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Optimal Lateral Transshipment Policies for a Two Location Inventory Problem with Multiple Demand Classes

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Abstract

We consider an inventory model for spare parts with two stockpoints, providing repairable parts for a critical component of advanced technical systems. As downtime costs for these systems are expensive, ready–for–use spare parts are kept in stock to be able to quickly respond to a breakdown of a system. We allow for lateral transshipments of parts between the stockpoints upon a demand arrival. Each stockpoint faces demands from multiple demand classes. We are interested in the optimal lateral transshipment policy. There are three ways in which a demand can by satisfied: from own stock, via a lateral transshipment, or via an emergency procedure. Using stochastic dynamic programming, we characterize and prove the structure of the optimal policy, that is, the policy for satisfying the demands which minimizes the average operating costs of the system. This optimal policy is a threshold type policy, with state-dependent thresholds at each stockpoint for every demand class. We show a partial ordering in these thresholds in the demand classes. In addition, we derive conditions under which the so-called *hold back* and *complete pooling* policies are optimal, two policies that are often assumed in the literature. Furthermore, we study several model extensions which fit in the same modeling framework.

Keywords: inventory, spare parts, lateral transshipments, optimal policy, multiple demand classes

1. Introduction

In this paper we study an inventory model with two stockpoints, which provide spare parts for advanced technical systems. These systems are typically used in the primary processes

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of their users. Hence, any downtime of these systems is extremely costly, mainly because of loss of production. Therefore, ready–for–use spare parts are kept in stock for the critical components of these systems. We focus on a single, repairable part, for which a repair–by–replacement strategy is executed: upon failure of a system, the defective part is replaced by a part from inventory. The defective part is returned to the stockpoint, where it is repaired and added to the inventory. We take the total number of spare parts (on-hand and in repair) at each location to be given.

The stockpoints service multiple groups of technical systems. Each group is assigned to one stockpoint. The cost of downtime of a system depends on the service contract. The systems are classed into groups based on these downtime cost. In case of a breakdown of one of the systems, it demands a spare part at its dedicated stockpoint. Hence, at each stockpoint, there are multiple demand classes, which differ in importance based on downtime cost. The demands form a Poisson process, possibly with different rates per stockpoint and per demand class. If a demand for a spare part is directly met at the stockpoint, we refer to this as a demand that is *directly* fulfilled. Otherwise, there are two possibilities. The first option is a *lateral transshipment*, which means that a part is shipped from the other stockpoint. In this case, the system is down while it is waiting for the part and extra transportation costs are incurred. The second option is an *emergency procedure*: the defective part is repaired in a fast repair procedure or a ready–for–use part is obtained from somewhere else via an emergency shipment. In both cases, high costs are incurred. Downtime costs heavily depend on the amount of time the system is down. Since the system is down for a longer period of time when an emergency procedure is applied, the emergency option is much more expensive than a lateral transshipment. Because of the large downtime costs, backordering of demands is not allowed. Notice that the use of these emergency shipments is standard in several industries (Kranenburg and Van Houtum, 2009; Grahovac and Chakravarty, 2001; Lee, 1987; Zhao et al., 2006). The failed parts are repaired at external repair shops, which are modeled as ample server stations where the service times are exponentially distributed (service times at these servers represent repair lead times).

When lateral transshipments are used efficiently in a spare parts provisioning inventory system, significant cost reductions can be achieved. This is shown by Kranenburg and Van Houtum (2009) for the company ASML, an original equipment manufacturer in the semiconductor industry. They consider the cost of providing spare parts when lateral transshipments are used, and compare these to the cost in the same setting when lateral transshipments are not allowed. They show that the costs of spare parts provisioning can be up to 50% lower when using lateral transshipments (without affecting the service level). Robinson (1990) shows that substantial costs savings can be realized by the use of lateral transshipments even when the transportation costs are high. Besides, Cohen and Lee (1990) show that stock pooling is an effective way to improve the service levels even with less on-hand inventory. Their conclusions are based on two case studies in the computer and automobile industry. Furthermore, Cohen et al. (2006) point out that the pooling of spare parts is one of the best ways for companies to realize cost reductions.

In this paper, we focus on the optimization of the lateral transshipment policy (under the assumptions as specified in Section 2) in a setting with multiple demand classes per stockpoint. That is, we determine the optimal decision on how to fulfill a demand to minimize the average operating costs of the system in the long-run: (i) directly from own stock, (ii) via a lateral transshipment, or (iii) via an emergency procedure. When is it beneficial to apply a lateral transshipment, and when is it better to apply an emergency procedure? A straightforward policy would be to always fulfill demands from the own stockpoint whenever possible, and otherwise via a lateral transshipment (if possible). This policy is known as *complete pooling* (or *full pooling*) of inventory.

Depending on the cost parameters, a complete pooling policy is suboptimal in certain cases. If, for example, a stockpoint has only one part left in stock, it could be beneficial to hold it back from a lateral transshipment request, for some or even for all demand classes. This situation can occur when the cost parameters for both stockpoints and all demand classes are equal (i.e., symmetric), but its effect may be even larger under asymmetric costs parameters. It could, in fact, be better to hold parts back *even in case of a demand at the stockpoint itself.* This may be wise to be able to respond to a future request from a more important demand class, or even for a future lateral transshipment request of the other stockpoint. This situation where stockpoints can hold back some inventory is known as *partial pooling.* A *hold back policy* (cf. Xu et al. 2003, see also Van Wijk et al. 2012) is a special case of this policy. Under that policy, one will directly fulfill a demand from the own stockpoint if possible and outgoing transshipments are limited by a threshold parameter on the stock level.

A considerable amount of work has been done on the use of lateral transshipments in various settings. Wong et al. (2006) and Paterson et al. (2011) provide good overviews. Most studies focus on (approximate) evaluation of performance characteristics or (approximate) optimization of parameters when the policy is given: Lee (1987); Axsäter (1990); Sherbrooke (1992); Tagaras and Cohen (1992); and Van Wijk et al. (2012). Optimal reorder policies in the presence of transshipments are derived in Robinson (1990) and Olsson (2009).

More relevant in relation to our work, is the literature on the *optimization* of the lateral transshipment rule, for which only limited results are known. Archibald et al. (1997) study a periodic-review model and prove the optimal transshipment policy in case of stock-outs. They allow demands to arise at any time epoch in a period. They find that a threshold type policy is optimal, where the thresholds depend on the remaining time in a period until the next review epoch (that is, the remaining time until the start of the next period). More specifically, in case of a demand at a location that is out-of-stock, the optimal action is to apply a lateral transshipment if the remaining time in the period is smaller than the threshold, and to apply an emergency procedure otherwise. They consider only a single customer class, assume zero-lead time replenishments, and allow only for lateral transshipments in case of a stock-out. Our results indicate that it is not always optimal to satisfy a demand directly from a location's own stock. Hu et al. (2008) consider a similar model, allowing for uncertainty in production capacity.

Periodic–review models for multiple locations include Archibald et al. (2007), who present a heuristic for the lateral transshipment rule, and Herer et al. (2006), who approximate the optimal transshipment rule. Van der Heide and Roodbergen (2013) (see also Van der Heide et al., 2018) consider a problem of redistributing stock between local depots, which shows similarities with lateral transshipments, and propose a heuristic that performs within close margin of the optimal solution. In Herer and Tzur (2001) an optimal transshipment policy is derived for pro-active lateral transshipments. Here lateral transshipments are only applied to balance stock because of different holding and replenishment costs at the locations. Further, Wee and Dada (2005) study the decision for a single period in a system with multiple locations and one central warehouse, providing different protocols for transshipment attempts in case of a stock-out. Xu et al. (2003) introduce a hold back parameter that limits the amount of outgoing transshipments. Only when the stock level is above the hold back level, a lateral transshipment is carried out. They approximately evaluate the performance characteristics of the model. Motivated by the retailing of seasonal goods, Abouee-Mehrizi et al. (2015) study a multi-period, finite horizon model. The structure of the optimal joint replenishment and transshipment policies is derived, such that the total expected cost over the season are minimized. They show that these optimal policies can be described by four switching curves. Amrani and Khmelnitsky (2017) consider a model with a central depot and multiple bases, with lateral transshipments allowed between the latter. They first focus on the optimal division of stock, and show that partial pooling often is the best strategy.

For optimal lateral transshipments rules in a continuous time review setting, few results are available. Zhao et al. (2005, 2006) prove the optimal transshipment policy for so-called *decentralized* networks, where the locations are independently owned and operated. They find a policy where two (Zhao et al., 2005) or three (Zhao et al., 2006) parameters determine when to send and when to accept a transshipment request. Evers (2001) provides two heuristics giving critical values for on-hand inventory, above which a stock transfers should be applied. Minner et al. (2003) improve these heuristics using an approach based on net present value. For the case of compound Poisson processes, Axsäter (2003a) presents a heuristic rule determining which part of a given demand should be covered by a lateral transshipment.

In Zhao et al. (2008), a two location make–to–stock system is considered for which the optimal production and optimal transshipment policy are derived, where the production units are modeled as exponential, *single-server* queues. They show that both policies can be described by a switching curve, i.e., by state-dependent thresholds. However, they do not allow inventory to be held back, whereas our results show that keeping stock back could be beneficial if the other location is (having a large risk of) facing a stock-out. The optimal lateral transshipment policy for a multi-location system is characterized in Van Wijk et al. (2013), under the assumption that only a single stockpoint (referred to as the quick response warehouse) is allowed to supply lateral transshipments to each of the other warehouses. The dynamic optimal transshipment policy is shown to be a state-dependent threshold policy.

By far most of the works mentioned before, assume a single demand class at each of the stockpoints. However, the demands faced by a stockpoint can be of different importance. Different groups of systems may have different downtime costs, because they are used by different types of companies or departments. Consequently, it might be of more or less importance to quickly satisfy a certain demand. Therefore demands are categorized in classes, the so-called *demand classes*. The costs for lateral transshipments and emergency procedures depend on the class the customer belongs to. When the on-hand inventory level becomes low, it might be beneficial to stop serving lower priority demands, in order to be able to satisfy future demands of higher priority. In this way, stock is reserved for demands of higher importance, which is called *stock rationing*. For a single stockpoint, the seminal works in this field are Veinott Jr (1965) and Topkis (1968), both showing that the optimal policy consists of *critical levels*: inventory levels at or below which only high(er) priority demands are served. These critical levels are non-increasing in the priority of the class. An overview of the literature on stock rationing is found in Teunter and Klein Haneveld (2008).

For the combination of multiple demand classes at different stockpoints with lateral transshipment, only few results seem to be available. Closely related to our setup is the work by Tiemessen et al. (2013), which proposes a heuristic dynamic rule for demand fulfillments based on cost approximations and only looking one step ahead. Jalil (2011) does look into a similar model with multiple periods, where the cost function is approximated under rather restrictive conditions. In numerical experiments it is shown that a dynamic rule significantly outperforms a static allocation rule, especially in the presence of major differences between the demand classes and when on-hand inventory levels are low.

Our main contributions are as follows. For the described model with two stockpoints and multiple demand classes, under the assumptions as specified, (a) we characterize and prove the structure of the optimal lateral transshipment policy. That is, we model the inventory problem as a Markov decision problem, and use stochastic dynamic programming to characterize the optimal policy structure as a threshold type policy, with state-dependent thresholds at each stockpoint for every demand class. We show a (partial) ordering in these thresholds in the demand classes. Next, (b) we give conditions under which the optimal policy simplifies to either a *hold back policy* or a *complete pooling policy*. The latter policy is often assumed in the literature considering multi-location models with lateral transshipments and with ample server assumptions for the repair/replenishment pipelines; see, e.g., Lee (1987); Tagaras (1989); Axsäter (1990); Sherbrooke (1992); Grahovac and Chakravarty (2001); Kukreja et al. (2001); Wong et al. (2006) and Zhao et al. (2008). Therefore we contribute to the literature by presenting conditions on the cost parameters under which this policy is indeed optimal, and we give relaxed conditions under which the more general hold back policy is optimal. Third, (c) we extend our results to systems with limited repair capacities, systems with substitutable inventories, and order fulfillment problems in e-commerce. We remark that our policy is optimal only under the assumptions made.

The outline of this paper is as follows. We start by describing the model in more detail, and introducing the notation in Section 2. We model the system as a Markov decision problem and introduce the technique of Event–Based Dynamic Programming. In Section 3, we give the structural properties of the event operators and of the value function. This leads to the characterization of the structure of the optimal policy of the Markov decision problem as a threshold type policy. Illustrated with examples, the ordering of the thresholds is shown, and conditions are given under which certain simple policies are optimal. Section 4 considers several generalizations and extensions to the model. Finally, we summarize the results and indicate possibilities for further research in Section 5. Appendix A contains all proofs.

2. Model and Notation

In Section 2.1, we introduce the problem, followed by its modeling as a Markov decision problem in Section 2.2. We introduce the *value function*, i.e., the n–period minimal expected cost function. Furthermore, we introduce the two types of event operators: for the demands and for the repairs. Using these, the value function can be recursively expressed.

2.1. Problem Description

We consider a spare parts inventory system consisting of two stockpoints. These provide spare parts for a single critical component of advanced technical systems. The time horizon is infinite, starting at time 0. Each stockpoint is assigned a predetermined number of spare parts, denoted by $S_i \in \mathbb{N} \cup \{0\}$ for stockpoint i, $i = 1, 2$. Each stockpoint serves multiple groups of systems, and these groups are ordered according to their importance. We refer to these groups as the (demand) classes of a stockpoint.

When a technical system breaks down, the critical component must be replaced by a spare part. Consequently, the system demands a spare part at its designated stockpoint. The demands arrive continuously and according to independent Poisson processes. The arrival rate is $\lambda_{ij} \geq 0$ at stockpoint i for a system of class $j \in \mathcal{J}_i := \{1, \dots, J_i\}$. A demand can be fulfilled in one of the following three ways: (i) directly from own stock, (ii) via a lateral transshipment (from the other stockpoint), or (iii) via an emergency procedure. If the demand is fulfilled from the own stock, then the downtime of the corresponding failed technical system is short. If the demand is fulfilled by a lateral transshipment, there is extra downtime because a part from the other stockpoint must be transported to the failed technical system. If the demand is fulfilled by an emergency procedure, there is an even longer downtime.

Under the options (i) and (ii), the failed part is returned to the stockpoint that supplied the requested spare part. In this way, the total number of parts at each location is constant. This assumption not only facilitates the analysis, but since in practice the total number of parts at a location is typically determined based on the size of the installed base that is served by this stockpoint, one typically prefers to keep the total number of parts at a location the same, instead of having it altered after a lateral transshipment has been applied. Next, the failed part is immediately sent into repair at an external repair shop. In practice, external repair shops generally have standard customer order lead times, and they have possibilities to adapt their repair capacity for peaks in their workload. Hence, at both stockpoints, the external repair shop is modeled as an ample server. The service times at this ample server correspond to the repair lead times of failed parts. To facilitate the analysis, we assume that these service times are exponentially distributed. The assumption of exponentially distributed service times is a common one when optimal policy structures are derived in a continuous-review setting; see also Zhao et al. (2008). The system performance is known to be rather insensitive to this distribution, which has been shown by Alfredsson and Verrijdt (1999) and Enders et al. (2014). The mean repair time is denoted by $1/\mu$ ($\mu > 0$), and is the same for both stockpoints. In Section 4.4, we investigate unequal repair rates. We assume that parts can be repaired an unlimited number of times, and that repaired parts attain their original quality.

In practice, the emergency procedure, option (iii), may reflect different options. It may represent that an external spare part is sent from another place/company, and that the failed part is sent back to that other place. Alternatively, the failed part may be repaired off-line, and placed back in the system after repair, after which the external part can be sent back to that other place/company. A third possibility is that no other part is provided, but that the failed part itself is repaired via a special fast repair procedure.

Our goal is to minimize the average costs in the long-run. The costs are composed of the costs for the lateral transshipments, emergency procedures, and the downtimes of the systems. We are only interested in the influence of the decisions on the costs. That is, we consider the *extra* costs a lateral transshipment or an emergency procedure causes, compared to a fulfillment of a demand directly from stock. The number of spare parts S_1 and S_2 is given, where S_i is the total of parts on-hand and in repair. Hence, we ignore the purchase costs of these parts. Neither do we take into account inventory holding or storage costs, because, under the above assumptions, we have a fixed circulating stock at both stockpoints. We investigate the case of non-zero holding costs in Section 4.1, where we extend the model to consumable parts. We set the costs when a demand is met directly from own stock to zero. These would be the costs for the downtime of the machine and for the shipment of the spare part to the system. Also replacing, shipping back, and repairing the broken part contribute to these costs. If a lateral transshipment is applied, higher transportation costs are incurred. Moreover, the system is down during the extra transportation time, so extra costs for loss of production are incurred too. All these costs for applying a lateral transshipment *to* stockpoint i for a system of class j constitute the penalty costs for a lateral transshipment, denoted by P_{ij}^{LT}. The third option for fulfilling a demand, is an emergency procedure. The extra costs for this procedure and the downtime constitute the penalty costs for an emergency procedure for a demand at stockpoint i for a system of class j, denoted by P_{ij}^{EP}. We assume $P_{ij}^{EP} > 0$ and $P_{ij}^{EP} \geq P_{ij}^{LT} \geq 0$, for all pairs (i, j), $i = 1, 2$, $j \in \mathcal{J}_i$.

We model the delays for lateral transshipments and emergency procedures entirely in the cost factors P_{ij}^{LT} and P_{ij}^{EP}. This is because, compared to the repair lead times (i.e., the service times at the ample servers), these delays are on a different time scale. From our work together with several companies in the spare parts industry, we know the typical orders of magnitude for repair lead times, lateral transshipment times, and emergency shipments. Repair lead times are typically in the order of multiple weeks or months, whereas lateral transshipments and emergency procedures are typically in the order of hours or, at most, one day, say. As a result, it is unlikely that a normal repair lead time of a part is completed in the few hours in which a lateral transshipment or an emergency procedure is executed. Hence, we model the lateral transshipments and emergency procedures to occur instantaneously.

