for each nonbasis S, and

$$t_B \det A(x_1, \ldots, x_N)_B = 1$$

for each basis B. Then due to Hilbert's Nullstellensatz, a Gröbner basis of $\mathcal{I}_\alpha(A)$ contains 1 if and only if $\mathcal{M}_\alpha$ is linear over K.

4.2. The neighbors condition. The following is straightforward.

Lemma 7.9. *Let $B = [I_r, A]$ be a $r \times n$ matrix representing a matroid M. Then $B^* := [-A^T, I_{n-r}]$ represents M^*.*

Proof. The row space of B^* is the orthogonal complement of the row space of B. □

Thus we may model conditions (8,9) as follows.

Lemma 7.10. *Let $\mathcal{M}$ be a matroid flock. Let $\mathcal{V}$ be a skeleton representation of $\mathcal{M}$. Let α and $\alpha + ke_I + l\mathbf{1}$ be two neighboring central points of $\mathcal{V}$. Suppose A is a matrix representing $\mathcal{V}_\alpha$, and B is a matrix representing $\mathcal{V}_{\alpha+ke_I+l\mathbf{1}}$. Then*

$$\begin{aligned} \mathcal{V}_\alpha/I = -l\mathbf{1}\mathcal{V}_{\alpha+ke_I+l\mathbf{1}}\backslash I \quad &\Leftrightarrow \quad (F^l[B]\backslash I)(A^*\backslash I)^T = 0; \\ -(k+l)\mathbf{1}\mathcal{V}_{\alpha+ke_I+l\mathbf{1}}/\overline{I} = \mathcal{V}_\alpha\backslash\overline{I} \quad &\Leftrightarrow \quad (A\backslash\overline{I})(F^{k+l}[B^*]\backslash\overline{I})^T = 0 \end{aligned}$$

Proof. The matrix of $\mathcal{V}_{\alpha+ke_I+l\mathbf{1}}\backslash I$ is $B\backslash I$; the matrix of $\mathcal{V}_\alpha/I$ is $(A^*\backslash I)^*$, as deletion is the dual operation of contraction. Thus $\mathcal{V}_{\alpha+ke_I}\backslash I = -l\mathbf{1}\mathcal{V}_{\alpha+ke_I+l\mathbf{1}}\backslash I$ and $\mathcal{V}_\alpha/I$ are the same if and only if they have the same dimension and the kernel of $(A^*\backslash I)^*$ is contained in the kernel of $-l\mathbf{1}B\backslash I = F^l[B]\backslash I$. The kernel of $(A^*\backslash I)^*$ is the column space of $(A^*\backslash I)^T$, so $(F^l[B]\backslash I)(A^*\backslash I)^T = 0$ if and only if the kernel of $(A^*\backslash I)^*$ is contained in the kernel of $F^l[B]\backslash I$. The dimensions of $\mathcal{V}_{\alpha+ke_I}\backslash I$ and $\mathcal{V}_\alpha/I$ are equal because α and $\alpha + ke_I$ are neighboring central points. So the first statement follows. The proof of the second statement is analogous. □

Hence the neighbors condition for an edge $\{\alpha, \beta\}$ of the skeleton graph of $\mathcal{M}$ is as follows. Pick general matrix representations $A(x_1, \ldots, x_N)$ of $\mathcal{M}_\alpha$ and $B(y_1, \ldots, y_N)$ of $\mathcal{M}_\beta$. Then each entry of the matrices

$$(F^l[B]\backslash I)(A^*\backslash I)^T$$

and

$$(A\backslash\overline{I})(F^{k+l}[B^*]\backslash\overline{I})^T$$

as in Lemma 7.10 should be 0. We may apply an appropriate Frobenius power to each of the entries to make sure the entries are elements of

$$K[x_1, \ldots, x_N, y_1, \ldots, y_N].$$

Let

$$\mathcal{I}_{\alpha,\beta}(A, B)$$

be the ideal generated by the polynomials in the entries of the matrices.

Now we have explained how to obtain a list of equations to check the existence of a skeleton representation. The algorithm that brings it all together is Algorithm 2.

Algorithm 2 FFlockRepresentable(ν, p)

---

input: a valuation ν of a connected matroid M, and a prime number p
output: a boolean stating whether $\mathcal{M}^\nu$ is Frobenius-flock representable
$\mathcal{C} \leftarrow \mathcal{C}(\mathcal{M}^\nu)$
for $\alpha \in \mathcal{C}$ **do**
  $A_\alpha \leftarrow$ a general matrix representation of $\mathcal{M}^\nu_\alpha$.
  if $\mathcal{M}^\nu_\alpha$ not linear over K **then**
    return False (linearity)
  end if
end for

$$\mathcal{I} \leftarrow \bigcap_\alpha \mathcal{I}_\alpha(A_\alpha) \cap \bigcap_{\{\alpha,\beta\}} \mathcal{I}_{\alpha,\beta}(A_\alpha, A_\beta)$$

$g \leftarrow$ GrobnerBasis($\mathcal{I}$)
return $\neg(1 \in g)$ (Gröbner)

Theorem 7.11. *Let ν be a matroid valuation of a connected matroid, and let p be prime. Then FFlockRepresentable(ν, p) (Algorithm 2) returns 'True (Gröbner)' if and only if $\mathcal{M}^\nu$ is Frobenius-flock representable.*

Proof. If $\mathcal{M}^\nu_\alpha$ is not linear over K, then the linearity condition fails and hence $\mathcal{M}^\nu$ is not linear-flock representable over K. Otherwise, $\mathcal{I}$ is the ideal containing all polynomials that describe the linearity and neighbors conditions on $\mathcal{V}_\alpha$ for $\alpha \in \mathcal{C}(\mathcal{M}^\nu)$. Hence $1 \notin \mathcal{I}$ if and only if $\mathcal{M}^\nu$ admits a skeleton representation $\mathcal{V}$. By Theorem 4.30, $\mathcal{V}$ then extends to a Frobenius flock of $\mathcal{M}^\nu$. □

It is computationally beneficial to reduce the number of variables for Gröbner basis computations. Therefore it helps to choose the general matrix representations A_α in row-reduced form, with respect to a basis with as few as possible adjacent bases. One of the matrices A_α only needs to be a general matrix representation up to scaling, since $\mathcal{V} : \alpha \mapsto \mathcal{V}_\alpha$ is a linear flock if and only if $\mathcal{V}' : \alpha \mapsto \mathcal{V}_\alpha \cdot (\alpha D)$ is a linear flock for some invertible $E \times E$ diagonal matrix D. This allows us to eliminate another $|E| - 1$ variables.

5. Deciding Frobenius-flock representability of matroids

A matroid M is linear-flock representable over a field (K, f) if there exists a valuation ν of M such that $\mathcal{M}^\nu$ is linear-flock representable over (K, f). Ideally, we wish to enumerate the valuations ν of M and then either find one that is linear-flock representable over (K, f), or find a reason for all of them to be not linear-flock representable. This is not always possible, but in many cases it is. We start by enumerating the cells of $\mathfrak{D}(M)$, and we check Frobenius-flock representability of one valuation for each cell. If a valuation ν is not Frobenius-flock representable due to nonlinearity of one of the matroids $\mathcal{M}^\nu_\alpha$, then we will see that no valuation of the same combinatorial type as ν is Frobenius-flock representable. If ν is not Frobenius-flock representable for a different reason, then we do not know in general whether any other valuation of the same combinatorial type is Frobenius-flock representable.

We first prove an invariant of valuations of the same combinatorial type, for which we need the following lemma.

Lemma 7.12. *Let ν be a matroid valuation of a matroid on E. Let O be an octahedron of M^ν. Then there exists $\alpha \in \mathbb{Z}^E$ such that O is an octahedron of $\mathcal{M}^\nu_\alpha$, if and only if $\kappa_\nu(O) = 0$.*

Proof. Let $B_{11}, \ldots, B_{32}$ be the bases of $\mathcal{M}^\nu_\alpha$ corresponding to O as described above. Then by definition of $\mathcal{M}^\nu_\alpha$, we have

$$\alpha \cdot e_{B_{11}} - \nu(B_{11}) = \ldots = \alpha \cdot e_{B_{32}} - \nu(B_{32}).$$

In particular, when we add the terms for the pairs (B_{i1}, B_{i2}), we get that

$$\alpha \cdot (e_{B_{i1}} + e_{B_{i2}}) - \nu(B_{i1}) - \nu(B_{i2})$$

is equal for each i. Since

$$e_{B_{i1}} + e_{B_{i2}} = 2e_S + e_a + e_b + e_c + e_d$$

for each i, it follows that $\kappa_\nu(O) = 0$.

Conversely, translating ν by the trivial valuation

$$-\nu(B_{11})e_b - \nu(B_{21})e_c - \nu(B_{31})e_d - \frac{1}{2}\nu(B_{12})(e_{bcd} - e_a),$$

we obtain a valuation ν' such that $\nu'(B_{11}) = \nu'(B_{21}) = \nu'(B_{31}) = \nu'(B_{12}) = 0$. As $\kappa_{\nu'}(U) = \kappa_\nu(U) = 0$, we also get $\nu'(B_{22}) = \nu'(B_{32}) = 0$. Then $B_{11}, \dots, B_{32}$ are all present as bases of $\mathcal{M}^{\nu'}_{ke_{S+a+b+c+d}}$ for k large enough, which can be translated back to a point α of $\mathcal{M}^\nu$. $\square$

The following theorem relates the images of $\mathcal{M}^\nu$ and $\mathcal{M}^\tau$ if ν and τ are valuations of the same combinatorial type, which is useful for analyzing linear-flock representability of all valuations of a combinatorial type.

Theorem 7.13. *Let M be a matroid. Let ν, ν' be rational valued valuations of M of the same combinatorial type. Then*

$$\{M^\nu_\alpha : \alpha \in \mathbb{R}^E\} = \{M^{\nu'}_\alpha : \alpha \in \mathbb{R}^E\}.$$

Proof. For $t \in [0, 1]$, consider

$$\tau_t := t\nu + (1 - t)\nu'.$$

It is straightforward to show that τ_t is a valuation of M with the same combinatorial type as ν and ν'. The polyhedral complex $\mathcal{D}^{\tau_t}$ changes continuously with t, since the defining equations of the polyhedra are continuous in ν: for any $C \in \mathcal{D}^{\tau_t}$ and an interior point β of C, we have

$$C = C_\beta = \{\alpha \in \mathbb{R}^E : B \in \arg\max_{B' \in \mathcal{B}^{\tau_t}} \{\alpha \cdot e_{B'} - \tau_t(B)\} \text{ for all } B \in \mathcal{B}^{\tau_t}_\beta\}.$$

As t changes, cells may split apart or be joined together, resulting in a difference in the set of matroids $\{\mathcal{M}^\nu_\alpha : \alpha \in \mathbb{R}^E\}$ and $\{\mathcal{M}^{\tau_t}_\alpha : \alpha \in \mathbb{R}^E\}$. We need to show that this splitting or joining together does not happen.

Let $C_\alpha \in \mathcal{D}^\nu$ be given, and suppose by contradiction that C_α splits at τ_t for some $t < 1$. Let $0 < \varepsilon < 1 - t$ be given such that C_α splits into two neighboring cells C_β and $C_{\beta+ke_I}$ in $\mathcal{D}^{\tau_{t+\varepsilon}}$ for some $\beta \in \mathbb{R}^E$, $k \geq 0$ and $I \subset E$. We show that then the combinatorial types of ν and $\tau_{t+\varepsilon}$ are different. To that end, we construct an octahedron of M on which κ_ν and $\kappa_{\tau_{t+\varepsilon}}$ are different, contradicting the assumption that $\kappa_\nu = \kappa_{\nu'}$. We use results from Chapter 3 in this construction, so we use scaling to obtain integral valuations, which correspond with matroid flocks.

