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Join-the-Shortest Queue Diffusion Limit in Halfin-Whitt Regime: Tail Asymptotics and Scaling of Extrema

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Abstract

Consider a system of N parallel single-server queues with unit-exponential service time distribution and a single dispatcher where tasks arrive as a Poisson process of rate $\lambda(N)$. When a task arrives, the dispatcher assigns it to one of the servers according to the Join-the-Shortest Queue (JSQ) policy. Eschenfeldt and Gamarnik (2015) established that in the Halfin-Whitt regime where $(N - \lambda(N))/\sqrt{N} \rightarrow \beta > 0$ as $N \rightarrow \infty$, appropriately scaled occupancy measure of the system under the JSQ policy converges weakly on any finite time interval to a certain diffusion process as $N \rightarrow \infty$. Recently, it was further established by Braverman (2018) that the convergence result extends to the steady state as well, i.e., stationary occupancy measure of the system converges weakly to the steady state of the diffusion process as $N \rightarrow \infty$, proving the interchange of limits result.

In this paper we perform a detailed analysis of the steady state of the above diffusion process. Specifically, we establish precise tail-asymptotics of the stationary distribution and scaling of extrema of the process on large time-interval. Our results imply that the asymptotic steady-state scaled number of servers with queue length two or larger exhibits an Exponential tail, whereas that for the number of idle servers turns out to be Gaussian. From the methodological point of view, the diffusion process under consideration goes beyond the state-of-the-art techniques in the study of the steady-state of diffusion processes. Lack of any closed form expression for the steady state and intricate interdependency of the process dynamics on its local times make the analysis significantly challenging. We develop a technique involving the theory of regenerative processes that provides a tractable form for the stationary measure, and in conjunction with several sharp hitting time estimates, acts as a key vehicle in establishing the results. The technique and the intermediate results might be of independent interest, and can possibly be used in understanding the bulk behavior of the process.

Keywords and phrases: Join the shortest queue; diffusion limit; steady state analysis; local time; non-elliptic diffusion; Halfin-Whitt regime; regenerative processes.

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1 Introduction

For any $\beta > 0$, consider the following diffusion process

$$\begin{aligned} Q_1(t) &= Q_1(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-Q_1(s) + Q_2(s))ds - L(t), \\ Q_2(t) &= Q_2(0) + L(t) - \int_0^t Q_2(s)ds. \end{aligned} \tag{1.1}$$

for $t \geq 0$, where W is the standard Brownian motion, L is the unique nondecreasing nonnegative process in $D_{\mathbb{R}}[0, \infty)$ satisfying $\int_0^\infty \mathbb{1}_{[Q_1(t) < 0]} dL(t) = 0$, and $(Q_1(0), Q_2(0)) \in (-\infty, 0] \times [0, \infty)$. In this paper we establish tail asymptotics of the stationary distribution of the above diffusion process and identify the scaling behavior of $\inf_{0 \leq s \leq t} Q_1(s)$ and $\sup_{0 \leq s \leq t} Q_2(s)$ for large t . The diffusion process in (1.1) arises as the weak limit of the sequence of scaled occupancy measure of systems under the Join-the-Shortest Queue (JSQ) policy, as the system size (number of servers in the system) becomes large. Specifically, consider a system with N parallel identical single-server queues and a single dispatcher. Tasks with unit-mean exponential service requirements arrive at the dispatcher as a Poisson process of rate $\lambda(N)$, and are instantaneously forwarded to one of the servers with the shortest queue length (ties are broken arbitrarily). For $t \geq 0$, let

$$\mathbf{Q}^N(t) := (Q_1^N(t), Q_2^N(t), \dots)$$

denote the system occupancy measure, where $Q_i^N(t)$ is the number of servers under the JSQ policy with a queue length of i or larger, at time t , including the possible task in service, $i = 1, 2, \dots$. Now consider an asymptotic regime where the number of servers grows large, and additionally assume that

$$\frac{N - \lambda(N)}{\sqrt{N}} \rightarrow \beta \quad \text{as } N \rightarrow \infty$$

for some positive coefficient $\beta > 0$, i.e., the load per server $\lambda(N)/N$ approaches unity as $1 - \beta/\sqrt{N}$, with $\beta > 0$ some positive coefficient. In terms of the aggregate traffic load and total service capacity, this scaling corresponds to the so-called Halfin-Whitt heavy-traffic regime which was introduced in the seminal paper [11] and has been extensively studied since. The set-up in [11], as well as the numerous model extensions in the literature (see [7, 8, 9, 11, 21, 22, 23], and the references therein), predominantly concerned a setting with a single centralized queue and server pool (M/M/N), rather than a scenario with parallel queues. Eschenfeldt and Gamarnik [6] initialed the study of the scaling behavior for parallel-server systems in the Halfin-Whitt heavy-traffic regime. Define the centered and scaled system occupancy states as $\bar{\mathbf{Q}}^N(t) = (\bar{Q}_1^N(t), \bar{Q}_2^N(t), \dots)$, with

$$\bar{Q}_1^N(t) = -\frac{N - Q_1^N(t)}{\sqrt{N}}, \quad \bar{Q}_i^N(t) = \frac{Q_i^N(t)}{\sqrt{N}}, \quad i = 2, 3, \dots$$

The reason why $Q_1^N(t)$ is centered around N while $Q_i^N(t)$, $i = 2, \dots$, are not, is because the fraction of servers at time t with a queue length of exactly one tends to 1, whereas the fraction of servers with a queue length of two or more tends to zero as $N \rightarrow \infty$. For each fixed N , $\bar{\mathbf{Q}}^N$ is a positive recurrent continuous time Markov chain, and there exists a stationary distribution for $\bar{\mathbf{Q}}^N(t)$ as $t \rightarrow \infty$. Denote by $\bar{\mathbf{Q}}^N(\infty)$ a random variable distributed as the steady

state of the process $\bar{Q}^N(t)$. Assuming $(\bar{Q}_i^N(0))_{i \geq 1} \xrightarrow{\mathcal{L}} (Q_i(0))_{i \geq 1}$ with $Q_i(0) = 0$ for $i \geq 3$, it was shown by Eschenfeldt and Gamarnik [6] that on any finite time interval $[0, T]$, the sequence of processes $\{(\bar{Q}_1^N(t), \bar{Q}_2^N(t), \dots)\}_{0 \leq t \leq T}$ converges weakly to the limit $\{(Q_1(t), Q_2(t), \dots)\}_{0 \leq t \leq T}$, where (Q_1, Q_2) is given by (1.1) and $Q_i(\cdot) \equiv 0$ for $i \geq 3$. Subsequently, a broad class of other schemes were shown to exhibit the same scaling behavior in this regime [14, 15, 16]. See [20] for a recent survey. In all these above works, the convergence of scaled occupancy measure was established in the transient regime on any finite time interval. The tightness of diffusion-scaled occupancy measure and the interchange of limits were open until recently, when Braverman [3] further established that the weak convergence result extends to the steady state as well, i.e., $\bar{Q}^N(\infty)$ converges weakly to $(Q_1(\infty), Q_2(\infty), 0, 0, \dots)$ as $N \rightarrow \infty$, where $(Q_1(\infty), Q_2(\infty))$ is distributed as the stationary distribution of the process (Q_1, Q_2) . Thus, the steady state of the diffusion process in (1.1) captures the asymptotic behaviors of large-scale systems under the JSQ policy.