Only in case that delays for lateral transshipments and emergency procedures are significant compared to the repair lead times, and that the repair lead times are, for example, deterministically distributed, there would be a benefit in keeping track of remaining lead times and considering this information first before deciding on the action taken to fulfill a demand. Such a model is studied in Howard et al. (2015). However, they are not able to prove any structural results on the optimal policy and have to resort to heuristics.

All costs, particularly for the down-times, are united in the factors P_{ij}^{LT} and P_{ij}^{EP}. We allow these to be non-equal at the two stockpoints and for each of the classes, to reflect for differences in down-time costs and transportation costs.

The class j of a demand represents the importance of the demand to the firm. For example, some systems might incur higher downtime cost per time unit than others, and therefore belong to a higher priority class. This can also be based on agreements made with the user of the systems of the various groups. This prioritization, which we assume to be solely cost based, is reflected as an ordering of the cost for lateral transshipments and emergency procedures. Denoting by 1 the class of the highest importance, i.e., with the highest costs, we impose the ordering, for $i = 1, 2$: $P_{i1}^{LT} \geq P_{i2}^{LT} \geq \ldots \geq P_{iJ_i}^{LT}$ and $P_{i1}^{EP} \geq P_{i2}^{EP} \geq \ldots \geq P_{iJ_i}^{EP}$.

As denoted already, under the above assumptions, we have a fixed circulating stock at each of the two stockpoints. That is, the inventory position (the total number of parts in stock and parts in repair) is constant at each stockpoint. It equals the initial amount of spare parts, which is S_i at stockpoint i. Our model fits also for consumables (spare parts or other goods) for which basestock control is used, with basestock level S_i at stockpoint i. The repair lead times are the equivalent of the replenishment/production lead times. The emergency repair procedure is equivalent with either lost sales or emergency shipments from outside.

2.2. Dynamic Programming Formulation

The state x of the system is given by the inventory levels at both stockpoints: $x = (x_1, x_2)$. Here $x_i \in \{0, 1, \ldots, S_i\}$ is the on-hand stock at stockpoint i. The state space $\mathcal{S}$ is given by all possible combinations of inventory levels: $\mathcal{S} = \{0, 1, \ldots, S_1\} \times \{0, 1, \ldots, S_2\}$. Upon a demand of class j at stockpoint i, a decision must be made how to fulfill it, in one of the following three ways: (0) directly from own stock, (1) via a lateral transshipment or (2) via an emergency procedure. The action taken for a demand of class j at stockpoint i when in state x, is denoted by $a_{ij}(x) \in \{0, 1, 2\}$. An optimal action is denoted by $a_{ij}^*(x)$. As backorders are not allowed, the decision space of $a_{ij}(x)$ consists of the decisions under which x_1 and x_2 remain greater than or equal to zero.

We apply uniformization (cf. Lippman, 1975) to convert the semi-Markov decision problem into an equivalent Markov decision problem (MDP). The existence of an average optimal policy is guaranteed by Theorem 8.4.5a of Puterman (1994): if the state space and action space for every state are finite, the costs are bounded, and the model is *unichain*, then there exists a stationary average optimal policy. A model is said to be unichain if the transition matrix of every (deterministic) stationary policy is unichain, that is, if it consists of a single recurrent class plus a possibly empty set of transient states. The current model is unichain, as the state (S_1, S_2) is accessible from every state $(x_1, x_2) \in \mathcal{S}$ for every stationary policy.

When facing a decision, we should take into account the direct costs for a decision as well as the future expected costs this decision brings along. For the expected costs from a state, we introduce the *value function* $V_n : \mathcal{S} \mapsto \mathbb{R}^+$. $V_n(x_1, x_2)$ is the minimum expected total cost when there are n events (demands or repairs) left when starting in state $(x_1, x_2) \in \mathcal{S}$. This V_n can be recursively expressed. The two types of operators it consists of (G_i for the repairs and H_{ij} for the demands) are defined below. V_n is given by:

$$V_{n+1}(x_1, x_2) = \frac{1}{\nu}\left(\sum_{i=1}^{2} \mu\, G_i\, V_n(x_1, x_2) + \sum_{i=1}^{2}\sum_{j=1}^{J_i} \lambda_{ij}\, H_{ij}\, V_n(x_1, x_2)\right), \text{ for } (x_1, x_2) \in \mathcal{S}, n \geq 0, \tag{1}$$

starting with $V_0 \equiv 0$. Here $\nu = (S_1 + S_2)\mu + \sum_{i=1}^{2}\sum_{j=1}^{J_i} \lambda_{ij}$ is the uniformization rate. Decisions are only made through fulfilling demands (in the operator H_{ij}). The decision is made each time a demand arrives and is based on the inventory levels and demand class. For the repairs no decisions are made.

The operator G_1 models the repairs at stockpoint 1, and is defined by

$$G_1 f(x_1, x_2) = \begin{cases} (S_1 - x_1) f(x_1 + 1, x_2) + x_1 f(x_1, x_2) & \text{if } x_1 < S_1, \\ S_1 f(x_1, x_2) & \text{if } x_1 = S_1, \end{cases} \tag{2}$$

for some arbitrary function $f : \mathcal{S} \mapsto \mathbb{R}^+$. G_2 is defined analogously. If the inventory level is x_1, there are $S_1 - x_1$ outstanding repairs. Hence, the repairs occur at a rate proportional to $S_1 - x_1$. Since we apply uniformization, we add the term $x_1 f(x_1, x_2)$, which corresponds to *fictitious* transitions. In this way, we assure that the total rate at which G_1 occurs is always equal to S_1.

The operator H_{1j} models the demands at stockpoint 1 from class $j \in \mathcal{J}_1$:

$$H_{1j} f(x_1, x_2)$$
$$= \begin{cases} P_{1j}^{EP} + f(x_1, x_2) & \text{if } x_1 = 0, x_2 = 0, \\ \min\{f(x_1 - 1, x_2), P_{1j}^{EP} + f(x_1, x_2)\} & \text{if } x_1 > 0, x_2 = 0, \\ \min\{P_{1j}^{LT} + f(x_1, x_2 - 1), P_{1j}^{EP} + f(x_1, x_2)\} & \text{if } x_1 = 0, x_2 > 0, \\ \min\{f(x_1 - 1, x_2), P_{1j}^{LT} + f(x_1, x_2 - 1), P_{1j}^{EP} + f(x_1, x_2)\} & \text{if } x_1 > 0, x_2 > 0. \end{cases} \quad (3)$$

H_{2j}, for $j \in \mathcal{J}_2$, is defined analogously. H_{ij} takes the costs-minimizing action when a demand arises. The costs consist of the direct costs for an action (0, P_{ij}^{LT}, or P_{ij}^{EP}) and the expected remaining costs from the state the system is in after taking that action, given by $f(\cdot)$. As stock levels cannot become negative, four cases are distinguished.

3. Structural Results

In this section we prove our main result: the structure of the optimal policy, for the Markov decision problem as presented in the previous section. For this, we first prove that the value function V_n satisfies certain structural properties, such as monotonicity and multimodularity. We show that all the operators of which V_n is composed, preserve these properties. Then, as V_0 satisfies them, it follows directly by induction that the properties hold for V_n for all $n \geq 0$. A framework for this was introduced by Koole (1998, 2006) as *Event–Based Dynamic Programming*. The main advantage of this approach is that one can prove the propagation of properties for each of the event operators separately. Hence the complexity of the problem is reduced.

In Section 3.1, we introduce the structural properties and prove that $G_1 + G_2$ and H_{ij} preserve these. It then follows that V_n, for all $n \geq 0$, satisfies them as well. From this we derive, in Section 3.2, the structure of the optimal lateral transshipment policy. This policy is a threshold type policy. The ordering of these thresholds in the demand classes is shown in Section 3.3. In Section 3.4, we derive conditions under which it reduces to a simple policy, such as a hold back or a complete pooling policy. Two examples are given in Section 3.5. Finally, Section 3.6 deals with the special case of symmetric system parameters. All proofs are given in Appendix A.

We remark that the optimal lateral transshipment as derived here is optimal for the problem under the assumptions as discussed in Section 2. That is, the repair completions are assumed to be one-for-one, and the repair lead times are exponentially distributed. When these assumptions are relaxed, the policy is not necessarily optimal anymore. In particular, if the repair lead times are non-exponential, the optimal policy most certainly has different characteristics, since in such a case decisions can depend on the state of the repair processes.

3.1. Properties of Operators and Value Function

Consider the following properties of a function f, defined for all (x_1, x_2) such that the states appearing in the right–hand and left–hand side of the inequalities exist in $\mathcal{S}$:

$$\text{Decr}(1): \quad f(x_1, x_2) \geq f(x_1 + 1, x_2), \tag{4}$$

$$\text{Decr}(2): \quad f(x_1, x_2) \geq f(x_1, x_2 + 1), \tag{5}$$

$$\text{Conv}(1): \quad f(x_1, x_2) + f(x_1 + 2, x_2) \geq 2f(x_1 + 1, x_2), \tag{6}$$

$$\text{Conv}(2): \quad f(x_1, x_2) + f(x_1, x_2 + 2) \geq 2f(x_1, x_2 + 1), \tag{7}$$

$$\text{Supermod}: \quad f(x_1, x_2) + f(x_1 + 1, x_2 + 1) \geq f(x_1 + 1, x_2) + f(x_1, x_2 + 1), \tag{8}$$

$$\text{SuperC}(1,2): \quad f(x_1 + 2, x_2) + f(x_1, x_2 + 1) \geq f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1), \tag{9}$$

$$\text{SuperC}(2,1): \quad f(x_1, x_2 + 2) + f(x_1 + 1, x_2) \geq f(x_1, x_2 + 1) + f(x_1 + 1, x_2 + 1). \tag{10}$$

Decr(i) stands for (non-strict) decreasingness of f in x_i. Conv(i) stands for convexity of f in x_i. This means that the difference $f(x) - f(x + e_i)$ is decreasing in x_i. Here, e_i denotes the unit vector consisting of all zeros except for a 1 at position i. Supermod stands for supermodularity, the definition of which is symmetric in x_1 and x_2. SuperC(i, j) stands for superconvexity, adopting the terminology of Koole (2006). It is a straightforward result that Supermod and SuperC(i, j) imply Conv(i). Decr stands for the combination of Decr(1) and Decr(2), i.e., Decr $=$ Decr(1) $\cap$ Decr(2). Similarly, Conv $=$ Conv(1) $\cap$ Conv(2) and SuperC $=$ SuperC(1, 2) $\cap$ SuperC(2, 1). Multimodularity (MM) (introduced by Hajek, 1985) is, for the case of a two-dimensional domain, equal to the combination of Supermod and SuperC:

$$\text{MM} = \text{Supermod} \cap \text{SuperC}. \tag{11}$$

The following two lemmas provide useful properties of the operators G_i and H_{ij}. These enable us to derive the structure of the optimal policy.

Lemma 1. *a) Operator G_i, $i = 1, 2$, preserves each of the following properties:*
(i) Decr; (ii) Conv; (iii) Supermod.
b) The sum of the operators $G_1 + G_2$ preserves each of the following properties:
(i) Decr; (ii) Conv; (iii) Supermod; (iv) SuperC; (v) MM.

For example, part $a)$ (i) of the lemma states the following: if a function f is Decr, then $G_i f$ is Decr as well, for $i = 1, 2$. Note that SuperC (and hence MM), is only preserved by the sum of the operators $G_1 + G_2$, and not by G_1 and G_2 separately. That is, if a function f is SuperC, then $(G_1 + G_2) f$ is SuperC as well, but that does not necessarily hold for $G_1 f$ and $G_2 f$. The reason is as follows: when $G_1 + G_2$ is applied to SuperC$(1, 2)$ (9) and SuperC$(2, 1)$ (10) respectively, some terms introduced by G_1 cancel out against terms introduced by G_2, and this is exploited in deriving the properties for $G_1 + G_2$.

Lemma 2. *Operator H_{ij}, $i = 1, 2$, $j \in \mathcal{J}_i$, preserves each of the following properties:*
(i) Decr, (ii) MM.

Note that, while H_{ij} preserves MM, it does not hold that H_{ij} preserves the parts Supermod, SuperC$(1, 2)$, and SuperC$(2, 1)$ individually. That is, if a function f is MM, then $H_{ij} f$ is MM as well. However, if f is Supermod, then $H_{ij} f$ does not necessarily have to be Supermod as well, which would have been the case if f is also SuperC.

By induction on n, and Lemmas 1 and 2, the next theorem immediately follows.

Theorem 3. *V_n satisfies* (4)–(10) *for all $n \geq 0$.*

The properties (4)–(10) of V_n are the key in classifying the structure of the optimal policy.

3.2. Structure of Optimal Policy

We now characterize the structure of the optimal policy in the following two theorems. We state the optimal policy for fulfilling a demand *of class j at stockpoint 1* (see Figure 1); for stockpoint 2, analogous results hold. First we give the results for x_2 fixed, next for x_1 fixed. Combining both results leads to the general structure as given in Figure 1.

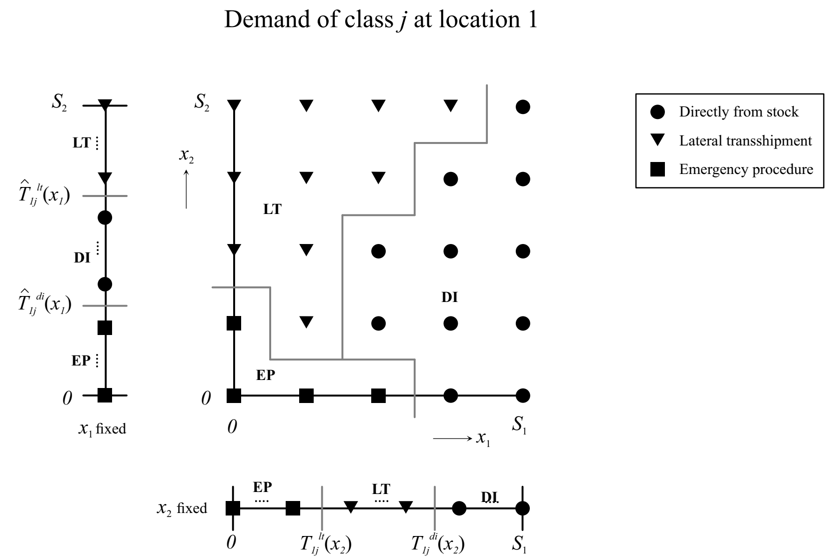


Figure 1: General structure of the optimal policy for a demand of class j at location 1. For fixed x_2, the optimal policy structure is indicated below the horizontal axis, for fixed x_1 next to the vertical axis.

Theorem 4. *The optimal policy for fulfilling a demand of class j at stockpoint 1 for fixed x_2 is a threshold type policy: for each $x_2 \in \{0, 1, \ldots, S_2\}$, there exist thresholds $T_{1j}^{lt}(x_2) \in \{0, 1, \ldots, S_1 + 1\}$ and $T_{1j}^{di}(x_2) \in \{1, \ldots, S_1 + 1\}$, with $T_{1j}^{lt}(x_2) \leq T_{1j}^{di}(x_2)$, such that:*

$$a_{1j}^*(x) = 2 \text{ (emergency procedure), for } 0 \leq x_1 \leq T_{1j}^{lt}(x_2) - 1;$$

$$a_{1j}^*(x) = 1 \text{ (lateral transshipment), for } T_{1j}^{lt}(x_2) \leq x_1 \leq T_{1j}^{di}(x_2) - 1;$$

$$a_{1j}^*(x) = 0 \text{ (directly from own stock), for } T_{1j}^{di}(x_2) \leq x_1 \leq S_1 + 1,$$

where $T_{1j}^{lt}(0) = T_{1j}^{di}(0) \geq 1$.
The analogous result holds for demands at stockpoint 2 under a fixed $x_1 \in \{0, 1, \ldots, S_1\}$.

This theorem follows by the supermodularity and superconvexity in (x_1, x_2) of the value function (cf. (8) and (9), respectively). The structure is graphically represented below the horizontal axis in Figure 1. For a given demand class j it holds that for each x_2, the thresholds divide the set $\{0, \ldots, S_1\}$ into (at most) three subsets. In the first subset, where x_1 is small, an emergency procedure is optimal; in the second one a lateral transshipment; and in the third one, where x_1 is large, it is optimal to satisfy a demand from the own stock. A threshold can be equal to $S_1 + 1$, hence, implying that for a given x_2 taking parts from stock or applying lateral transshipments is never optimal. A special case is $x_2 = 0$: as lateral transshipments are not possible at stockpoint 1, we have $T^{lt}_{1j}(0) = T^{di}_{1j}(0)$, where $T^{di}_{1j}(0) \geq 1$, for all j. In this case, there are (at most) two subsets: an emergency procedure is applied for $0 \leq x_1 < T^{di}_{1j}(0)$, and a demand is directly delivered from the own stock for $T^{di}_{1j}(0) \leq x_1 \leq S_1 + 1$.

The intuition behind this theorem is as follows. If the stock level x_1 is high, one is willing to take a part from the own stock as there are still many remaining afterward. But if the stock level is low, one might, depending on the costs parameters, decide to hold some parts back, either for future request for higher priority demands, or for future lateral transshipment requests of the other stockpoint. If $x_1 = 0$, one is forced to apply either an emergency procedure or a lateral transshipment, and the optimal choice depends on the stock level at the other stockpoint. A similar characterization of the optimal policy can be made for fixed x_1, which is given in the following theorem.