As ν, ν' are rational valuations, we may assume by scaling (Lemma 2.33) that ν, $\tau_{t+\varepsilon}$ and ν' are integral. Then $\mathcal{M} := \mathcal{M}^{\tau_{t+\varepsilon}}$ is a matroid flock due to Theorem 3.3. Moreover, by scaling we may assume that $k > 1$ so that $\gamma := \beta + e_I$ is an interior point of the cell between C_β and $C_{\beta+ke_I}$.

So now, we have integral points β, $\beta + ke_I$ and γ in a matroid flock for each of the three cells C_β, $C_{\beta+ke_I}$ and C_γ respectively. Denote $\mathcal{M} := \mathcal{M}^{\tau_{t+\varepsilon}}$. Due to Lemma 3.46,

$$\mathcal{M}_\beta \boxdot I = \mathcal{M}_\gamma = \mathcal{M}_{\beta+ke_I} \boxdot \overline{I}.$$

Using Lemma 3.33, we find that since $\mathcal{M}_\gamma = \mathcal{M}_{\beta+e_I} \not\geq \mathcal{M}_\beta$, we have $r_\beta(I) > r_\gamma(I)$. Similarly, $r_{\beta+ke_I}(\overline{I}) > r_\gamma(\overline{I})$. Now consider any basis B of $\mathcal{M}_\gamma$. Since $\lambda_\gamma(I) = 0$, we have $|B \cap I| = r_\gamma(I)$ and $|B \cap \overline{I}| = r_\gamma(\overline{I})$. So there exist $a \in I$ and $c \in \overline{I}$ such that $r_\beta((B+a+c) \cap I) = r_\gamma(I)+1$ and $r_{\beta+ke_I}((B+a+c) \cap \overline{I}) = r_\gamma(\overline{I}) + 1$. Now there is a circuit C_I of $\mathcal{M}_\gamma$ contained in $(B + a + c) \cap I$ and similarly there is a circuit $C_{\overline{I}}$ of $\mathcal{M}_\gamma$ contained in $(B+a+c) \cap \overline{I}$. Pick $b \in C_I - a$ and $d \in C_{\overline{I}} - c$. Then $B - b - d + a + c$ is a basis of $\mathcal{M}_\gamma$. Denoting $S = B - b - c$, we find that the bases

$$S + a + b, S + a + c, S + a + d, S + b + c, S + b + d, S + c + d$$

describe an octahedron O of M, and in particular all of the bases are contained in $\mathcal{B}_\beta \cup \mathcal{B}_{\beta+ke_I} = \mathcal{B}^\nu_\alpha$. By Lemma 7.12, we find that $\kappa_\nu(O) = 0$, while $\kappa_{\tau_{t+\varepsilon}}(O) \neq 0$; contradiction. □

The theorem is also valid for real-valued valuations, but in order to prove it using our available matroid flock results, we needed to be able to scale the valuations up to integral valuations.

When checking linear-flock representability of a given $\mathcal{M}^\nu$ over (K, f), there are three possible outcomes:

(1) $\mathcal{M}^\nu$ is linear-flock representable;
(2) $\mathcal{M}^\nu$ is not linear-flock representable and there exists $\alpha \in \mathbb{Z}^E$ such that $\mathcal{M}^\nu_\alpha$ is not linear over K;
(3) $\mathcal{M}^\nu$ is not linear-flock representable and for all $\alpha \in \mathbb{Z}^E$, $\mathcal{M}^\nu_\alpha$ is linear over K.

In case (2), due to Theorem 7.13 each valuation of the same combinatorial type as ν is not linear-flock representable either. In case (3), there might be another valuation of the same combinatorial type that is linear-flock representable over (K, f). We do not know yet whether checking a finite number of valuations of this combinatorial type could, in general, lead to the conclusion that all of the valuations of this combinatorial type are not linear-flock representable over (K, f).

If case (3) does not occur, and each combinatorial type has a valuation of case (2), we may hence conclude that M is not linear-flock representable.

Theorem 7.14. *Let M be a matroid and K a field. Suppose M has t combinatorial types of valuations, represented by $(\nu_1, \ldots, \nu_t)$ respectively. If for all i there exists $\alpha(i) \in \mathbb{Z}^E$ such that $\mathcal{M}^{\nu_i}_{\alpha(i)}$ is not linear over K, then M is not linear-flock representable over (K, f) for any automorphism f.*

This theorem yields a necessary (but not sufficient) condition for algebraicity of a matroid as follows:

Definition 7.15. *A matroid M satisfies the* flock condition in characteristic p *if and only if there exists a valuation ν of M such that all matroids in* $\mathrm{Im}(\mathcal{M}^\nu)$ *are linearly representable in characteristic p.*

5.1. The quaternary butterfly matroid. In this section we present the quaternary butterfly matroid, which is linear-flock representable only over fields of characteristic 2. The quaternary butterfly is the matroid represented geometrically in Figure 1. We denote this matroid by $\mathfrak{Q}$. We show that in characteristic > 2, $\mathfrak{Q}$ is not linear-flock representable, despite satisfying the flock condition in all of these characteristics. The proof is computer-assisted, as enumeration of the combinatorial types of $\mathfrak{Q}$ was not done manually.

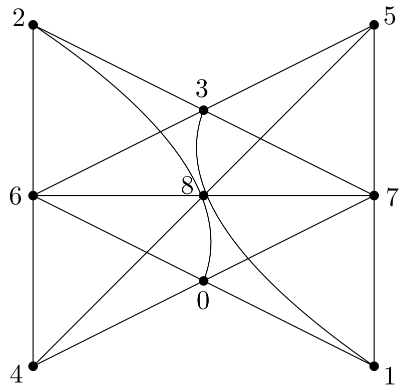


FIGURE 1. The quaternary butterfly, a rank 3 matroid on 9 elements. A triple is a non-basis if and only if it is connected by a line.

In this section we prove the following theorem.

Theorem 7.16. *The matroid $\mathfrak{Q}$ is only algebraic in characteristic 2.*

A matroid is *quaternary* if it is linearly representable over $GF(4)$.

Lemma 7.17. *$\mathfrak{Q}$ is a quaternary matroid, and $\mathfrak{Q}$ is only linear in characteristic 2.*

Proof. Let K be a field. Up to row reduction and scaling, the matrix of a linear representation of $\mathfrak{Q}$ over K is

$$\begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x_0 & 0 & x_2 & 1 & 0 & x_6 \\ 0 & 1 & 0 & x_1 & 1 & x_3 & x_4 & x_5 & 0 \end{pmatrix} \end{array},$$

where $x_0, \dots, x_6 \in K \setminus \{0\}$. As $\{0, 1, 6\}$ is dependent, $x_4 = 1$. Due to dependence of $\{2, 3, 7\}$, we obtain $x_5 = x_1$. Since $\{4, 5, 8\}$ is dependent, we get $x_6 = x_2$. By dependency of $\{6, 7, 8\}$, now $x_4 x_6 + x_5 = 0$, and hence $x_6 = -x_5$.

$$\begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x_0 & 0 & -x_1 & 1 & 0 & -x_1 \\ 0 & 1 & 0 & x_1 & 1 & x_3 & 1 & x_1 & 0 \end{pmatrix} \end{array}.$$

As $\{1, 5, 7\}$ is dependent, we obtain $x_3 + x_1 - x_1^2 - x_1 = 0$, so $x_3 = x_1^2$. By dependence of $\{1, 3, 8\}$, we have $x_1 - x_0 + x_1^2 - x_1 = 0$ and hence $x_0 = x_1^2$.

$$\begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x_1^2 & 0 & -x_1 & 1 & 0 & -x_1 \\ 0 & 1 & 0 & x_1 & 1 & x_1^2 & 1 & x_1 & 0 \end{pmatrix} \end{array}.$$

Finally, due to dependence of $\{3, 5, 6\}$, we find $-x_1 - x_1^2 - x_1^2 - x_1 = 0$, and thus $2x_1(x_1 + 1) = 0$. We cannot take $x_1 = -1$, as that would make $\{1, 5\}$ a dependent set. Moreover x_1 is nonzero, so we must have $2 = 0$ in K. Hence $\mathrm{char}(K) = 2$.

On the other hand, if $K = \mathrm{GF}(4)$ and $\zeta \in K$ satisfies $\zeta^2 + \zeta = 1$, then taking $x_1 = \zeta$ yields a linear representation of $\mathfrak{Q}$:

$$\begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \zeta^2 & 0 & \zeta & 1 & 0 & \zeta \\ 0 & 1 & 0 & \zeta & 1 & \zeta^2 & 1 & \zeta & 0 \end{pmatrix} \end{array}.$$

In particular, $\mathfrak{Q}$ is quaternary. □

To show that $\mathfrak{Q}$ is non-algebraic over K if the characteristic of K is nonzero, we enumerate the combinatorial types of valuations of $\mathfrak{Q}$ and investigate them separately. These valuations are given in Table 1.

Clearly $0_{\mathfrak{Q}}$ is not Frobenius-flock representable over fields of characteristic other than 2. For $\rho_{\mathfrak{Q}}$ and $\tau_{\mathfrak{Q}}$, there exists $\alpha \in \mathbb{Z}^E$ such that the simplification of $\mathcal{M}_\alpha$ is the Fano matroid. Thus valuations of these combinatorial types

$\mathcal{B}$	$0_{\mathfrak{Q}}$	$\rho_{\mathfrak{Q}}$	$\tau_{\mathfrak{Q}}$	$\nu_{\mathfrak{Q}}$	$\mathcal{B}$	$0_{\mathfrak{Q}}$	$\rho_{\mathfrak{Q}}$	$\tau_{\mathfrak{Q}}$	$\nu_{\mathfrak{Q}}$
{0, 1, 2}	0	0	0	1	{1, 4, 7}	0	0	0	1
{0, 1, 3}	0	0	0	1	{1, 4, 8}	0	0	1	0
{0, 1, 4}	0	0	0	1	{1, 5, 6}	0	1	0	0
{0, 1, 5}	0	0	0	1	{1, 5, 8}	0	1	0	0
{0, 1, 7}	0	0	0	1	{1, 6, 7}	0	1	0	0
{0, 1, 8}	0	0	0	1	{1, 6, 8}	0	1	0	0
{0, 2, 3}	0	0	0	1	{1, 7, 8}	0	1	0	0
{0, 2, 4}	0	0	1	0	{2, 3, 4}	0	0	0	1
{0, 2, 5}	0	1	0	0	{2, 3, 5}	0	0	0	1
{0, 2, 6}	0	0	1	0	{2, 3, 6}	0	0	0	1
{0, 2, 7}	0	0	1	0	{2, 3, 8}	0	0	0	1
{0, 3, 4}	0	1	0	0	{2, 4, 5}	0	0	0	1
{0, 3, 5}	0	0	1	0	{2, 4, 7}	0	0	1	0
{0, 3, 6}	0	0	1	0	{2, 4, 8}	0	0	1	0
{0, 3, 7}	0	1	0	0	{2, 5, 6}	0	0	0	1
{0, 3, 8}	0	0	0	1	{2, 5, 7}	0	0	1	0
{0, 4, 5}	0	0	0	1	{2, 5, 8}	0	1	0	0
{0, 4, 6}	0	0	1	0	{2, 6, 7}	0	0	1	0
{0, 4, 8}	0	0	1	0	{2, 6, 8}	0	0	1	0
{0, 5, 6}	0	0	1	0	{2, 7, 8}	0	0	1	0
{0, 5, 7}	0	0	0	1	{3, 4, 5}	0	0	0	1
{0, 5, 8}	0	1	0	0	{3, 4, 6}	0	0	0	1
{0, 6, 7}	0	0	1	0	{3, 4, 7}	0	1	0	0
{0, 6, 8}	0	0	1	0	{3, 4, 8}	0	0	1	0
{0, 7, 8}	0	0	1	0	{3, 5, 7}	0	1	0	0
{1, 2, 3}	0	0	0	1	{3, 5, 8}	0	1	0	0
{1, 2, 4}	0	1	0	0	{3, 6, 7}	0	1	0	0
{1, 2, 5}	0	0	1	0	{3, 6, 8}	0	1	0	0
{1, 2, 6}	0	1	0	0	{3, 7, 8}	0	1	0	0
{1, 2, 7}	0	0	1	0	{4, 5, 6}	0	0	0	1
{1, 2, 8}	0	0	0	1	{4, 5, 7}	0	0	0	1
{1, 3, 4}	0	0	1	0	{4, 6, 7}	0	0	1	0
{1, 3, 5}	0	1	0	0	{4, 6, 8}	0	0	1	0
{1, 3, 6}	0	1	0	0	{4, 7, 8}	0	0	1	0
{1, 3, 7}	0	1	0	0	{5, 6, 7}	0	1	0	0
{1, 4, 5}	0	0	0	1	{5, 6, 8}	0	1	0	0
{1, 4, 6}	0	1	0	0	{5, 7, 8}	0	1	0	0