The steady-state of the diffusion process in (1.1) is technically hard to analyze. In fact, even establishing its ergodicity is non-trivial. The standard method employed in studying steady-state behavior of diffusions [1, 4, 5, 10] is to construct a suitable Lyapunov function which shows that the diffusion has a strong drift towards a compact set. Inside the compact set, some irreducibility condition, like uniform ellipticity (as in [1, 4, 5]) or hypoellipticity (as in [10]), is used to show positive recurrence, and consequently, existence and uniqueness of the stationary distribution and ergodicity of the diffusion process. The construction of the Lyapunov function usually involves establishing stability of the associated noiseless dynamical system and having tractable bounds on hitting times for this deterministic system. In our setup, even the noiseless system requires non-trivial analysis (see Section 4.1 of [3]). In [3] a Lyapunov function is obtained via a generator expansion framework using the Stein's method that establishes exponential ergodicity of (Q_1, Q_2) . Although this approach gives a good handle on the rate of convergence to stationarity, the non-trivial dynamics of the noiseless system results in a complicated form for the Lyapunov function which sheds little light on the form of the stationary distribution itself. Moreover, the diffusion in (1.1) (without the reflection term) is not hypoelliptic and this complicates things even further. It is also worth pointing out here that we obtain different tail behavior for Q_1 and Q_2 (Gaussian and Exponential, respectively) and get explicit dependence of β in the exponents, which is hard to obtain using the Lyapunov function methods known in the literature.

This demands a fundamentally different characterization of the stationary distribution. For that we take resort to the theory of regenerative processes (see Chapter 10 of [19]) to obtain a tractable representation of the steady state. A variant of this method was first used in [2] to study a diffusion process with inert drift, although the stationary distribution in that case had an explicit product form that facilitated the analysis, as opposed to the current scenario. First, we show that the diffusion $\{(Q_1(t), Q_2(t))\}_{t \geq 0}$ can be decomposed into i.i.d. renewal cycles between carefully constructed regeneration times having good moment bounds. This decomposition gives an alternative, more transparent proof of ergodicity, and also shows that the diffusion falls in the category of classical regenerative processes. Loosely speaking, regeneration times are random times when the process *starts afresh*, and the theory of classical regenerative processes can be used to conclude that the stationary behavior of a process is same as the behavior within one renewal cycle (i.e., between two successive regeneration times). The regenerative process representation

enables us to obtain a form for the stationary distribution that is amenable to analysis (see Theorem 3.3). Tail estimates for the stationary measure are then obtained by analyzing this form and are presented in Theorem 2.1. Moreover, in Theorem 2.2, we obtain precise almost sure scaling behavior of the extrema of the process sample paths.

The regenerative structure of the diffusion process and the intermediate results might be of independent interest. In fact, they might also be used to provide detailed result about the behavior of the stationary measure near the center (bulk behavior) and produce sharp estimates on the stationary mean of Q_2 .

Rest of the article is arranged as follows. In Section 2, we describe the two main results of this paper. In the Section 3, we establish (Q_1, Q_2) as a classical regenerative process and state several crucial hitting time estimates that are required to prove the main results. In Section 4, we obtain a tail estimate for the regeneration time which, in particular, implies that it has a finite first moment. This, in turn, implies the ergodicity of the diffusion process and gives a tractable form for the stationary distribution. In Section 5, we obtain fluctuation estimates of the paths of Q_1 and Q_2 between two successive regeneration times, which are used in the proofs of Theorems 2.1 and 2.2. In Section 6, we combine the results in Sections 3, 4 and 5 to prove Theorems 2.1 and 2.2.

2 Main results

In this section we will state the main results, and discuss their ramifications. Recall the diffusion process $\{(Q_1(t), Q_2(t))\}_{t \geq 0}$ as defined by Equation (1.1). As mentioned in the introduction, it is known [3] that for any $\beta > 0$, (Q_1, Q_2) is an ergodic continuous-time Markov process. Let $(Q_1(\infty), Q_2(\infty))$ denote a random variable distributed as the unique stationary distribution of the process. Then the next theorem gives a precise characterization of the tail of the stationary distribution.

Theorem 2.1. *For any $\beta > 0$ there exist positive constants C_1, C_2, D_1, D_2 not depending on β and positive constants $C^l(\beta), C^u(\beta), D^l(\beta), D^u(\beta), C_R(\beta), D_R(\beta)$ depending only on β such that*

$$\begin{aligned} C^l(\beta)e^{-C_1x^2} &\leq \pi(Q_1(\infty) < -x) \leq C^u(\beta)e^{-C_2x^2}, \quad x \geq C_R(\beta) \\ D^l(\beta)e^{-D_1\beta y} &\leq \pi(Q_2(\infty) > y) \leq D^u(\beta)e^{-D_2\beta y}, \quad y \geq D_R(\beta). \end{aligned} \quad (2.1)$$

The dependence on β of the tail-exponents is precisely captured in the above theorem. Note that $Q_1(\infty)$ has a Gaussian tail, and the tail exponent is uniformly bounded by constants which do not depend on β , whereas $Q_2(\infty)$ has an exponentially decaying tail, and the coefficient in the exponent is linear in β .

Remark 1. Let us now discuss a further implication of Theorem 2.1. Recall that $Q_i^N(t)$ denotes the number of servers in the N -th system with queue length i or larger at time t . Let $S^N(t) := \sum_{i \geq 1} Q_i^N(t)$ denote the total number of tasks in the system. Then [3, Theorem 5] implies that $(S^N(\infty) - N)/\sqrt{N}$ converges weakly to $S(\infty) \stackrel{d}{=} Q_1(\infty) + Q_2(\infty)$. In that case, Theorem 2.1 implies that $S(\infty)$ has an Exponential upper tail (large positive deviation) and a Gaussian lower tail (large negative deviation).

Remark 2. It is worth mentioning that in case of M/M/N systems in the Halfin-Whitt heavy-traffic regime [11, Theorem 2], the centered and scaled total number of tasks in the system $(\bar{S}^N(t) - N)/\sqrt{N}$ converges weakly to a diffusion process $\{\bar{S}(t)\}_{t \geq 0}$ having the infinitesimal generator $A = (\sigma^2(x)/2)(d^2/dx^2) + m(x)(d/dx)$ with

$$m(x) = \begin{cases} -\beta & \text{if } x > 0 \\ -(x + \beta) & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad \sigma^2(x) = 2.$$

Note that since this is a simple combination of a Brownian motion with a negative drift (when all servers are fully occupied) and an Ornstein Uhlenbeck process (when there are idle servers), the steady-state distribution $\bar{S}(\infty)$ can be *computed explicitly*, and is a combination of Exponential (from the Brownian motion with a negative drift) and Gaussian (from the OU process). Although in terms of tail asymptotics, $\bar{S}(\infty)$ and $S(\infty)$ in Remark 1 behave somewhat similarly, there are some fundamental differences between the two processes, that not only make the analysis of the JSQ policy much harder, but also lead to several completely different qualitative behavior.

- (i) Observe that in case of M/M/N, whenever there are some waiting tasks (equivalent to Q_2 being positive in our case), the queue length has a constant negative drift towards zero. This leads to the Exponential upper tail of $\bar{S}(\infty)$, by comparing with the stationary distribution of reflected Brownian motion with constant negative drift. In our case, the rate of decrease of Q_2 is always proportional to itself, which makes it somewhat counter-intuitive that its stationary distribution has an Exponential tail.
- (ii) Further, from (1.1), Q_2 *never* hits zero. Thus, in the steady state, there is no mass at $Q_2 = 0$, and the system always has waiting tasks. This is in sharp contrast with the M/M/N case, where with positive probability the steady-state system has no waiting task.
- (iii) In the M/M/N setup, given that a task faces a non-zero wait, *the steady-state waiting time is of order $1/\sqrt{N}$* unlike in our case, where it is of *constant order* (the time till the service of the task ahead of it in its queue finishes). Moreover, in the current scenario, it is easy to see that Q_1 (the limit of the scaled number of idle servers) spends zero time at the origin, i.e., in steady state the fraction of arriving tasks that find all servers busy vanishes in the large- N limit. Consequently, JSQ achieves an *asymptotically vanishing steady-state probability of non-zero wait* (in fact, this is of order $1/\sqrt{N}$, see [3]). This is another sharp contrast with the M/M/N case, where the *asymptotic steady-state probability of non-zero wait is strictly positive*.
- (iv) In the M/M/N setup, the number of idle servers can be non-zero only when the number of waiting tasks is zero. Thus, the dynamics of both the number of idle servers and the number of waiting tasks are completely captured by the one-dimensional process S^N and by the one-dimensional diffusion \bar{S} in the limit. But in our case, Q_2 is never zero, and the dynamics of (Q_1, Q_2) is truly two-dimensional (although the diffusion is non-elliptic) with Q_1 and Q_2 interacting with each other in an intricate manner.