Theorem 5. *For the optimal policy for fulfilling a demand of class j at stockpoint 1 for fixed $x_1 \in \{0, 1, \ldots, S_1\}$, there exist $\hat{T}^{di}_{1j}(x_1) \in \{0, 1, \ldots, S_2 + 1\}$ and $\hat{T}^{lt}_{1j}(x_1) \in \{1, \ldots, S_2 + 1\}$, with $\hat{T}^{di}_{1j}(x_1) \leq \hat{T}^{lt}_{1j}(x_1)$, such that:*

$$a^*_{1j}(x) = 2 \text{ (emergency procedure), for } 0 \leq x_2 \leq \hat{T}^{di}_{1j}(x_1) - 1;$$
$$a^*_{1j}(x) = 0 \text{ (direct from own stock), for } \hat{T}^{di}_{1j}(x_1) \leq x_2 \leq \hat{T}^{lt}_{1j}(x_1) - 1;$$
$$a^*_{1j}(x) = 1 \text{ (lateral transshipment), for } \hat{T}^{lt}_{1j}(x_1) \leq x_2 \leq S_2 + 1,$$

where $\hat{T}^{di}_{1j}(0) = \hat{T}^{lt}_{1j}(0) \geq 1$.
The analogous result holds for demands at stockpoint 2 under a fixed $x_2 \in \{0, 1, \ldots, S_2\}$.

This theorem follows by the supermodularity and superconvexity in (x_2, x_1) of the value function (cf. (8) and (10), respectively). The structure is graphically represented next to the vertical axis in Figure 1. For a given demand class j it holds that for each x_1, the set $\{0, \ldots, S_2\}$ is divided into (at most) three subsets, such that in each subset one decision is optimal. Again, a $\hat{T}^{di}_{1j}(x_1)$ or $\hat{T}^{lt}_{1j}(x_1)$ larger than the maximum stock level indicates that a certain subset is empty, hence, that decision is never optimal. A special case is $x_1 = 0$, when it is not possible to deliver a demand directly from stock. Hence $\hat{T}^{di}_{1j}(0) = \hat{T}^{lt}_{1j}(0)$, where $\hat{T}^{lt}_{1j}(0) \geq 1$ for all j.

If the stock level at the other stockpoint, x_2, is high, a lateral transshipment can be a good option as there are still plenty of parts remaining after the transshipment is carried out.

When x_2 decreases, lateral transshipments are less likely to become the best option. If x_2 is low, or even zero, stockpoint 1 might hold stock back by applying emergency procedures, which can be optimal if there are emergency costs at stockpoint 2 that are much higher than those of 1. We note that this is the general form of the structure. It is unlikely that it turns out to be optimal to take parts via lateral transshipments when x_2 is large, but hold parts back for stockpoint 2 when x_2 is small. In Section 3.4, we discuss conditions under which this general optimal policy structure simplifies.

By combining Theorem 4 and Theorem 5, we can show that for each j the optimal policy for fulfilling demands at stockpoint 1 is described by two switching curves. Define the sets EP_j, LT_j, and DI_j, as the areas where it is optimal to satisfy a demand of class j from the own stock, to apply a lateral transshipment, and to apply the emergency procedure, respectively:

$$\mathrm{DI}_j = \{x \in \mathcal{S} \mid a^*_{1j}(x) = 0\}, \quad \mathrm{LT}_j = \{x \in \mathcal{S} \mid a^*_{1j}(x) = 1\}, \quad \mathrm{EP}_j = \{x \in \mathcal{S} \mid a^*_{1j}(x) = 2\}.$$

The set EP_j consists of connected states in the lower left corner in Figure 1. This follows from the property that $a^*_{1j}(\tilde{x}) = 2$ for some $\tilde{x}$ implies $a^*_{1j}(x) = 2$ for all x with $x_1 \leq \tilde{x}_1$ and $x_2 = \tilde{x}_2$ (by Theorem 4), and for all x with $x_1 = \tilde{x}_1$ and $x_2 \leq \tilde{x}_2$ (by Theorem 5). Hence, EP_j consists of all states below a first switching curve $k_{1j}(x_1) = \hat{T}^{di}_{1j}(x_1)$, $x_1 \in \{0, \ldots, S_1\}$, which is non-increasing in x_1. The remaining states are split by a second switching curve $k_{2j}(x_2) = T^{di}_{1j}(x_2)$ defined for all $x_2 \in \{0, \ldots, S_2 \mid T^{di}_{1j}(x_2) > T^{lt}_{1j}(x_2)\}$. By Theorem 5, it holds that: (i) if $T^{di}_{1j}(\tilde{x}_2) > T^{lt}_{1j}(\tilde{x}_2)$ for a given $\tilde{x}_2$, then $T^{di}_{1j}(x_2) > T^{lt}_{1j}(x_2)$ for all $x_2 \geq \tilde{x}_2$ (the point $(T^{lt}_{1j}(\tilde{x}_2), \tilde{x}_2)$ belongs to LT_j and hence also all points right above that point); (ii) $k_{2j}(x_2)$ is non-decreasing as a function of x_2. Hence, $k_{2j}(x_2)$ is a curve that starts at the first switching curve $k_{1j}(x_1)$ and ends at the line $x_2 = S_2$. The set LT_j consists of all states $x \in \mathcal{S} \setminus \mathrm{EP}_j$ that are to the left of $k_{2j}(x_2)$ (excluding states at $k_{2j}(x_2)$ itself). All other states belong to the set DI_j.

Theorem 6. *The optimal policy for fulfilling a demand of class j at stockpoint 1 is described by the switching curves $k_{1j}(x_1)$ and $k_{2j}(x_2)$ as defined above. EP_j consists of all states below $k_{1j}(x_1)$, LT_j consists of all other states that are to the left of $k_{2j}(x_2)$, and DI_j consists of all remaining states. The analogous structure holds for demands at stockpoint 2.*

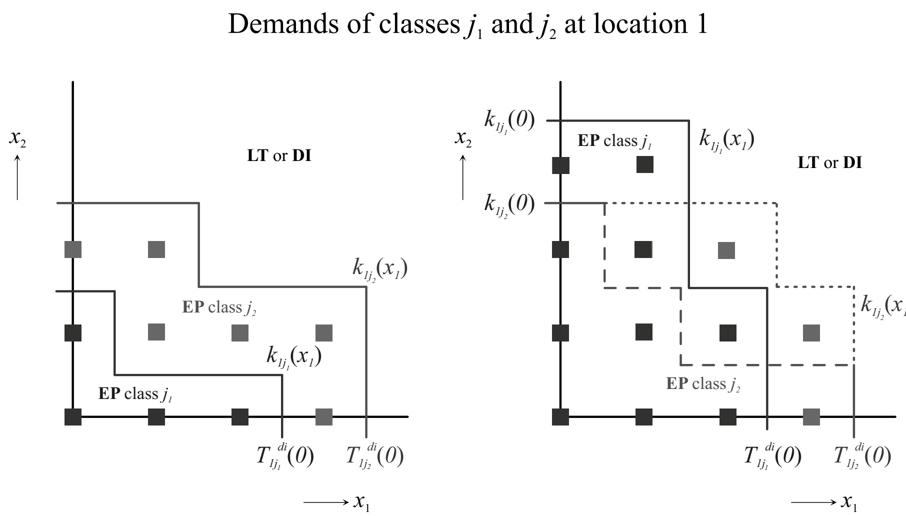


Figure 2: Ordering of the switching curves between demand classes, when $j_1 < j_2$, as in Lemma 7. Note that part a) of Lemma 7 always holds, i.e., $T^{di}_{1j_1}(0) \leq T^{di}_{1j_2}(0)$. Left: Case *b.1)* of Lemma 7, resulting in $k_{1j_1}(x_1) \leq k_{1j_2}(x_1)$ for all $x_1 \in \{0, \ldots, S_1\}$. Right: Case *b.2)* of Lemma 7, resulting in $k_{1j_1}(0) \geq k_{1j_2}(0)$. Here, two possible curves of $k_{1j_2}(x_1)$ are drawn.

3.3. Ordering of Demand Classes

The result that the optimal policy can be described by two switching curves can be seen as an extension of the classical results in stock rationing (Veinott Jr, 1965; Topkis, 1968). This result states that for a single location, multiclass model, the optimal policy for fulfilling demands is described by critical levels, one per demand class, such that a demand of class j is only satisfied if the on-hand inventory level is at or above the critical level for j, say c_j. These critical levels are ordered in the priority of the class: $0 = c_1 \leq c_2 \leq \ldots \leq c_J$, assuming J classes. That is, for the highest priority demands ($j = 1$, largest penalty costs) no stock is ever kept back, and the lower the priority of the class, the more stock is kept back from it.

In the two-location model, these critical levels also depend on the stock level at the other location, hence, transforming these into switching curves. For the ordering of the switching curves, only a partial characterization can be made, as stated in the following lemma and illustrated in Figure 2.

Lemma 7. *Assuming $J_1 \geq 2$, let $j_1, j_2 \in \mathcal{J}_1$, such that $j_1 < j_2$, then*

a) $T^{di}_{1j_1}(0) \leq T^{di}_{1j_2}(0)$

b.1) If $P^{EP}_{1j_1} - P^{LT}_{1j_1} \geq P^{EP}_{1j_2} - P^{LT}_{1j_2}$*, then* $k_{1j_1}(x_1) \leq k_{1j_2}(x_1)$ *for all* $x_1 \in \{0, \ldots, S_1\}$*;*

b.2) If $P^{EP}_{1j_1} - P^{LT}_{1j_1} \leq P^{EP}_{1j_2} - P^{LT}_{1j_2}$*, then* $k_{1j_1}(0) \geq k_{1j_2}(0)$*;*

The analogous results hold for demands at stockpoint 2, assuming $J_2 \geq 2$.

Part a) states that the higher the priority of the demand, the lower the hold back level when the other stockpoint is out-of-stock (i.e., $x_2 = 0$). Part b.1) states that when the *difference* in cost between an emergency procedure and a lateral transshipment for a certain demand class is larger than that difference for a lower priority demand class, then an emergency procedures will be the non-preferred option, for all levels x_1. If the difference is smaller (cf. part b.2), in case of a stock-out (i.e., $x_1 = 0$), a lateral transshipment is preferred over an emergency procedure at lower stock-levels at the other stockpoint for the lower priority demand class. In this case, no general result can be obtained for the ordering of the curves $k_{1j_1}(x_1)$ and $k_{1j_2}(x_1)$ when $x_1 > 0$.

That only limited results on the ordering can be obtained is a result of the fact that the optimal policy and its switching curves originate from three hidden, underlying switching curves. For given x, i, and j, in every state, there is an ordering of the costs of the actions DI, LT, and EP (assuming costs are infinite when an action is not possible). For the *optimal action*, it is only of importance which of these three costs is smallest. However, a closer consideration of the ordering in these costs in each state, results in a better understanding why the optimal policy has this structure. When the cost for each of the three actions are compared in pairs of two (DI vs EP, DI vs LT, and LT vs EP), there exist three switching curves that define the boundaries between sets of states where one action is preferred over the other. Denote these curves for class j at location 1 by $k_{1j}^{DI\text{-}EP}(x_1)$, $k_{1j}^{DI\text{-}LT}(x_1)$, and $k_{1j}^{LT\text{-}EP}(x_1)$. Note that the existence (and monotonicity) of these switching curves results from Theorem 4. There are six possible orderings for costs of these actions (assuming no ties, for the sake of this argument). Using the three curves, the state space can be subdivided into six mutually exclusive subsets, where one of the six orderings of cost occur. This is illustrated in Figures 3 and 4 for an example. Figure 3(a) shows how the state space is subdivided into six subsets, where in each set one of the ordering in costs of DI, LT, and EP occurs (in state $(0, 0)$ there is a tie in costs for DI and LT, as both are infinite). The optimal policy, as shown in Figure 3(b) only takes into account which option results in the smallest cost. The three switching curves are shown in Figure 4. These define the areas of Figure 3(a), and the switching curves $k_{1j}(x_1)$ and $k_{2j}(x_2)$ of the optimal policy (as in Figure 3(b), cf. Theorem 6) are parts of these three switching curves.

There exists a (partial) ordering in these underlying switching curves in the demand

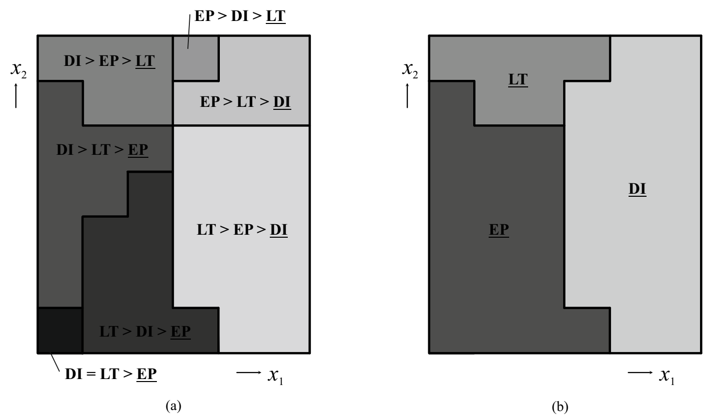


Figure 3: Optimal policy for a demand of class j at location 1. (a): Ordering in costs. (b): Resulting optimal policy. The optimal actions are underlined.

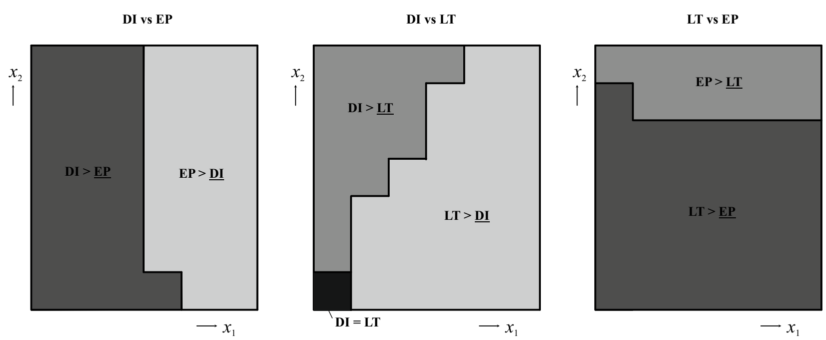


Figure 4: Three switching curves, for the same case as in Figure 3. The optimal actions are underlined.

classes, as stated in the following lemma. Despite this result, this does in general not imply that the switching curves $k_{1j}(x_2)$ and $k_{2j}(x_2)$ which describe the optimal policy (cf. Theorem 6) will be ordered as well in the demand classes (see Example 1 in Section 3.5, classes 2 and 3 at location 1).

Lemma 8. *Assuming $J_1 \geq 2$, let $j_1, j_2 \in \mathcal{J}_1$, such that $j_1 \leq j_2$, then, for all x_1:*
a) $k_{1j_1}^{\text{DI-EP}}(x_1) \leq k_{1j_2}^{\text{DI-EP}}(x_1)$;
b) $k_{1j_1}^{\text{DI-LT}}(x_1) \geq k_{1j_2}^{\text{DI-LT}}(x_1)$;
c.1) If $\left(P_{1j_1}^{EP} - P_{1j_1}^{LT}\right) - \left(P_{1j_2}^{EP} - P_{1j_2}^{LT}\right) \geq 0$, *then* $k_{1j_1}^{\text{LT-EP}}(x_1) \leq k_{1j_2}^{\text{LT-EP}}(x_1)$;
c.2) If $\left(P_{1j_1}^{EP} - P_{1j_1}^{LT}\right) - \left(P_{1j_2}^{EP} - P_{1j_2}^{LT}\right) \leq 0$, *then* $k_{1j_1}^{\text{LT-EP}}(x_1) \geq k_{1j_2}^{\text{LT-EP}}(x_1)$.

3.4. Conditions Simplifying the Optimal Policy

A special instance of the model presented is the case in which at each stockpoint, there is only a single customer class (i.e., $J_1 = J_2 = 1$). In this setting, under simple, sufficient conditions for the cost parameters, the structure of the optimal policy is simplified. We give two such conditions: under the first one, (i) it is optimal to fulfill a demand directly from own stock, whenever possible. However, a parameter limits the amount of outgoing lateral transshipment. This parameter is called the *hold back level*, and indicates the amount of stock that is held back from a transshipment request. Hence, we refer to this policy as a *hold back policy* (see Xu et al., 2003; Van Wijk et al., 2012). Furthermore, under the second condition, (ii) it is optimal to fulfill a demand directly from own stock, whenever possible, and otherwise to apply a lateral transshipment, whenever possible. For an individual stockpoint, we call this a *zero hold back policy*, as the hold back level equals zero. When both stockpoints execute this policy, this is called a *complete pooling* policy.

The following theorem states conditions under which it is optimal to always fulfill a demand directly from own stock. Since in this section we focus on the setting with a single demand class per stockpoint, we drop the subscript j for clarity of exposition.

Theorem 9. *1a) If*

$$P_2^{EP} \leq P_2^{LT} + \left(1 + \frac{\mu}{\lambda_2}\right) P_1^{EP}, \tag{12}$$

then $T_1^{di}(x_2) = 1$ *for all* $x_2 \in \{0, 1, \ldots, S_2\}$*, i.e., a hold back policy is optimal at stockpoint 1.*
b) If

$$P_1^{EP} \leq P_1^{LT} + \left(1 + \frac{\mu}{\lambda_1}\right) P_2^{EP}, \tag{13}$$

then $T_2^{di}(x_1) = 1$ *for all* $x_1 \in \{0, 1, \ldots, S_1\}$*, i.e., a hold back policy is optimal at stockpoint 2.*
2) If (12) *and* (13) *hold, then it is optimal for both stockpoints to execute a hold back policy.*

Under condition (12), whenever there are items in stock at stockpoint 1, they should always be used in case of a demand at stockpoint 1, see Figure 5(a). However, stock can possibly be held back from lateral transshipment requests. If both stockpoints execute a hold back policy, the entire policy is prescribed by only 2 parameters ($\hat{T}_1^{lt}(0)$ and $\hat{T}_2^{lt}(0)$).