TABLE 1. A representative valuation for each combinatorial type of of $\mathfrak{Q}$

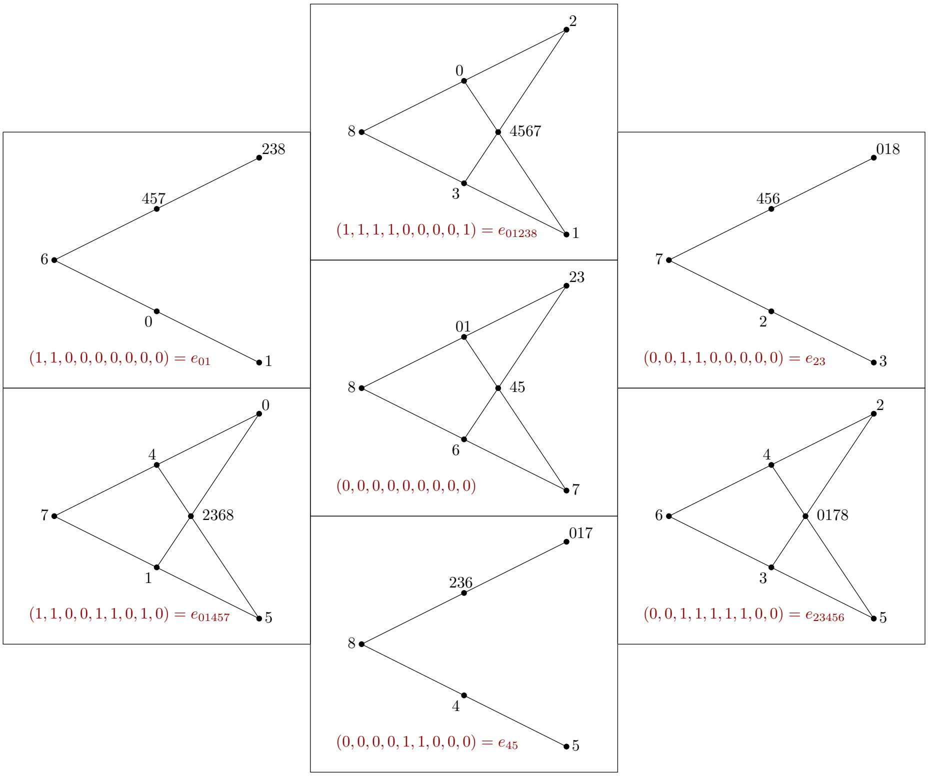


FIGURE 2. The central points of $\mathcal{M}^{\nu_{\mathfrak{Q}}}$.

are not Frobenius-flock representable over fields of characteristic other than 2 either.

The combinatorial type of $\nu_{\mathfrak{Q}}$ remains. See Figure 2.

Each of the central matroids is linearly representable over any field, and even uniquely so, up to column scaling. However, in order for a Frobenius-flock representation to exist, the column scalars must be fixed for all of these representations in such a way that the flock axioms hold between neighbors. As it turns out, this is infeasible.

$$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$(1,1,1,1,0,0,0,0,1) = e_{01238}$

$$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$(1,1,0,0,0,0,0,0,0) = e_{01}$

$$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$(0,0,0,0,0,0,0,0,0)$

$$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$(0,0,1,1,0,0,0,0,0) = e_{23}$

$$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$(1,1,0,0,1,1,0,1,0) = e_{01457}$

$$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$(0,0,0,0,1,1,0,0,0) = e_{45}$

$$\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$(0,0,1,1,1,1,1,0,0) = e_{23456}$

FIGURE 3. Linear representations of the matroids in Figure 2.

Theorem 7.18. *Suppose K is a field of characteristic not 2. Then $\mathcal{M}^{\nu_\Omega}$ admits no linear-flock representation over K.*

PROOF. We have to show there exist no nonzero column scalars for each column of each matrix in Figure 3, in such a way that for each neighboring pair $\alpha, \beta \in \{0, e_{01}, e_{23}, e_{45}, e_{01238}, e_{01457}, e_{23456}\}$, we have $\mathcal{V}_\alpha/I = \mathcal{V}_\beta \backslash I$ and $\mathcal{V}_\beta/(E-I) = \mathcal{V}_\alpha \backslash (E-I)$, where $\beta = \alpha + e_I$. Since $\mathcal{M}^{\nu_\Omega}$ is a matroid flock and all of the matrices are reduced with respect to the same basis $\{0,2,4\}$, these conditions hold if and only if for each neighboring pair of matrices all entries that are nonzero in both matrices agree up to row scaling.

It is straightforward to check that these conditions are all satisfied in Figure 3, except between e_{45} and e_{23456}.

By scaling the columns of all matrices simultaneously, we may assume all column scalars at the point 0 are equal to 1. This has the effect that the scalars for the matrix at e_I are split into two parts, namely I and $E-I$, in which the scalars must be equal. Furthermore, one can easily verify that scaling one of the parts by a has the same effect as scaling the other part by a^{-1}. Hence we may also assume the scalars of $E-I$ at e_I are 1. The freedom of choosing the scalars λ_I of I remains at each point e_I.

Now suppose $\lambda_{23456} = a$. This fixes the scalars of the remainder of the points, which we address in counter-clockwise direction in Figure 3. It follows that $\lambda_{23} = a$, implying $\lambda_{01238} = a$. Hence $\lambda_{01} = a$ and moreover $\lambda_{01457} = a$, implying $\lambda_{45} = a$. Closing the cycle, we find that $\lambda_{23456} = -a$, and since a is nonzero and K has characteristic not 2, this yields a contradiction. $\square$

The proof does not use the automorphism f at all; it only uses the structure of $\mathcal{M}^{\nu_{\mathfrak{Q}}}$. This structure is the same for any valuation of the same combinatorial type as $\nu_{\mathfrak{Q}}$. Therefore we obtain the following result.

Theorem 7.19. *Suppose K is a field of characteristic not 2. Then $\mathcal{M}^{\nu}$ admits no linear-flock representation over K for any ν of the same combinatorial type as $\nu_{\mathfrak{Q}}$.*

So no valuation of $\mathfrak{Q}$ can be Frobenius-flock representable. Theorem 7.16 follows.

Aart Blokhuis recognised $\mathfrak{Q}$ as a hyperoval on the points $\{0, 1, 2, 3, 4, 5\}$ together with a line $\{6, 7, 8\}$ embedded in the projective plane $PG(2, 4)$ [**4**]. To 'complete' the matroid, there should be two more elements on the line such that each pair of points on the hyperoval is collinear with a point on the line.

6. Computing the Lindström valuation of an algebraic representation

Let K be a field of characteristic $p > 0$. As mentioned below Corollary 5.15, Cartwright [**9**] found a direct construction of the Lindström valuation of an algebraic representation of a matroid over K:

$$\nu(B) = \log_p[L : K(\phi(B))^{\mathrm{sep}(L)}].$$

Algebraic matroids can be represented in several different ways [**26**]: as a set of elements in a field extension like in Definition 2.8, as a closed, irreducible algebraic variety like in Chapter 5, and as the (prime) ideal of this variety. To compute the Lindström valuation of an algebraic representation, we follow Cartwright in using the ideal representation. Thus, an algebraic representation of a matroid M on E is a prime ideal I in the polynomial ring $K[x_e : e \in E]$. The circuits of M are the $C \subseteq E$ such that $I \cap K[x_e : e \in C]$ is a principal ideal with support $\{x_e : e \in C\}$. The (monic) generating polynomial f_C of this principal ideal is called the *circuit polynomial* of C.

For a circuit C, let $k = k(C) \in (\mathbb{Z} \cup \{\infty\})^E$ be such that f_C is a polynomial in $(x_e^{p^{k_e}})_{e \in C}$ where each k_e is maximal, and $k_f = \infty$ for all $f \in \overline{C}$. Then Cartwright shows that the set of vectors

$$\mathcal{C} = \{k(C) + \lambda\mathbf{1} : C \text{ circuit}, \lambda \in \mathbb{Z}\} \subseteq (\mathbb{Z} \cup \{\infty\})^E$$

is a *circuit valuation* of M. I will not give the definition of circuit valuations here, but I will note that due to Murota and Tamura [**40**, Theorem 3.3], circuit valuations are in one-to-one correspondence with basis valuations (up to a constant) via a simple formula. Let ν be the basis valuation corresponding to $\mathcal{C}$.

Theorem 7.20 (Cartwright [9]). *The Lindström valuation of the ideal I is ν.*

Thus we know how to compute the Lindström valuation from the set of circuit polynomials of I. What remains is finding all of the circuit polynomials. For small matroids it suffices to naively compute the elimination ideal $I \cap K[x_e : e \in S]$ for each S of the appropriate size, and check if this ideal is principal with full support. This can be done with a Gröbner basis computation and we implemented it in Sage.

7. Computing the Frobenius flock from an algebraic representation

Let K be an algebraically closed field of characteristic $p > 0$. Let $X \subseteq K^E$ be an irreducible closed subvariety. Due to Theorem 5.17, $\mathcal{V} : \alpha \mapsto T_{\alpha v}\alpha X$ is a Frobenius flock over K for some general point $v \in X$. The goal of this section is to find such a general point v, and to compute $T_{\alpha v}\alpha X$ for given α.

Algorithm 3 computes the Jacobi matrix A of αX. This is a matrix with entries in $K[y_1, \ldots, y_n]$ which can be seen as representing the dual of the matroid of the tangent space of αX at the generic point $y_1, \ldots, y_n$. If v is given, then $T_{\alpha v}\alpha X = A(v)^{\perp}$. Otherwise, A yields conditions on the point v. Let $\mathcal{M} = \mathcal{M}(X)$ and let $\mathcal{B}_\alpha$ be the basis set of $\mathcal{M}_\alpha$. Then for a general v it must hold that for all $B \in \mathcal{B}_\alpha$, we have $\det A(v)_{\overline{B}} \neq 0$. In other words, in terminology used by Rosen [**45**], v avoids the *NM-locus* of A, where $A(v)$ does not represent $\mathcal{M}_\alpha$. Due to Theorem 4.30, it suffices for v to avoid the NM-locus of the Jacobi matrices of αX at the central points α of $\mathcal{M}$. By Lemma 4.31, it suffices to avoid it for one point per central cell of $\mathcal{M}$.

Algorithm 4 finds the ideal I that a general point $v \in X$ needs to avoid. If $K = \overline{GF(p)}$, v has coordinates in a finite subfield of K, and it can be found by simply enumerating the points on X of which the coordinates lie in a large enough finite subfield of K, and checking whether v lies in I. For other fields K I have not implemented a method to find a general point.