The next theorem establishes scaling behavior of the extrema of the process $\{(Q_1(t), Q_2(t))\}_{t \geq 0}$ on large time intervals.

Theorem 2.2. *There exists a positive constant \mathcal{C}^* not depending on β such that the following hold almost surely along any sample path:*

$$\begin{aligned} -2\sqrt{2} &\leq \liminf_{t \rightarrow \infty} \frac{Q_1(t)}{\sqrt{\log t}} \leq -1, \\ \frac{1}{\beta} &\leq \limsup_{t \rightarrow \infty} \frac{Q_2(t)}{\log t} \leq \frac{2}{\mathcal{C}^* \beta}. \end{aligned}$$

Again, Theorem 2.2 captures the explicit dependence on β of the width of the fluctuation window of Q_1 and Q_2 . Specifically, note that the width of fluctuation of Q_1 does not depend on the value of β , whereas that of Q_2 is linear in β^{-1} .

Remark 3. Our proof of Theorem 2.1 provides explicit values of the constants $C_1, C_2, D_1, D_2, \mathcal{C}^*$. We are not explicit about them in the statements of the theorems since these estimates are not sharp in the constants.

3 Regenerative process view of the diffusion

As mentioned in the introduction, the key challenge in analyzing the steady state of the diffusion process in (1.1) stems from its lack of explicit characterization. In order to obtain sharp estimates for the stationary distribution we take resort to the theory of regenerative processes. Loosely speaking, a stochastic process is called *classical regenerative* if it starts anew at random times (called *regeneration times*), independent of the past. See [19, Chapter 10] for a rigorous treatment of regenerative processes. The regeneration times split the process into renewal cycles that are independent and identically distributed, possibly except the first cycle. Consequently, the behavior inside a specific renewal cycle characterizes the steady-state behavior.

In case of recurrent discrete state-space Markov chains regeneration times can be defined as hitting times of a fixed state. Although the diffusion process in (1.1) is two dimensional, we will show that it actually *exhibits point recurrence* and we can define regeneration times in terms of hitting times as follows.

First we introduce the following notations.

$$\tau_i(z) := \inf\{t \geq 0 : Q_i(t) = z\}, \quad i = 1, 2, \quad \text{and} \quad \sigma(t) := \inf\{s \geq t : Q_1(t) = 0\}.$$

We now define the renewal cycles as follows. Fix any $B > 0$. For $k \geq 0$, define the stopping times

$$\alpha_{2k+1} := \inf\{t \geq \alpha_{2k} : Q_2(t) = B\}, \quad \alpha_{2k+2} := \inf\{t > \alpha_{2k+1} : Q_2(t) = 2B\}, \quad \Xi_k := \alpha_{2k+2}, \quad (3.1)$$

with the convention that $\alpha_0 = 0$ and $\Xi_{-1} = 0$. The dependence of B in the above stopping times is suppressed for convenience in notation. Hereafter we will assume $B > 0$ to be fixed unless mentioned otherwise. The next lemma describes the diffusion process as an appropriate classical regenerative process.

Lemma 3.1. *The process $\{Q_1(t), Q_2(t)\}_{t \geq 0}$ is a classical regenerative process with regeneration times given by $\{\Xi_k\}_{k \geq 0}$.*

Proof. Note that it is enough to prove that $Q_1(\alpha_{2k}) = 0$ for all $k \geq 1$. Indeed, this ensures that for all $k \geq 0$, $(Q_1(\Xi_k), Q_2(\Xi_k)) = (0, 2B)$, and the Markov process naturally regenerates at time Ξ_k .

Fix any $k \geq 1$. Assume, if possible, $Q_1(\alpha_{2k}) < 0$. In that case, the path-continuity of Q_1 implies that the local time L is constant in a small neighborhood of α_{2k} . Consequently, Q_2 must be strictly decreasing in an open time interval containing α_{2k} . This contradicts the fact that α_{2k} is the hitting time of a level *from below* by the process Q_2 . \square

The above lemma implies that the regenerative cycles given by $\{(Q_1(t), Q_2(t))\}_{\Xi_k \leq t < \Xi_{k+1}}$ form an i.i.d. sequence for $k \geq 0$. The time-intervals $\{\Xi_{k+1} - \Xi_k\}_{k \geq 0}$ are called the *inter-regeneration times*. In order to characterize the steady-state distribution using regenerative approach, we first show that the initial *delay length* Ξ_0 (time to enter into the regenerative cycles starting from an arbitrary state) as well as inter-regeneration times have finite expectations. In fact, the next proposition establishes detailed tail asymptotics for the delay length Ξ_0 and thus, in particular, for the inter-regeneration times.

Proposition 3.2. *Let $(Q_1(0), Q_2(0)) = (x, y)$ with $x \leq 0, y > 0$. There exist constants $c_{\Xi}^{(1)}, c_{\Xi}^{(2)}, t_{\Xi} > 0$, possibly depending on x, y, B, β , such that for all $t \geq t_{\Xi}$,*

$$\mathbb{P}_{(x,y)}(\Xi_0 > t) \leq c_{\Xi}^{(1)} \exp(-c_{\Xi}^{(2)} t^{1/6}).$$

In particular, $\mathbb{E}_{(x,y)}\Xi_0 < \infty$.

Proposition 3.2 is proved in Section 4.

Proposition 3.2 yields the existence and uniqueness of the stationary distribution and ergodicity of the process as stated in Theorem 3.3 below. We note that the geometric ergodicity has already been proved in [3]. The principal importance of Theorem 3.3 lies in the fact that it provides an explicit form of the stationary measure which will be the key vehicle in the study of the tail asymptotics and the fluctuation window, as stated in Theorems 2.1 and 2.2.

Theorem 3.3. *Fix any $B > 0$. The process described by Equation (1.1) has a unique stationary distribution π which can be represented as*

$$\pi(A) = \frac{\mathbb{E}_{(0,2B)} \left(\int_0^{\Xi_0} \mathbb{1}_{[(Q_1(s), Q_2(s)) \in A]} ds \right)}{\mathbb{E}_{(0,2B)}(\Xi_0)}$$

for any measurable set $A \subseteq (-\infty, 0] \times (0, \infty)$. Moreover, the process is ergodic in the sense that for any measurable function f satisfying $\mathbb{E}_{(0,2B)} \left(\int_0^{\Xi_0} f((Q_1(s), Q_2(s))) ds \right) < \infty$,

$$\frac{1}{t} \int_0^t f((Q_1(s), Q_2(s))) ds \longrightarrow \int_{(-\infty, 0] \times (0, \infty)} f d\pi = \frac{\mathbb{E}_{(0,2B)} \left(\int_0^{\Xi_0} f((Q_1(s), Q_2(s))) ds \right)}{\mathbb{E}_{(0,2B)}(\Xi_0)} \quad (3.2)$$

almost surely as $t \rightarrow \infty$.

