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# Optimizing the Throughput of Particulate Streams Subject to Blocking

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**Abstract.** Filtration, flow in narrow channels and traffic flow are examples of processes subject to blocking when the channel conveying the particles becomes too crowded. If the blockage is temporary, which means that after a finite time the channel is flushed and reopened, one expects to observe a maximum throughput for a finite intensity of entering particles. We investigate this phenomenon by introducing a queueing theory inspired, circular Markov model. Particles enter a channel with intensity $\lambda$ and exit at a rate $\mu$. If $N$ particles are present at the same time in the channel, the system becomes blocked and no more particles can enter. After an exponentially distributed time with rate $\mu^*$ the blockage is cleared and the system resets to an empty channel. We obtain an exact expression for the steady state throughput (including the exiting blocked particles) for all values of $N$. For $N = 2$ we show that the throughput assumes a maximum value for finite $\lambda$ if $\mu^*/\mu < 1/4$. The time-dependent throughput either monotonically approaches the steady state value, or reaches a maximum value at finite time. We demonstrate that, in the steady state, this model can be mapped to a previously introduced non-Markovian model with fixed transit and blockage times.

We also examine an irreversible, non-Markovian blockage process with constant transit time exposed to an entering flux of fixed intensity for a finite time and we show that the first and second moments of the number of exiting particles are maximized for a finite intensity.

**Keywords:** exact results; non-equilibrium processes

## 1. Introduction

Whenever the carrying capacity of a channel is limited, blockage is a possibility and indeed the phenomenon is commonly observed over a range of length scales [1]. Examples include filtration [2,3] vehicular and pedestrian traffic flow [4–6], granular systems, the flow of macro-molecules through micro or nanochannels [7–9] and in other applications like internet attacks (DoS) [10]. The blockage may be temporary, in which case the flow resumes after a certain amount of time has elapsed, or it may be irreversible with no reopening of the channel possible.

In the first situation, if the entering flux is constant, a steady state with alternating open and blocked states will eventually be reached [11]. Increasing the intensity of entering particles would, in the absence of blockage, lead to a proportionate increase in the throughput, or rate of exiting particles. With blockage present, however, increasing the intensity increases the probability of blockage that disrupts the throughput. Thus, one can expect that, under certain conditions, the throughput will be maximized for a finite intensity of entering particles.

When the blockage is irreversible, the total number of exiting particles can be continually increased by reducing the intensity to values approaching zero [12]. This will, of course, require an ever-increasing amount of time. A more interesting question is to consider a situation in which the entering flux ceases at a giving stopping time. Here we expect that the total number of exiting particles can be optimized for a finite intensity.

Such considerations also arise in queueing theory that has long been used to analyze service operations performed on units arriving according to a given distribution [13,14]. Traditional applications include industrial engineering, telecommunications and traffic flow. More recently, queueing theory has been used in biophysics, for example to model enzymatic servers [15]. In many of these applications, the throughput of serviced jobs is a crucial quantity. An example taken from industrial engineering is a closed loop conveyor system with homogeneous servers [16].

The existence of a maximum throughput is an example of the *slower-is-faster* (equivalently *faster-is-slower*) effect first described by Helbing, Farkas and Vicsek [17]. In pedestrian traffic flow, overly competitivive individuals attempting to exit a narrow door typically leads to clogging that reduces the rate of egress and may [18] or may not [19] lead to a faster-is-slower effect. Other settings in which the effect occurs include multilayer networks with finite storage capacity [20], self-healing materials [21], as well as several others [22].

Here we introduce a circular Markov chain model with a blocked state that arises when $N$ particles are simultaneously present in the system. The blockage is subject to removal at a constant rate. We demonstrate that, depending on the rates of transit and deblockage, the throughput may be maximized (i) at a finite intensity of entering particles in the steady state and (ii) as a function of time in the transient regime. A maximum throughput in the steady state guarantees the existence of a maximum as a
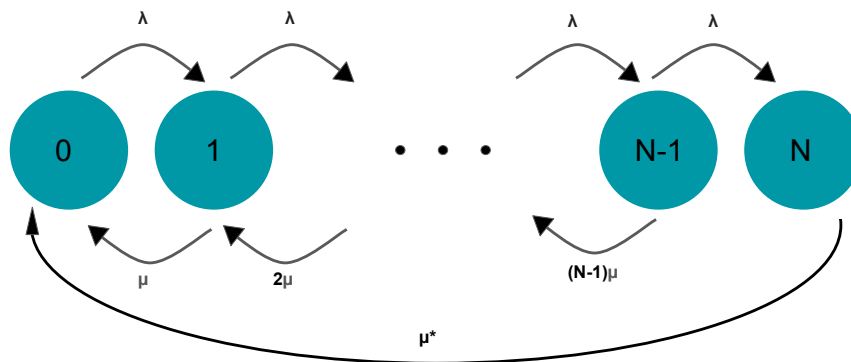
**Figure 1.** Circular Markov chain model with $N+1$ states corresponding to the number of particles in the system. The arrows indicate the possible interstate transitions and their associated rates.

function of time, but the reverse is not true. We also compare the model to a previously introduced semi-deterministic process with constant transit and blockage times as well as with the Erlang loss formula.

## 2. Circular Markov Model of Reversible Blockage

Particles enter an initially empty channel with a finite capacity according to a Poisson process with intensity $\lambda$. The channel remains open if it contains fewer than $N$ particles. In the open state, any particle that is present exits the channel at a rate $\mu$, independent of the time already spent inside. When $N$ particles are present the channel is blocked and, all newly arriving particles are rejected. After a finite duration of time the blockage is released, and all $N$ particles are simultaneously ejected, at a rate $\mu^* < \mu$. See Fig. 1. The number of particles in the system is a Markov process with state space $S = \{0, 1, \ldots, N\}$ and infinitesimal generator

$$\mathbf{Q_N} = \begin{pmatrix} -\lambda & \lambda & & & 0 \\ \mu & -(\lambda + \mu) & \lambda & & \\ & \ddots & \ddots & & \ddots \\ & & (N-1)\mu & -(\lambda + (N-1)\mu) & \lambda \\ \mu^* & & & 0 & -\mu^* \end{pmatrix}. \tag{1}$$

The system evolves according to the forward Kolmogorov equation

$$\frac{d\mathbf{P_N}}{dt} = \mathbf{P_N}\mathbf{Q_N}. \tag{2}$$

where $\mathbf{P_N}$ denotes the state vector $[\pi_0(t), \pi_1(t), \ldots \pi_n(t)]$ with $\pi_k(t)$ being the probability that the system is in state $k$ at time $t$.

The system is an example of a circular Markov processes, in that all but the Nth state can only transition to a neighboring state. The steady-state probabilities of the circular processes are explicitly calculated in Adan and Resing [23]. In the steady state, i.e. $\dot{\mathbf{P}}_{\mathbf{N}} = \mathbf{0}$, the probabilities are given by $\pi_N = C_N \lambda^N$ and, for $k = 1, 2, \ldots, N$,

$$\pi_{N-k} = C_N \mu^* \sum_{j=1}^{k} \left( \lambda^{N-j} \mu^{j-1} \prod_{i=1}^{j-1} [N - k + i] \right), \tag{3}$$

where $C_N$ is a normalization constant so that $\sum_{i=0}^{N} \pi_i = 1$. The explicit formula is

$$C_N = \left( \lambda^N + \mu^* \sum_{j=0}^{N-1} \frac{N!}{(j+1)(N-j-1)!} \mu^j \lambda^{N-1-j} \right)^{-1}. \tag{4}$$

Following the analysis presented by Cohen [24] we find the following formula for the mean first passage time from the empty state 0 to the blocked state with $N$ particles

$$\nu_{0,N} = \frac{1}{\lambda} \sum_{m=0}^{N-1} m! \sum_{k=0}^{m} \frac{1}{k!} \left( \frac{\lambda}{\mu} \right)^{k-m}. \tag{5}$$

(see Appendix).