In the case of symmetric costs at both stockpoints, i.e., $P_1^{LT} = P_2^{LT}$ and $P_1^{EP} = P_2^{EP}$, one clearly satisfies conditions (12) and (13). The conditions (12) and (13) are also satisfied when $\lambda_1 \downarrow 0$ and $\lambda_2 \downarrow 0$, i.e., under sufficiently low demand rates. Furthermore, in case the cost structure for a stockpoint satisfies a 'triangle inequality' $P_i^{EP} \leq P_k^{EP} + P_i^{LT}$ (that is, it is cheaper to immediately apply an emergency procedure at i, than to apply an emergency procedure at the other stockpoint k and then ship that part by lateral transshipment from k

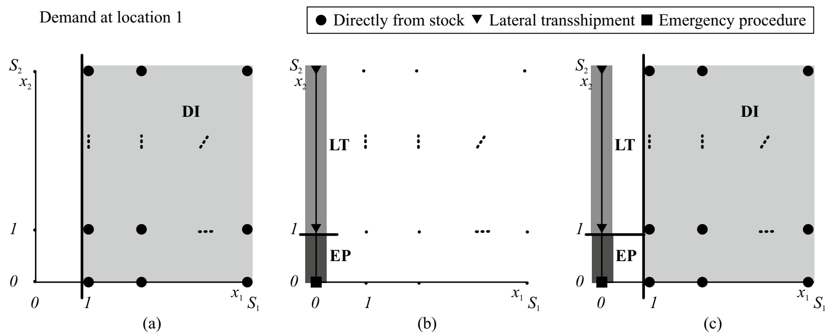


Figure 5: Optimal policy structure of Theorems 9 and 10: (a) Always from stock if $x_1 > 0$ (hold back policy, Theorem 9); (b) Always lateral transshipment if $x_1 = 0$ (Theorem 10); (c) Combination of (a) and (b): zero hold back policy (again Theorem 10, as it implies Theorem 9)

to i), the condition of Theorem 9 at this stockpoint will always be true.

Next we give conditions under which a zero hold back policy is optimal. The required conditions are stronger versions of the conditions in Theorem 9.

Theorem 10. *1a) If*

$$P_1^{LT} + \frac{\lambda_2}{\lambda_2 + \mu} P_2^{EP} \leq P_1^{EP}, \tag{14}$$

then $T_1^{di}(x_2) = 1$ *for all* $x_2 \in \{0, 1, \ldots, S_2\}$ *and* $\hat{T}_1^{lt}(0) = 1$*, i.e., a zero hold back policy is optimal at stockpoint 1.*

1b) If

$$P_2^{LT} + \frac{\lambda_1}{\lambda_1 + \mu} P_1^{EP} \leq P_2^{EP}, \tag{15}$$

then $T_2^{di}(x_1) = 1$ *for all* $x_1 \in \{0, 1, \ldots, S_1\}$ *and* $\hat{T}_2^{lt}(0) = 1$*, i.e., a zero hold back policy is optimal at stockpoint 2.*

2) If (14) *and* (15) *hold, then a complete pooling policy is optimal.*

Under condition (14), stockpoint 2 should not hold back stock if stockpoint 1 requests a lateral transshipment when it is out–of–stock, see Figure 5(b). As condition (14) is stronger than condition (12), it follows that under condition (14) a zero hold back pooling policy is optimal, see Figure 5(c). When both conditions (14) and (15) are satisfied, we obtain a complete pooling policy as optimal policy. Notice that this policy is often assumed in the literature (e.g. Lee, 1987; Tagaras, 1989; Axsäter, 1990; Sherbrooke, 1992; Alfredsson and Verrijdt, 1999; Grahovac and Chakravarty, 2001; Kukreja et al., 2001; Wong et al., 2006; Zhao et al., 2008, to mention only a few). Theorem 10 contributes to a better understanding of when it is justified to assume complete pooling.

The conditions (14) and (15) are always satisfied when $1/\mu \downarrow 0$, i.e., under sufficiently small repair lead times. From a practical point of view: if the part taken for the lateral transshipment can be expected to be added back to stock again (almost) immediately, a lateral transshipment is preferred over an emergency procedure.

Furthermore, we can partially understand the structure of the conditions by considering the following scenario: a demand arises at stock point 1 when in state $x_1 = 0$, $x_2 = 1$, followed by a demand at stock point 2. For the first demand (at stock point 1), applying an emergency procedure yields direct cost P_1^{EP} while leaving the state unchanged, and in that case, the second demand (at stock point 2) can be satisfied directly from stock at no additional extra cost. Alternatively, the first demand can be satisfied by a lateral transshipment, at direct cost P_1^{LT}, after which the state becomes $x_1 = x_2 = 0$. Now, if the second demand occurs before the part used in this lateral transshipment is added back to stock again, which happens with a probability that is at most $\lambda_2/(\lambda_2 + \mu)$ (if $S_1 = 0$, $S_2 = 1$), and the extra cost for an emergency procedure are P_2^{EP} (and are zero otherwise). Comparing the total expected costs for this scenario yields that a lateral transshipment for the first demand, i.e., the demand at stockpoint 1, is preferred if the expected costs of this option, which are at most $P_1^{LT} + \big(\lambda_2/(\lambda_2+\mu)\big)P_2^{EP}$, are smaller than the cost of applying an emergency procedure, P_1^{EP}. This is exactly condition (14). A similar reasoning holds for condition (15) when the stockpoints are interchanged.

The conditions in Theorems 9 and 10 are, in general, sufficient, but not necessary. For the cases $S_1 = 1, S_2 = 0$, respectively $S_1 = 0, S_2 = 1$, the conditions are necessary *and* sufficient. There exist examples *not* satisfying these conditions, in which case, the optimal policy is neither a hold back nor a zero hold back pooling policy (see Example 2a of Section 3.5).

Combining these conditions for both stockpoints leads to the following result.

Corollary 11. *The optimal lateral transshipment policy is*

1. *either a hold back policy at both locations;*
2. *or a zero hold back policy for at least one location.*

In the second case, the optimal policy for one location is a zero hold back policy, and the optimal policy for the other location can be a hold back policy, a zero hold back policy, or neither of the two.

3.5. Examples

Example 1. Consider the following example: $S_1 = 5, S_2 = 6$, and $J_1 = 3$, $J_2 = 3$, $\lambda_{11} = 2$, $\lambda_{12} = 0.5$, $\lambda_{13} = 0.25$, $\lambda_{21} = 0.5$, $\lambda_{22} = 0.5$, $\lambda_{23} = 1$, $\mu = 1/3$, and cost parameters given by $P_{11}^{EP} = 13$, $P_{12}^{EP} = 7$, $P_{13}^{EP} = 6$, $P_{21}^{EP} = 23$, $P_{22}^{EP} = 18$, $P_{23}^{EP} = 9$, and $P_{11}^{LT} = 6$, $P_{12}^{LT} = 1$, $P_{13}^{LT} = 1$, $P_{21}^{LT} = 15$, $P_{22}^{LT} = 10$, $P_{23}^{LT} = 1$. The optimal policy for fulfilling demands at location 1 is given in Figure 6.

For the highest priority class at location 1, orders are always satisfied if on-hand stock is available. In case of a stock-out, a lateral transshipments is applied in case location 2 has three or more parts on-hand. The lower the priority class, the more often the choice is made for an emergency procedure or lateral transshipment (which are less costly than these actions for a high priority demand). In particular, for the lowest demand class at location 1, the threshold above which demands are directly fulfilled is high. Note that since $P_{1j_1}^{EP} - P_{1j_1}^{LT} > P_{1j_2}^{EP} - P_{1j_2}^{LT} > P_{1j_3}^{EP} - P_{1j_3}^{LT}$, we have $\text{EP}_{j_1} \subseteq \text{EP}_{j_2} \subseteq \text{EP}_{j_3}$ (as in Lemma 7, part b.1). At location 2, stock is almost always used to fulfill demand directly. Only for the lowest priority demand class, parts are kept back. Although the parameters are different, the optimal policies for the two highest priority classes here are equal.

Example 2a. We now consider an example with only a single demand class per stockpoint, that is, $J_1 = J_2 = 1$ (as in Section 3.4). In this way, we can fully concentrate on the use of lateral transshipment in the optimal policy. Again, the subscript j is dropped for clarity of exposition. The parameters are set to: $S_1 = S_2 = 4$, and $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu = 1/3$, and for the costs: $P_1^{EP} = 25$, $P_1^{LT} = 5$ and $P_2^{EP} = 10$, $P_2^{LT} = 2$. Hence, an emergency procedure is five times as expensive as a lateral transshipment, and at location 1, the demand rate as well as the costs are higher. The optimal policy is given in Figure 7. At stockpoint 1 a zero hold back policy is optimal. This structure is implied by the fact that the parameters satisfy condition (14), and hence, part 1a) of Theorem 10 holds. For stockpoint 2, demands are only fulfilled directly from stock if the sum of the inventory levels at both locations is large enough. That is, if $x_1 + x_2 \geq 3$ (and $x_2 > 0$) a part is taken from stock, and otherwise an emergency procedure is applied. This can be explained in the following way. The costs for lateral transshipments to and emergency procedures at stockpoint 1 are much higher than those at stockpoint 2. This results in the fact that stockpoint 2 will hold back parts, even when it faces a demand. By holding back parts, the expensive costs for an emergency procedure

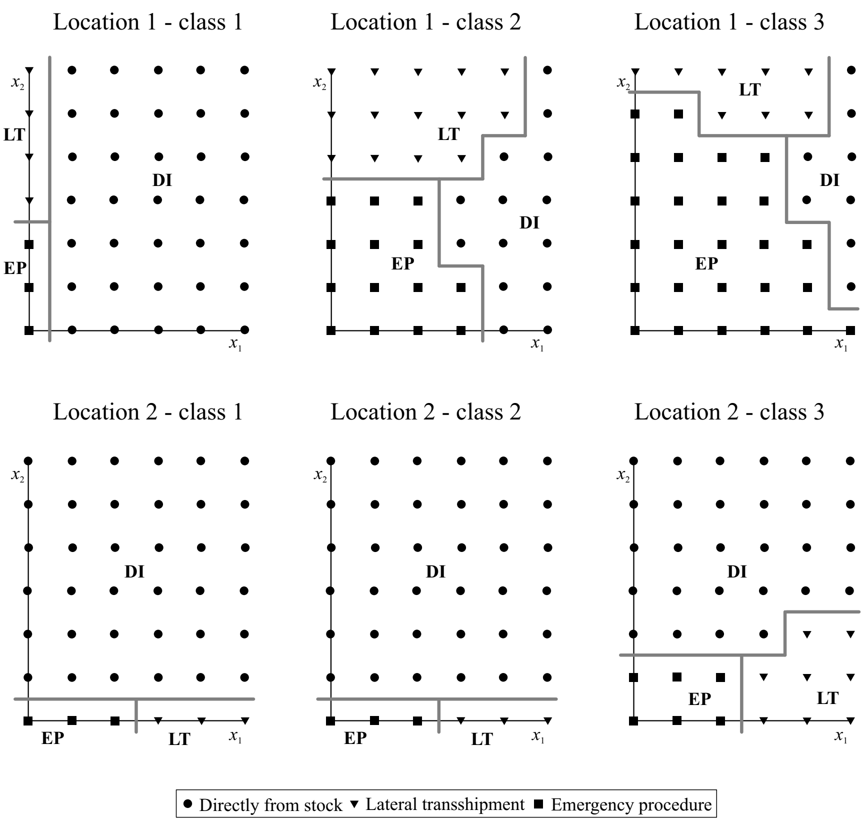


Figure 6: Example 1: Optimal policy for fulfilling demands of each of the classes at locations 1 and 2.

at stockpoint 1 are saved in case of a demand there, when it is stocked-out. This is at the expense of a lateral transshipment from 2 and possibly one or more emergency procedures at 2. The option of holding back parts at 2, however, turns out to be less costly, on average. The optimal policy at location 2 resembles a critical level policy where the sum $x_1 + x_2$ acts as a critical level determining whether a demand at location 2 is directly satisfied.

The optimal policy gives expected average costs per time unit of 18.2. Without lateral transshipments, these costs would be 25.5; hence, the optimal policy reduces this by almost 29%. A complete pooling policy has expected average costs per time unit of 20.0. So, the optimal policy reduces these by 9.4%.

Example 2b. In Example 2a, condition (14) (and hence, condition (12)) was satisfied for stockpoint 1, but not for stockpoint 2. By doubling the penalty costs at 2, into $P_2^{EP} = 20$ and $P_2^{LT} = 4$, condition (13) is satisfied as well. Hence, by Theorem 9, this results in the optimality of a hold back policy at both locations (with still zero hold back at 1). The optimal policy is given in Figure 8. The two hold back levels $\hat{T}_1^{lt}(0) = 1$ and $\hat{T}_2^{lt}(0) = 2$ determine the

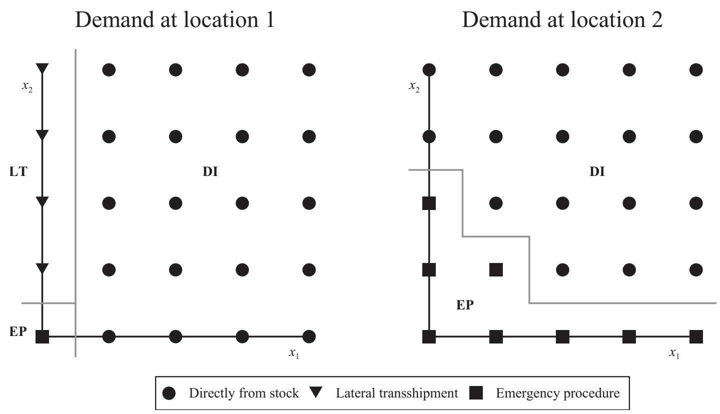


Figure 7: Example 2a: Optimal policy for the case with $S_1 = S_2 = 4, \lambda_1 = 2, \lambda_2 = 1$, $\mu = 1/3$ and penalty costs $P_1^{EP} = 25, P_1^{LT} = 5, P_2^{EP} = 10, P_2^{LT} = 2$.

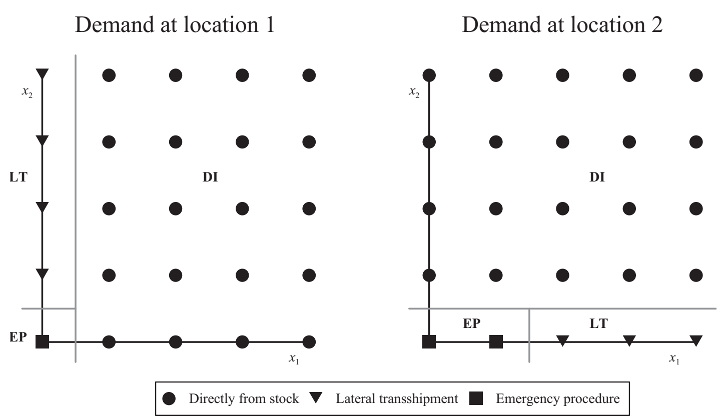
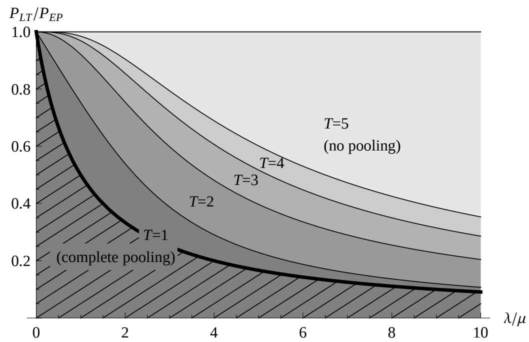


Figure 8: Example 2b: Optimal policy. Note that the difference in parameters with Example 2a is $P_2^{EP} = 20$ and $P_2^{LT} = 4$.

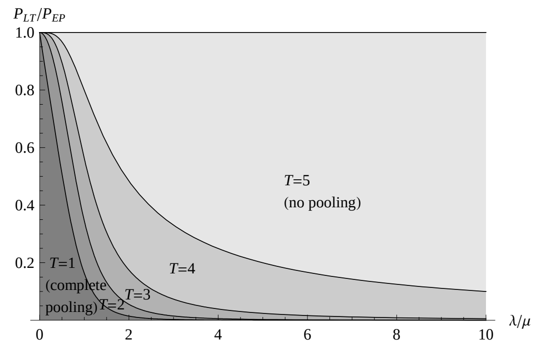
entire policy. These are the inventory levels from which on lateral transshipments ($a_i^* = 1$) are applied instead of emergency procedures ($a_i^* = 2$). The expected average costs per time unit are 22.9. For a policy without lateral transshipments these would be 27.6 (almost 17% reduction for the optimal policy), and complete pooling would give 23.2. This is only a 1.4% reduction, but this policy differs from the optimal policy only in $a_2(1, 0)$.

3.6. Symmetric Parameters

A special case is the system in which all parameters are symmetric, i.e., in which all parameters for both stockpoints are equal: $S_1 = S_2 =: S, \lambda_1 = \lambda_2 =: \lambda, P_1^{LT} = P_2^{LT} =: P^{LT}, P_1^{EP} = P_2^{EP} =: P^{EP}$ (still assuming $J_1 = J_2 = 1$). It is straightforward that in this case there exists a symmetric optimal policy. As the conditions of Theorem 9 are clearly satisfied, it follows that for both stockpoints a hold back policy is optimal. The entire policy can now



(a) Ample repair capacity



(b) Single repair server at each of the locations

Figure 9: For $S = 4$ and symmetric system parameters, the $S + 1$ regions where each of the thresholds T is optimal, in the cases of ample repair capacity (left) and in case of a single repair server at each of the locations (right). For ample repair capacity, the region where Condition (16) assures the optimal policy to be complete pooling is indicated as well (marked area below the bold line).

be described by a single (for both stockpoints equal) hold back level $\hat{T}_1^{lt}(0) = \hat{T}_2^{lt}(0) =: T \in \{1, 2, \ldots, S+1\}$: $T = 1$ indicates a zero hold back policy, $T = 2$ indicates that one part is held back, and so on. Finally, $T = S + 1$ indicates that no stock in shared in any way, i.e., there is no interaction between the stockpoints. Hence, there are $S + 1$ possible optimal policies.