The above theorem follows using [19, Chapter 10, Theorem 2.1], details of which are deferred till Section 4.

Remark 4. We note that it can be shown by soft arguments involving Girsanov theorem and the theory of Lévy processes that the distribution of $\Xi_1 - \Xi_0$ has a density with respect to the Lebesgue measure, see the proof of Lemma 7.1 in [2]. This implies that the inter-regeneration time $\Xi_{k+1} - \Xi_k$ is *spread-out* (see Section 3.5 of Chapter 10 in [19]). Consequently, the total variation convergence of the diffusion process at time t to the stationary distribution as $t \rightarrow \infty$, can be obtained using Theorem 3.3 of Chapter 10 in [19]. However, we skip this argument, since geometric ergodicity has already been established in [3, Theorem 3].

In light of Theorem 3.3, observe that establishing tail asymptotics of the stationary distribution reduces to studying the amount of time spent by the diffusion in a certain region in one particular renewal cycle. The next theorem provides several important hitting time estimates that will play a crucial role in the proofs of Theorems 2.1 and 2.2. Define

$$l_0(\beta) := \max \left\{ \beta, \beta^{-1}, \frac{1}{\beta} \log \frac{1}{\beta} \right\}. \quad (3.3)$$

Theorem 3.4. *There exists a positive constant R_0 such that with $B = R_0 l_0(\beta)$ in (3.1), the following hold:*

(i) *There exist constants $C_1^*, C_2^* > 0$ that do not depend on β such that for all $y \geq 4B$,*

$$\mathbb{P}_{(0,2B)}(\tau_2(y) \leq \Xi_0) \leq C_1^* e^{-C_2^* \beta (y-\beta)/2}.$$

(ii) *For all $y \geq 2B$,*

$$\mathbb{P}_{(0,2B)}(\tau_2(y) \leq \Xi_0) \geq (1 - e^{-\beta R_0 l_0(\beta)}) e^{-\beta (y-2R_0 l_0(\beta))} e^{-(x-2\beta)^2/8}.$$

(iii) *There exists a constant $C^*(\beta) > 0$ depending on β such that for any $x \geq 18B$,*

$$\mathbb{P}_{(0,2B)}(\tau_1(-x) \leq \Xi_0) \leq C^*(\beta) e^{-(x-2\beta)^2/8}.$$

(iv) *There exists a constant $C^{**}(\beta) > 0$ depending on β such that for any $x \geq \beta$,*

$$\mathbb{P}_{(0,2B)} \left(\inf_{t \leq \Xi_0} Q_1(t) < -x \right) \geq C^{**}(\beta) e^{-x^2}.$$

Theorem 3.4 is proved in Section 5 where we analyze the behavior of the process (Q_1, Q_2) between two successive regeneration times. Results in Theorem 3.4 in conjunction with Proposition 3.2 and Theorem 3.3 are used to prove Theorems 2.1 and 2.2, which is presented in Section 6.

4 Analysis of regeneration times

In this section we will prove Proposition 3.2 and Theorem 3.3. The proof of Proposition 3.2 consists of several steps. The first step is to analyze the down-crossings of Q_2 , where we establish various hitting time estimates in the time-interval $[\alpha_{2k}, \alpha_{2k+1}]$, $k \geq 0$. In particular, we prove the following lemma.

Lemma 4.1. *Fix $(Q_1(0), Q_2(0)) = (x, y)$ with $x \leq 0, y > 0$. There exist $c_{\alpha_1}, c'_{\alpha_1}, t_{\alpha_1} > 0$ possibly depending on (x, y) , B , and β , such that for all $t \geq t_{\alpha_1}$,*

$$\mathbb{P}_{(x,y)}(\alpha_1 > t) \leq c'_{\alpha_1} \exp(-c_{\alpha_1} t^{1/6}).$$

As before, note that setting $(x, y) = (0, 2B)$ furnishes the corresponding probabilities when α_1 is replaced by $\alpha_{2k+1} - \alpha_{2k}$. Lemma 4.1 is proved in Subsection 4.1. Next we consider the up-crossings of Q_2 , where we establish various hitting time estimates in the time-interval $[\alpha_{2k+1}, \alpha_{2k+2}]$, $k \geq 0$. Specifically, we establish the following.

Lemma 4.2. *Fix $(Q_1(0), Q_2(0)) = (x, y)$ with $x \leq 0, y > 0$. There exist $c_{\alpha_2}, c'_{\alpha_2}, t_{\alpha_2} > 0$ possibly depending on (x, y) , B , and β , such that for all $t \geq t_{\alpha_2}$,*

$$\mathbb{P}_{(x,y)}(\alpha_2 - \alpha_1 > t) \leq c'_{\alpha_2} \exp(-c_{\alpha_2} t^{1/6}).$$

Lemma 4.2 is proved in Subsection 4.2. Now observe that Lemmas 4.1 and 4.2 together complete the proof of Proposition 3.2. \square

Proof of Theorem 3.3. Due to Proposition 3.2, the fact that π defined in the theorem is stationary follows from [19, Chapter 10, Theorem 2.1]. Now, we will prove the ergodicity result (3.2) which will also yield uniqueness. Take any starting point (x, y) with $x \leq 0$ and $y > 0$ and recall $\Xi_{-1} = 0$. Take any measurable function f satisfying $\mathbb{E}_{(0,2B)} \left(\int_0^{\Xi_0} f((Q_1(s), Q_2(s))) ds \right) < \infty$. Let $N_t = \sup\{k \geq -1 : \Xi_k \leq t\}$. Assume without loss of generality that f is non-negative (for general f , consider the positive and negative parts of f separately). We can write

$$\begin{aligned} \int_0^{\Xi_0 \wedge t} f((Q_1(s), Q_2(s))) ds + \mathbb{1}_{[\Xi_1 \leq t]} \sum_{k=1}^{N_t} \int_{\Xi_{k-1}}^{\Xi_k} f((Q_1(s), Q_2(s))) ds &\leq \int_0^t f((Q_1(s), Q_2(s))) ds \\ &\leq \int_0^{\Xi_0} f((Q_1(s), Q_2(s))) ds + \sum_{k=1}^{N_t+1} \int_{\Xi_{k-1}}^{\Xi_k} f((Q_1(s), Q_2(s))) ds. \end{aligned}$$

Clearly, $t^{-1} \int_0^{\Xi_0} f((Q_1(s), Q_2(s))) ds \rightarrow 0$ as $t \rightarrow \infty$. By Proposition 7.3 of [18],

$$t^{-1} \sum_{k=1}^{N_t} \int_{\Xi_k}^{\Xi_{k+1}} f((Q_1(s), Q_2(s))) ds \rightarrow \frac{\mathbb{E}_{(0,2B)} \left(\int_0^{\Xi_0} f((Q_1(s), Q_2(s))) ds \right)}{\mathbb{E}_{(0,2B)}(\Xi_0)}$$

and

$$t^{-1} \sum_{k=1}^{N_t+1} \int_{\Xi_k}^{\Xi_{k+1}} f((Q_1(s), Q_2(s))) ds \rightarrow \frac{\mathbb{E}_{(0,2B)} \left(\int_0^{\Xi_0} f((Q_1(s), Q_2(s))) ds \right)}{\mathbb{E}_{(0,2B)}(\Xi_0)}$$

almost surely as $t \rightarrow \infty$. This proves (3.2), and consequently uniqueness of the stationary distribution. \square

4.1 Down-crossings of Q_2 and tightness estimates

In this subsection, we will prove tail-asymptotics for the distribution of α_1 as stated in Lemma 4.1. This will require a crucial tightness estimate for the process Q_2 , which is given in Lemma 4.7 below. Loosely speaking, we need to have sharp estimates for the time Q_2 takes to hit the level B starting from a large initial state. This, in turn, amounts to estimating the time integral of the Q_1 process when Q_2 is large, which is furnished by Lemma 4.6. The tail estimates presented in Lemmas 4.3 and 4.5 will be used in the proof of Lemma 4.6.