The throughput can be calculated by noting that state $k$, $0 < k < N$, contributes $k\mu\pi_k(t)$ particles per unit time, which is the rate of exiting particles $k\mu$ times the probability that the channel is in state $k$. There is an additional contribution from the blocked state whose $N$ particles are simultaneously ejected at the rate $\mu^*$, which gives a contribution of $N\mu^*\pi_N(t)$. The general expression of the time dependent throughput is thus the sum of these contributions

$$j_N(\lambda, t) = \left( \sum_{k=1}^{N-1} k\mu\pi_k(t) \right) + N\mu^*\pi_N(t). \tag{6}$$

In the steady state the rate of exiting particles also corresponds to the incident particulate flux minus the part that is rejected when the system is in the blocked state:

$$j_N(\lambda) = \lambda(1 - \pi_N). \tag{7}$$

From $\pi_N = C_N \lambda^N$ and Eq. (4) we find the following explicit expression

$$j_N(\lambda) = \frac{\lambda}{\left[ \mu^* \sum_{j=0}^{N-1} \frac{N!}{(j+1)(N-j-1)!} \mu^j \lambda^{-(j+1)} \right]^{-1} + 1}. \tag{8}$$

We confirm the expected result that, in the large intensity limit

$$j_N(\lambda \to \infty) = N\mu^*. \tag{9}$$

which corresponds to the situation where no particle can cross the channel without blockage. The flux is given by the number of particles trapped in each blockage, $N$, times the rate of the channel release, $\mu^*$. At low intensity the throughput is given by

$$j_N(\lambda) = \lambda - \frac{\lambda^{N+1}}{(N-1)! \mu^{N-1} \mu^*} + O(\lambda^{N+2}). \tag{10}$$

In this limit almost all particles cross the channel, which corresponds to the leading term $\lambda$. The decrease of the flux corresponding to the term of order $\lambda^{N+1}$ is due to the
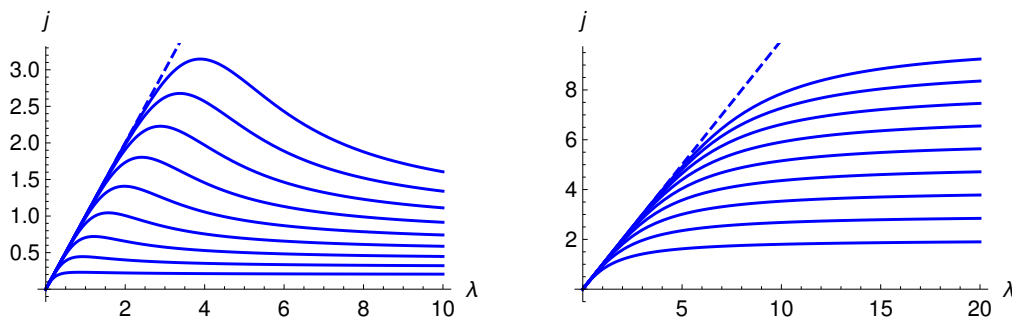
**Figure 2.** (Left) Steady state throughput, $j$, Eq. (7), as a function of the intensity, $\lambda$, for $\mu = 1, \mu* = 0.1$ and $N = 2, 3, ..., 10$ bottom to top. The dashed line shows the behavior in the limit of small intensity, $j = \lambda$. The limiting value at large intensity is given by $N\mu^*$. (Right) Steady state throughput of the $M/M/N/N$ queue as a function of the intensity, $\lambda$, for $\mu = 1$. The limiting value at large intensity is given by $N\mu$.

rare events where the channel is blocked and is given by $\frac{\lambda}{\mu^*\tau}$, where $\tau$ is the mean time of blockage at low intensity [25], $\tau = (N-1)!\frac{\mu^{N-1}}{\lambda^N}$. The outgoing flux as a function of the intensity is shown for several values of $N$ and for $\mu = 1, \mu^* = 0.1$ in Fig. 2. One notes the initial linear regime corresponding to no loss of the incoming flux, which increases in importance with increasing $N$. This is followed by a maximum throughput at a finite intensity.

Let us compare our model with the well-known $M/M/N/N$ queue [23], i.e., one with exponentially distributed arrival and service times with rates $\lambda$ and $\mu$, respectively, $N$ servers and no waiting line [14]. The Markov chain of this model is similar to Fig. 1 except that there is no direct transition from state $N$ to state 0. Instead there is a transition from state $N$ to state $N - 1$ with rate $N\mu$. An arriving unit is lost to the system if all $N$ servers are busy. The probability of this event is given by the Erlang loss formula

$$\pi_N = \frac{\frac{1}{N!}\left(\frac{\lambda}{\mu}\right)^N}{\sum_{i=0}^{N}\frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i} \tag{11}$$

The throughput, given by substituting this probability in Eq. (7), always increases monotonically towards the maximum value $N\mu$ as $\lambda$ increases: See Fig. 2.

In the following we examine in more detail the behavior of the systems $N = 2, 3$. For simplicity, and without loss of generality, we will often set $\mu = 1$. This is equivalent to taking the unit of time as $\mu^{-1}$.

### 2.1. N=2

The explicit equations describing the evolution of the three state probabilities, with $\mu = 1$, are;

$$\dot{\pi}_0 = -\lambda\pi_0(t) + \mu\pi_1(t) + \mu^*\pi_2(t),$$
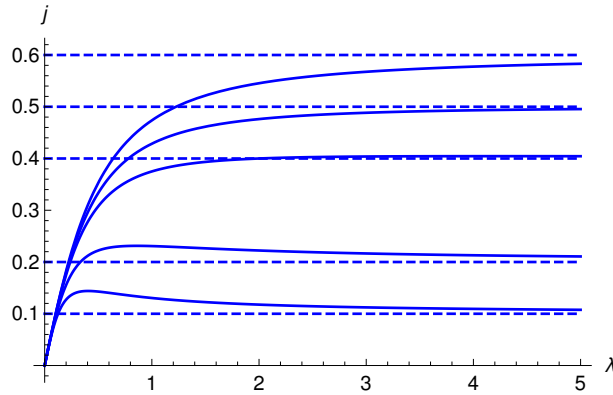
**Figure 3.** $N = 2$. Steady state throughput, $j$ as a function of intensity, $\lambda$ for $\mu = 1, \mu* = 0.3, 0.25, 0.2, 0.1, 0.05$ top to bottom. The dashed lines show the limiting value, $2\mu^*$. Global maxima are evident for certain values of $\mu^*$ (see text).

$$\dot{\pi}_1 = \lambda \pi_0(t) - (\mu + \lambda)\pi_1(t),$$
$$\dot{\pi}_2 = \lambda \pi_1(t) - \mu^* \pi_2(t). \tag{12}$$

and the time dependent throughput is

$$j(\lambda, t) = \mu \pi_1(t) + 2\mu^* \pi_2(t). \tag{13}$$

As the system evolves towards a steady state the throughput approaches a constant value.