Given the ratio λ/μ, it turns out that the optimal policy is determined by only the ratio P^{LT}/P^{EP}. For $S = 4$ it is indicated in Figure 9a when each of the five ($= S + 1$) possible policies is optimal. These areas are determined by solving the steady-state distribution of the Markov process for each of the policies and deriving the average costs of a policy. For the symmetric case, Theorem 10 reduces to the following corollary, which also holds when $S_1 \neq S_2$. The interpretation of the condition in this theorem is similar to the interpretation of conditions (14) and (15).

Corollary 12. *In case of symmetric system parameters λ, P^{EP} and P^{LT}, a complete pooling policy is optimal if*

$$P^{LT} \leq \frac{\mu}{\lambda + \mu} P^{EP}. \tag{16}$$

In Figure 9a the curve $P^{LT}/P^{EP} = \mu/(\lambda + \mu)$ is plotted as well. Below this curve, by Corollary 12, complete pooling is optimal. From this figure it turns out that this condition, although only sufficient, covers a large part of the total, exact area.

4. Model Extensions

In this section, we consider several model extensions, which include consumables (4.1), order fulfillment problems (4.2), and substitutable items (4.3). Typically, with possibly some minor adjustments, all results derived in the previous sections remain to hold. We also show that in case of unequal repair rates, the structural results might fail to hold (4.4). However, when one limits the repair capacity, the results remain valid, even when the repair rates are unequal (4.5).

4.1. Consumables

We have presented the model for an inventory system with repairables. However, the model is also suitable for consumable parts. Instead of having S_i circulating repairable spare parts at location i, we assume that a basestock policy is executed, where the basestock level is S_i. Replenishments are used to increase the stock level, where the replenishment lead times are modeled by the operator G_i, in the same way as the repair lead times were modeled.

Holding cost. For a system with circulating stock, one can without loss of generality charge holding costs for items in repair. Hence the holdings costs are constant and can therefore be left out. However, when considering consumables, typically holding costs are charged *only* for parts in stock. This can be incorporated in the model, by adding the term $\sum_{i=1}^{2} h_i(x_i)$ in the value function, where $h_i : \{0, 1, \ldots, S_i\} \mapsto \mathbb{R}$ denotes the holding costs at stockpoint i per part per time unit. We assume that $h_i(0) = 0$ and that $h_i(x_i)$ is non-decreasing and convex in x_i. The latter is required to ensure that the value function remains convex in every iteration.

We can again use Lemmas 1 and 2. Instead of Theorem 3, the value function now satisfies (6)–(10) for all $n \geq 0$. Decr is not satisfied, as the holding costs are increasing in x_1 and x_2. Consequently, Theorems 4, 5, and 6 still hold (as Decr is not used in the proofs). When focusing on a single demand class per stockpoint, in the conditions under which the (zero) hold back policy is optimal, however, an extra term incorporating the holding costs should be added. In Theorem 9, condition (12) changes into $P_2^{EP} \leq P_2^{LT} + (1 + \mu/\lambda_2)\, P_1^{EP} + h_1(1)/(\nu\, \lambda_2)$, and similarly for condition (13). In Theorem 10, condition (14) changes into $P_1^{LT} + (\lambda_2\, P_2^{EP})/(\lambda_2 + \mu) \leq P_1^{EP} + h_2(1)/(\nu\, (\lambda_2 + \mu))$, and similarly for condition (15). Under these changed conditions, both theorems hold.

Backordering. Since we consider a model in which high cost are incurred for downtime of systems, we did not allow for backorders. However, it is straightforward to include this option, as long as we limit ourselves to a single demand class per stock point. That is, we allow parts to be 'taken directly from stock' at location i, even if $x_i \leq 0$, which means that a demand is backordered. The negative part of x_i, which is $x_i^- = \max\{-x_i, 0\}$, denotes the number of outstanding backorders at location i. For technical reasons, we limit the maximum number of outstanding backorders at location i to be $B_i \geq 0$, to ensure that one can still apply uniformization (this requires a finite rate out of each state). Let $b_i : \{0, 1, \ldots, B_i\} \mapsto \mathbb{R}$ denote the backordering costs at stockpoint i per backordered demand per time unit. We assume that $b_i(0) = 0$ and that $b_i(x_i)$ is non-increasing and convex in x_i. We add the term $\sum_{i=1}^{2} b_i(x_i^-)$ to the value function. The state space becomes $\mathcal{S}^b = \{-B_1, \ldots, -1, 0, 1, \ldots, S_1\} \times \{-B_2, \ldots, -1, 0, 1, \ldots, S_2\}$, where the superscripts b indicates the case of backordering. Furthermore, for the operator H_1 the states for which a certain action can be taken have to be adjusted. In the definition (3), $x_1 = 0$ and $x_1 > 0$ are replaced by $x_1 = -B_1$ and $x_1 > -B_1$, respectively, and $x_2 = 0$ becomes $x_2 \leq 0$. Similar adjustments are made for H_2. Denote the new operators by H_i^b, $i = 1, 2$. Note that a demand at location i can only be backordered at location i. Assuming that the replenishment rate is linear in the number of outstanding orders, the operator G_i remains the same. The uniformization rate becomes $\nu^b = (S_1 + S_2 + B_1 + B_2)\mu + \sum_{i=1}^{2} \lambda_i$.

It is straightforward to derive that Lemma 2 still holds for the operator H_i^b on the state space $\mathcal{S}^b$. This is the case, as in fact the extension to backordering only expands the state space and then shifts the origin from $(0, 0)$ to $(-B_1, -B_2)$. For the same reason, Lemma 1 remains to hold as well. Analogously to the case with holding cost, instead of Theorem 3, the value function now satisfies (6)–(10) for all $n \geq 0$ and state space $\mathcal{S}^b$. The structural results of Theorems 4, 5, and 6 still hold, when the domains of x_1 and x_2 (and accordingly, those of the thresholds) are adjusted to $\mathcal{S}^b$. In Theorem 9, if condition (12) is changed into

$$P_2^{EP} \leq P_2^{LT} + \left(1 + \frac{\mu}{\lambda_2}\right) P_1^{EP} + \frac{b_1(-B_1) - b_1(-B_1 + 1)}{\nu^b \lambda_2},$$

and similarly for condition (13), the theorem still holds, with an adjusted definition of a hold back policy. That is, a hold back policy now means that in case of a demand, whenever there is the possibility to take a part from stock *or* to backorder the demand, this option is preferred over a lateral transshipment and over an emergency procedure. Note that the condition given, simplifies if $b_1(\cdot)$ is linear in the number of outstanding backorders. Theorem 10 remains to hold, with unchanged conditions, if the definitions of a zero hold back policy and complete pooling policy are adjusted similarly to the adjustment of the definition of a hold back policy for Theorem 9.

4.2. Order Fulfillment Problems

The lateral transshipment model also exactly fits a so-called order fulfillment problem in e-commerce: a retailer has multiple warehouses and customers differ in revenue and criticality regarding shipment times (see, e.g., Acimovic and Graves, 2014). When a customer orders a product via a website or by phone, the retailer might have that product stocked at multiple of its warehouses. Moreover, the customer might choose from different options for shipping: the faster, the higher the cost, with these options also differing in revenue for the retailer. Once the demand is placed, the retailer has to decide on the optimal way to fulfill the demand. This could be from the warehouse closest to the customer, or from another warehouse.

The customers are assigned to the warehouses based on, e.g., their geographical location. The shipment time chosen for their order defines the demand classes it has at a warehouse, where the shorter the shipment time, the higher the priority of the customer. A direct demand fulfillment equals the customer receiving the ordered product directly from the closest warehouse. A lateral transshipment equals the action that the order is fulfilled from another warehouse. Higher transportation costs are incurred, however, stock is kept back at the closest warehouse for future high priority demands. A shipment from a central depot, having large shipment cost, reflects an emergency procedure. As this problem fits into the lateral transshipment model (for a two warehouse setting), all results of Section 3.2 remain to hold, and so will the orderings and the conditions of Sections 3.3 and 3.4.

4.3. Substitutions and Unidirectional Transshipments

The presented model is also suitable for a single-location inventory models with two types of products. Then the state (x_1, x_2) denotes the inventory levels of each type of the product. One type can serve as a substitute for the other in case of a stock-out. However, extra costs for such a substitution might be incurred, which are then the cost factors P_{1j}^{LT} and P_{2j}^{LT}. Applications of these models are found in, e.g., car rentals and storage box rentals. For example, x_1 denotes the number of available large cars, and x_2 of small cars. A customer wishing to rent a small car, can be assigned a large car if $x_2 = 0$. This is called a substitution. The extra costs this brings along, are taken into account in P_{1j}^{LT}. Since some customers might be more affected by a substitution than others, the cost differ per demand class.

In the model we allow transshipments (substitutions) in both ways. A simplification of this is an *unidirectional transshipment* model (cf. Axsäter, 2003b). In that case, a lateral transshipment (substitution) can only be applied in one way, e.g., from 2 to 1 only, and not the other way around. Other reasons for a substitution in one way might be due to higher quality or product specifications. The restriction to unidirectional transshipments (from 2 to 1) can be achieved by setting $P_{2j}^{LT} = P_{2j}^{EP}$ for all $j \in \mathcal{J}_2$. In this way a lateral transshipment will never be an optimal action for a demand at stockpoint 2 (as it is as expensive as an emergency procedure but does not reduce the stock level at 1). Obviously, all structural results will remain to hold.

For a single demand class per stockpoint, note that by putting $P_2^{LT} = P_2^{EP}$, inequality (12) is always satisfied. Consequently, a hold back policy is optimal for 1, which was to be expected. Note that although (15) is never satisfied (unless $P_1^{EP} = 0$), we can *not* conclude that a zero hold back policy is suboptimal at stockpoint 2. This is because the conditions in Theorems 9 and 10 are sufficient but not necessary. As for a one-way substitution model, typically $P_2^{EP} \geq P_1^{EP}$ (it is more costly not satisfying a demand for a larger car than for a smaller car), inequality (13) will be satisfied as well. Hence, a demand at 2 is always satisfied directly from stock.

4.4. Asymmetric Repair Rates

As both stockpoints stock the same repairable part, we have assumed that both stockpoints have the same repair lead time distribution. In fact, we need this assumption in order to derive the structural results. This is because according to Lemma 1 only $G_1 + G_2$ preserves MM, and not G_1 and G_2 individually. This assumption is sufficient but not necessary. One can easily construct examples with unequal repair rates for which the structural results *do* hold. However, there are also examples with unequal repair rates for which the structural results fail to hold, as shown in the following example.

Let $S_1 = 1, S_2 = 2, J_1 = J_2 = 1, \lambda_1 = \lambda_2 = 1$, and $\mu_1 = 1/3 \neq \mu_2 = 1$, denoting by μ_i the repair rate at stockpoint i. Furthermore, let $P_1^{EP} = 1000, P_1^{LT} = 175$ and $P_2^{EP} = P_2^{LT} = 10$.

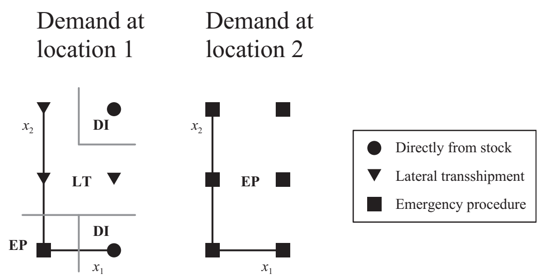


Figure 10: Example in which $\mu_1 \neq \mu_2$: the structural properties of the optimal policy does not hold.

The (unique) optimal policy is given in Figure 10. Clearly, for demands at stockpoint 1 when $x_1 = 1$, the structure of the optimal policy is not as described by Theorem 6. This example illustrates that under $\mu_1 \neq \mu_2$ the structural results do not necessarily hold. At location 1 when $x_1 = 1$, a demand is directly satisfied from stock if $x_2 = 0$ or $x_2 = 2$, however, if $x_2 = 1$ a lateral transshipment is the (only) optimal action.

By applying a lateral transshipment at location 1 when in state $x_1 = x_2 = 1$, one exploits the faster repair times at location 2, at the expense of P_1^{LT}. Namely, after applying a lateral transshipment in this state, the expected time till any part is added back to stock is 0.25, whereas it would be twice as long, i.e., 0.50, if a part is taken from stock at location 1 when in this state. Given the high cost for an emergency procedure at 1 in this example, the reduction in expected repair time (and hence, the reduction in expected future cost) outweighs the extra cost P_1^{LT}. Note that this situation cannot happen when the repair rates are equal, since then there is no advantage to apply a lateral transshipment in such a situation, only to decrease the expected time till a part is added back to stock.

The parameters in this example are such that in state $x_1 = 1$, $x_2 = 2$ the smaller expected repair lead time does not lead to sufficient savings on expected future costs to justify the extra cost of a lateral transshipment at location 1 when in this state. Only if P_1^{LT} is reduced from 175 to (approximately) 165.2, the optimal action in this state becomes a lateral transshipment. If P_1^{LT} is increased to (approximately) 196.0, it will be optimal in state $x_1 = 1$, $x_2 = 1$ to take a part directly from stock.

4.5. Limited Repair Capacity

A variant of the described system is a system in which there is limited repair capacity: at each stockpoint there is only one server to repair the failed returned parts. The repair times remain exponentially distributed with mean $1/\mu_i$ at stockpoint i, where, for generality, we allow for non-identical repair rates at both locations. We only have to change the operator G_i into, say, $\tilde{G}_i$, defined by

$$\tilde{G}_1 f(x_1, x_2) = \begin{cases} f(x_1 + 1, x_2) & \text{if } x_1 < S_1, \\ f(x_1, x_2) & \text{if } x_1 = S_1, \end{cases} \tag{17}$$

and $\tilde{G}_2$ analogously. In the value function, one replaces G_i by $\tilde{G}_i$, $i = 1, 2$ and updates ν into $\nu = \sum_{i=1}^{2} \sum_{j=1}^{J_i} \lambda_{ij} + \mu_1 + \mu_2$, the remainder being unchanged.

Theorem 13. *In case of a single repair server at each of the stockpoints, the same structural results for the optimal policy hold. That is, Theorems 4, 5, and 6 hold.*

For Theorem 13 we do not need equal μ_i's, as $\tilde{G}_1$ and $\tilde{G}_2$ separately preserve MM, and not only the sum of both. For symmetric system parameters, we compare the optimal policy for a single repair server (see Figure 9b) with the case of ample repair capacity (see Figure 9a). From the graphs it follows that the set of system parameters where one can benefit from lateral transshipments, is much smaller in the case of a single repair server.

5. Conclusion

In this paper, we proved that the structure of the optimal lateral transshipment policy (under the given assumptions) is a threshold type policy. We also gave sufficient conditions under which a (zero) hold back policy or a complete pooling policy is optimal. We studied a number of model extensions fitting within the same framework.

Interesting problems for further research would be the extension to three or more stockpoints; variations in the repair time distribution (such as Erlang-k distributed repair lead times, or state-dependent repair rates); and the incorporation of so-called pro-active lateral transshipments, i.e., rebalancing of the stock levels triggered by a replenishment.

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Supplementary Materials
Optimal Lateral Transshipment Policies for a Two Location Inventory Problem with Multiple Demand Classes

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A. APPENDIX - Proofs

A.1. Proof of Lemma 1

Proof. a) We give the proofs for the operator G_1. By interchanging the numbering of the locations, the results directly follow for the operator G_2 as well.

(i) It is straightforward to check that if f is Decr(1) (cf. (4)), then $G_1 f$ is Decr(1) as well, i.e., if $f(x_1, x_2) \geq f(x_1+1, x_2)$, then $G_1 f(x_1, x_2) \geq G_1 f(x_1+1, x_2)$, for all (x_1, x_2) such that the states appearing exists $\in \mathcal{S}$. Along the same lines it follows that if f is Decr(2) (cf. (5)), then $G_1 f$ is Decr(2) as well, i.e., then $G_1 f(x_1, x_2) \geq G_1 f(x_1, x_2+1)$. Combining this proves that the operator G_1 preserves Decr.

(ii) Assume that f is Conv(1) (cf. (6)), then we show that $G_1 f$ is Conv(1) as well. For $x_1 + 2 < S_1$:

$$
\begin{aligned}
&G_1 f(x_1, x_2) + G_1 f(x_1+2, x_2) \\
&= (S_1 - x_1) f(x_1+1, x_2) + x_1 f(x_1, x_2) \\
&\quad + (S_1 - x_1 - 2) f(x_1+3, x_2) + (x_1+2) f(x_1+2, x_2) \\
&= (S_1 - x_1 - 2)\Big[f(x_1+1, x_2) + f(x_1+3, x_2)\Big] + x_1\Big[f(x_1, x_2) + f(x_1+2, x_2)\Big] \\
&\quad + 2f(x_1+1, x_2) + 2f(x_1+2, x_2) \\
&\geq 2\,(S_1 - x_1 - 2) f(x_1+2, x_2) + 2\,x_1 f(x_1+1, x_2) + 2f(x_1+1, x_2) + 2f(x_1+2, x_2) \\
&= 2\,(S_1 - x_1 - 1) f(x_1+2, x_2) + 2\,(x_1+1) f(x_1+1, x_2) \\
&= 2\,G_1 f(x_1+1, x_2),
\end{aligned}
$$

where the inequality holds by applying that f is Conv(1) on the parts between brackets. And for

$x_1 + 2 = S_1$:

$$\begin{aligned}
&G_1 f(S_1-2,x_2) + G_1 f(S_1,x_2)\\
&= 2f(S_1-1,x_2) + (S_1-2)f(S_1-2,x_2) + S_1 f(S_1,x_2)\\
&= 2f(S_1-1,x_2) + (S_1-2)[f(S_1-2,x_2) + f(S_1,x_2)] + 2\,f(S_1,x_2)\\
&\geq 2f(S_1-1,x_2) + 2\,(S_1-2)f(S_1-1,x_2) + 2\,f(S_1,x_2)\\
&= 2\,f(S_1,x_2) + 2\,(S_1-1)f(S_1-1,x_2)\\
&= 2\,G_1 f(S_1-1,x_2),
\end{aligned}$$

where again the inequality holds by applying that f is Conv(1) on the part between brackets.