Fix any $M > 0$ and $\varepsilon > 0$. Observe that if $\inf_{0 \leq s \leq t} Q_2(s) > M + \beta$, then the process $\{Q_1(s)\}_{0 \leq s \leq t}$ is bounded below by the process $\{\eta(s)\}_{0 \leq s \leq t}$, where

$$\eta(t) = Q_1(0) + \sqrt{2}W(t) + Mt - L_\eta(t),$$

with L_η being the local time of η given by $L_\eta(t) = \sup_{s \leq t} \{Q_1(0) + \sqrt{2}W(s) + Ms\}$, and W being the standard Brownian motion. Note that the dependence of M in η is suppressed for convenience in notation. For $i \geq 1$ define

$$\begin{aligned} T_{2i-1} &:= \inf \{t > T_{2i-2} : \eta(t) = -\varepsilon\}, & T_{2i} &:= \inf \{t > T_{2i-1} : \eta(t) = -\varepsilon/2\}, \\ \xi_i &:= T_{2i} - T_{2i-1}, & \zeta_i &:= T_{2i+1} - T_{2i}, & u_i &:= \sup_{T_{2i-1} \leq t \leq T_{2i}} (-\eta(t)), \\ N_t &:= \inf \{n \geq 1 : T_{2n} \geq t\}. \end{aligned}$$

with the convention that $T_0 \equiv 0$. Further, for $i \geq 1$, let T_i^W denote the corresponding stopping times when the process η is replaced by the process W_R described as

$$W_R(t) = Q_1(0) + \sqrt{2}W(t) - L_W(t)$$

with L_W being the local time of W_R given by $L_W(t) = \sup_{s \leq t} \{Q_1(0) + \sqrt{2}W(s)\}$. Also, similarly denote $\xi_i^W := T_{2i}^W - T_{2i-1}^W$ and $\zeta_i^W := T_{2i+1}^W - T_{2i}^W$.

Lemma 4.3. *Assume that $Q_1(0) \in [-\varepsilon, 0]$. Then the following hold:*

(i) For $i \geq 1$, $\zeta_i^W \leq_{st} \zeta_i$.

(ii) There exist constants $c_\zeta, c_W > 0$ not depending on M, ε such that for $t \geq \varepsilon^2$

$$(a) \mathbb{P}(\zeta_1 > t) \geq \exp(-c_\zeta t / \varepsilon^2) \quad \text{and} \quad (b) \mathbb{P}(\zeta_1^W > t) \leq \exp(-c_W t / \varepsilon^2). \quad (4.1)$$

(iii) For all $x \geq \varepsilon$, $\mathbb{P}(u_1 > x) \leq \exp(-M(x - \varepsilon))$,

(iv) For all $t \geq \varepsilon/M$, $\mathbb{P}(\xi_1 > t) \leq \frac{2}{\sqrt{\pi M \sqrt{t}}} \exp(-M^2 t / 16)$.

(v) There exist constants $b, c_N^{(1)} > 0$ not depending on M, ε , such that for $t \geq \varepsilon^2/b$

$$\mathbb{P}(N_t > b\varepsilon^{-2}t) \leq 2 \exp(-c_N^{(1)} t / \varepsilon^2).$$

Proof. (i) This is an immediate consequence of the fact that $\{\eta(s)\}_{0 \leq s \leq t} \geq_{st} \{W_R(s)\}_{0 \leq s \leq t}$.

(ii) Take $\varepsilon = 1$. Using the Markov property for reflected Brownian motion, it is easy to see that there exist constants $c_\zeta, c_W > 0$ such that $\exp(-c_W t) \geq \mathbb{P}(\zeta_1^W > t) \geq \exp(-c_\zeta t)$ for $t \geq 1$. (ii.a) now follows from (i) and Brownian scaling. (ii.b) is also an immediate consequence of Brownian scaling.

(iii) Observe that

$$\mathbb{P}(u_1 > x) \leq \mathbb{P}\left(\inf_{s < \infty} (-\varepsilon + \sqrt{2}W(s) + Ms) < -x\right) = \exp(-M(x - \varepsilon)),$$

since $-\inf_{s < \infty} (\sqrt{2}W(s) + Ms)$ follows an exponential random variable with mean $1/M$.

(iv) Note that

$$\begin{aligned} \mathbb{P}(\xi_1 > t) &= \mathbb{P}\left(\sup_{s \leq t} (-\varepsilon + \sqrt{2}W(s) + Ms) \leq -\varepsilon/2\right) \\ &\leq \mathbb{P}\left(\sqrt{2}W(t) + Mt \leq \varepsilon/2\right) \leq \frac{2}{\sqrt{\pi}M\sqrt{t}} \exp(-M^2t/16) \quad \forall t \geq \varepsilon/M. \end{aligned}$$

(v) Observe that

$$\begin{aligned} \mathbb{P}(N_t > b\varepsilon^{-2}t) &\leq \mathbb{P}\left(\sum_{i=1}^{\lfloor b\varepsilon^{-2}t \rfloor} \zeta_i \leq t\right) \leq \mathbb{P}\left(\sum_{i=1}^{\lfloor b\varepsilon^{-2}t \rfloor} \zeta_i^W \leq t\right), \text{ by part (i),} \\ &\leq \mathbb{P}\left(\sum_{i=1}^{\lfloor b\varepsilon^{-2}t \rfloor} \varepsilon^{-2}(\zeta_i^W - \mathbb{E}(\zeta_i^W)) \leq -\left(\frac{b}{2}\varepsilon^{-2}\mathbb{E}\zeta_1^W - 1\right)t\varepsilon^{-2}\right) \\ &\leq 2 \exp(-c_N^{(1)}t/\varepsilon^2) \quad [\text{choosing } b = 4\varepsilon^2/\mathbb{E}(\zeta_1^W)], \end{aligned}$$

where the last step follows from part (ii), which shows that $\varepsilon^{-2}(\zeta_i^W - \mathbb{E}(\zeta_i^W))$ are sub-exponential random variables, and then using the Chernoff's inequality (see [13, Pg. 16, Equation (2.2)]) to the sum $\sum_{i=1}^{\lfloor b\varepsilon^{-2}t \rfloor} \varepsilon^{-2}(\zeta_i^W - \mathbb{E}(\zeta_i^W))$. Here, note that by Brownian scaling, b chosen above does not depend on ε . \square

The next technical lemma establishes a useful concentration inequality that will be crucial in obtaining tail probabilities for $\sum_{i=1}^{N_t} u_i \xi_i$.

Lemma 4.4. Fix $\varepsilon > 0$ and $M \geq \frac{1}{\varepsilon}$. Let Φ_i 's be iid nonnegative random variables with

$$\mathbb{P}(\Phi_1 > z) \leq \exp(-c'M^{3/2}\sqrt{z}) \quad \text{for all } z \geq 4\varepsilon^2/M,$$

and $\mathbb{E}(\Phi_1) \leq c_{11}\varepsilon^2/M$ where c', c_{11} are positive constants not depending on M, ε . Then

$$\mathbb{P}\left(\sum_{i=1}^n \Phi_i \geq 4c_{11}n\frac{\varepsilon^2}{M}\right) \leq \left(1 + c_1\frac{1}{n^{2/5}(\varepsilon M)^{8/5}}\right) \exp\left(-c_2(\varepsilon M)^{4/5}n^{1/5}\right),$$

for $n \geq c_3\varepsilon M$, where c_1, c_2, c_3 are positive constants not depending on M, ε .