**Steady State Behavior.** In the steady state, the stationary probabilities are

$$[\pi_0, \pi_1, \pi_2] = C_2[(\lambda + \mu)\mu^*, \lambda\mu^*, \lambda^2]. \tag{14}$$

with $C_2 = ((2\lambda + \mu)\mu^* + \lambda^2)^{-1}$. As expected Eq. (6) and Eq. (7) yield the same result for the steady state throughput:

$$j(\lambda) = \lambda \frac{2\lambda + \mu}{\lambda^2/\mu^* + 2\lambda + \mu} \tag{15}$$

Figure 3 displays the steady state throughput as a function of the intensity, $\lambda$. The limiting values of the steady state throughput at small and large intensity are

$$j(\lambda) = \lambda - \frac{\lambda^3}{\mu\mu^*} + O(\lambda^4), \tag{16}$$
$$\lim_{\lambda \to \infty} j(\lambda) = 2\mu^*. \tag{17}$$

We search for a non-trivial maximum throughput at finite $\lambda$ by seeking solutions of $dj/d\lambda = 0$ which requires $4\lambda\mu\mu^* + \mu^2\mu^* - \lambda^2(\mu - 4\mu^*) = 0$. The solutions can be written as

$$\frac{\lambda}{\mu} = \frac{\sqrt{r} + 2r}{1 - 4r} \geq 0, \tag{18}$$
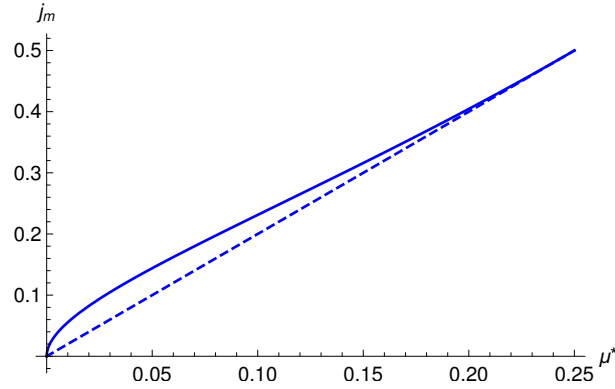
**Figure 4.** $N = 2$. Maximum throughput, $j_{\max}$, as a function of $\mu^*$ ($\mu = 1$). The dashed lines shows the limiting value, $2\mu^*$.

where $r = \mu^*/\mu$. Thus, solutions exist for $r < r_c = 1/4$. If $r > 1/4$ there is no maximum at finite $\lambda$. If $r < 1/4$ the maximum throughput is given by

$$j_{\max} = \frac{\mu\sqrt{r}}{2(1 - \sqrt{r})} = \frac{\mu^*}{2(1 - \sqrt{r})\sqrt{r}} \tag{19}$$

Figure 4 shows that this approaches the limiting value $2\mu^*$ as $\mu \to 1/4$.

**Mean Time to First Blockage.** From Eq. (5) we have

$$\nu_{0,2} = \frac{2}{\lambda} + \frac{\mu}{\lambda^2}. \tag{20}$$

Following [11] we note that in the steady state there is an alternation of open and blocked states with average durations $\nu_{0,2}$ and $1/\mu^*$, respectively whose sum defines a cycle with period $1/\mu^* + \nu_{0,2}$. This provides an alternative route to the probability that the system is in the blocked state, $\pi_2$ which is the fraction of the cycle spent in the blocked state: $\pi_2 = (1/\mu^*)/((1/\mu^*) + \nu_{0,2}) = \lambda^2/((\mu + 2\lambda)\mu^* + \lambda^2)$, as obtained previously.

**Kinetic Behavior.** The system of equations Eq. (12) may be solved analytically. The eigenvalues of the matrix $Q_3$ are 0 (associated with the conservation of the total probability), and two real negative values $\gamma_{1,2}$ given by

$$\gamma_{1,2} = -\frac{\mu + \mu^* + 2\lambda \pm \beta}{2} \tag{21}$$

where $\beta = \sqrt{(\mu - \mu^*)^2 + 4\lambda(\mu - \mu^*)}$. The probabilities are given by

$$\pi_i(t) = \pi_i + a_i e^{\gamma_1 t} + b_i e^{\gamma_2 t} \tag{22}$$

where $\pi$ are the stationary values and $a_i$ and $b_i$ are determined by the initial conditions for $i = 0, 1, 2$. One easily obtains that

$$a_0 = \frac{\gamma_2}{\beta}(1 - \pi_0) + \frac{\lambda}{\beta}, b_0 = -\frac{\gamma_1}{\beta}(1 - \pi_0) - \frac{\lambda}{\beta} \tag{23}$$

$$a_1 = -\frac{\gamma_2}{\beta}\pi_1 - \frac{\lambda}{\beta}, \qquad b_1 = \frac{\gamma_1}{\beta}\pi_1 + \frac{\lambda}{\beta} \tag{24}$$

$$a_2 = -\frac{\gamma_2}{\beta}\pi_2, \qquad b_2 = \frac{\gamma_1}{\beta}\pi_2. \tag{25}$$

Figure 5 illustrates the behavior of the time dependent throughput for different parameter values. We note two distinct behaviors: either the throughput increases monotonically to the steady state value, or it displays a maximum at a finite time before decreasing to the steady state value.

By using Eq.(13), one obtains

$$\frac{\partial j(t)}{\partial t} = \gamma_1(\mu a_1 + 2\mu^* a_2)e^{\gamma_1 t} + \gamma_2(\mu b_1 + 2\mu^* b_2)e^{\gamma_2 t} \qquad (26)$$

The solution for $\frac{\partial j(t)}{\partial t} = 0$ is given by

$$t_{\max} = \frac{1}{\beta}\ln\left(\frac{\mu^*(\mu + 2\lambda) + \gamma_1\mu}{\mu^*(\mu + 2\lambda) + \gamma_2\mu}\right). \qquad (27)$$

For a solution to exist the argument of the logarithm must be finite and positive. Setting the denominator equal to zero thus gives the boundary of the set of parameter values at which $j(t)$ displays a maximum:

$$\mu_b^*(\lambda) = \frac{\mu}{2}\left(1 - \sqrt{\frac{\mu}{2\lambda + \mu}}\right), \qquad (28)$$

It is the limiting value of $\mu^*$, for a given value of $\lambda$, at which the nontrivial maxima of the time dependent throughput exists. We note that if $r > 1/2$ no maximum at finite time exists; rather the throughput approaches the steady state value from below. The kinetic and steady state behaviors are shown in Fig. 6.