Algorithm 3 FlockJacobian(α, X)

input: An irreducible closed subvariety $X \subseteq K^E$ and $\alpha \in \mathbb{Z}^E$
output: The Jacobi matrix of αX
output:
$I \leftarrow \text{Ideal}(X)$
▷ I is an ideal in the polynomial ring $K[x_1, \ldots, x_n]$
$twists \leftarrow \emptyset$
for $i \in \{1, \ldots, n\}$ **do**
  $k \leftarrow \alpha_i$
  if $k \geq 0$ **then**
    $twists \leftarrow twists \cup \{x_i - y_i^{p^k}\}$
  else
    $twists \leftarrow twists \cup \{y_i - x_i^{p^{-k}}\}$
  end if
end for
$J \leftarrow I + \langle twists \rangle$
▷ J is an ideal in the polynomial ring $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$
$I' \leftarrow$ the elimination ideal of J by $\{x_1, \ldots, x_n\}$
▷ I' is an ideal in the polynomial ring $K[y_1, \ldots, y_n]$
$Y \leftarrow$ an irreducible component of the variety of I'
$A \leftarrow$ the Jacobian matrix of Y
▷ Sage has built-in functions for all of the above
return A

Algorithm 4 NMloci(X)

input: An irreducible closed subvariety $X \subseteq K^E$
output: The intersection of the NM-loci of the Jacobi-matrices for all central points of $\mathcal{M}(X)$
$L \leftarrow$ LindstromValuation(X)
$M \leftarrow M(L)$
$\mathcal{B} \leftarrow$ the bases of M
$cp \leftarrow$ CentralPoints(M, L)
$I \leftarrow K[y_1, \ldots, y_n]$
for $\alpha \in cp$ **do**
  $A_\alpha \leftarrow$ FlockJacobian(α, X)
  for $B \in \mathcal{B}$ **do**
    $I \leftarrow I + \langle \det A_{\overline{B}} \rangle$
  end for
end for
return I

CHAPTER 8

Small matroids

1. Introduction

In this chapter, I provide an account of the algebraicity of matroids on at most 9 elements in characteristic 2. The complete set of matroids (up to isomorphism) on at most 9 elements was generated by Mayhew and Royle [**38**], and later by Matsumoto, Moriyama, Imai and Bremner [**37**]. The number of matroids for each combination of rank and size is listed in Table 1.

Earlier in the thesis, some conditions were discussed under which a matroid is algebraic or non-algebraic. Linear matroids are algebraic. Matroids that are linear over the endomorphism ring of a one-dimensional algebraic group are algebraic (section 6.3). Specifically for characteristic 2, this means that that we are looking for matroids representable over $\mathbb{Q}$, skew fields in characteristic 2, imaginary quadratic number fields arising from elliptic curves in characteristic 2, and quaternion algebras arising from elliptic curves in characteristic 2.

r, n	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9
2			1	3	7	13	23	37	58	87
3				1	4	13	38	108	325	1275
4					1	5	23	108	940	190214
5						1	6	37	325	190214
6							1	7	58	1275
7								1	8	87
8									1	9
9										1

TABLE 1. Total numbers of matroids of rank r on n elements.

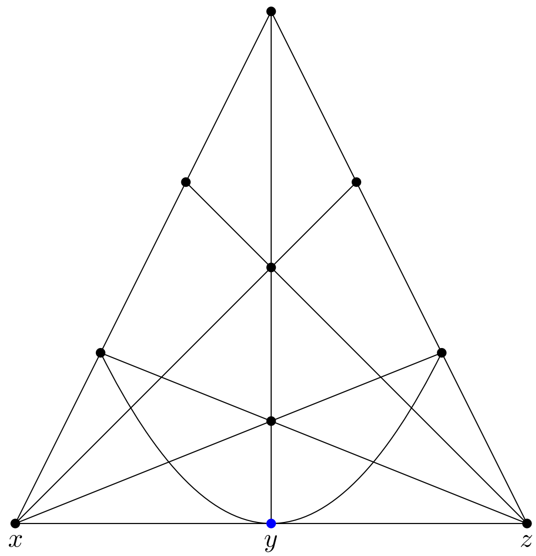


FIGURE 1. A non-algebraic matroid on 10 elements; y does not have a unique harmonic conjugate with respect to x and z.

Conversely, algebraic matroids in general must satisfy the Dress-Lovász condition, which will be discussed in this chapter. Moreover, algebraic matroids in characteristic 2 must be Frobenius-flock representable in characteristic 2.

Lindström's criterion that *harmonic points* must have a unique harmonic conjugate hardly applies to such small matroids. Suppose x, y, z are three collinear points in a rank 3 matroid M. Then y is *harmonic* with respect to x and z if there is a Fano or non-Fano minor of M containing x, y, z. As Lindström showed [**32**], if M is algebraic, then y then has a unique *harmonic conjugate* with respect to x and z.

So in order to disprove algebraicity of a matroid using this argument, a (non-)Fano minor must be present, or somehow forced to be present in an extension (such as the *harmonic closure* of M [**17**]), in order to have a harmonic point in a matroid. Then a different (non-)Fano minor should be present to make the harmonic conjugate ambiguous. The smallest known matroid where this phenomenon occurs is shown in Figure 1, and has 10 elements.

In this chapter I list the results of testing the above-mentioned conditions on the set of matroids on at most 9 elements. The results are compiled in a

'Matroid Encyclopedia' in the shape of a Sage dictionary, which can be found at www.github.com/gpbollen/Algebraicity-of-Matroids-and-Frobenius-Flocks. For each tuple (rank,size), the matroids are numbered from 0 to the number of matroids of that type minus 1. Then `M[rank,size][i]` contains a dictionary of properties of matroid i of type (rank,size). Using various data, which can also be found in the GitHub project, the encyclopedia is filled from scratch in the file 'Matroid Encyclopedia.ipynb'.

2. Linear matroids

Linear matroids over a field K are algebraic over K. It is relatively straightforward to check whether a matroid M is linear over the algebraic closure $\overline{K}$ of K. Let r be the rank of M and let E be the ground set of M. Take a general matrix representation A of M.

Theorem 8.1. *Let M be a matroid and let K be an algebraically closed field. Then it is decidable whether a M is linear over K.*

PROOF. Due to Lemma 7.8, linear representability of M can be described by a system of polynomial equations. Due to Hilbert's Nullstellensatz, there is a solution over $\overline{K} = K$ if and only if 1 is not in the Gröbner basis generated by these equations. Hence we have an algorithm that determines whether M is linear over K. □

Gröbner basis computations are generally slow. Even for the ideals from matroids on 9 elements, despite the reduction of variables that can be obtained due to Lemmas 7.6 and 7.7. So in order to classify which matroids are linear in characteristic 2, I used faster techniques for the majority of matroids on at most 9 elements. To prove a matroid is linear, it suffices to find a linear representation of the matroid. Most of the linear matroids in characteristic 2 were found by generating random matrices. For the few remaining linear matroids, I extracted a representation from the Gröbner basis computation. Nonlinearity for most matroids can be proven by means of the Dress-Wenzel condition in characteristic 2 from the next subsection.

2.1. The Dress-Wenzel condition.

Let M be a matroid. Theorem 2.14 yields a necessary condition for linearity of M as follows:

Definition 8.2. *A matroid M is said to satisfy the* Dress-Wenzel condition *if for every non-degenerate pure quadrangle $\{S+a+b, S+c+d, S+a+d, S+b+c\}$, we have*

$$\frac{[b_1,\ldots,b_{r-2},a,b][b_1,\ldots,b_{r-2},c,d]}{[b_1,\ldots,b_{r-2},a,d][b_1,\ldots,b_{r-2},c,b]} \neq 1 \textit{ in } \mathbb{T}_M,$$

where $S = \{b_1, \ldots, b_{r-2}\}$.

By Theorem 2.14, if M does not satisfy the Dress-Wenzel condition, then M is not linear over any field.

If K has characteristic 2, then for each $r \times E$ matrix A over K linearly representing M, we have $\tilde{\varphi}_A(\varepsilon) = -1 = 1$. Let $\mathbb{T}_M^{(2)} := \mathbb{T}_M / \langle \varepsilon \rangle$. Then if K has characteristic 2, $\tilde{\varphi}_A$ induces a homomorphism $\tilde{\varphi}_A^{(2)} : \mathbb{T}_M^{(2)} \to K^*$. Theorem 2.14 specializes to the following theorem.

Theorem 8.3. *Let K be a field of characteristic 2. Let M be a matroid on E that is linear over K. Let $S = \{b_1, \ldots, b_{r-2}\}$ be given. Suppose $Q = \{S+a+b, S+c+d, S+a+d, S+b+c\}$ is a pure quadrangle of M and suppose $S + a + c \in \mathcal{B}$. The following are equivalent:*

(1) *for all $r \times E$ matrices A over K linearly representing M we have*

$$\tilde{\varphi}_A^{(2)} \left(\frac{[b_1, \ldots, b_{r-2}, a, b][b_1, \ldots, b_{r-2}, c, d]}{[b_1, \ldots, b_{r-2}, a, d][b_1, \ldots, b_{r-2}, c, b]} \right) \neq 1;$$

(2)

$$\frac{[b_1, \ldots, b_{r-2}, a, b][b_1, \ldots, b_{r-2}, c, d]}{[b_1, \ldots, b_{r-2}, a, d][b_1, \ldots, b_{r-2}, c, b]} \neq 1 \text{ in } \mathbb{T}_M^{(2)};$$

(3) *Q is non-degenerate.*

Proof. Denote Theorem 2.14(1) by (1'), and define (2') and (3') similarly. Clearly (2) $\Rightarrow$ (2'), as (2) is stronger. By Theorem 2.14, we have (2') $\Rightarrow$ (1'). As K has characteristic 2, for each A we have $\tilde{\varphi}_A^{(2)} = \tilde{\varphi}_A$, so that (1') $\Leftrightarrow$ (1). The implication (1) $\Rightarrow$ (2) follows by applying $\tilde{\varphi}_A^{(2)}$ on both sides of (2). We conclude that (1) $\Leftrightarrow$ (2). The equivalence (1) $\Leftrightarrow$ (3') $\Leftrightarrow$ (3) follows from Theorem 2.14. $\square$

This theorem allows us to formulate a condition similar to the Dress-Wenzel condition that is necessary for linearity in characteristic 2.

Definition 8.4. *A matroid M is said to satisfy the* Dress-Wenzel condition in characteristic 2 *if for every non-degenerate pure quadrangle $\{S + a + b, S + c + d, S + a + d, S + b + c\}$, we have*

$$\frac{[b_1, \ldots, b_{r-2}, a, b][b_1, \ldots, b_{r-2}, c, d]}{[b_1, \ldots, b_{r-2}, a, d][b_1, \ldots, b_{r-2}, c, b]} \neq 1 \text{ in } \mathbb{T}_M^{(2)},$$

where $S = \{b_1, \ldots, b_{r-2}\}$.

By Theorem 8.3, if a matroid M does not satisfy the Dress-Wenzel condition in characteristic 2, then M is not linear over any field of characteristic 2.

$r\backslash n$	9	8	7	6
3	4	0	0	0
4	37988	44		

TABLE 2. Number of matroids failing the Dress-Wenzel condition.

$r\backslash n$	9	8	7	6
3	67	7	1	0
4	64395	100		

TABLE 3. Number of matroids failing the Dress-Wenzel condition in characteristic 2.

$r\backslash n$	9	8	7
3	1208	318	107
4	125692	840	107
5	125692	318	
6	1208		

TABLE 4. Total numbers of linear matroids in characteristic 2 of rank r on n elements; entries are omitted if all matroids in them are linear.

The numbers of matroids that satisfy both versions of the Dress-Wenzel condition are depicted in Tables 2 and 3. Only the cases where $r \leq n$ are listed, because satisfying the Dress-Wenzel condition is closed under duality.

Clearly, all matroids failing the regular Dress-Wenzel condition also fail the regular Dress-Wenzel condition in characteristic 2.