Proof. For some $A \geq 4\varepsilon^2/M$ to be chosen later, define

$$\Phi_i^* := \Phi_i \mathbb{1}_{[\Phi_i \geq A]} \quad \text{and} \quad \Phi_i^{**} := \Phi_i \mathbb{1}_{[\Phi_i < A]}.$$

Thus, $\Phi_i = \Phi_i^* + \Phi_i^{**}$. Note that

$$\begin{aligned} \mathbb{E}(\Phi_i^*)^2 &= \int_{A^2}^{\infty} \mathbb{P}(\Phi_i > \sqrt{z}) dz = \int_A^{\infty} 2z\mathbb{P}(\Phi_i > z) dz \\ &\leq \int_A^{\infty} 2z \exp(-c'M^{3/2}\sqrt{z}) dz = \int_{\sqrt{A}}^{\infty} 4z^3 \exp(-c'M^{3/2}z) dz \\ &= \frac{4}{M^6} \int_{M^{3/2}\sqrt{A}}^{\infty} z^3 \exp(-c'z) dz \leq c'' \frac{A^{3/2}}{M^{3/2}} \exp(-c'M^{3/2}\sqrt{A}), \end{aligned}$$

where the constant c'' does not depend on M, A . Thus, using Chebyshev's inequality

$$\mathbb{P} \left(\sum_{i=1}^n \Phi_i^* > 2c_{11}n \frac{\varepsilon^2}{M} \right) \leq \frac{c''M^{1/2}A^{3/2} \exp(-c'M^{3/2}\sqrt{A})}{4nc_{11}^2\varepsilon^4}. \quad (4.2)$$

Further note that Φ_i^{**} 's are bounded random variables. Therefore using Azuma-Hoeffding inequality we obtain,

$$\begin{aligned} \mathbb{P} \left(\sum_{i=1}^n \Phi_i^{**} > 2c_{11}n \frac{\varepsilon^2}{M} \right) &= \mathbb{P} \left(\sum_{i=1}^n (\Phi_i^{**} - \mathbb{E}(\Phi_i^{**})) > c_{11}n \frac{\varepsilon^2}{M} \right) \\ &\leq \exp \left(- \left(\frac{c_{11}n\varepsilon^2}{M} \right)^2 / (8A^2n) \right) = \exp(-c_{11}^2n\varepsilon^4 / (8A^2M^2)) \end{aligned} \quad (4.3)$$

Equating the exponents of equations (4.2) and (4.3), and solving for A , we get

$$A = \left(\frac{c_{11}^2}{8c'} \right)^{2/5} \left(\frac{\varepsilon^{8/5}n^{2/5}}{M^{7/5}} \right).$$

The condition $A \geq 4\varepsilon^2/M$ implies $n \geq 2^5 \left(\frac{8c'}{c_{11}^2} \right) \varepsilon M$. This choice for A yields the bound claimed in the lemma. \square

Lemma 4.5. Fix any $\varepsilon > 0$ and $M \geq \frac{1}{\varepsilon}$.

(i) There exist positive constants c', c_{11} not depending on M, ε , such that

$$\begin{aligned} \text{(a)} \quad &\mathbb{P}(\mathbf{u}_1 \xi_1 > x) \leq \exp(-c'M^{3/2}\sqrt{x}) \quad \forall x \geq 4\varepsilon^2/M, \\ \text{(b)} \quad &\mathbb{E}(\mathbf{u}_1 \xi_1) \leq c_{11} \frac{\varepsilon^2}{M}. \end{aligned}$$

(ii) Let b, c_{11} be the constants in Lemma 4.3 (v) and Lemma 4.5 (i) respectively. There exist constants c_1, c_2, c_3 not depending on ε, M such that

$$\mathbb{P} \left(\sum_{i=1}^{N_t} \mathbf{u}_i \xi_i > 4 \frac{bc_{11}t}{M} \right) \leq c_1 \exp(-c_2(\varepsilon M)^{4/5}(t/\varepsilon^2)^{1/5})$$

for $t \geq c_3\varepsilon^3M$.

Proof. (i.a) Recall that $M \geq \frac{1}{\varepsilon}$. By Lemma 4.3 (iii), we obtain for $x \geq 4\varepsilon^2/M$,

$$\begin{aligned} \mathbb{P}(\mathbf{u}_1 \xi_1 > x) &\leq \mathbb{P}(\mathbf{u}_1 > \sqrt{Mx}) + \mathbb{P}(\mathbf{u}_1 \xi_1 > x, \mathbf{u}_1 \leq \sqrt{Mx}) \\ &\leq \mathbb{P}(\mathbf{u}_1 > \sqrt{Mx}) + \mathbb{P}\left(\xi_1 > \frac{\sqrt{x}}{\sqrt{M}}\right) \\ &\leq \exp(-M(\sqrt{Mx} - \varepsilon)) + \frac{2}{\sqrt{\pi}M^{3/4}x^{1/4}} \exp(-M^{3/2}\sqrt{x}/16) \\ &\leq \exp(-M^{3/2}\sqrt{x}/2) + \frac{2}{\sqrt{\pi}M^{3/4}x^{1/4}} \exp(-M^{3/2}\sqrt{x}/16) \\ &\leq \exp(-c'M^{3/2}\sqrt{x}), \end{aligned}$$

where the last line is a consequence of the fact that for $x \geq 4\varepsilon^2/M$ and $M \geq \frac{1}{\varepsilon}$, $M^{3/4}x^{1/4} \geq \sqrt{2M\varepsilon} > 1$.

(i.b) As a consequence of part (i.a) we obtain

$$\begin{aligned} \mathbb{E}(u_1 \xi_1) &\leq \int_0^{4\varepsilon^2/M} dx + \frac{1}{M^3} \int_{4\varepsilon^2/M}^{\infty} \exp(-c'M^{3/2}\sqrt{x})M^3 dx \\ &\leq \frac{4\varepsilon^2}{M} + \frac{c'''}{M^3} \leq c_{11} \frac{\varepsilon^2}{M}, \end{aligned}$$

where we again used $M \geq \frac{1}{\varepsilon}$ to obtain $\frac{1}{M^3} \leq \frac{\varepsilon^2}{M}$.

(ii) Observe that due to Lemma 4.3 (v) and Lemma 4.4,

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{N_t} u_i \xi_i > 4\frac{bc_{11}t}{M}\right) &\leq \mathbb{P}(N_t > b\varepsilon^{-2}t) + \mathbb{P}\left(\sum_{i=1}^{\lfloor b\varepsilon^{-2}t \rfloor} u_i \xi_i > 4\frac{bc_{11}t}{M}\right) \\ &\leq \exp(-c_N^{(1)}t/\varepsilon^2) + C_1 \exp(-C_2(\varepsilon M)^{4/5}(t/\varepsilon^2)^{1/5}) \end{aligned}$$

for $t \geq C_3\varepsilon^3M$, where C_1, C_2, C_3 can be chosen to be independent of M, ε . This completes the proof. \square

We are now in a position to state and prove Lemma 4.6 that provides us with a crucial estimate for the time-integral of the Q_1 process when Q_2 is large.