### 2.2. N=3

The master equations describing the evolution of the four state probabilities are;

$$\begin{aligned}
\dot{\pi}_0 &= -\lambda\pi_0(t) + \mu\pi_1(t) + \mu^*\pi_3(t), \\
\dot{\pi}_1 &= \lambda\pi_0(t) - (\mu + \lambda)\pi_1(t) + 2\mu\pi_2(t), \\
\dot{\pi}_2 &= \lambda\pi_1(t) - (2\mu + \lambda)\pi_2(t), \\
\dot{\pi}_3 &= \lambda\pi_2(t) - \mu^*\pi_3(t).
\end{aligned} \qquad (29)$$

And the time dependent throughput is;

$$j(\lambda, t) = \mu\pi_1(t) + 2\mu\pi_2(t) + 3\mu^*\pi_3(t). \qquad (30)$$

**Steady State Behavior.** The steady state probabilities are

$$[\pi_0, \pi_1, \pi_2, \pi_3] = C_3[(\lambda^2 + \lambda\mu + 2\mu^2)\mu^*, \lambda(\lambda + 2\mu)\mu^*, \lambda^2\mu^*, \lambda^3]. \qquad (31)$$

with $C_3 = (\lambda^3 + 3\mu^*\lambda^2 + 3\mu\mu^*\lambda + 2\mu^2\mu^*)^{-1}$. And the throughput is

$$j(\lambda) = \lambda(1 - \pi_3) = \frac{\lambda(3\lambda^2 + 3\lambda\mu + 2\mu^2)\mu^*}{\lambda^3 + 3\mu^*\lambda^2 + 3\mu\mu^*\lambda + 2\mu^2\mu^*} \qquad (32)$$

with the limiting values at small and large intensities

$$j(\lambda) \qquad = \lambda - \frac{\lambda^4}{2\mu^2\mu^*} + O(\lambda^5), \qquad (33)$$

$$\lim_{\lambda \to \infty} j(\lambda) = 3\mu^*, \qquad (34)$$
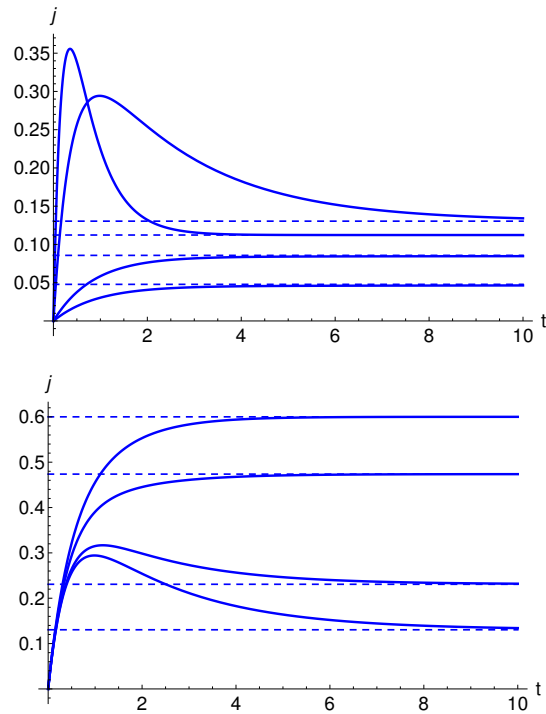
**Figure 5.** Time evolution of $j$ for $N = 2$. Top: $\mu^* = 0.05$, $\lambda = 3$, 1, 0.1, 0.05, top to bottom. Bottom: $\lambda = 1$, $\mu^* = 0.05, 0.1, 0.3, 0.5$ bottom to top. The dashed lines show the steady state value, Eq. (15).
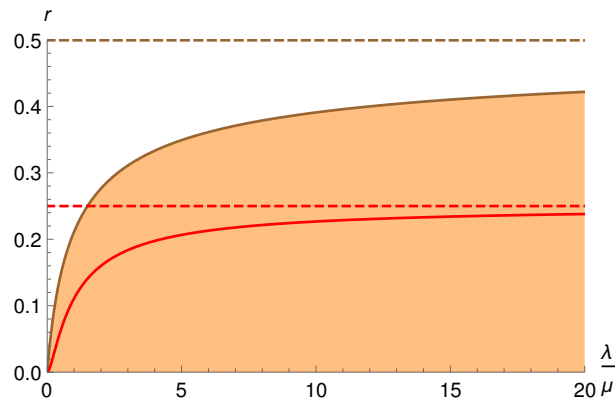


**Figure 6.** State diagram for $N = 2$. The shaded region bounded by the upper solid line, Eq. (28), corresponds to the parameter space in which a maximum throughput occurs at a finite time given by Eq. (27). No maximum at any time occurs if $r > 1/2$ (upper dashed line). The solid, red line, Eq. (18), corresponds to the intensity that maximizes the steady state throughput for a given value of $r = \mu^*/\mu$. No maximum occurs for finite $\lambda$ if $r > 1/4$ (lower dashed line).
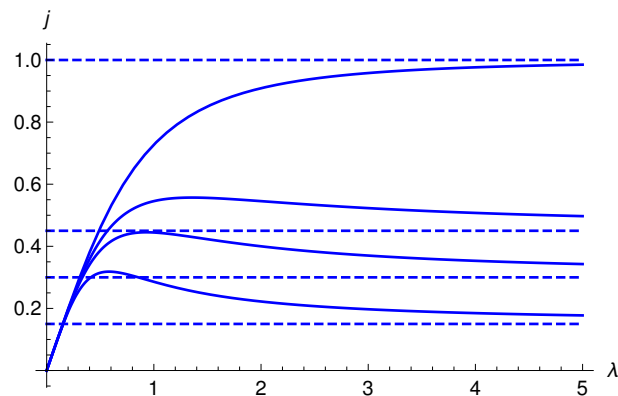
**Figure 7.** $N = 3$. Steady state throughput, $j$, as a function of $\lambda$ for $\mu = 1, \mu* = 0.3333, 0.15, 0.1, 0.05$ top to bottom. The dashed lines show the limiting value, $3\mu^*$.
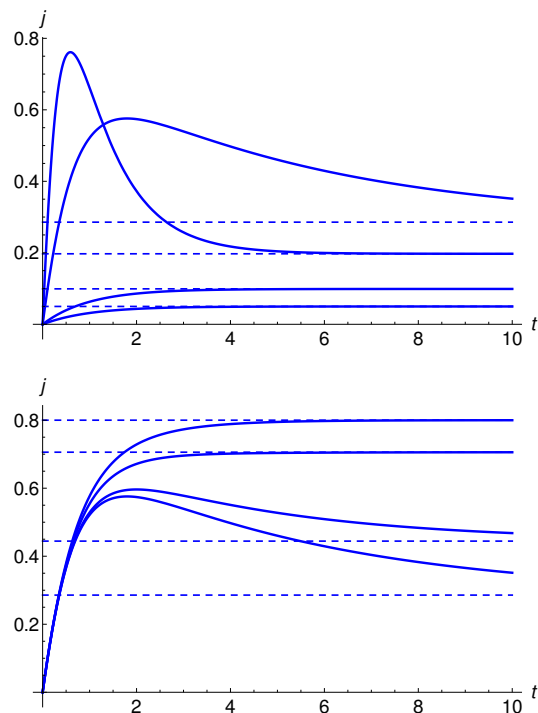


**Figure 8.** $N = 3$. Time evolution of $j(t)$. Top: $\mu^* = 0.05$ fixed, for $\lambda = 3, 1, 0.1, 0.05$, top to bottom. Bottom: $\lambda = 1$ fixed; $\mu^* = 0.05, .1, 0.3, 0.5$, bottom to top. The dashed lines show the steady state value, Eq. (32).

respectively.

Some examples are shown in Fig. 7. For $r < r_c = 1/3$ there is a maximum throughput at finite intensity.

**Kinetic Behavior.** To explore the transient behavior, we solved Eqs. (29) numerically. Some results are shown in Fig. 8. The same two distinct behaviors, as remarked for $N = 2$, are present. No maximum at finite time is observed if $\mu^* > 0.75$.

*2.3. Comparison with Reversible Model with Constant Transit and Blockage Times*

The model introduced above is fully stochastic in the sense that the interval between particle entries, transit and blockage times are all sampled from exponential distributions with given rates. While this choice can lead to a considerable simplification of the mathematics, it may be unrealistic in certain physical applications. A particle cannot traverse a channel in zero time, while according to the exponential distribution this is the most likely outcome. It therefore seems reasonable to consider alternative models with constant transit and/or blockage times [11].