2.2. The number of linear matroids. We have been able to determine exactly how many matroids on at most 9 elements are linear in characteristic 2; see Table 4. Most non-linear matroids are excluded by the Dress-Wenzel condition. For the remaining matroids we either found a representation, or a Gröbner basis calculation ensured no representation exists.

3. Matroids from one-dimensional algebraic groups

3.1. Rational matroids. Since $\mathbb{Q}$ is not algebraically closed, Theorem 8.1 cannot be applied. So in order to find which matroids are rational, we have fewer tools at our disposal than for algebraically closed fields.

Theorem 2.14 applies to $\mathbb{Q}$, so we can use the failing of the Dress-Wenzel condition as a certificate of non-rationality of matroids. Conversely, a given rational representation is a certificate for rationality. As with linearity in characteristic 2, we can also generate random $\mathbb{Q}$-matrices to find represented matroids. I generated a $r \times n$ matrix A by picking $A_{ij} \in Q$ randomly for each i, j for some finite set $Q \subset \mathbb{Q}$. The choice of Q matters greatly for the diversity of represented matroids that are generated. In particular, if Q is too large, then a random set of columns is likely to be independent, and hence the resulting represented matroids are likely close to uniform. A particularly fertile source of diverse represented matroids turned out to be $Q = \{-1, 0, 1\}$.

3.2. Matroids linear over an imaginary quadratic number field. Let K be the number field $\mathbb{Q}(\sqrt{D})$ for some $D < 0$. As K is commutative, Theorem 2.14 applies. On the other hand, K is not algebraically closed and thus Theorem 8.1 cannot be applied.

However, there is still hope in gathering information on the extension K of $\mathbb{Q}$ over which a matroid is representable. Take the ideal I from Theorem 8.1, and intersect it with $\mathbb{Q}[x_{ij}]$ for each entry variable x_{ij}. Then the obtained ideal is either zero or principal. If it is nonzero, then at least one of the roots of the generating polynomial must lie in K. If there is no $D < 0$ and choice of roots for each x_{ij} such that all of them lie in $\mathbb{Q}(\sqrt{D})$, then the matroid is not linear over an imaginary quadratic number field. Conversely, if $I \cap \mathbb{Q}[x_{ij}]$ is nonzero for each entry variable x_{ij}, and the roots of the respective generating polynomials lie in a common imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$, then we may find a representation of the matroid over this number field. If moreover D is the discriminant of an elliptic curve in characteristic 2, then the matroid is algebraic due to Theorem 6.4.

In Table 5 we list one elliptic curve in characteristic 2 for each (square-free part of) D we found among the matroids on up to 9 elements.

We now give an example of a matroid that is algebraic in characteristic 2 over an elliptic curve, but not over any other connected one-dimensional algebraic group. Like the Non-Pappus matroid, it is nonlinear in characteristic 2.

Example 8.5. *Consider the matroid M from Figure 2.*

We construct a general matrix over a skew field S such that each dependent set of M is a dependent column in the matrix.

D	Field	Elliptic curve
-1	GF(2)	$y^2 + y = x^3 + x^2 + 1$
-2	GF(2)	$y^2 + y = x^3$
-3	GF(4)	$y^2 + \omega y = x^3 + x^2$
-7	GF(2)	$y^2 + xy = x^3 + 1$
-15	GF(4)	$y^2 + xy = x^3 + \omega$

TABLE 5. Elliptic curves in characteristic 2 of which the square-free part of the discriminant is D. Here ω satisfies $\omega^2+\omega+1 = 0$. Note that the first three curves are supersingular, and hence their endomorphism rings are embedded in a quaternion algebra containing $\mathbb{Q}(\sqrt{D})$.

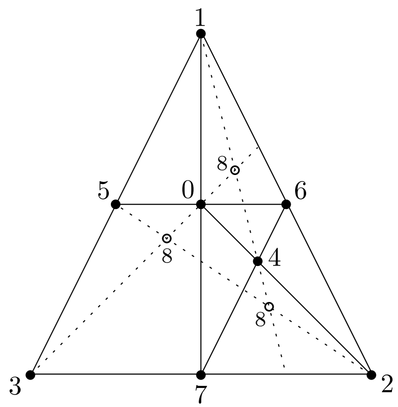


FIGURE 2. A matroid of rank 3 on 9 elements. A triple is collinear if and only if it is dependent in the matroid. The point 8 is the common intersection of the lines through $\{0, 3\}$, $\{1, 4\}$ and $\{2, 5\}$.

$$NS = \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & a & 1 & 1 & a \\ 0 & 0 & 1 & a & -1 & a & 1 & 0 & -1 \end{pmatrix} \end{array},$$

where $a \in K$ satisfies $a^2 = -1$. After choosing a basis and fixing row and column scalars, the remaining entries were chosen as freely as possible given *the dependent triples of M. As it turns out, the only freedom that is left is the choice of a.*

No assumptions on the characteristic or commutativity of S have been made at this point. If S has characteristic 2, then (among others) the rows corresponding to the basis $\{3, 4, 5\}$ become dependent, so that N cannot be a representation of M. So S must have characteristic $\neq 2$. Indeed, if $S = \mathbb{Q}(i)$ and $a = i$, then a subset of rows of NS is dependent if and only if it is dependent in M. Hence the column space of NS is a representation of M over $\mathbb{Q}(i)$.

Now suppose K is a field of characteristic 2, and G is a connected one-dimensional algebraic group over K. Let X be a closed, connected subgroup of G^n with $M(X) = M$. Then there is a linear representation of M over $\mathrm{End}(G)$. *Due to the above, M is not representable over $\mathbb{Q}$, nor over $K(F)$. Hence G cannot be the additive or multiplicative group. So G must be an elliptic curve.*

Consider the elliptic curve G over $GF(2)$ given by the equation $y^2 + y = x^3 + x^2 + 1$. As shown in Table 5, the curve embeds into $Q(i)$. By Theorem 6.4, M is then algebraic over this curve.

We proceed to construct the Lindström valuation of the corresponding algebraic representation, following Theorem 6.5. For bases B of M, define $\nu(B)$ to be the 2-adic valuation of $d\bar{d}$, where $\bar{d}$ denotes the conjugate of d and where $d = \mathrm{Ddet} NS_B = \det NS_B$. Then it can be shown that ν is the corresponding Lindström valuation of M. ♙

3.3. Matroids linear over skew fields. It is not as easy to find out whether a matroid is linear over a skew field as it is over a field. One can try to construct representations by filling a matrix with random elements of a skew field, like over $\mathbb{Q}$ or $\overline{GF(2)}$. However, it is not as easy to find a suitable set Q of skew field elements to pick from. Even for small Q, picking the matrix entries $A_{ij} \in Q$ randomly is likely to yield represented matroids close to the uniform matroid.

It is possible to describe linearity of a matroid over a skew field by a system of polynomial equations over the skew field, in terms of the *quasi-Plücker coordinates* [**44**]. However, such a system has many variables, and checking whether a solution exists within the given skew field is much harder than in the commutative case.

Despite these difficulties, Lindström managed to find all (or rather, both) of the matroids of rank 4 on 8 elements that are linear over a skew field, but not over any field [**29, 33**]. They appear several times in Figure 3, most of which Lindström drew in his paper [**33**]. One of them is given by the points $\{1, 4, 7, 8, 10, 11, 13, 14\}$ and the other by the points $\{3, 4, 7, 8, 9, 10, 13, 14\}$. All minors of the matroid in Figure 3 are representable over $GF(4)(F)$ and thus

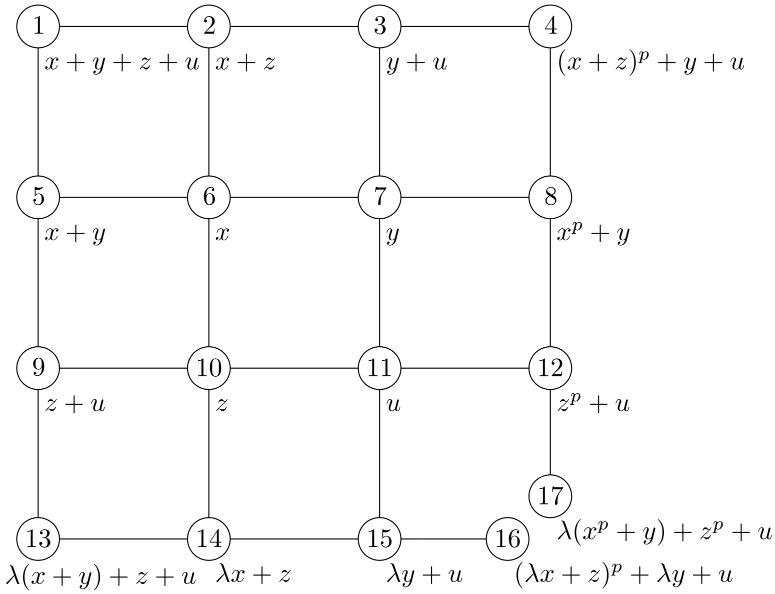


FIGURE 3. A skew-field representable matroid on 17 elements.

algebraic due to Theorem 6.4. Computations show that of the 1212 pairwise non-isomorphic minors of rank 4 on 9 elements, 57 matroids are not linear over any field. These matroids are nevertheless algebraic due to skew-field representability.

In order to find more matroids representable over a skew field, I used a heuristic method inspired by the following example.

Example 8.6. *Consider the matroid M given by the points*

$$E = \{1, 4, 7, 8, 10, 11, 13, 14\}$$

in Figure 3. The nonbases of this matroid are:

$$\begin{gathered}\{1,4,7,11\}\\ \{4,8,10,11\}\\ \{1,7,8,13\}\\ \{1,10,11,13\}\\ \{1,4,10,14\}\\ \{7,8,10,14\}\\ \{7,11,13,14\}.\end{gathered}$$

Let $\mathbb{H}$ *be the skew field over* $\mathbb{Q}$ *generated by* $1, i, j, k$ *with the relations* $i^2 = j^2 = k^2 = ijk = -1$*, the rational Hamilton quaternions. We try to construct a linear representation of* M *over* $\mathbb{H}$ *as a left* $\mathbb{H}$*-module. That is, we construct an* $4 \times E$ *matrix* A *over* $\mathbb{H}$ *such that the columns of* A *corresponding to a set* $B \in \binom{E}{4}$ *are independent if and only if* B *is a basis of* M*.*

By row reduction, we may assume A *has an identity submatrix at the basis* $B = \{1,7,10,11\}$ *(compare Lemma 7.6 for the commutative case). We display* A *as an augmented matrix, leaving out the columns corresponding to* B *and instead labeling the rows by* B*. Independence of the columns of* A *corresponding to* S *for some* $S \in \binom{E}{4}$ *is then equivalent to invertibility of the submatrix of the augmented matrix obtained by deleting the rows* $B \cap S$ *and restricting to the columns* $\overline{B} \cap S$*.*

By row and column scaling we may assume some entries of A *are* 1 *(compare Lemma 7.7 for the commutative case). Now we have*

$$A = \begin{array}{c} \\ 1 \\ 7 \\ 10 \\ 11 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 4 & 8 & 13 & 14 \end{array} \\ \begin{pmatrix} & 1 & 1 & 1 \\ 1 & & 0 & 1 \\ 0 & & & 1 \\ & & & 1 \end{pmatrix} \end{array},$$

where the blank spaces still need to be filled, and where the zeroes follow from the fact that $\{1,4,7,11\}$ *and* $\{1,10,11,13\}$ *are nonbases respectively. Next, for each non-basis containing* 14*, the augmented submatrix looks like*

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

Hence it follows that each of these blank spots needs to be filled with 1*, and we obtain the following matrix:*

$$A = \begin{array}{c c} & \begin{array}{cccc} 4 & 8 & 13 & 14 \end{array} \\ \begin{array}{c} 1 \\ 7 \\ 10 \\ 11 \end{array} & \begin{pmatrix} & 1 & 1 & 1 \\ 1 & & 0 & 1 \\ 0 & & 1 & 1 \\ 1 & 1 & & 1 \end{pmatrix} \end{array}.$$

Two nonbases remain. The corresponding augmented submatrices look like

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Hence the two opposite elements must be each other's inverse in both cases. Thus any representation of M over $\mathbb{H}$ must be of the form

$$A = \begin{array}{c c} & \begin{array}{cccc} 4 & 8 & 13 & 14 \end{array} \\ \begin{array}{c} 1 \\ 7 \\ 10 \\ 11 \end{array} & \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a^{-1} & 0 & 1 \\ 0 & b^{-1} & 1 & 1 \\ 1 & 1 & b & 1 \end{pmatrix} \end{array}$$

for some $a, b \in \mathbb{H} \setminus \{0\}$, up to row reduction and scaling.