It is straightforward to check that if f is Conv(2) (cf. (7)), then $G_1 f$ is Conv(2) as well, i.e., then $G_1 f(x_1,x_2) + G_1 f(x_1,x_2+2) \geq 2\,G_1 f(x_1,x_2+1)$. Combining this proves that the operator G_1 preserves Conv.

(iii) Along the same lines of the proof of (ii) one can prove that if f is Supermod (cf. (8)), then $G_1 f$ is Supermod as well. Hence the operator G_1 preserves Supermod.

b) (i)–(iii) trivially follow from part a).

(iv) We show that $G_1 + G_2$ preserves SuperC$(1,2)$; then SuperC$(2,1)$ follows by interchanging the numbering of the locations. Assume that f is SuperC$(1,2)$ (cf. (9)), then, for $x_1 + 2 < S_1$ and $x_2 + 1 < S_2$:

$$\begin{aligned}
&(G_1+G_2)f(x_1,x_2+1) + (G_1+G_2)f(x_1+2,x_2)\\
&= (S_1-x_1-2)\Big[f(x_1+1,x_2+1) + f(x_1+3,x_2)\Big] + 2f(x_1+1,x_2+1)\\
&\quad + x_1\Big[f(x_1,x_2+1) + f(x_1+2,x_2)\Big] + 2f(x_1+2,x_2)\\
&\quad + (S_2-x_2-1)\Big[f(x_1,x_2+2) + f(x_1+2,x_2+1)\Big] + f(x_1+2,x_2+1)\\
&\quad + (x_2+1)\Big[f(x_1,x_2+1) + f(x_1+2,x_2)\Big] - f(x_1+2,x_2).
\end{aligned}$$

Now we use that f is SuperC$(1,2)$, and apply this to the terms between brackets. This gives

$$\begin{aligned}
&(G_1+G_2)f(x_1,x_2+1)+(G_1+G_2)f(x_1+2,x_2)\\
&\geq (S_1-x_1-2)\Big[f(x_1+2,x_2)+f(x_1+2,x_2+1)\Big]+2f(x_1+1,x_2+1)\\
&\quad +x_1\Big[f(x_1+1,x_2)+f(x_1+1,x_2+1)\Big]+2f(x_1+2,x_2)\\
&\quad +(S_2-x_2-1)\Big[f(x_1+1,x_2+1)+f(x_1+1,x_2+2)\Big]+f(x_1+2,x_2+1)\\
&\quad +(x_2+1)\Big[f(x_1+1,x_2)+f(x_1+1,x_2+1)\Big]-f(x_1+2,x_2)\\
&=(S_1-x_1-1)f(x_1+2,x_2)+(x_1+1)f(x_1+1,x_2)\\
&\quad +(S_1-x_1-1)f(x_1+2,x_2+1)+(x_1+1)f(x_1+1,x_2+1)\\
&\quad +(S_2-x_2)f(x_1+1,x_2+1)+x_2f(x_1+1,x_2)\\
&\quad +(S_2-x_2-1)f(x_1+1,x_2+2)+(x_2+1)f(x_1+1,x_2+1)\\
&=(G_1+G_2)f(x_1+1,x_2)+(G_1+G_2)f(x_1+1,x_2+1).
\end{aligned}$$

The cases $x_1+2=S_1$ and/or $x_2+1=S_2$ are along the same lines.

(v) As MM = Supermod $\cap$ SuperC (cf. (11)), it directly follows from parts (iii) and (iv) that G_1+G_2 preserves MM. ■

A.2. Proof of Lemma 2

Proof. (i) It is straightforward to check that if f is Decr(k), then $H_{ij}f$ is Decr(k), for (i,j), where $i=1,2$, $j=1,2,\dots,J_i$, and $k=1,2$.

(ii) In order to prove that H_{ij} preserves MM, we prove (cf. (11)) that it preserves Supermod, SuperC$(1,2)$ and SuperC$(2,1)$ (cf. (8)–(10)) together, that is, given that f is Supermod, SuperC$(1,2)$ and SuperC$(2,1)$, we show that $H_{ij}f$ is Supermod, SuperC$(1,2)$ and SuperC$(2,1)$ as well. We show this for H_{1j}; then for H_{j2} it follows by interchanging the numbering of the locations. Recall that Supermod and SuperC(i,k) imply Conv(i) (cf. (6) and (7)).

The proofs come down to case checking: applying H_{1j} to $f(x)$ introduces a minimization over three terms, so the sum of two gives a total of $3\times 3=9$ possibilities, which we all check separately. For this we use the trivial result:

$$a\geq \min\{a,b\},\ \forall a,b\in\mathbb{R}.$$

The proofs are given for $x_1>0, x_2>0$, but it is straightforward to check that they also hold for the cases $x_1=0, x_2>0$, and $x_1>0, x_2=0$, and $x_1=0, x_2=0$.

Assume that f is Supermod, SuperC$(1,2)$ and SuperC$(2,1)$, which implies that f is also Conv(1) and Conv(2). Below we prove that H_{1j}, for all $j\in\mathcal{J}_1$, preserves (i) Supermod, (ii)

SuperC(1, 2), and (iii) SuperC(2, 1).

(i) Supermod

For $x_1 > 0, x_2 > 0$:

$$
\begin{aligned}
&H_{1j}f(x_1,x_2) + H_{1j}f(x_1+1,x_2+1)\\
&= \min\Big\{f(x_1-1,x_2),\ f(x_1,x_2-1)+P_{1j}^{LT},\ f(x_1,x_2)+P_{1j}^{EP}\Big\}\\
&\quad + \min\Big\{f(x_1,x_2+1),\ f(x_1+1,x_2)+P_{1j}^{LT},\ f(x_1+1,x_2+1)+P_{1j}^{EP}\Big\}\\
&= \min\Big\{f(x_1-1,x_2)+f(x_1,x_2+1),\ f(x_1-1,x_2)+f(x_1+1,x_2)+P_{1j}^{LT},\\
&\qquad f(x_1-1,x_2)+f(x_1+1,x_2+1)+P_{1j}^{EP},\ f(x_1,x_2-1)+P_{1j}^{LT}+f(x_1,x_2+1),\\
&\qquad f(x_1,x_2-1)+P_{1j}^{LT}+f(x_1+1,x_2)+P_{1j}^{LT},\ f(x_1,x_2-1)+P_{1j}^{LT}+f(x_1+1,x_2+1)+P_{1j}^{EP},\\
&\qquad f(x_1,x_2)+P_{1j}^{EP}+f(x_1,x_2+1),\ f(x_1,x_2)+P_{1j}^{EP}+f(x_1+1,x_2)+P_{1j}^{LT},\\
&\qquad f(x_1,x_2)+P_{1j}^{EP}+f(x_1+1,x_2+1)+P_{1j}^{EP}\Big\}.
\end{aligned}
$$

It holds that:

$$
\begin{aligned}
f(x_1-1,x_2)+f(x_1,x_2+1) &\geq f(x_1,x_2)+f(x_1-1,x_2+1) \text{ (by (8))},\\
f(x_1-1,x_2)+f(x_1+1,x_2)+P_{1j}^{LT} &\geq 2\,f(x_1,x_2)+P_{1j}^{LT} \text{ (by (6))},\\
f(x_1-1,x_2)+f(x_1+1,x_2+1)+P_{1j}^{EP} &\geq 2\,f(x_1,x_2)-f(x_1+1,x_2)+f(x_1+1,x_2+1)+P_{1j}^{EP}\\
&\geq f(x_1,x_2)+f(x_1,x_2+1)+P_{1j}^{EP} \text{ (by (6), resp. (8))},\\
f(x_1,x_2-1)+P_{1j}^{LT}+f(x_1,x_2+1) &\geq 2\,f(x_1,x_2)+P_{1j}^{LT} \text{ (by (7))},\\
f(x_1,x_2-1)+P_{1j}^{LT}+f(x_1+1,x_2)+P_{1j}^{LT} &\geq f(x_1,x_2)+P_{1j}^{LT}+f(x_1+1,x_2-1)+P_{1j}^{LT} \text{ (by (8))},\\
f(x_1,x_2-1)+P_{1j}^{LT}+f(x_1+1,x_2+1)+P_{1j}^{EP} &\geq 2\,f(x_1,x_2)-f(x_1,x_2+1)+P_{1j}^{LT}+f(x_1+1,x_2+1)+P_{1j}^{EP}\\
&\geq f(x_1,x_2)+P_{1j}^{LT}+f(x_1+1,x_2)+P_{1j}^{EP} \text{ (by (7), resp. (8))},\\
f(x_1,x_2)+P_{1j}^{EP}+f(x_1+1,x_2+1)+P_{1j}^{EP} &\geq f(x_1+1,x_2)+P_{1j}^{EP}+f(x_1,x_2+1)+P_{1j}^{EP} \text{ (by (8))}.
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
&H_{1j}f(x_1,x_2)+H_{1j}f(x_1+1,x_2+1)\\
&\geq \min\Big\{f(x_1,x_2)+f(x_1-1,x_2+1),\ 2\,f(x_1,x_2)+P_{1j}^{LT},\\
&\qquad f(x_1,x_2)+f(x_1,x_2+1)+P_{1j}^{EP},\ f(x_1,x_2)+f(x_1+1,x_2-1)+2\,P_{1j}^{LT},\\
&\qquad f(x_1,x_2)+P_{1j}^{LT}+f(x_1+1,x_2)+P_{1j}^{EP},\ f(x_1+1,x_2)+f(x_1,x_2+1)+2\,P_{1j}^{EP}\Big\}\\
&\geq \min\Big\{f(x_1,x_2),\ f(x_1+1,x_2-1)+P_{1j}^{LT},\ f(x_1+1,x_2)+P_{1j}^{EP}\Big\}\\
&\quad+\min\Big\{f(x_1-1,x_2+1),\ f(x_1,x_2)+P_{1j}^{LT},\ f(x_1,x_2+1)+P_{1j}^{EP}\Big\}\\
&=H_{1j}f(x_1+1,x_2)+H_{1j}f(x_1,x_2+1).
\end{aligned}
$$

(ii) SuperC(1,2)

For $x_1>0,x_2>0$:

$$
\begin{aligned}
&H_{1j}f(x_1+2,x_2)+H_{1j}f(x_1,x_2+1)\\
&=\min\Big\{f(x_1+1,x_2),\ f(x_1+2,x_2-1)+P_{1j}^{LT},\ f(x_1+2,x_2)+P_{1j}^{EP}\Big\}\\
&\quad+\min\Big\{f(x_1-1,x_2+1),\ f(x_1,x_2)+P_{1j}^{LT},\ f(x_1,x_2+1)+P_{1j}^{EP}\Big\}\\
&=\min\Big\{f(x_1+1,x_2)+f(x_1-1,x_2+1),\ f(x_1+1,x_2)+f(x_1,x_2)+P_{1j}^{LT},\\
&\qquad f(x_1+1,x_2)+f(x_1,x_2+1)+P_{1j}^{EP},\ f(x_1+2,x_2-1)+P_{1j}^{LT}+f(x_1-1,x_2+1),\\
&\qquad f(x_1+2,x_2-1)+P_{1j}^{LT}+f(x_1,x_2)+P_{1j}^{LT},\ f(x_1+2,x_2-1)+P_{1j}^{LT}+f(x_1,x_2+1)+P_{1j}^{EP},\\
&\qquad f(x_1+2,x_2)+P_{1j}^{EP}+f(x_1-1,x_2+1),\ f(x_1+2,x_2)+P_{1j}^{EP}+f(x_1,x_2)+P_{1j}^{LT},\\
&\qquad f(x_1+2,x_2)+P_{1j}^{EP}+f(x_1,x_2+1)+P_{1j}^{EP}\Big\}
\end{aligned}
$$

It holds that:

$$\begin{aligned}
f(x_1+1,x_2)+f(x_1-1,x_2+1) &\geq f(x_1,x_2)+f(x_1,x_2+1) \text{ (by (9))},\\
f(x_1+2,x_2-1)+P_{1j}^{LT}+f(x_1-1,x_2+1) &\geq f(x_1+1,x_2-1)+f(x_1+1,x_2)-f(x_1,x_2)\\
&\qquad +P_{1j}^{LT}+f(x_1-1,x_2+1)\\
&\geq f(x_1+1,x_2-1)+P_{1j}^{LT}+f(x_1,x_2+1) \text{ (by twice (9))},\\
f(x_1+2,x_2-1)+P_{1j}^{LT}+f(x_1,x_2)+P_{1j}^{LT} &\geq f(x_1+1,x_2-1)+P_{1j}^{LT}+f(x_1+1,x_2)+P_{1j}^{LT} \text{ (by (9))},\\
f(x_1+2,x_2-1)+P_{1j}^{LT}+f(x_1,x_2+1)+P_{1j}^{EP} &\geq f(x_1+1,x_2-1)+f(x_1+1,x_2)-f(x_1,x_2)+P_{1j}^{LT}\\
&\qquad +f(x_1,x_2+1)+P_{1j}^{EP}\\
&\geq 2\,f(x_1+1,x_2)+P_{1j}^{LT}+P_{1j}^{EP} \text{ (by (9), resp. (10))},\\
f(x_1+2,x_2)+P_{1j}^{EP}+f(x_1-1,x_2+1) &\geq f(x_1+1,x_2)+f(x_1+1,x_2+1)-f(x_1,x_2+1)+P_{1j}^{EP}\\
&\qquad +f(x_1-1,x_2+1)\\
&\geq f(x_1+1,x_2)+f(x_1,x_2+1)+P_{1j}^{EP} \text{ (by (9), resp. (6))},\\
f(x_1+2,x_2)+P_{1j}^{EP}+f(x_1,x_2)+P_{1j}^{LT} &\geq 2\,f(x_1+1,x_2)+P_{1j}^{EP}+P_{1j}^{LT} \text{ (by (6))},\\
f(x_1+2,x_2)+P_{1j}^{EP}+f(x_1,x_2+1)+P_{1j}^{EP} &\geq f(x_1+1,x_2)+P_{1j}^{EP}+f(x_1+1,x_2+1)+P_{1j}^{EP} \text{ (by (9))}.
\end{aligned}$$

This implies that:

$$\begin{aligned}
&H_{1j}f(x_1+2,x_2)+H_{1j}f(x_1,x_2+1)\\
&\geq \min\Big\{f(x_1,x_2)+f(x_1,x_2+1),\ f(x_1+1,x_2)+f(x_1,x_2)+P_{1j}^{LT},\\
&\qquad f(x_1+1,x_2-1)+f(x_1,x_2+1)+P_{1j}^{LT},\ f(x_1+1,x_2-1)+f(x_1+1,x_2)+2\,P_{1j}^{LT},\\
&\qquad f(x_1+1,x_2)+f(x_1,x_2+1)+P_{1j}^{EP},\ 2\,f(x_1+1,x_2)+P_{1j}^{LT}+P_{1j}^{EP},\\
&\qquad f(x_1+1,x_2)+f(x_1+1,x_2+1)+2\,P_{1j}^{EP}\Big\}\\
&\geq \min\Big\{f(x_1,x_2),\ f(x_1+1,x_2-1)+P_{1j}^{LT},\ f(x_1+1,x_2)+P_{1j}^{EP}\Big\}\\
&\quad +\min\Big\{f(x_1,x_2+1),\ f(x_1+1,x_2)+P_{1j}^{LT},\ f(x_1+1,x_2+1)+P_{1j}^{EP}\Big\}\\
&= H_{1j}f(x_1+1,x_2)+H_{1j}f(x_1+1,x_2+1).
\end{aligned}$$

(iii) SuperC(2,1)

For $x_1 > 0, x_2 > 0$:

$$
\begin{aligned}
&H_{1j}f(x_1, x_2+2) + H_{1j}f(x_1+1, x_2)\\
&= \min\Big\{ f(x_1-1, x_2+2),\ f(x_1, x_2+1) + P_{1j}^{LT},\ f(x_1, x_2+2) + P_{1j}^{EP} \Big\}\\
&\quad + \min\Big\{ f(x_1, x_2),\ f(x_1+1, x_2-1) + P_{1j}^{LT},\ f(x_1+1, x_2) + P_{1j}^{EP} \Big\}\\
&= \min\Big\{ f(x_1-1, x_2+2) + f(x_1, x_2),\ f(x_1-1, x_2+2) + f(x_1+1, x_2-1) + P_{1j}^{LT},\\
&\qquad f(x_1-1, x_2+2) + f(x_1+1, x_2) + P_{1j}^{EP},\ f(x_1, x_2+1) + P_{1j}^{LT} + f(x_1, x_2),\\
&\qquad f(x_1, x_2+1) + P_{1j}^{LT} + f(x_1+1, x_2-1) + P_{1j}^{LT},\ f(x_1, x_2+1) + P_{1j}^{LT} + f(x_1+1, x_2) + P_{1j}^{EP},\\
&\qquad f(x_1, x_2+2) + P_{1j}^{EP} + f(x_1, x_2),\ f(x_1, x_2+2) + P_{1j}^{EP} + f(x_1+1, x_2-1) + P_{1j}^{LT},\\
&\qquad f(x_1, x_2+2) + P_{1j}^{EP} + f(x_1+1, x_2) + P_{1j}^{EP} \Big\}.
\end{aligned}
$$