For $N = 1$ (that is the channel is blocked by the entry of the first particle) it is easy to show that the steady state properties for the model with an exponentially distributed blocking time with rate $\mu^*$ are the same as for a model with a constant blocking time, $\tau_b$, if the deblocking rate is equal to the inverse of the (constant) blocking time. For example the probabilities that the channel is open are $p_o = \mu^*/(\mu^* + \lambda)$ and $p_o = 1/(1 + \lambda\tau_b)$, respectively. These are the same if $\tau_b = 1/\mu^*$. The kinetics of the two models are, however, different (For example, when the blockage time is constant the exiting flux is strictly zero for $t < \tau_b$, while for an exponentially distributed blockage time the mean flux is finite for all $t > 0$. We examined the corresponding ($N = 2$) model in [11], i.e., one with a constant transit and blockage times $\tau$ and $\tau_b$, respectively. In the steady state the exiting flux is given by

$$j_\infty = \frac{\lambda(2 - e^{-\lambda\tau})}{\lambda\tau_b(1 - e^{-\lambda\tau}) + 2 - e^{-\lambda\tau}}. \tag{35}$$

Comparing this to the throughput of the first model above with $N = 2$, Eq. (15), we see that there is no simple mapping when one replaces $\tau$ with $1/\mu$ and $\tau_b$ with $1/\mu^*$. If, however, we introduce an effective exit (service) rate we can map the two systems:

$$\mu^* = \frac{1}{\tau_b},$$
$$\mu = \frac{\lambda e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}. \tag{36}$$

That is, by substituting these equations in Eq. (15) we obtain Eq. (35). Thus, we expect the steady state behavior to be qualitatively similar. In particular the exiting flux, Eq. (35) may be maximized for a finite value of the intensity if $\tau/\tau_b < 0.16$.

## 3. Irreversible Blockage Model with a Source of Finite Duration

This model, originally introduced by Gabrielli et al. [26], considers a stream of particles entering a channel according to a Poisson process of intensity $\lambda$ with the channel carrying capacity set to $N = 2$. A single particle exits in a time $\tau$, but if ever two particles are present in the channel it blocks irreversibly. Properties of interest, including the survival probability at time $t$, mean blockage time, exiting particle flux, and the total number of exiting particles can be calculated exactly [12, 26]. Here we consider the problem of maximizing the total number of exiting particles in a given finite time.

The average number of particles that exit in the time interval $(0, t_s)$ can be computed by integrating the exiting particle flux,

$$m(t_s) = \int_1^{t_s} j(t) dt \tag{37}$$

We take the lower limit of the integral $t = \tau = 1$, as no particle can exit before this time (assuming that the channel is empty at $t = 0$.) For $t_s \to \infty$ we can show that

$$m(\infty) = \tilde{j}(u = 0) = \frac{1}{e^\lambda - 1} \tag{38}$$

where $\tilde{j}(u) = \int_0^\infty e^{-ut} j(t) dt$ is the Laplace transform. The number of exiting particles tends to infinity as $\lambda \to 0$ and to zero as $\lambda \to \infty$. More interesting, however, is the situation for finite $t_s$. In this case there is clearly a finite entering intensity that optimizes the total number of exiting particles. If the intensity is too small, blocking is unlikely but few particles enter, while if $\lambda$ is too large more particles are injected but blocking is highly probable.

The explicit equation for the flux at time $t$ [12] is given by

$$j(t) = e^{-\lambda t} \sum_{k=1}^{\lfloor t \rfloor} \frac{\lambda^k (t-k)^{k-1}}{(k-1)!}. \tag{39}$$

Substituting this in Eq. (37) gives

$$m(t_s) = \sum_{k=1}^{\lfloor t_s \rfloor} \exp(-\lambda k) \left[ 1 - \frac{\Gamma(k, \lambda(t_s - k))}{(k-1)!} \right] = \sum_{k=1}^{\lfloor t_s \rfloor} \exp(-\lambda k) \frac{\gamma(k, \lambda(t_s - k))}{(k-1)!}, \tag{40}$$

where $\Gamma(k, x)$ and $\gamma(k, x)$ are the upper and lower incomplete gamma functions, respectively. Some results are shown in Fig. 9. It can be seen that the expected number of exiting particles displays a maximum at finite intensity for a finite stopping time, $t_s$. We observe that the value of $\lambda$ that maximizes the output increases as $t_s$ decreases and that the maximum sharpens as $t_s$ increases.

The explicit expression for the intensity that maximizes the output is a piece-wise function. For $1 \leq t_s \leq 2$,

$$m(t_s) = e^{-\lambda} - e^{-\lambda t_s}. \tag{41}$$

It is easy to show that a maximum occurs for

$$\lambda = \frac{\ln(t_s)}{t_s - 1} \tag{42}$$

which gives $\lambda = \ln(2)$ when $t_s = 2$.

For $2 \leq t_s \leq 3$

$$m(t_s) = e^{-\lambda} - e^{-\lambda t_s} + e^{-2\lambda} - e^{-\lambda t_s}(1 + \lambda(t_s - 2)). \tag{43}$$

$m(t)$ being a piece-wise continuous function, one easily checks that for $t_s = 2$, the value is the same as that given by Eq. (41). There is no analytic expression of the value of $\lambda$ that maximizes this expression for $t > 2$, but numerical solutions are, however,
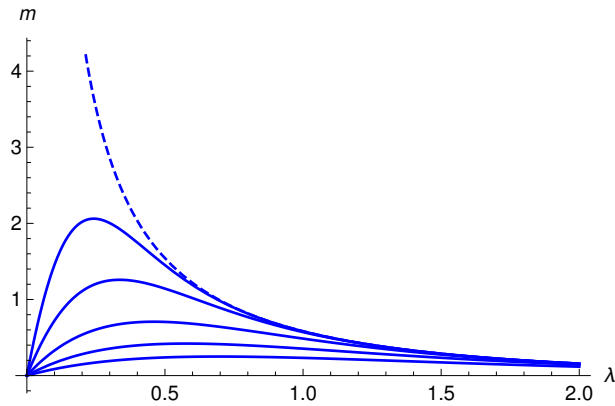
**Figure 9.** Average number of exiting particles as a function of the intensity for different stopping times, $t_s = 32, 16, 8, 4, 2, 1.5$ top to bottom. The dashed line shows the long-time limit, Eq. (38).

straightforward. Moreover, for large $t_s$, which corresponds to small intensity $\lambda$, one can obtain an asymptotic solution. By using the Laplace's method [12], $j(t)$ decays as

$$j(t) \simeq \lambda e^{-\lambda^2 t}. \tag{44}$$

Integrating Eq. (44), one obtains that $m(t_s)$ is given by

$$m(t_s) \simeq \frac{1}{\lambda}(1 - e^{-\lambda^2 t_s}). \tag{45}$$

Differentiating Eq. (45) with respect of $\lambda$, one obtains a maximum of $m(t_s)$ when

$$\lambda \simeq \frac{C}{\sqrt{t_s}}. \tag{46}$$

where $C$ is a constant.