Now all minors of A corresponding to the nonbases are zero, and it remains to find values of a and b such that the remaining minors, which correspond to bases, are all nonzero. Observe that if a and b commute, then the columns corresponding to the basis $\{4, 8, 13, 14\}$ are dependent. However, if we pick $a = i$ and $b = j$, then we find that A represents M. ♙

My method for finding a representation for a matroid M over a skew field K is the following. First, pick a basis B and a feasible set of entries to be scaled to 1. Now for each nonbasis S, we find a set of polynomial equations in the entries of A that admits a solution if and only if the submatrix corresponding to S is invertible. For 3×3 or 4×4 matrices, this might not be possible, but we can overcome this problem by distinguishing several cases, as I will explain later.

Once we have obtained a set of polynomials from all of the nonbases, we try to find a common solution over K. Note that the variables are not assumed to commute with each other, nor with K. As a consequence, it is hard to determine whether there exists a common solution. We repeatedly apply the following simple procedure until we have a common solution of all polynomials:

(1) For each polynomial in one variable, try to solve it algebraically. Polynomials of the form $axb - c$ for $a, b, c \in K$ are solved by taking $x = a^{-1}cb^{-1}$; more complicated polynomials, such as $ax + xb - c$, we

try to solve by trying randomly chosen $x \in K$. If we fail to find a solution, we stop. We then substitute this solution for x into all other polynomials. If this substitution yields a nonzero constant for one of the polynomials, we also stop.
(2) If each remaining unsolved polynomial has more than one variable, then we pick one of the remaining variables randomly in K.

If we manage to find a common solution, then we have found a matrix A over K representing some matroid $M' \leq M$. Finally, we check whether $M' = M$. If M admits a linear representation over K and if we can find a common solution to the polynomials given by the nonbases of M, then it is likely that A represents M over K.

We proceed to show how to obtain the polynomials asserting non-invertibility of a matrix for sizes 2×2, 3×3 and 4×4.

Lemma 8.7. *Let K be a skew field and let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a matrix over K. Then A is not invertible if and only if

(1) $a = 0$ *and* $bc = 0$*; or*
(2) $a \neq 0$ *and* $d = ca^{-1}b$.

PROOF. In the first case, since K has no zero divisors, there is a zero row or column, and hence A is not invertible.

In the second case, suppose (x, y) is a nonzero vector such that $ax + by = 0$. Then $y \neq 0$, so we may assume $y = 1$. Due to the first row of A, we find $x = -a^{-1}b$. Now A is not invertible if and only if (x, y) is in the right-kernel of A, which is the case if and only if $cx + dy = d - ca^{-1}b = 0$. $\square$

For 3×3 and 4×4 matrices, the zero pattern plays an important role. Fortunately we precisely know the zero pattern of a linear representation of M, as every entry corresponds to a basis (nonzero) or nonbasis (zero) at distance 1 from B.

If any row or column of a 3×3 matrix contains 2 or more zeroes, then invertibility can be derived from a smaller submatrix. Two cases remain.

Lemma 8.8. *Let K be a skew field and let*

$$A = \begin{pmatrix} 0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}$$

be a matrix over K such that a_1 and b_0 are nonzero. Then A is not invertible if and only if

$$c_0 b_0^{-1}(b_2 - b_1 a_1^{-1} a_2) + c_1 a_1^{-1} a_2 - c_2 = 0.$$

PROOF. The first two rows force that any nonzero vector that is right-perpendicular to the first two rows is a scalar multiple of

$$v = (b_0^{-1}(b_2 - b_1 a_1^{-1} a_2), a_1^{-1} a_2, -1).$$

Hence A is not invertible if and only if v is also right-perpendicular to the third row. □

Lemma 8.9. *Let K be a skew field and let*

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}$$

be a matrix over K such that b_0 is nonzero. Consider the matrix

$$A' = \begin{pmatrix} 0 & a_1' & a_2' \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}$$

where $a_1' = a_1 - a_0 b_0^{-1} b_1$ and $a_2' = a_2 - a_0 b_0^{-1} b_2$. Then A is not invertible if and only if

(1) *$a_1' = a_2' = 0$; or*
(2) *$a_1' = 0$, $a_2' \neq 0$ and*

$$\begin{pmatrix} b_0 & b_1 \\ c_0 & c_1 \end{pmatrix}$$

is not invertible; or

(3) *$a_1' \neq 0$ and*

$$c_0 b_0^{-1}(b_2 - b_1 (a_1')^{-1} a_2') + c_1 (a_1')^{-1} a_2' - c_2 = 0.$$

PROOF. Since A differs from A' by only an elementary row operation, A is invertible if and only if A' is.

If $a_1' = a_2' = 0$, then A' has a zero row and is hence not invertible.

If $a_1' = 0$ and $a_2' \neq 0$, then any vector in the right-kernel of A' has a zero in its third entry. Hence then A' is invertible if and only if the given submatrix is invertible.

If $a_1' \neq 0$, then apply Lemma 8.8. □

For 4×4 matrices, we use essentially the same ideas. If a row or column contains 3 or more zeroes, then invertibility can again be derived from a smaller submatrix. Similarly, if the matrix contains a 2×2 zero submatrix, then invertibility can be derived from two 2×2 submatrices. Next, a matrix

$$A = \begin{pmatrix} 0 & 0 & + & * \\ 0 & + & * & * \\ + & * & * & * \\ * & * & * & * \end{pmatrix},$$

where '$+$' indicates a nonzero entry and '$*$' indicates any entry, gives rise to a polynomial that is zero if and only if A is not invertible analogously to Lemma 8.8.

If A does not have any of the above-mentioned zero patterns up to row and column permutations, then by row reduction and case distinction in a similar way to Lemma 8.9, such a zero pattern can be achieved. These are tedious but straightforward computations, and are omitted.

We see that the choice of basis B matters greatly for the simplicity of this method of finding a linear representation over a skew field. In Example 8.6, we chose a basis such that all nonbases correspond to 2×2-submatrices. As a consequence, no case distinction is required, and all polynomials obtained from the nonbases are relatively simple. As we apply the method to matroids of rank 4 on 9 elements, we will often be unable to choose a B that has no nonbases at distance 4. In order to still get a computationally pleasant basis, we pick a basis B minimizing

$$\sum_{S \in \binom{E}{r} \setminus \mathcal{B}} |B \setminus S|^2.$$

The choice of entries to scale as in Lemma 7.7 also matters. It is beneficial to have many ones in the submatrices corresponding to nonbases. Therefore we define a weight function on the entries, assigning to an entry (e, f) the number of nonbasis submatrices containing (e, f). Then the entries we will scale are those in a maximum weight spanning tree of the bipartite graph from Lemma 7.7.

In conclusion, we have developed a method to check linear representability of a rank at most 4 matroid over a skew field. If there is a basis B for which the number of submatrices requiring case distinction is small, then the method is generally very fast. The method would be less effective on larger rank 4 matroids, since it is then impossible to avoid having several nonbases at distance 4 of B, which generally leads to a lot of case distinctions. Moreover, the success chance of the method would be greatly enhanced with a better way of solving equations in one variable.

4. The Ingleton-Main and Dress-Lovász conditions

Ingleton and Main discovered an extension property of algebraic matroids [**25**].

Definition 8.10. *Let M be a matroid. Suppose l_1, l_2, l_3 are three lines of M that are pairwise coplanar, but do not all lie in the same plane. Then an extension of M by a non-loop element e such that $e \in l_1 \cap l_2 \cap l_3$ is called an* Ingleton-Main extension *of M with respect to l_1, l_2, l_3.*

Theorem 8.11. *Let M be an algebraic matroid. Then for each triple of lines l_1, l_2, l_3 of M that are pairwise coplanar, but do not all lie in the same plane, there exists an Ingleton-Main extension of M with respect to l_1, l_2, l_3 that is algebraic.*

The above theorem yields a necessary condition for algebraicity of matroids as follows.

Definition 8.12. *A matroid M is said to satisfy the* Ingleton-Main condition *if for each triple of lines l_1, l_2, l_3 of M that are pairwise coplanar, but do not all lie in the same plane, M admits an Ingleton-Main extension with respect to l_1, l_2, l_3.*

The Vámos matroid does not satisfy the Ingleton-Main condition, and hence the Vámos matroid is non-algebraic. There are more matroids that are non-algebraic for this reason. Dress and Lovász found a generalisation of Theorem 8.11 [**13**, Corollary 1.4].

Definition 8.13. *Let M be a matroid on E with rank function r. A set $S \subseteq E$ is called a* double circuit *of M if $r(S) = |S| - 2$ and for each $e \in S$, $r(S - e) = |S| - 2$.*

Dress and Lovász characterized the structure of double circuits in the following lemma.

Lemma 8.14. *Let M be a matroid and let S be a double circuit of M. Then S has a partition $S = S_1 \cup \ldots \cup S_k$ such that the following holds: C is a circuit of M contained in S if and only if $C = S \backslash S_i$ for some i.*

The number k from the above lemma is the *degree* of the double circuit S. We now state the generalization of Theorem 8.11.

Definition 8.15. *Let M be a matroid. Suppose S is a double circuit in M of degree $k \geq 3$. Then an extension N of M by a set of elements T with*

$|T| = k-2$ such that $r^N(T) = |T|$, and $T \subseteq \mathrm{cl}_\mathrm{N}(C)$ for each circuit $C \subset S$ is called a Dress-Lovász extension *of M with respect to S.*

Theorem 8.16. *Let M be an algebraic matroid. Then for each double circuit S in M of degree ≥ 3, there exists a Dress-Lovász extension of M with respect to S that is algebraic.*

This theorem is a corollary of a more general theorem by Dress and Lovász, the series reduction theorem [**13**, Theorem 1.3], which makes use of more general extensions than what we call a Dress-Lovász extension here. Theorem 8.16 gives a necessary condition on algebraicity of matroids as follows.

Definition 8.17. *A matroid M is said to satisfy the* Dress-Lovász condition *if for each double circuit S in M of degree ≥ 3, M admits a Dress-Lovász extension with respect to S.*

Note that the Dress-Lovász condition is stronger than the Ingleton-Main condition in general. But in rank 4, the conditions are equivalent.

For some matroids it takes several extensions before the Dress-Lovász or Ingleton-Main conditions fail. This is the case for the dual of the Tic-Tac-Toe matroid [**27, 22**], in which after two Ingleton-Main extensions a Vámos-minor arises, when the Ingleton-Main condition fails. There are recursive versions of the Ingleton-Main and Dress-Lovász conditions as follows.