It holds that:

$$
\begin{aligned}
f(x_1-1, x_2+2) + f(x_1, x_2) &\geq f(x_1-1, x_2+1) + f(x_1, x_2+1) \text{ (by (10))},\\
f(x_1-1, x_2+2) + f(x_1+1, x_2-1) + P_{1j}^{LT} &\geq f(x_1-1, x_2+1) + f(x_1, x_2+1) - f(x_1, x_2)\\
&\qquad + f(x_1+1, x_2-1) + P_{1j}^{LT}\\
&\geq f(x_1-1, x_2+1) + f(x_1+1, x_2) + P_{1j}^{LT} \text{ (by twice (10))},\\
f(x_1-1, x_2+2) + f(x_1+1, x_2) + P_{1j}^{EP} &\geq f(x_1-1, x_2+1) + f(x_1, x_2+1) - f(x_1, x_2)\\
&\qquad + f(x_1+1, x_2) + P_{1j}^{EP}\\
&\geq 2\, f(x_1, x_2+1) + P_{1j}^{EP} \text{ (by (10), resp. (9))},\\
f(x_1, x_2+1) + P_{1j}^{LT} + f(x_1+1, x_2-1) + P_{1j}^{LT} &\geq f(x_1, x_2) + f(x_1+1, x_2) + 2\, P_{1j}^{LT} \text{ (by (10))},\\
f(x_1, x_2+2) + P_{1j}^{EP} + f(x_1, x_2) &\geq 2\, f(x_1, x_2+1) + P_{1j}^{EP} \text{ (by (7))},\\
f(x_1, x_2+2) + P_{1j}^{EP} + f(x_1+1, x_2-1) + P_{1j}^{LT} &\geq f(x_1, x_2+1) + f(x_1+1, x_2+1) - f(x_1+1, x_2) + P_{1j}^{EP}\\
&\qquad + f(x_1+1, x_2-1) + P_{1j}^{LT}\\
&\geq f(x_1, x_2+1) + P_{1j}^{LT} + f(x_1+1, x_2) + P_{1j}^{EP}\\
&\qquad \text{(by (10), resp. (7))},\\
f(x_1, x_2+2) + P_{1j}^{EP} + f(x_1+1, x_2) + P_{1j}^{EP} &\geq f(x_1, x_2+1) + f(x_1+1, x_2+1) + 2\, P_{1j}^{EP} \text{ (by (10))}.
\end{aligned}
$$

This implies that:

$$\begin{aligned}
&H_{1j}f(x_1,x_2+2)+H_{1j}f(x_1+1,x_2)\\
&\geq \min\Big\{f(x_1-1,x_2+1)+f(x_1,x_2+1),\ f(x_1-1,x_2+1)+f(x_1+1,x_2)+P_{1j}^{LT},\\
&\qquad f(x_1,x_2+1)+P_{1j}^{LT}+f(x_1,x_2),\ f(x_1,x_2)+f(x_1+1,x_2)+2\,P_{1j}^{LT},\\
&\qquad 2\,f(x_1,x_2+1)+P_{1j}^{EP},\ f(x_1,x_2+1)+P_{1j}^{LT}+f(x_1+1,x_2)+P_{1j}^{EP},\\
&\qquad f(x_1,x_2+1)+f(x_1+1,x_2+1)+2\,P_{1j}^{EP}\Big\}\\
&\geq \min\Big\{f(x_1-1,x_2+1),\ f(x_1,x_2)+P_{1j}^{LT},\ f(x_1,x_2+1)+P_{1j}^{EP}\Big\}\\
&\quad +\min\Big\{f(x_1,x_2+1),\ f(x_1+1,x_2)+P_{1j}^{LT},\ f(x_1+1,x_2+1)+P_{1j}^{EP}\Big\}\\
&= H_{1j}f(x_1,x_2+1)+H_{1j}f(x_1+1,x_2+1).
\end{aligned}$$

■

A.3. Proof of Theorem 4

Proof. Consider a demand at stockpoint 1. For $j \in \mathcal{J}_1$, $(x_1,x_2)\in\mathcal{S}$ and $u\in\{0,1,2\}$, define

$$w^j(u,x_1,x_2) := \begin{cases} V_n(x_1-1,x_2) & \text{if } u=0,\\ V_n(x_1,x_2-1)+P_{1j}^{LT} & \text{if } u=1,\\ V_n(x_1,x_2)+P_{1j}^{EP} & \text{if } u=2,\end{cases} \tag{A.1}$$

where $V_n(x_1,x_2):=\infty$ if $(x_1,x_2)\notin\mathcal{S}$. Hence $H_{1j}V_n(x_1,x_2)=\min_{u\in\{0,1,2\}} w^j(u,x_1,x_2)$. Define, for $u\in\{0,1,2\}$ and $x_1\in\{0,1,\dots S_1-1\}$, $x_2\in\{0,1,\dots S_2\}$:

$$\Delta w^j_{x_1}(u,x_1,x_2) := w^j(u,x_1+1,x_2)-w^j(u,x_1,x_2).$$

Then for each $n\geq 0$, and for $x_2>0$:

$$\Delta w^j_{x_1}(1,x_1,x_2)-\Delta w^j_{x_1}(0,x_1,x_2) = V_n(x_1+1,x_2-1)-V_n(x_1,x_2-1)-V_n(x_1,x_2)+V_n(x_1-1,x_2)\geq 0$$

(as, by Theorem 3, V_n is SuperC$(1,2)$), and:

$$\Delta w^j_{x_1}(2,x_1,x_2)-\Delta w^j_{x_1}(1,x_1,x_2) = V_n(x_1+1,x_2)-V_n(x_1,x_2)-V_n(x_1+1,x_2-1)+V_n(x_1,x_2-1)\geq 0$$

(as V_n is Supermod). So, for $x_2>0$, $\Delta w^j_{x_1}(u,x_1,x_2)$ is increasing in u:

$$\Delta w^j_{x_1}(2,x_1,x_2)\geq \Delta w^j_{x_1}(1,x_1,x_2)\geq \Delta w^j_{x_1}(0,x_1,x_2).$$

This implies that, for every $n \geq 0$, there exists a threshold for the inventory level x_1, which can depend on x_2, say $T^{di}_{n,1j}(x_2)$, from which on it is optimal to fulfill demands directly from stock. Next there exists a threshold, say $T^{lt}_{n,1j}(x_2)$, such that $T^{lt}_{n,1j}(x_2) \leq T^{di}_{n,1j}(x_2)$, from which on (until $T^{di}_{n,1j}(x_2) - 1$) it is optimal to fulfill demands via a lateral transshipment, and on the interval $x_1 = 0$ up till $T^{lt}_{n,1j}(x_2) - 1$ an emergency procedure is optimal. Hence, if f_{n+1} is the minimizing policy in (1), then f_{n+1} is a threshold policy. Note that the transition probability matrix of every stationary policy is unichain (since every state can access (S_1, S_2)) and aperiodic (since the transition probability from state (S_1, S_2) to itself is positive). Then, by Theorem 8.5.4 of Puterman (1994), the long run average costs under the stationary policy f_{n+1} converges to the minimal long run average costs as n tends to infinity. Since there are only finitely many stationary threshold policies, this implies that there exists an optimal stationary policy that is a threshold type policy.

For $x_2 = 0$, lateral transshipments ($u = 1$) are not possible, and we have, for each $n \geq 0$:

$$\Delta w^j_{x_1}(2, x_1, 0) - \Delta w^j_{x_1}(0, x_1, 0) = V_n(x_1 + 1, 0) - V_n(x_1, 0) - V_n(x_1, 0) + V_n(x_1 - 1, 0) \geq 0,$$

(as V_n is Conv(1)). Hence $\Delta w^j_{x_1}(2, x_1, 0) \geq \Delta w^j_{x_1}(0, x_1, 0)$, and so, for the special case $x_2 = 0$, there exists only one threshold. By the analogous reasoning as for $x_2 > 0$, it follows that there exists a $T^{di}_{1j}(0)$ (which is equal to $T^{lt}_{1j}(0)$). As it is only possible to deliver directly from stock if $x_1 \geq 1$, it follows that $T^{di}_{1j}(0) \geq 1$.

By interchanging the numbering of the stockpoints, the analogous result for stockpoint 2 directly follows. ■

A.4. Proof of Theorem 5

Proof. Analogously to the proof of Theorem 4, consider a demand at stockpoint 1, and define:

$$\Delta w^j_{x_2}(u, x_1, x_2) := w^j(u, x_1, x_2 + 1) - w^j(u, x_1, x_2),$$

where $w^j(u, x_1, x_2)$ is as defined in (A.1). Then for each $n \geq 0$, and for $x_1 > 0$:

$$\Delta w^j_{x_2}(0, x_1, x_2) - \Delta w^j_{x_2}(1, x_1, x_2) = V_n(x_1 - 1, x_2 + 1) - V_n(x_1 - 1, x_2) - V_n(x_1, x_2) + V_n(x_1, x_2 - 1) \geq 0$$

(as, by Theorem 3, V_n is SuperC$(2, 1)$), and:

$$\Delta w^j_{x_2}(2, x_1, x_2) - \Delta w^j_{x_2}(0, x_1, x_2) = V_n(x_1, x_2 + 1) - V_n(x_1, x_2) - V_n(x_1 - 1, x_2 + 1) + V_n(x_1 - 1, x_2) \geq 0$$

(as V_n is Supermod). Hence, for $x_1 > 0$:

$$\Delta w^j_{x_2}(2, x_1, x_2) \geq \Delta w^j_{x_2}(0, x_1, x_2) \geq \Delta w^j_{x_2}(1, x_1, x_2).$$

Analogously to the reasoning in the proof of Theorem 4, it now follows that, for n to infinity, there exist two thresholds $\hat{T}^{di}_{1j}(x_1)$ and $\hat{T}^{lt}_{1j}(x_1)$, where $\hat{T}^{di}_{1j}(x_1) \leq \hat{T}^{lt}_{1j}(x_1)$, such that from $\hat{T}^{lt}_{1j}(x_1)$ lateral transshipments are optimal, from $\hat{T}^{di}_{1j}(x_1)$ to $\hat{T}^{lt}_{1j}(x_1) - 1$ direct delivering from stock is optimal, and from 0 to $\hat{T}^{di}_{1j}(x_1) - 1$ emergency procedures are optimal.

For $x_1 = 0$, directly satisfying a demand from stock ($u = 0$) is not possible, and we have, for each $n \geq 0$:

$$\Delta w^j_{x_2}(2, 0, x_2) - \Delta w^j_{x_2}(1, 0, x_2) = V_n(0, x_2 + 1) - V_n(0, x_2) - V_n(0, x_2) + V_n(0, x_2 - 1) \geq 0,$$

(as V_n is Conv(2)). Hence $\Delta w^j_{x_1}(2, 0, x_2) \geq \Delta w^j_{x_1}(1, 0, x_2)$, and so, for the special case $x_1 = 0$, there exists only one threshold: $\hat{T}^{lt}_{1j}(0)$ (which is equal to $\hat{T}^{di}_{1j}(0)$). As it is only possible to apply a lateral transshipment if $x_2 \geq 1$, it follows that $\hat{T}^{lt}_{1j}(0) \geq 1$.

By interchanging the numbering of the stockpoints, the analogous result for stockpoint 2 directly follows. ■

A.5. Proof of Lemma 8

Proof. We can use the following:
Assuming $J_1 \geq 2$, let $j_1, j_2 \in \mathcal{J}_1$, such that $j_1 \leq j_2$. The following results hold for $x_1 \in \{0, 1, \ldots S_1\}, x_2 \in \{0, 1, \ldots S_2\}$, and each $n \geq 0$:

a) $$\big(w^{j_1}(2, x_1, x_2) - w^{j_1}(0, x_1, x_2)\big) - \big(w^{j_2}(2, x_1, x_2) - w^{j_2}(0, x_1, x_2)\big) = P^{EP}_{1j_1} - P^{EP}_{1j_2} \geq 0,$$

by assumption;

b) $$\big(w^{j_1}(1, x_1, x_2) - w^{j_1}(0, x_1, x_2)\big) - \big(w^{j_2}(1, x_1, x_2) - w^{j_2}(0, x_1, x_2)\big) = P^{LT}_{1j_1} - P^{LT}_{1j_2} \geq 0,$$

by assumption;

c) $$\big(w^{j_1}(2, x_1, x_2) - w^{j_1}(1, x_1, x_2)\big) - \big(w^{j_2}(2, x_1, x_2) - w^{j_2}(1, x_1, x_2)\big) = \big(P^{EP}_{1j_1} - P^{LT}_{1j_1}\big) - \big(P^{EP}_{1j_2} - P^{LT}_{1j_2}\big) \geq 0,$$

given that this ordering holds, and otherwise the other way around. ■

A.6. Proof of Theorem 9

Proof. We prove part 1a). Part 1b) then directly follows by interchanging the stockpoints, and 2) is a trivial consequence of 1a) and 1b).

For 1a), we prove that $a_1^*(1, x_2) = 0$ for all $x_2 \in \{0, 1, \dots, S_2\}$, then it follows by Theorem 4 that $T_1^{di}(x_2) = 1$ for all x_2. It suffices to prove that, for all $n \geq 0$:

$$V_n(1, x_2) + P_1^{EP} \geq V_n(0, x_2), \quad \text{for } x_2 \in \{0, \dots, S_2\}, \tag{A.2}$$

$$V_n(1, x_2 - 1) + P_1^{LT} \geq V_n(0, x_2), \quad \text{for } x_2 \in \{1, \dots, S_2\}. \tag{A.3}$$

For $S_1 = 0$ trivially $T_1^{di}(x_2) = 1$ for all x_2, and for $S_1 > 0, S_2 = 0$ we only have to prove (A.2).

We prove the inequalities by induction, using that, by Theorem 3, V_n satisfies (4)–(10). For $V_0 \equiv 0$ both inequalities trivially hold. We first prove (i) the induction step of (A.2), then (ii) that of (A.3), both for $S_1 > 0$. All given inequalities hold by the induction hypothesis, unless stated otherwise.

(i) Assume that (A.2) holds for a given n (induction hypothesis), and let $S_1 > 0$. We have to show that $V_{n+1}(1, x_2) + P_1^{EP} \geq V_{n+1}(0, x_2)$. The left-hand side can be written as

$$\begin{aligned} &\frac{\lambda_1}{\nu}[H_1 V_n(1, x_2) + P_1^{EP}] + \frac{\lambda_2}{\nu}[H_2 V_n(1, x_2) + P_1^{EP}] \\ &\qquad + \frac{\mu}{\nu}[G_1 V_n(1, x_2) + (S_1 - 1) P_1^{EP}] + \frac{\mu}{\nu}[G_2 V_n(1, x_2) + S_2 P_1^{EP}] + \frac{\mu}{\nu} P_1^{EP}, \end{aligned}$$

since $\nu = \lambda_1 + \lambda_2 + \mu S_1 + \mu S_2$. Here, the term P_1^{EP} has been distributed such that each of the four expressions in brackets (for the operators H_1, H_2, G_1, and G_2) can be considered separately.

For $x_2 = 0$:

$$\begin{aligned} H_1 V_n(1, 0) + P_1^{EP} &= \min\{P_1^{EP} + V_n(0, 0), 2\, P_1^{EP} + V_n(1, 0)\} \\ &\geq \min\{P_1^{EP} + V_n(0, 0), P_1^{EP} + V_n(0, 0)\} \\ &= P_1^{EP} + V_n(0, 0) = H_1 V_n(0, 0); \end{aligned}$$

and for $x_2 \in \{1, 2, \dots, S_2\}$:

$$\begin{aligned} H_1 V_n(1, x_2) + P_1^{EP} &= \min\{P_1^{EP} + V_n(0, x_2), P_1^{EP} + P_1^{LT} + V_n(1, x_2 - 1), 2\, P_1^{EP} + V_n(1, x_2)\} \\ &\geq \min\{P_1^{LT} + V_n(0, x_2 - 1), P_1^{EP} + V_n(0, x_2)\} = H_1 V_n(0, x_2). \end{aligned}$$

For $x_2 = 0$:

$$\begin{aligned} H_2 V_n(1, 0) + P_1^{EP} &= \min\{P_1^{EP} + P_2^{LT} + V_n(0, 0), P_1^{EP} + P_2^{EP} + V_n(1, 0)\} \\ &\geq \min\{P_1^{EP} + P_2^{LT} - P_2^{EP} + H_2 V_n(0, 0), H_2 V_n(0, 0)\} \\ &= H_2 V_n(0, 0) + \min\{P_1^{EP} + P_2^{LT} - P_2^{EP}, 0\}; \end{aligned}$$

and for $x_2 \in \{1, 2, \ldots, S_2\}$:

$$\begin{aligned}
H_2 V_n(1, x_2) + P_1^{EP} &= \min\{P_1^{EP} + V_n(1, x_2 - 1), P_1^{EP} + P_2^{LT} + V_n(0, x_2), P_1^{EP} + P_2^{EP} + V_n(1, x_2)\} \\
&\geq \min\{V_n(0, x_2 - 1), P_1^{EP} + P_2^{LT} + V_n(0, x_2), P_2^{EP} + V_n(0, x_2)\} \\
&\geq H_2 V_n(0, x_2) + \min\{P_1^{EP} + P_2^{LT} - P_2^{EP}, 0\},
\end{aligned}$$

as $H_2 V_n(0, x_2) = \min\{V_n(0, x_2 - 1), P_2^{EP} + V_n(0, x_2)\}$.