The limitation of the flux route is that it does not allow us to calculate higher moments of the exiting particle distribution, but only the mean value. To go further, we therefore introduce the function $f(m,t)$ giving the probability that $m$ particles have exited at time $t$, regardless of the state of the channel (open or closed) at time $t$ (the joint probabilities that $m$ particles have exited at time $t$ and the channel is still open are discussed in the Appendix). The equations describing the evolution of these functions are as follows:

$$\frac{df(0,t)}{dt} = -\lambda e^{-\lambda} q_s(0, t-1) \tag{47}$$

$$\frac{df(1,t)}{dt} = \lambda e^{-\lambda} q_s(0, t-1) - \lambda e^{-2\lambda} q_s(1, t-2) \tag{48}$$

$$\frac{df(n,t)}{dt} = \lambda e^{-2\lambda} q_s(n-1, t-2) - \lambda e^{-2\lambda} q_s(n, t-2), \ n > 1, \tag{49}$$

where $q_s(n,t)$ is the joint probability that $n$ particles have entered the channel and the channel still open [12]. The loss term for the evolution of $f(0,t)$ is the result of a particle exiting the channel at time $t$ that had previously entered the channel at $t-1$. This term

is also the gain term for the evolution of $f(1, t)$. For $n > 1$ the gain term consists of the entry of a particle at $t - 1$ that exits at $t$. For this to be possible the channel must have been open at $t - 2$ with $n - 1$ particles entering in the interval $(0, t - 2)$, which occurs with probability $q_s(n - 1, t - 2)$, and no particle must enter in the intervals $(t - 2, t - 1)$ and $(t - 1, t)$ giving rise to the factor of $e^{-2\lambda}$. The loss term is similar except that $n$ particles must have entered in the interval $(0, t - 2)$ with an additional particle entering at $t - 1$ and exiting at $t$. The gain term for $n = 1$ is slightly different as the particle entering the channel at $t - 1$ is the first one so the channel is certainly empty at this time.

We note the telescopic structure of equations (47), (48) and (49) is consistent with the conservation of probability, $\sum_{n=0}^{\infty} df(n, t)/dt = 0$.

We can obtain the complete solution by introducing the generating function

$$G_f(x, t) = \sum_{n=0}^{\infty} z^n f(n, t). \tag{50}$$

Taking the time derivative and substituting the above expressions for $f(n, t)$ we obtain

$$\frac{\partial G_f(z, t)}{\partial t} = \lambda(z - 1)(e^{-\lambda} q_s(0, t - 1) + e^{-2\lambda}(G(z, t - 2) - q_s(0, t - 2))), \tag{51}$$

where $G(z, t) = \sum_{n=0}^{\infty} z^n q_s(n, t)$. Taking the Laplace transform and using the initial condition $G_f(z, 0) = 1$ we finally obtain

$$\tilde{G}_f(z, u) = \frac{u + \lambda - \lambda e^{-(u+\lambda)}}{u(u + \lambda - \lambda z e^{-(u+\lambda)})}. \tag{52}$$

The individual functions can be recovered from

$$\tilde{f}(n, u) = \frac{1}{n!} \frac{\partial^n \tilde{G}_f(z, u)}{\partial z^n} \bigg|_{z=0}. \tag{53}$$

The first two are

$$f(0, t) = 1 + (e^{-\lambda t} - e^{-\lambda})\theta(t - 1) \tag{54}$$

and

$$f(1, t) = ((e^{-\lambda t}(1 + \lambda(t - 2)) - e^{-2\lambda})\theta(t - 2) - (e^{-\lambda t} - e^{-\lambda})\theta(t - 1), \tag{55}$$

where $\theta(t)$ is the Heaviside function. These results can also be obtained by direct solution of Eqs. (47) and (48), respectively. The first two moments are

$$\langle \tilde{m}(u) \rangle = \frac{\partial \tilde{G}_f(u, z)}{\partial z} \bigg|_{z=1} = \frac{\lambda e^{-(u+\lambda)}}{u(u + \lambda - \lambda e^{-(u+\lambda)})}, \tag{56}$$

and

$$\begin{aligned} \langle \tilde{m}^2(u) \rangle &= \frac{\partial^2 \tilde{G}_f(u, z)}{\partial z^2} \bigg|_{z=1} + \frac{\partial \tilde{G}_f(u, z)}{\partial z} \bigg|_{z=1} \\ &= \frac{\lambda e^{-(u+\lambda)}(\lambda + (u + \lambda)e^{-(u+\lambda)})}{u(\lambda - (u + \lambda)e^{-(u+\lambda)})^2}. \end{aligned} \tag{57}$$

from which the first and second moments at infinite time may be obtained as

$$\langle m \rangle = \lim_{u \to 0} u \langle \tilde{m}(u) \rangle = \frac{1}{e^\lambda - 1} \tag{58}$$

and

$$\langle m^2 \rangle = \lim_{u \to 0} u \langle \tilde{m}^2(u) \rangle = \frac{1 + e^\lambda}{(1 - e^\lambda)^2}, \tag{59}$$

giving for the variance

$$\langle m^2 \rangle - \langle m \rangle^2 = \frac{e^\lambda}{(e^\lambda - 1)^2}. \tag{60}$$

At small intensity, this behaves as $\lambda^{-2}$, while for large intensity it approaches zero as $e^{-\lambda}$.

In order to obtain the variance of $m$ at time $t$, we have to invert $\langle m^2(u) \rangle$ and $\langle m(u) \rangle$. By using Eq.(56), one has

$$\langle m(u) \rangle = \frac{e^{-(u+\lambda)}}{1 - e^{-(u+\lambda)}} \left[ \frac{1}{u} - \frac{1}{u + \lambda - \lambda e^{-(u+\lambda)}} \right] \tag{61}$$

$$= \sum_{n \geq 1} e^{-n(u+\lambda)} \left[ \frac{1}{u} - \sum_{k \geq 0} \frac{(\lambda e^{-(u+\lambda)})^k}{(u + \lambda)^{k+1}} \right], \tag{62}$$

which gives

$$\langle m(t) \rangle = \sum_{n \geq 1} \left( e^{-\lambda n} \theta(t - n) - e^{-\lambda t} \sum_{k=0} \frac{\lambda^k (t - k - n)^k \theta(t - k - n)}{k!} \right). \tag{63}$$

This expression corresponds to that obtained from the survival probability [11].

Similarly, by using Eq.(84), one obtains

$$\langle m^2(u) \rangle = \frac{e^{-(u+\lambda)}(1 + e^{-(u+\lambda)})}{(1 - e^{-(u+\lambda)})^2} \left[ \frac{1}{u} - \frac{1}{u + \lambda - \lambda e^{-(u+\lambda)}} \right]$$

$$- 2 \frac{\lambda e^{-2(u+\lambda)}}{1 - e^{-(u+\lambda)}} \frac{1}{(u + \lambda - \lambda e^{-(u+\lambda)})^2}. \tag{64}$$

By using the identity $\frac{1}{(1-a)^2} = \sum_{n=0}(n + 1)a^n$, one has

$$\langle m^2(u) \rangle = \sum_{n=0}(n + 1)(1 + e^{-(u+\lambda)})e^{-(n+1)(u+\lambda)} \left[ \frac{1}{u} - \sum_{k=0} \frac{(\lambda e^{-(u+\lambda)})^k}{(u + \lambda)^{k+1}} \right]$$

$$- 2 \sum_{n=0} \lambda e^{-(n+2)(u+\lambda)} \sum_{k=0}(k + 1) \frac{(\lambda e^{-(u+\lambda)})^k}{(u + \lambda)^{k+1}}. \tag{65}$$

that can be inverted to give the second moment as a function of time:

$$\langle m^2(t) \rangle = \sum_{n \geq 0}(1 + n)e^{-(2+n)\lambda} \left( \theta(t - n - 2) + e^\lambda \theta(t - n - 1) \right)$$

$$- \sum_{n \geq 0} \sum_{k \geq 0} \frac{e^{-t\lambda}(1 + n)\lambda^k}{k!} \left( (t - n - k - 2)^k \theta(t - n - k - 2) + (t - n - k - 1)^k \theta(t - n - k - 1) \right)$$

$$- \sum_{n \geq 0} \sum_{k \geq 0} \frac{2e^{-k\lambda - (2+n)\lambda - (t-n-k-2)\lambda} \lambda^{k+1}}{(k + 1)!}(1 + k)(t - n - k - 2)^{1+k} \theta(t - n - k - 2). \tag{66}$$