Definition 8.18. *Let d be a natural number. A matroid M is said to satisfy the* Ingleton-Main condition at depth d *if $d > 1$ and for each triple of lines l_1, l_2, l_3 of M that are pairwise coplanar, but do not all lie in the same plane, there exists an Ingleton-Main extension of M with respect to l_1, l_2, l_3 that satisfies the Ingleton-Main condition at depth $d-1$, or if $d = 1$ and M satisfies the Ingleton-Main condition.*

Definition 8.19. *Let d be a natural number. A matroid M is said to satisfy the* Dress-Lovász condition at depth d *if $d > 1$ and for each double circuit S in M of degree ≥ 3 there exists a Dress-Lovász extension of M with respect to S that satisfies the Dress-Lovász condition at depth $d-1$, or if $d = 1$ and M satisfies the Dress-Lovász condition.*

Due to Theorems 8.11 and 8.16, these conditions are both necessary for algebraicity of M. I implemented checks of the Dress-Lovász and Ingleton-Main conditions and their recursive versions in Sage. In rank 5, depth 3 already seemed to be too much to handle for checking the Dress-Lovász condition. Not only is number of double circuits that requires checking generally larger than in rank 4, but there are also more Dress-Lovász extensions as the double circuit degree increases. As a consequence, I was unable to check the recursive Dress-Lovász condition for depth greater than 2 for a large number of matroids

(r,n)	(4,8)	(4,9)	(5,9)
DL	39	27137	27137
DL depth 2	39	27137	27137
IM depth 3	39	28418	27144
IM depth 4	39	30171	27442
IM depth 5	39	30658	27500
IM depth 6	39	30756*	?

Table 6. Number of matroids failing the Dress-Lovász and Ingleton-Main conditions of the given depth. The computations at depth 6 are unfinished; the actual number is likely higher.

FF-representable	(4,8)	(3,9)	(4,9)	(5,9)	(6,9)
true	938	1274	185346	185348	1274
false	2	1	4551	4551	1
unknown	0	0	317	315	0

Table 7. Number of matroids that are Frobenius-flock representable in characteristic 2. In all other types of matroids on 9 at most elements, all matroids are Frobenius-flock representable.

of rank 5 on 9 elements. Therefore at depth > 2 I computed the recursive Ingleton-Main condition instead.

Table 6 contains the numbers of non-algebraic matroids due to the (recursive) Dress-Lovász and Ingleton-Main conditions. It is interesting to see that the number of matroids failing the Dress-Lovász condition is the same for rank 4 and 5 on 9 elements. In fact, the (5,9)-matroids are the duals of the (4,9)-matroids. One would not expect the Dress-Lovász condition to be closed under duality, as it is an extension property. However, in this case, the (4,9)- and (5,9)-matroids are exactly the (co-)extensions of the (4,8)-matroids that do not satisfy the Dress-Lovász condition. Indeed, satisfying the Dress-Lovász condition is closed under taking minors. There are simply no other matroids that fail the Dress-Lovász condition. At depth 3, this ‘closedness under duality’ vanishes, witnessed by (among others) the Tic-Tac-Toe matroid and its dual.

Algebraic	(4,8)	(3,9)	(4,9)	(5,9)	(6,9)
true	897	1271	148822	148822	1271
false	40	1	31820	29263	1
unknown	3	3	9572	12129	3

TABLE 8. Number of algebraic matroids in characteristic 2.

5. The flock condition

Table 7 contains the results of my computations with respect to Frobenius-flock representability. I checked the flock condition (see Definition 7.15) for all matroids. The matroids that satisfied the flock condition, I tried to find a Frobenius-flock representable valuation. Of the matroids for which I did not succeed in finding a Frobenius-flock representable valuation, we do not know whether they are Frobenius-flock representable. The matroids that fail the flock condition are not Frobenius-flock representable. Then neither are their duals, because the flock condition is closed under duality.

There is a small difference between the numbers of matroids for which we know they are Frobenius-flock representable in the cases (4,9) and (5,9). So there exist matroids that are Frobenius-flock representable, but their duals might not be. Indeed, these matroids all admit one or several valuations ν such that $\mathcal{M}^\nu$ is Frobenius-flock representable, but $\mathcal{M}^{\nu^*}$ is not Frobenius-flock representable. So we get the following result.

Theorem 8.20. *Frobenius-flock representability of matroid flocks is not closed under duality.*

Some caution is in order here, as the proof of this theorem depends strongly on computer calculations. The cases for which $\mathcal{M}^\nu$ is Frobenius-flock representable, but $\mathcal{M}^{\nu^*}$ is not, are too large to check by hand.

I believe that this result extends to matroids.

Conjecture 8.21. *Frobenius-flock representability of matroids is not closed under duality.*

The matroids for which the dual gave a different output will not settle the question whether algebraic matroids are closed under duality. All of these matroids and their duals are already non-algebraic due to the Dress-Lovász condition.

6. Algebraic matroids in characteristic 2

Table 8 contains the numbers of matroids per type that are algebraic, non-algebraic, or unknown to be algebraic. We elaborate on our findings.

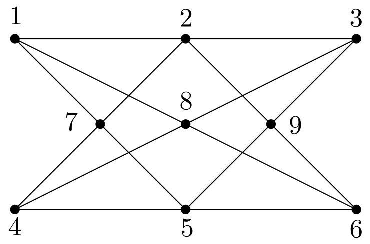


FIGURE 4. The Non-Pappus matroid.

6.1. Type (4,8). The non-algebraic matroids of rank 4 on 8 elements contain the 39 matroids that are non-algebraic due to Dress-Lovász. The other non-algebraic matroid is the matroid 'T8' [**43**], which is the Lazarson matroid representable in characteristic 3 with the element z from its representation in equation 6 removed. It was shown to be non-algebraic in characteristic 2 by Lindström [**31**], and it is also one of the 2 matroids of type (4,8) that is not flock-representable in characteristic 2.

Furthermore, there are 5 matroids that satisfy the Dress-Lovász condition, but not the Dress-Wenzel condition. Two of them are representable over a skew field [**33, 29**] (Figure 3) and thus algebraic.

For the remaining three matroids, algebraicity is unknown. Two of them, which are dual to each other, are obtained from the matroid in Figure 3 algebraically represented by the points $\{1,4,7,8,10,11,13,14\}$ by turning one of the bases $\{1,8,11,14\}$ and $\{4,7,10,13\}$ into a circuit. To obtain the third matroid, both of these bases are turned into circuits.

6.2. Types (3,9) and (6,9). Of the four matroids that do not satisfy the Dress-Wenzel condition, one is representable over any skew field and thus algebraic in characteristic 2. This matroid is the Non-Pappus matroid (Figure 4).

The 3 matroids for which algebraicity is unknown are obtained from the Non-Pappus matroid by turning any nonempty subset of the set of bases

$$\{\{1,4,9\},\{2,5,8\},\{3,6,7\}\}$$

into circuits (the opposite operation of circuit hyperplane relaxation). The duals of all of these matroids suffer the same fate.

The single non-algebraic matroid in characteristic 2 of type (3,9) is a ternary Reid geometry, and hence only algebraic in characteristic 3 (just like its dual).

	(4,9)	(5,9)
algebraic	148822	148822
Linear in characteristic 2	125692	125692
Rational	20469	20469
GF(4)(F)-representable	98	98
Number field representable	284	284
Quaternion representable	2279	2279
non-algebraic	31820	29263
Ingleton-Main condition	30756	27500
Frobenius-flock representability	1064	1763
unknown	9572	12129

TABLE 9. Numbers of algebraic, non-algebraic and unknown matroids, and their first found certificate (checked from top to bottom).

6.3. Types (4,9) and (5,9). In Table 9 we show the certificates for (non-)algebraicity. The algebraic matroids we found are all due to reasons that are closed under duality, namely linearity or representability over the endomorphism ring of a connected one-dimensional algebraic group. This explains the equality of the numbers of algebraic matroids for (4,9) and (5,9). The numbers for non-algebraicity are different, however. This is purely due to the fact that the Ingleton-Main and Dress-Lovász conditions are not closed under duality, and that checking the Dress-Lovász condition at depth 3 is already much harder for matroids of type (5,9) than for matroids of type (4,9).

The matroids for which algebraicity is unknown can be divided into several categories:

- (co)extensions of the 3 matroids of type (4,8) for which algebraicity is unknown (1059 and 1825 matroids for (4,9) and (5,9) respectively);
- matroids linear over the endomorphism ring of a connected one-dimensional algebraic group (not all of them have likely been found);
- matroids failing the Dress-Lovász condition at greater depth than computed;
- the remainder.

6.4. The impact of Frobenius-flock methods. Only for types (4,9) and (5,9) Frobenius-flock methods provided new results on algebraicity. Of the 4551 matroids of type (4,9) that have been found to be non-Frobenius-flock representable, 1064 matroids could not be proven to be non-algebraic by any other means. The number of non-Frobenius-flock representable matroids of type (5,9) that could not be shown to be non-algebraic by other means is 1763. We conclude that Frobenius-flock methods constitute an effective new method for matroids of rank 4 or 5 on 9 elements. The smaller matroids that fail the flock condition had already been found by Lindström and others.

CHAPTER 9

Discussion and further work

In this chapter we revisit the main results of the thesis and indicate possible directions for further research.

1. Results

1.1. Matroid flocks and valuations. A matroid flock consists of a collection of matroids on the integer lattice $\mathbb{Z}^E$ which are locally related by the two simple axioms (MF1) and (MF2). A matroid valuation, too, is defined by local structure: not in an integer lattice, but on the set of bases of the matroid. Matroid valuations with integral values and matroid flocks were united by a simple global function: for each matroid flock $\mathcal{M}$ there exists an integral matroid valuation ν such that $\mathcal{M}_\alpha = \mathcal{M}^\nu_\alpha$ for each $\alpha \in \mathbb{Z}^E$, and vice versa (Theorem 3.3). As a consequence, the well-known correspondence between matroid valuations and polyhedral complexes extends to matroid flocks (Theorem 3.17). These insights were then used to analyse the structure of matroid flocks: taking minors, duality and circuit-hyperplane relaxation come to mind, as well as the behaviour of matroid flocks along lines.

1.2. Linear flocks. Similarly to matroid flocks, a linear flock consists of a collection of vector spaces over a field K of nonzero characteristic on the integer lattice $\mathbb{Z}^E$. The vector spaces are locally related by two axioms (LF1) and (LF2), of which the latter includes the action of an automorphism f of K on a vector space. A linear flock can be seen as a representation of a matroid flock, assigning a linear representation to each $\mathcal{M}_\alpha$. Similarly, a linear flock is a representation of a matroid, namely the valuated matroid of the underlying matroid flock. Due to Theorem 4.2, a linear flock is completely determined by a finite number of vector spaces at its central points. While the class of linear-flock representable matroids over (K, f) is probably not closed under duality, we show that it is closed under circuit-hyperplane relaxation (Theorem 4.37). By considering spike matroids, we concluded that the class of linear-flock representable matroids is asymptotically large (Theorem 4.3).

1.3. Frobenius flocks and algebraic matroids. A special case of a linear flock is one over (K, F), where F is the Frobenius endomorphism of K, and K is algebraically closed. Due to Theorem 5.13, each algebraic representation X of a matroid, together with a very general point $v \in X$, give rise to the Frobenius flock $\mathcal{V} : \alpha \mapsto T_{\alpha v}\alpha X$. This Frobenius flock is then a Frobenius-flock representation of $M(X)$. Due to Theorem 5.17, a very general v always exists in X, so that each algebraic representation induces a Frobenius flock. This yields a criterion that algebraic matroids need to satisfy: they need to be Frobenius-flock representable. However, Frobenius-flock representability is not sufficient for a matroid to be algebraic: the Vámos matroid is Frobenius-flock representable (Corollary 4.39), but not algebraic. An algorithm to compute the Frobenius flock of an algebraic representation has been implemented (Chapter 7, Section 7).

1.4. Algebraic equivalence of algebraic representations. We defined a notion of algebraic equivalence of algebraic representations in which two algebraic representations of a matroid M are equivalent if they can be combined into an algebraic representation of M^2. All algebraic representations of the uniform matroid $U_{1,2}$ are algebraically equivalent (Theorem 5.29). However, there are inequivalent algebraic representations of $U_{2,3}$ (Theorem 5.31), and we do not even know how many equivalence classes there are. Due to Theorem 5.30 equivalent algebraic representations have the same Lindström valuation up to translation. Finally we showed that algebraically equivalent of linear representations must be field-equivalent (Corollary 5.37).