Lemma 4.6. *There exist $c'_1, c'_2, c'_3 > 0$, not depending on β such that for any $y > c'_1(\beta \vee \beta^{-1}) + \beta$,*

$$\begin{aligned} \mathbb{P}_{(0,y)}\left(\int_0^t (-Q_1(s))ds > (\beta \wedge \beta^{-1}) \frac{t}{2}, \inf_{s \leq t} Q_2(s) \geq c'_1(\beta \vee \beta^{-1}) + \beta\right) \\ \leq \exp\left(-c'_2 t^{1/5} (\beta \vee \beta^{-1})^{2/5}\right) \quad \text{for } t \geq c'_3 (\beta \wedge \beta^{-1})^2. \end{aligned}$$

Proof. Recall the constants b and c_{11} from Lemma 4.3 (v) and Lemma 4.5 (i) respectively. As c_{11} appears in the upper bound of $\mathbb{E}(u_1 \xi_1)$ in Lemma 4.5 (i), we can take $c_{11} > b^{-1} \vee 1$. First we consider the case $\beta \in (0, 1)$. Take $\varepsilon = \beta/4$. Choose $M = 16c_{11}b/\beta$, since in that case

$$\varepsilon = \frac{\beta}{4} = \frac{4c_{11}b}{M}.$$

Observe that

$$\begin{aligned} \mathbb{P}_{(0,y)}\left(\int_0^t (-Q_1(s))ds > \frac{\beta t}{2}, \inf_{s \leq t} Q_2(s) \geq M + \beta\right) \\ \leq \mathbb{P}_{(0,y)}\left(\sum_{i=1}^{N_t} \int_{T_{2i-1}}^{T_{2i}} (-Q_1(s))ds > \frac{4c_{11}b}{M}t, \inf_{s \leq t} Q_2(s) \geq M + \beta\right) \\ \leq \mathbb{P}\left(\sum_{i=1}^{N_t} u_i \xi_i > \frac{4c_{11}b}{M}t\right) \\ \leq \exp\left(-c''_2(\beta M)^{4/5}(t/\beta^2)^{1/5}\right) \\ \leq \exp\left(-c'_2(t/\beta^2)^{1/5}\right) \quad \text{for } t \geq c''_3 \beta^3 M = c'_3 \beta^2, \quad \text{due to Lemma 4.5 (ii),} \end{aligned}$$

where the constants c'_2, c''_2, c'_3, c''_3 do not depend on β, M . Next, for the case $\beta > 1$, we take $\varepsilon = \frac{1}{4\beta}$ and $M = 16c_{11}\beta$ so that

$$\varepsilon = \frac{1}{4\beta} = \frac{4c_{11}b}{M},$$

and then apply the same argument. This completes the proof. \square

Lemma 4.7. *There exist positive constants c'_1, c'_2, c'_3, c'_4 not depending on β such that the following hold:*

(i) *For $\beta \geq 1$ and any $y \geq 1$, for all $t \geq c'_4 y / \beta$*

$$\mathbb{P}_{(0, y+c'_1\beta)} \left(\inf_{s \leq t} Q_2(s) > c'_1\beta \right) \leq c'_3 \exp(-c'_2\beta^{2/5}t^{1/5}).$$

(ii) *For $\beta \in (0, 1)$ and any $y \geq 1$, for all $t \geq c'_4(y\beta^{-1} \vee \beta^{-2})$*

$$\mathbb{P}_{(0, y+c'_1\beta^{-1})} \left(\inf_{s \leq t} Q_2(s) > \frac{c'_1}{\beta} \right) \leq c'_3 \left(\exp(-c'_2\beta^{-2/5}t^{1/5}) + \exp(-c'_2\beta^2t) + \beta^{-2} \exp(-c'_2t) \right).$$

Proof. Let us denote the following events

$$\begin{aligned} \mathcal{E}_t &:= \left[\inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta \right], \\ \mathcal{E}_t^1 &:= \left[\int_0^t (-Q_1(s)) ds > \frac{\beta t}{2}, \quad \inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta \right], \\ \mathcal{E}_t^2 &:= \left[\int_0^t (-Q_1(s)) ds \leq \frac{\beta t}{2}, \quad \inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta \right]. \end{aligned}$$

Note that if $(Q_1(0), Q_2(0)) = (0, y + c'_1(\beta \vee \beta^{-1}) + \beta)$, then from the evolution equation of the diffusion in (1.1), the event \mathcal{E}_t^2 implies the event

$$\tilde{\mathcal{E}}_t^2 := \left[Q_1(t) + Q_2(t) \leq y + c'_1(\beta \vee \beta^{-1}) + \beta + \sqrt{2}W(t) - \frac{\beta t}{2}, \quad \inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta \right].$$

Therefore,

$$\mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}(\mathcal{E}_t) \leq \mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}(\mathcal{E}_t^1) + \mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}(\tilde{\mathcal{E}}_t^2). \quad (4.4)$$

Now, choose c'_1, c'_2 as in Lemma 4.6. Then for any $y \geq 1$,

$$\mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}(\mathcal{E}_t^1) \leq \exp(-c'_2t^{1/5}(\beta \vee \beta^{-1})^{2/5}). \quad (4.5)$$

Also, note that

$$\begin{aligned} &\mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}(\tilde{\mathcal{E}}_t^2) \\ &\leq \mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}\left(Q_1(t) \leq y + \sqrt{2}W(t) - \frac{\beta t}{2}, \quad \inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta\right) \\ &\leq \mathbb{P}\left(\sqrt{2}W(t) > \frac{\beta t}{4}\right) + \mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}\left(Q_1(t) \leq y - \frac{\beta t}{4}, \quad \inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta\right). \end{aligned} \quad (4.6)$$

Due to Brownian scaling we have

$$\mathbb{P}\left(\sqrt{2}W(t) > \frac{\beta t}{4}\right) \leq c \exp(-c'\beta^2 t) \text{ for } t \geq \beta^{-2}, \quad (4.7)$$

where c, c' do not depend on β . Moreover, choosing $t > 8y/\beta$, and applying Lemma 4.3 (iii) and Lemma 4.3 (v) with $\varepsilon = (\beta \wedge \beta^{-1})/4$ and $M = c'_1(\beta \vee \beta^{-1})$,

$$\begin{aligned} & \mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}(Q_1(t) \leq y - \frac{\beta t}{4}, \inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta) \\ & \leq \mathbb{P}_{(0, y+c'_1(\beta \vee \beta^{-1})+\beta)}(Q_1(t) \leq -\frac{\beta t}{8}, \inf_{s \leq t} Q_2(s) > c'_1(\beta \vee \beta^{-1}) + \beta) \\ & \leq \mathbb{P}\left(\sup_{1 \leq i \leq N_t} u_i > \frac{\beta t}{8}\right) \\ & \leq \mathbb{P}\left(N_t > 16b(\beta \vee \beta^{-1})^2 t\right) + 16b(\beta \vee \beta^{-1})^2 t \mathbb{P}\left(u_1 > \frac{\beta t}{8}\right) \\ & \leq \exp(-c(\beta \vee \beta^{-1})^2 t) + 16b(\beta \vee \beta^{-1})^2 t \exp\left(-(\beta \vee \beta^{-1})\left(\frac{\beta t}{8} - \frac{\beta \wedge \beta^{-1}}{4}\right)\right) \\ & \leq \exp(-c(\beta \vee \beta^{-1})^2 t) + 16b(\beta \vee \beta^{-1})^2 t \exp\left(-(\beta \vee \beta^{-1})\left(\frac{\beta t}{16}\right)\right), \end{aligned} \quad (4.8)$$

where b, c do not depend on β . Combining Equations (4.4) – (4.8) completes the proof of the lemma. \square

We now have all the necessary results to prove Lemma 4.1.