Numerical results for the second moment and the variance as a function of $\lambda$ are shown in Figs. 10 and 11, respectively. Like the first moment (Fig. 9), the second moment displays a maximum at a finite value of $\lambda$. The family of curves at different
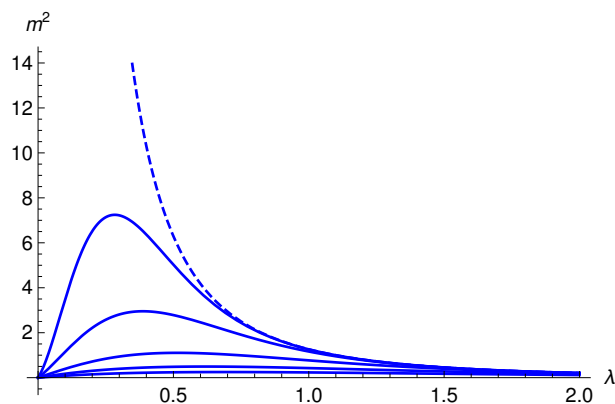
**Figure 10.** Second moment of the number of exiting particles as a function of the intensity for different stopping times, $t_s = 20, 10, 5, 3, 2$ top to bottom. The dashed line shows the long time limit, Eq. (59).
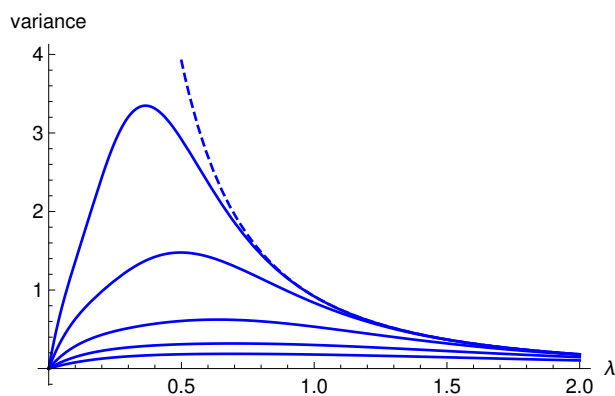


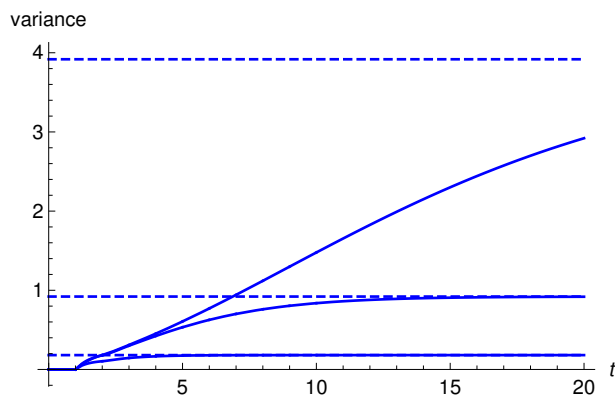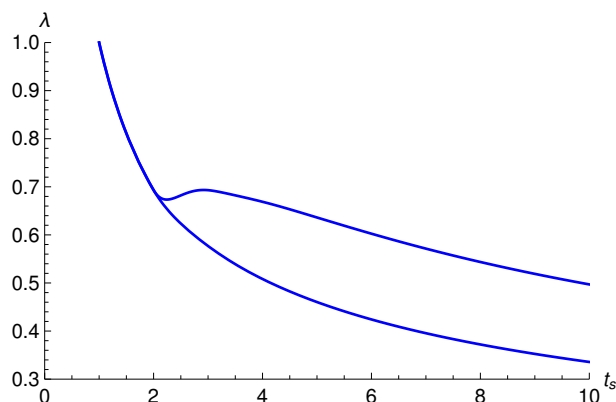**Figure 11.** Variance of the number of exiting particles as a function of the intensity for different stopping times, $t_s = 10, 6, 4, 2, 1.5$ top to bottom. The dashed line shows the long time limit, Eq. (60).



**Figure 12.** Variance of the number of exiting particles as a function of time for different intensities, $\lambda = 0.1, 0.5, 1, 2$. The dashed lines show the long time limit, Eq. (60).

**Figure 13.** Intensity that maximizes the variance (upper curve) and mean of number of exiting particles as a function of the stopping time.

stopping times approaches the long-time limit, Eq. (59) as time increases. The variance also displays a maximum as a function of intensity, see Fig. 11, but the location of the maximum for a given stopping time is displaced to a higher value of the intensity. The time dependent variance for different values of $\lambda$ is shown in Fig. 12. Finally in Fig. 13 we plot the value of $\lambda$ that maximizes the mean and variance as a function of the stopping time. We observe that the intensity that maximizes the mean number of exiting particles is a strictly decreasing function of the stopping time. The intensity that maximizes the variance of the number of exiting particles, however, has a non-trivial behavior. It is the same that maximizes the intensity for $1 \leq t_s \leq 2$, but it then increases to a maximum value for $t_s \approx 3$ and then decreases. For $t_s > 2$, the variance is maximized at a higher value of $\lambda$ than that which maximizes the mean number.

## 4. Conclusion

We have studied the optimization of the throughput of a stream of particles subject to blocking using a circular Markov chain model. The sojourn time of a particle contained in an open channel is exponentially distributed with rate $\mu$. If $N$ particles are simultaneously present the channel is blocked and all newly arriving particles are rejected. After an exponentially distributed blockage time with rate $\mu^*$ all particles forming the blockage simultaneously exit the channel. We presented general expressions for the steady state probabilities and throughput. For $N = 2$ we showed that the steady state throughput assumes a maximum value at finite intensity if $\mu^*/\mu < 1/4$. The time dependent throughput may also display a maximum if $\mu^*/\mu < 1/2$. We showed that this behavior is qualitatively different from the well-known $M/M/N/N$ queue whose steady state throughput always increases monotonically with the intensity of entering particles. We also compared the new model with a previously introduced one with deterministic transit and blockage times. For $N = 2$ we found an exact mapping between the two in the steady state. In future work we plan to apply the circular Markov chain model

to multi-channel systems where the entering flux is evenly distributed over the open channels [27].

In the second part of the article we examined an irreversible blockage model with capacity $N = 2$, fixed transit time and an input of constant intensity that is switched off after a given time, $t_s$. For small stopping times, the mean and variance of the number of exiting particles are maximized at the same value of the intensity $\lambda$. If $t_s/\tau > 2$, the maximum value of the variance occurs at a higher value of the intensity than the mean value.