1.5. Non-algebraicity of matroids due to non-Frobenius-flock representability. Frobenius-flock methods for determining whether a matroid is non-algebraic are generally most effective on matroids with few valuations. An extreme case is when the matroid is both rigid and non-linear over K, when due to Theorem 5.19 the matroid is non-algebraic over K. We show that Lazarson matroids are only linear-flock representable in a single characteristic (Theorem 6.8), implying results from Lindström [**31**]. Similarly, we show that Reid geometries are only linear-flock representable in a single characteristic (Theorem 6.11), generalising results from Gordon [**18**].

1.6. Computing algebraicity of matroids. Algorithms for checking certain properties related to algebraicity of matroids were implemented. We check the Ingleton-Main and Dress-Lovász conditions, and their versions at greater depth, which are necessary for algebraicity. We also check Frobenius-flock representability of a matroid, where possible. That requires enumerating the valuations of a matroid (Algorithm 1) and checking whether the matroid flock of a valuation is Frobenius-flock representable (Algorithm 2).

The question whether the class of algebraic matroids is closed under duality remains open. However due to Theorem 8.20, Frobenius-flock representability of matroid flocks is not closed under duality. Frobenius-flock methods might provide a new angle to try to solve this question, but I expect it could only be successful for matroids on (much) more than 9 elements.

1.7. Matroids on at most 9 elements in characteristic 2. We applied our algorithms to all matroids on at most 9 elements in characteristic 2. It turns out that Frobenius-flock methods provide no new results on matroids on 8 elements or fewer, or on matroids on 9 elements of rank 3 and 6. However, much less was previously known about matroids on 9 elements of rank 4 and 5. In both cases, 4551 matroids were shown to be not Frobenius-flock representable. Of these matroids, non-algebraicity of 1064 matroids could not be determined by different methods in the $(4, 9)$ case, and 1763 in the $(5, 9)$ case. For 317 matroids in the $(4, 9)$ case, Frobenius-flock representability remains unknown, and similarly for 315 matroids in the $(5, 9)$ case. While these numbers are different, the corresponding matroids are all non-algebraic due to the recursive Dress-Lovász condition. So while they might show that the class of Frobenius-flock representable matroids is not closed under duality, a similar statement for algebraic matroids cannot be derived. The final tally is that there are still 21710 matroids on at most 9 elements of which algebraicity is unknown. Strategies to find out whether they are algebraic include

- computing the Dress-Lovász condition at greater depth;
- finding rational representations or representations over skew fields coming from a connected one-dimensional algebraic group.

Frobenius-flock methods will not get us any further, since all of these matroids are Frobenius-flock representable.

2. Further work

2.1. Flock-representability of all valuations of a combinatorial type. We were unable to determine Frobenius-flock representability of all matroids of rank 4 and 5 on 9 elements. Suppose all matroids in a matroid flock are linear, but the matroid flock admits no Frobenius-flock representation. Then this does not generally imply anything about Frobenius-flock representability of other matroid flocks of the same combinatorial type. In some cases, there exists another matroid flock of the same combinatorial type that is Frobenius-flock representable. In other cases, such as the 'quaternary butterfly', there does not (Theorem 7.19). We do not know whether it is decidable if there exists a valuation of such a combinatorial type that is Frobenius-flock representable.

Conjecture 9.1. *Let K be an algebraically closed field. It is decidable whether a matroid is Frobenius-flock representable over K.*

2.2. Frobenius-flock methods for larger matroids. For matroids on more than 9 elements, Frobenius-flock methods are still feasible, but I expect the computations to become more time-consuming the more valuations a matroid has. Already for some matroids of types (4,9) and (5,9) there were some matroids for which the Frobenius-flock computations did not finish within a reasonable time frame, and had to be aborted. However, due to Theorem 4.38, these matroids could still shown to be Frobenius-flock representable. I expect the current implementation of Frobenius-flock methods to be too slow to determine Frobenius-flock representability of all matroids of type (4,10). On the other hand, if the number of valuations of a matroid is very small, then checking the flock condition is generally fast, even for matroids up to 20 elements. Since matroids with few valuations are most prone to failing the flock condition, it is still likely that Frobenius-flock methods are able to yield meaningful results for larger matroids.

Frobenius-flock methods also work over fields of characteristic > 2, albeit a little slower, since the degrees of the polynomials are higher.

2.3. Smallest matroids for which algebraicity is unknown. Six matroids stand out as the smallest matroids (in terms of size and rank) for which we still do not know whether they are algebraic or not. Three of them are of rank 3 on 9 elements and are shown in Figure 1. They are obtained from the Non-Pappus matroid by removing any subset of the three bases

$$\{\{1,4,9\},\{2,5,8\},\{3,6,7\}\}$$

from the basis set of the matroid. While the Non-Pappus matroid itself is linear over a skew field, these three 'circuit-hyperplane derelaxations' are all nonlinear over any skew field in any characteristic. Conversely, they are Frobenius-flock representable, and the rank of these matroids is too low to fail the Dress-Lovász condition.

The other three matroids are of rank 4 on 8 elements. Consider the matroid M in Figure 2. Like the Non-Pappus matroid, it is linear over a skew field, but not over any field. Removing any subset of the two bases

$$\{\{1,4,6,8\},\{2,3,5,7\}\}$$

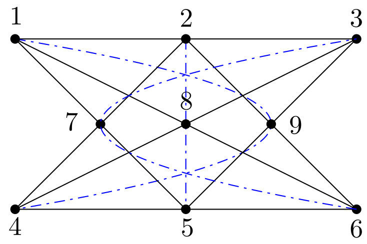


FIGURE 1. The Non-Pappus matroid and the weaker matroids for which algebraicity is unknown.

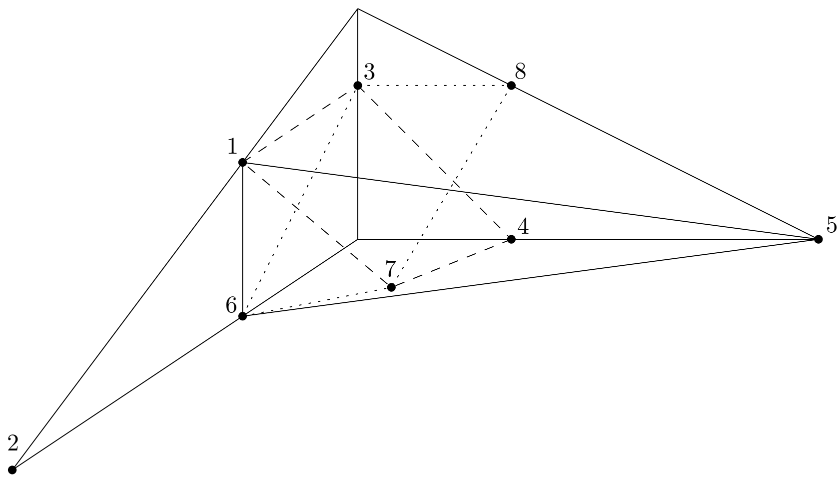


FIGURE 2. A self-dual matroid with nonbases $\{1,2,3,6\}$, $\{1,2,5,8\}$, $\{1,3,4,7\}$, $\{1,5,6,7\}$, $\{2,4,5,6\}$, $\{3,4,5,8\}$, $\{3,6,7,8\}$, discovered by Lindstrom [**33**] to be only linear over a skew field, but algebraic in any characteristic.

from the basis set of M yields one of the three matroids for which algebraicity is unknown. The two matroids obtained by 'derelaxing' precisely one of the two bases are dual to each other.

2.4. The Tic-Tac-Toe matroid and its siblings. The Tic-Tac-Toe matroid appeared in the paper of Alfter and Hochstättler [**1**] as a matroid which is closed under Dress-Lovász extensions, but whose dual is non-algebraic due to the Dress-Lovász condition at depth 3. In our computations we found that there are 14 matroids of rank 5 on 9 elements with this property. The Tic-Tac-Toe matroid is the one with the most bases among these 14.

Using the methods in this thesis, there is no hope of finding an algebraic representation of any of these 14 matroids: our positive results rely on linearity over some skew field, which is a property that is closed under duality. On the other hand, the methods in this thesis are unable to rule out that these matroids are algebraic. All of them are Frobenius-flock representable and satisfy the Dress-Lovász condition at arbitrary depth.

The question whether algebraic representability is closed under duality remains unanswered.

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Curriculum Vitae

Guus Bollen was born on April 27, 1990 in Eindhoven, The Netherlands. He obtained both his bachelor's degree (2011) and his master's degree (2014) in Industrial and Applied Mathematics at Eindhoven University of Technology. His master's thesis *An online version of Rota's basis conjecture* was supervised by Jan Draisma. During his enrollment in Eindhoven, he joined the board of E.S.T.T.V. TAVERES for one year in 2009.

In 2014, he stayed in Eindhoven to work as a doctoral candidate under supervision of Jan Draisma and Rudi Pendavingh. The position was funded by Stichting Computer Algebra Nederland. The results of this work are presented in this dissertation. During his time as a doctoral candidate, he joined the board of Schaakvereniging Woensel-Lichttoren Combinatie in 2015. He was also involved in the organisation of the Nederlandse Wiskunde Olympiade.

Summary

Frobenius Flocks and Algebraicity of Matroids

Matroids are abstract objects consisting of a finite set of elements and a collection of subsets of a fixed size. These subsets are called the bases of the matroid, and must satisfy the condition that for each pair of bases, each element of the first basis can be exchanged with an element of the second basis in such a way that after the exchange, both subsets remain bases. Matroids arise from graphs, where the edges constitute the set of elements, and the spanning trees are the bases. These are the graphic matroids. Furthermore, a finite set of vectors in a vector space forms the set of elements of a matroid, where the bases of the matroid are the sets of vectors that are bases of the vector space. This type of matroid is called a linear matroid. A third type of matroids is the main subject of this thesis: algebraic matroids. The elements of an algebraic matroid are elements of a field. A subset of elements is a basis if it is maximal with the property that there is no polynomial relation between the elements over a given subfield.

While graphic and linear matroids are relatively well understood, the class of algebraic matroids is much more arduous to describe. All linear matroids are algebraic, and over fields of characteristic zero the converse is also true. But in prime characteristic, there is no known algorithm that decides whether a general matroid is algebraic. Even among matroids with 9 elements, there is a significant portion of the matroids for which the known methods fail to determine algebraicity.

In this thesis, a new method for analyzing the class of algebraic matroids is introduced. By twisting the elements of an algebraic matroid by the Frobenius automorphism of the field, and then linearizing the result, a collection of linearly represented matroids is obtained, which is a *Frobenius flock*. Each Frobenius flock of a matroid gives rise to a matroid valuation of the matroid. By enumerating the valuations of a matroid and checking which valuations could emerge from a Frobenius flock, restrictions on algebraic representations of the matroid are obtained. Sometimes none of the valuations of a matroid emerge from a Frobenius flock, and then the matroid is hence not algebraic. This approach for determining whether a matroid is not algebraic generalizes a method first used by Bernt Lindström on a small number of (classes of) matroids. Frobenius-flock methods are applied to the classes of matroids that were done with the method of Lindström. Moreover, some classes of algebraic matroids are investigated, such as matroids over certain skew-fields and matroids from elliptic curves.

This thesis also covers the computational aspects of algebraic matroids. Among others, the following algorithms have been implemented in Sage and are documented in this thesis:

- enumerating the valuations of a matroid;
- checking whether a given matroid valuation emerges from a Frobenius flock in a given characteristic;
- computing the Frobenius flock and its corresponding valuation from an algebraic representation of a matroid.

These algorithms, along with previously known methods, are then applied to the set of all matroids on 9 elements in order to provide an as accurate as possible status quo of which matroids are known to be (non-)algebraic over some field of characteristic 2.