For the operator G_1 we obtain:

$$\begin{aligned}
G_1 V_n(1, x_2) + (S_1 - 1)P_1^{EP} &= (S_1 - 1)V_n(2, x_2) + V_n(1, x_2) + (S_1 - 1)P_1^{EP} \\
&= (S_1 - 1)[V_n(2, x_2) - V_n(1, x_2)] + S_1 V_n(1, x_2) + (S_1 - 1)P_1^{EP} \\
&\geq (S_1 - 1)[V_n(1, x_2) - V_n(0, x_2)] + S_1 V_n(1, x_2) + (S_1 - 1)P_1^{EP} \\
&\geq S_1 V_n(1, x_2) = G_1 V_n(0, x_2),
\end{aligned}$$

where the first inequality holds as V_n is Conv(1) (cf. Theorem 3).

For $x_2 \in \{0, 1, \ldots, S_2 - 1\}$ we obtain:

$$\begin{aligned}
G_2 V_n(1, x_2) + S_2 P_1^{EP} &= (S_2 - x_2)V_n(1, x_2 + 1) + x_2 V_n(1, x_2) + S_2 P_1^{EP} \\
&\geq (S_2 - x_2)V_n(0, x_2 + 1) + x_2 V_n(0, x_2) = G_2 V_n(0, x_2);
\end{aligned}$$

and for $x_2 = S_2$ trivially:

$$G_2 V_n(1, S_2) + S_2 P_1^{EP} = S_2 V_n(1, S_2) + S_2 P_1^{EP} \geq S_2 V_n(0, S_2) = G_2 V_n(0, S_2).$$

Combining these give, for all x_2 (recall $\nu = \lambda_1 + \lambda_2 + \mu S_1 + \mu S_2$):

$$\begin{aligned}
&\nu(V_{n+1}(1, x_2) + P_1^{EP}) \\
&= \lambda_1 H_1 V_n(1, x_2) + \lambda_2 H_2 V_n(1, x_2) + \mu G_1 V_n(1, x_2) + \mu G_2 V_n(1, x_2) + \nu\, P_1^{EP} \\
&= \lambda_1 [H_1 V_n(1, x_2) + P_1^{EP}] + \lambda_2 [H_2 V_n(1, x_2) + P_1^{EP}] + \mu[G_1 V_n(1, x_2) + (S_1 - 1)P_1^{EP}] \\
&\qquad\qquad + \mu[G_2 V_n(1, x_2) + S_2 P_1^{EP}] + \mu P_1^{EP} \\
&\geq \lambda_1 H_1 V_n(0, x_2) + \lambda_2 [H_2 V_n(0, x_2) + \min\{P_1^{EP} + P_2^{LT} - P_2^{EP}, 0\}] + \mu G_1 V_n(0, x_2) \\
&\qquad\qquad + \mu G_2 V_n(0, x_2) + \mu P_1^{EP} \\
&= \nu V_{n+1}(0, x_2) + \lambda_2 \, \min\{P_1^{EP} + P_2^{LT} - P_2^{EP}, 0\} + \mu P_1^{EP} \\
&\geq \nu V_{n+1}(0, x_2), \qquad (A.4)
\end{aligned}$$

where the last inequality holds by condition (12), since that can be rewritten as

$$\lambda_2 \, [P_1^{EP} + P_2^{LT} - P_2^{EP}] + \mu P_1^{EP} \geq 0.$$

This completes the induction step, and hence (A.2) holds for all $n \geq 0$.

(ii) Assume that (A.3) holds for a given n (induction hypothesis), and let $S_1, S_2 > 0$. We consider the operators H_1, H_2 and $G_1 + G_2$ separately:

For $x_2 \in \{2, \ldots, S_2\}$:

$$\begin{aligned} & H_1 V_n(1, x_2 - 1) + P_1^{LT} \\ & = \min\{P_1^{LT} + V_n(0, x_2 - 1), 2\, P_1^{LT} + V_n(1, x_2 - 2), P_1^{LT} + P_1^{EP} + V_n(1, x_2 - 1)\} \\ & \geq \min\{P_1^{LT} + V_n(0, x_2 - 1), P_1^{EP} + V_n(0, x_2)\} = H_1 V_n(0, x_2); \end{aligned}$$

and for $x_2 = 1$:

$$\begin{aligned} & H_1 V_n(1, 0) + P_1^{LT} \\ & = \min\{P_1^{LT} + V_n(0, 0), P_1^{LT} + P_1^{EP} + V_n(1, 0)\} \\ & \geq \min\{P_1^{LT} + V_n(0, 0), P_1^{EP} + V_n(0, 1)\} = H_1 V_n(0, 1). \end{aligned}$$

For $x_2 \in \{2, \ldots, S_2\}$:

$$\begin{aligned} & H_2 V_n(1, x_2 - 1) + P_1^{LT} \\ & = \min\{P_1^{LT} + V_n(1, x_2 - 2), P_1^{LT} + P_2^{LT} + V_n(0, x_2 - 1), P_1^{LT} + P_2^{EP} + V_n(1, x_2 - 1)\} \\ & \geq \min\{V_n(0, x_2 - 1), P_2^{EP} + V_n(0, x_2)\} = H_2 V_n(0, x_2); \end{aligned}$$

and for $x_2 = 1$:

$$\begin{aligned} & H_2 V_n(1, 0) + P_1^{LT} \\ & = \min\{P_1^{LT} + P_2^{LT} + V_n(0, 0), P_1^{LT} + P_2^{EP} + V_n(1, 0)\} \\ & \geq \min\{V_n(0, 0), P_2^{EP} + V_n(0, 1)\} = H_2 V_n(0, 1). \end{aligned}$$

For $x_2 \in \{1, \dots, S_2 - 1\}$:

$$
\begin{aligned}
&(G_1 + G_2)V_n(1, x_2 - 1) + (S_1 + S_2)P_1^{LT} \\
&= (S_1 - 1)V_n(2, x_2 - 1) + V_n(1, x_2 - 1) \\
&\quad + (S_2 - x_2 + 1)V_n(1, x_2) + (x_2 - 1)V_n(1, x_2 - 1) + (S_1 + S_2)P_1^{LT} \\
&= (S_1 - 1)[V_n(2, x_2 - 1) - V_n(1, x_2)] + (S_1 - 1)V_n(1, x_2) + V_n(1, x_2 - 1) \\
&\quad + (S_2 - x_2)[V_n(1, x_2) - V_n(0, x_2 + 1)] + V_n(1, x_2) + (S_2 - x_2)V_n(0, x_2 + 1) \\
&\quad + x_2[V_n(1, x_2 - 1) - V_n(0, x_2)] - V_n(1, x_2 - 1) + x_2 V_n(0, x_2) + (S_1 + S_2)P_1^{LT} \\
&\geq (S_1 - 1)[V_n(1, x_2 - 1) - V_n(0, x_2)] + S_1 V_n(1, x_2) \\
&\quad + (S_2 - x_2)[V_n(1, x_2) - V_n(0, x_2 + 1)] + (S_2 - x_2)V_n(0, x_2 + 1) \\
&\quad + x_2[V_n(1, x_2 - 1) - V_n(0, x_2)] + x_2 V_n(0, x_2) + (S_1 + S_2)P_1^{LT} \\
&\geq S_1 V_n(1, x_2) + (S_2 - x_2)V_n(0, x_2 + 1) + x_2 V_n(0, x_2) + P_1^{LT} \\
&= (G_1 + G_2)V_n(0, x_2) + P_1^{LT},
\end{aligned}
$$

where the first inequality holds as V_n is SuperC(1,2) (cf. Theorem 3). For $x_2 = S_2$:

$$
\begin{aligned}
&(G_1 + G_2)V_n(1, S_2 - 1) + (S_1 + S_2)P_1^{LT} \\
&= (S_1 - 1)V_n(2, S_2 - 1) + V_n(1, S_2 - 1) \\
&\quad + V_n(1, S_2) + (S_2 - 1)V_n(1, S_2 - 1) + (S_1 + S_2)P_1^{LT} \\
&= (S_1 - 1)[V_n(2, S_2 - 1) - V_n(1, S_2)] + S_1 V_n(1, S_2) \\
&\quad + S_2[V_n(1, S_2 - 1) - V_n(0, S_2)] + S_2 V_n(0, S_2) + (S_1 + S_2)P_1^{LT} \\
&\geq (S_1 - 1)[V_n(1, S_2 - 1) - V_n(0, S_2)] + S_1 V_n(1, S_2) \\
&\quad + S_2[V_n(1, S_2 - 1) - V_n(0, S_2)] + S_2 V_n(0, S_2) + (S_1 + S_2)P_1^{LT} \\
&\geq S_1 V_n(1, S_2) + S_2 V_n(0, S_2) + P_1^{LT} = (G_1 + G_2)V_n(0, S_2) + P_1^{LT},
\end{aligned}
$$

where the first inequality again holds as V_n is SuperC(1,2).

Combining these gives, for all $x_2 \in \{1, \dots, S_2\}$:

$$
\begin{aligned}
&\nu(V_{n+1}(1, x_2 - 1) + P_1^{LT}) \\
&= \lambda_1 H_1 V_n(1, x_2 - 1) + \lambda_2 H_2 V_n(1, x_2 - 1) + \mu(G_1 + G_2)V_n(1, x_2 - 1) + \nu\, P_1^{LT} \\
&= \lambda_1[H_1 V_n(1, x_2 - 1) + P_1^{LT}] + \lambda_2[H_2 V_n(1, x_2 - 1) + P_1^{LT}] + \mu[(G_1 + G_2)V_n(1, x_2 - 1) + (S_1 + S_2)P_1^{LT}] \\
&\geq \lambda_1 H_1 V_n(0, x_2) + \lambda_2 H_2 V_n(0, x_2) + \mu(G_1 + G_2)V_n(0, x_2) = \nu V_{n+1}(0, x_2),
\end{aligned}
$$

which completes the induction step, and hence (A.3) holds for all $n \geq 0$. ■

A.7. Proof of Theorem 10

Proof. We again prove only part 1a), as again part 1b) directly follows by interchanging the stockpoints, and 2) is a trivial consequence of 1a) and 1b).

For 1a), we first show that condition (14) implies (12). Rewriting (14) gives $\frac{\lambda_2}{\lambda_2+\mu}P_2^{EP} \leq P_1^{EP} - P_1^{LT}$. This implies $\frac{\lambda_2}{\lambda_2+\mu}P_2^{EP} \leq P_1^{EP} + \frac{\lambda_2}{\lambda_2+\mu}P_2^{LT}$ (as both P_1^{LT} and $\frac{\lambda_2}{\lambda_2+\mu}P_2^{LT}$ are nonnegative), which is equivalent to (12). Hence, $T_1^{di}(x_2) = 1$ for all $x_2 \in \{0, 1, \ldots, S_2\}$. Next, analogously to the proof of Theorem 9, we prove that $a_1^*(0,1) = 1$; then it follows by Theorem 5 that $\hat{T}_1^{lt}(0) = 1$. By induction, we prove that, for all $n \geq 0$:

$$V_n(0,1) + P_1^{EP} \geq V_n(0,0) + P_1^{LT}. \tag{A.5}$$

For $V_0 \equiv 0$ this trivially holds.

Assume that (A.5) holds for a given n (induction hypothesis), and we consider the operators H_1, H_2, G_1 and G_2 separately:

$$\begin{aligned} H_1V_n(0,1) + P_1^{EP} &= \min\{P_1^{EP} + P_1^{LT} + V_n(0,0), 2\,P_1^{EP} + V_n(0,1)\} \\ &\geq P_1^{EP} + P_1^{LT} + V_n(0,0) = H_1V_n(0,0) + P_1^{LT}; \end{aligned}$$

$$\begin{aligned} H_2V_n(0,1) + P_1^{EP} &= \min\{P_1^{EP} + V_n(0,0), P_1^{EP} + P_2^{EP} + V_n(0,1)\} \\ &\geq \min\{P_1^{EP} - P_2^{EP} + P_2^{EP} + V_n(0,0), P_2^{EP} + V_n(0,0) + P_1^{LT}\} \\ &= H_2V_n(0,0) + \min\{P_1^{EP} - P_2^{EP}, P_1^{LT}\}, \end{aligned}$$

as $H_2V_n(0,0) = P_2^{EP} + V_n(0,0)$;

$$\begin{aligned} G_1V_n(0,1) + S_1P_1^{EP} &= S_1[V_n(1,1) - V_n(1,0) + V_n(1,0) + P_1^{EP}] \\ &\geq S_1[V_n(0,1) - V_n(0,0) + V_n(1,0) + P_1^{EP}] \\ &\geq S_1[V_n(1,0) + P_1^{LT}] = G_1V_n(0,0) + S_1P_1^{LT}, \end{aligned}$$

where the first inequality holds as V_n is Supermod;

$$\begin{aligned} G_2V_n(0,1) + (S_2-1)P_1^{EP} &= (S_2-1)[V_n(0,2) - V_n(0,1) + P_1^{EP}] + S_2V_n(0,1) \\ &\geq (S_2-1)[V_n(0,1) - V_n(0,0) + P_1^{EP}] + S_2V_n(0,1) \\ &= S_2V_n(0,1) + (S_2-1)P_1^{LT} = G_2V_n(0,0) + (S_2-1)P_1^{LT}, \end{aligned}$$

where the first inequality holds as V_n is Conv(2).

Combining these, using condition (14), gives, analogously to (A.4), the induction step, and hence (A.5) holds for all $n \geq 0$. ■

A.8. Proof of Corollary 11

Proof. Either condition (12), or condition (15) holds (or both hold); and either condition (13), or condition (14) holds (or both hold). These statements can be derived in the following way (e.g. for the first one): (i) if condition (12) does *not* hold, then surely condition (15) holds; and (ii) if condition (15) does *not* hold, then surely condition (12) holds. This follows by rewriting the conditions: for (i) we have that if (12) does not hold, then $P_2^{EP} \geq P_2^{LT} + \frac{\lambda_2+\mu}{\lambda_2} P_1^{EP}$, but this implies $P_2^{EP} \geq P_2^{LT} + \frac{\lambda_1}{\lambda_1+\mu} P_1^{EP}$ (as $\frac{\lambda_2+\mu}{\lambda_2} \geq 1$, but $\frac{\lambda_1}{\lambda_1+\mu} \leq 1$), which is exactly (15); and (ii) follows as $\frac{\lambda_1}{\lambda_1+\mu} \leq 1$ in (15), but $1 + \frac{\mu}{\lambda_2} \geq 1$ in (12). The analogous reasoning holds for conditions (13) and (14).

Combined with the properties that (14) implies (12), and that (15) implies (13), this immediately leads to the statement of the corollary. ■

A.9. Proof of Theorem 13

Proof. The following holds for $\tilde{G}_k$:

Lemma 13.1. *The operator $\tilde{G}_k$, $k = 1, 2$, preserves each of the following properties: (i) Decr; (ii) Conv and Decr(k); (iii) Supermod; (iv) SuperC and Conv($k+1$); (v) MM and Conv($k+1$).*

Here $k+1$ should be read as $k+1$ mod 2. So, *(ii)* states that if f is Conv *and* Decr(k), then $\tilde{G}_k f$ is so as well. It follows that the valuefunction satisfies (4)–(10) for all $n \geq 0$. The theorem is a direct consequence of this. ■

A.10. Proof of Lemma 13.1

Proof. We give the proofs for the operator $\tilde{G}_1$. By interchanging the numbering of the locations, the results directly follow for the operator $\tilde{G}_2$ as well.

(i) It is straightforward to check that if f is Decr(1) (cf. (4)), then $\tilde{G}_1 f$ is Decr(1) as well, and if f is Decr(2) (cf. (5)), then $\tilde{G}_1 f$ is Decr(2) as well. Combining this proves that $\tilde{G}_1$ preserves Decr.

(ii) Assume that f is Conv(1) (cf. (6)), then we show that $\tilde{G}_1 f$ is Conv(1) as well. For $x_1 + 2 < S_1$ this is straightforward to check, for the case $x_1 + 2 = S_1$ we need Decr(1):

$$\begin{aligned} \tilde{G}_1(f(x_1, x_2) + f(x_1+2, x_2)) &= f(x_1+1, x_2) + f(x_1+2, x_2) \\ &\geq f(x_1+2, x_2) + f(x_1+2, x_2) = 2\,\tilde{G}_1 f(x_1+1, x_2). \end{aligned}$$

The preservation of Conv(2) (cf. (7)) is again straightforward to check, and hence $\tilde{G}_1$ preserves Conv.

(iii) It is straightforward to check that if f is Supermod (cf. (8)), then $\tilde{G}_1 f$ is Supermod as well, hence $\tilde{G}_1$ preserves Supermod.

(iv) It is straightforward to check that if f is SuperC(1,2) (cf. (9)), then $\tilde{G}_1 f$ is SuperC(1,2) as well, hence $\tilde{G}_1$ preserves SuperC(1,2). Assume that f is SuperC(2,1) (cf. (9)), then we show that $\tilde{G}_1 f$ is SuperC(2,1) as well. For $x_1 + 1 < S_1$ this is straightforward to check, for the case $x_1 + 1 = S_1$ we need Conv(2):

$$\begin{aligned}\tilde{G}_1(f(x_1, x_2 + 2) + f(x_1 + 1, x_2)) &= f(x_1 + 1, x_2 + 2) + f(x_1 + 1, x_2)\\ &\geq f(x_1 + 1, x_2 + 1) + f(x_1 + 1, x_2 + 1)\\ &= \tilde{G}_1(f(x_1, x_2 + 1) + f(x_1 + 1, x_2 + 1)).\end{aligned}$$

(v) By (11), this is a direct consequence of parts (iii) and (iv). ■