## 5. Appendices

### 5.1. Derivation of the Mean First Passage time to the blocked state

In Cohen [24] the following formulas can be found for the average first hitting times $\nu_{i,j}$ of hitting level $j$, starting from level $i$, in a birth-death process with birth rates $\lambda_n$ and death rates $\mu_n$ in state $n$:

For $i < j$,

$$\nu_{i,j} = \sum_{n=i}^{j-1} \frac{1}{\lambda_n \pi_n} \sum_{k=0}^{n} \pi_k. \tag{67}$$

For $i = j$ (= average first return time of state $j$),

$$\nu_{i,j} = \frac{\sum_{k=0}^{\infty} \pi_k}{(\lambda_j + \mu_j)\pi_j}. \tag{68}$$

For $i > j$,

$$\nu_{i,j} = \sum_{n=j}^{i-1} \frac{1}{\lambda_n \pi_n} \sum_{k=n+1}^{\infty} \pi_k. \tag{69}$$

Here,

$$\pi_0 = 1, \quad \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}. \tag{70}$$

In the special case of an $M/M/\infty$ queue we have $\lambda_n = \lambda$ and $\mu_n = n\mu$, leading to $\pi_n = \frac{\lambda^n}{n!\mu^n} = \frac{\rho^n}{n!}$, where $\rho = \lambda/\mu$. In this case, we obtain

$$\nu_{0,j} = \sum_{n=0}^{j-1} \frac{1}{\lambda \frac{\rho^n}{n!}} \sum_{k=0}^{n} \frac{\rho^k}{k!} = \frac{1}{\lambda} \sum_{n=0}^{j-1} n! \sum_{k=0}^{n} \frac{\rho^{k-n}}{k!}. \tag{71}$$

that is Eq. (5).

### 5.2. Joint probabilities for irreversible blockage at finite time

In [12] we considered the joint probability that $m$ particles have exited at time $t$ *and* the system is blocked, which we denoted as $h(m, t)$. Let us consider the joint probability

$g(m,t)$ that $m$ particles have exited the channel at time $t$ *and* that the channel is still open. Clearly

$$m(t) = \sum_{k=1}^{\infty} k(g(k,t) + h(k,t)) \tag{72}$$

and $f(k,t)$ introduced in Sec. 3 is simply $f(k,t) = g(k,t) + h(k,t)$.

The time evolution of $g(n,t)$ is given by

$$\frac{dg(0,t)}{dt} = -\lambda \int_0^{\min(t,1)} dt_1 \lambda e^{-\lambda t_1} q_s(0, t - t_1) - \lambda e^{-\lambda} q_s(0, t - 1)$$

$$\frac{dg(1,t)}{dt} = \lambda e^{-\lambda} q_s(0, t - 1) - \lambda \int_0^{\min(t-1,1)} dt_1 \lambda e^{-\lambda(t_1+1)} q_s(1, t - 1 - t_1) - \lambda e^{-2\lambda} q_s(1, t - 2)$$

$$\frac{dg(n,t)}{dt} = \lambda e^{-2\lambda} q_s(n-1, t-2) - \lambda \int_0^{\min(t-1,1)} dt_1 \lambda e^{-\lambda(t_1+1)} q_s(n, t - 1 - t_1) - \lambda e^{-2\lambda} q_s(n, t - 2). \tag{73}$$

The time derivative of $g(n,t)$ is given as the sum of a gain term and two loss terms. The gain term is the probability density that the *nth* particle exits at time $t$ and that the channel is still open. This corresponds to the event where the *nth* particle enters at $t - 1$ and that $n - 1$ particles have already exited the channel. The first loss term corresponds to a particle which blocks the channel at time $t$ knowing that $n$ particles already exited. This means that a particle is still in the channel at $t$ and a new one entering at time $t$ blocks the channel. The last term corresponds to the exit of the *nth* particle at time $t$ with a channel still open. The boundary term for $n = 0$ does not require a time lag in the two loss terms because for $t > 1$ a particle can enter without clogging the channel. A similar argument applies for $n = 1$ to the gain term. Defining the Laplace transform as

$$\tilde{g}(u) = \int_0^{\infty} dt\, e^{-ut} g(t), \tag{74}$$

the differential equations become

$$u\tilde{g}(0,u) - 1 = -\lambda \frac{\lambda + ue^{-(\lambda+u)}}{\lambda + u} \tilde{q}_s(0, u). \tag{75}$$

Knowing that $\tilde{q}_s(0,u) = \frac{1}{\lambda+u}$, the inverse Laplace transform of $\tilde{g}(0,u)$ is

$$\tilde{g}(0,u) = \frac{1}{u} - \frac{\lambda}{u} \left( \frac{\lambda + ue^{-(\lambda+u)}}{(\lambda + u)^2} \right), \tag{76}$$

which gives

$$g(0,t) = (1 + \lambda t - \theta(t-1)\lambda(t-1))e^{-\lambda t}. \tag{77}$$

To go further, let us recall that the generating function for $h(k,t)$, $G_h(z,t) = \sum_k z^k h(k,t)$, is given in Laplace space by (see [12])

$$\tilde{G}_h(z,u) = \frac{\lambda^2(1 - e^{-(\lambda+u)})}{u(\lambda + u)(u + \lambda(1 - ze^{-(\lambda+u)}))}. \tag{78}$$

Introducing a generating function for $g(k,t), G_g(z,t) = \sum_k z^k g(k,t)$, one can express the number of exiting particles as

$$m(t) = \frac{\partial G_g}{\partial z}(1,t) + \frac{\partial G_h}{\partial z}(1,t).$$ (79)

By combining the differential equations of $g(k,t)$ and $h(k,t)$, one obtains

$$\frac{\partial(G_g(z,t) + G_h(z,t))}{\partial t} = (z-1)\lambda e^{-\lambda}q_s(0,t-1) + (z-1)\lambda e^{-2\lambda}(G(z,t-2) - q_s(0,t-2)),$$ (80)

where $G(z,t)$ is the generating function of $q_s(k,t)$. By taking the Laplace transform of Eq.(80), one has

$$u(\tilde{G}_g(z,u) + \tilde{G}_h(z,u)) - 1 = (z-1)\frac{\lambda e^{-(\lambda+u)}}{\lambda+u} + (z-1)\lambda e^{-2(\lambda+u)}(\tilde{G}(z,u) - \frac{1}{\lambda+u}).$$ (81)

Taking the partial derivative of Eq.(81) with respect to $z$ and using Eq.(79), one obtains

$$\tilde{m}(u) = \frac{1}{u}\left(\frac{\lambda}{\lambda+u}e^{-(\lambda+u)}(1 - e^{-(\lambda+u)}) + \lambda e^{-2(\lambda+u)}\tilde{G}(1,u)\right).$$ (82)

Let us recall that (see Ref. [12])

$$\tilde{G}(1,u) = \frac{1}{\lambda+u}\left[1 + \frac{\lambda}{\lambda+u-\lambda e^{-(\lambda+u)}}\right].$$ (83)

Finally, inserting Eq. (83) in Eq. (82), the Laplace transform of $m(t)$ is given by

$$\tilde{m}(u) = \frac{\lambda}{u}\frac{e^{-(\lambda+u)}}{\lambda+u-\lambda e^{-(\lambda+u)}}.$$ (84)

Note that Eq. (84) can be written as

$$\tilde{m}(u) = \frac{\tilde{j}(u)}{u}$$ (85)

and one now recovers that $m(t) = \int_1^t dt' j(t')$. One also checks that $m(\infty) = \lim_{u\to 0} u\tilde{m}(u) = \frac{e^{-\lambda}}{1-e^{-\lambda}}$ as expected.

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