Proof of Lemma 4.1. From Lemma 4.7, for any $\beta > 0$, we obtain $M^* > 2B, t^* > 0$ such that for all $t \geq t^*$,

$$\mathbb{P}_{(0, 2M^*)}(\tau_2(M^*) > t) \leq C_1 \exp(-C_2 t^{1/5}), \quad (4.9)$$

where the constants $C_1, C_2 > 0$ depend on β, M^* . Set the starting state to be $(Q_1(0), Q_2(0)) = (0, y)$ where $M^* \geq y \geq B$. It will be clear from the proof that the same argument works for general starting points (x, y) with $x \leq 0, y > 0$. For $k \geq 0$, define the following stopping times:

$$\begin{aligned} \alpha_{2k+1}^* &= \inf\left\{t \geq \alpha_{2k}^* : Q_2(t) = 2M^* \text{ or } Q_2(t) = B\right\}, \\ \alpha_{2k+2}^* &= \inf\left\{t > \alpha_{2k+1}^* : Q_2(t) = M^* \text{ or } Q_2(t) = B\right\}, \end{aligned}$$

where by convention, we take $\alpha_0^* = 0$. Let $\mathcal{N}' := \inf\{k \geq 0 : Q_2(\alpha_{2k}^*) = B\}$.

We will first prove the following: for some positive constant $p(M^*)$ that depends only on M^*

$$\inf_{z \in [B, M^*]} \mathbb{P}_{(0, z)}(\tau_2(B) < \tau_2(2M^*)) \geq p(M^*) > 0. \quad (4.10)$$

To see this, recall $S(t) = Q_1(t) + Q_2(t)$ and note that for $t \leq \tau_1(-\beta/2)$,

$$S(0) + \sqrt{2}W(t) - \beta t/2 \geq S(t) \geq Q_1(t).$$

Further, note that $S(t) \leq Q_2(t)$. Moreover, due to arguments similar to Lemma 3.1, we know $Q_1(\tau_2(2M^*)) = 0$, and hence, $S(\tau_2(2M^*)) = Q_2(\tau_2(2M^*))$. Combining these facts, we obtain for any $z \in [B, M^*]$,

$$\begin{aligned} \mathbb{P}_{(0,z)}(\tau_2(2M^*) \leq \tau_1(-\beta/2)) &\leq \mathbb{P}(S(t) \text{ hits } 2M^* \text{ before } -\beta/2) \\ &\leq \mathbb{P}(S(0) + \sqrt{2}W(t) - \beta t/2 \text{ hits } 2M^* \text{ before } -\beta/2) \\ &\leq \mathbb{P}(\sqrt{2}W(t) - \beta t/2 \text{ hits } M^* \text{ before } -(M^* + \beta/2)) \\ &\leq e^{-\beta M^*/2} < 1, \end{aligned} \tag{4.11}$$

where we used the fact that the scale function (see [17, V.46]) for $\sqrt{2}W(t) - \beta t/2$ is $s(x) = \exp(\beta x/2)$.

Now we will show that if the process (Q_1, Q_2) starts with the initial state $(-\beta/2, z)$ with $z \leq 2M^*$, then with positive probability $Q_1(t) < 0$ for all $t \leq \log(2M^*B^{-1})$. This in turn implies that Q_2 hits the level B before time $\log(2M^*B^{-1})$, since for $t \leq \tau_1(0)$, $(d/dt)Q_2(t) = -Q_2(t)$.

Construct the Ornstein-Uhlenbeck process Q_1^+ on the same probability space as Q_1 as follow

$$Q_1^+(t) = Q_1(0) + \sqrt{2}W(t) + \int_0^t (-Q_1^+(s) + (2M^* - \beta)) ds,$$

where the driving Brownian motion W is the same as that for Q_1 . By [12, Proposition 2.18], $Q_1(t) \leq Q_1^+(t)$ for all $t \leq \tau_1(0)$. Now define the following event

$$\mathcal{E}(M^*) := \left\{ Q_1^+(t) < 0 \text{ for all } t \leq \log(2M^*B^{-1}) \right\}.$$

Note that $\mathcal{E}(M^*)$ does not depend on z . It follows from the Doob representation for Ornstein-Uhlenbeck processes that $\mathbb{P}(\mathcal{E}(M^*)) > 0$. Thus,

$$\begin{aligned} \inf_{z \in [B, 2M^*]} \mathbb{P}_{(-\beta/2, z)} \left(\tau_2(B) \leq \log(2M^*B^{-1}) < \tau_2(2M^*) \right) \\ \geq \inf_{z \in [B, 2M^*]} \mathbb{P}_{(-\beta/2, z)} (\tau_1(0) \geq \log(2M^*B^{-1})) \geq \mathbb{P}(\mathcal{E}(M^*)) > 0. \end{aligned} \tag{4.12}$$

The strong Markov property in combination with (4.11) and (4.12) now produces the following bound

$$\inf_{z \in [B, M^*]} \mathbb{P}_{(0,z)}(\tau_2(B) < \tau_2(2M^*)) \geq (1 - e^{-\beta M^*/2}) \mathbb{P}(\mathcal{E}(M^*)) > 0$$

which proves (4.10). By virtue of (4.10), we have the following for $n \geq 1$,

$$\mathbb{P}(\mathcal{N}' > n) \leq (1 - p(M^*))^n. \tag{4.13}$$

Now, let $T(M^*)$ be a number large enough such that

$$\mathbb{P} \left(\sqrt{2}W(T(M^*)) \geq \beta T(M^*)/2 - (2M^* + \beta/2) \right) \leq \mathbb{P}(\mathcal{E}(M^*))/2. \tag{4.14}$$

Then,

$$\begin{aligned}
& \sup_{z \in [B, 2M^*]} \mathbb{P}_{(0,z)} (\tau_2(B) \wedge \tau_2(2M^*) > T(M^*) + \log(2M^*B^{-1})) \\
& \leq \sup_{z \in [B, 2M^*]} \mathbb{P}_{(0,z)} (\tau_1(-\beta/2) < T(M^*), \tau_2(B) \wedge \tau_2(2M^*) > T(M^*) + \log(2M^*B^{-1})) \\
& \quad + \sup_{z \in [B, 2M^*]} \mathbb{P}_{(0,z)} (\tau_1(-\beta/2) \geq T(M^*)) \tag{4.15} \\
& \leq \sup_{z \in [B, 2M^*]} \mathbb{P}_{(-\beta/2, z)} (\tau_2(B) \wedge \tau_2(2M^*) > \log(2M^*B^{-1})) \\
& \quad + \sup_{z \in [B, 2M^*]} \mathbb{P}_{(0,z)} (\tau_1(-\beta/2) \geq T(M^*))
\end{aligned}$$

where we have used the strong Markov property in the last step. By (4.12),

$$\sup_{z \in [B, 2M^*]} \mathbb{P}_{(-\beta/2, z)} (\tau_2(B) \wedge \tau_2(2M^*) > \log(2M^*B^{-1})) \leq 1 - \mathbb{P}(\mathcal{E}(M^*)).$$

By using $S(0) + \sqrt{2}W(t) - \beta t/2 \geq S(t)$ for $t \leq \tau_1(-\beta/2)$ and $Q_1(t) \leq S(t) \leq Q_2(t)$ for $t \geq 0$,

$$\begin{aligned}
\sup_{z \in [B, 2M^*]} \mathbb{P}_{(0,z)} (\tau_1(-\beta/2) \geq T(M^*)) & \leq \mathbb{P} \left(\inf_{t \leq T(M^*)} (\sqrt{2}W(t) - \beta t/2) \geq -(2M^* + \beta/2) \right) \\
& \leq \mathbb{P} \left(\sqrt{2}W(T(M^*)) \geq \beta T(M^*)/2 - (2M^* + \beta/2) \right) \leq \mathbb{P}(\mathcal{E}(M^*))/2.
\end{aligned}$$

Using these bounds in (4.15), we obtain

$$\sup_{z \in [B, 2M^*]} \mathbb{P}_{(0,z)} (\tau_2(B) \wedge \tau_2(2M^*) > T(M^*) + \log(2M^*B^{-1})) \leq 1 - \frac{\mathbb{P}(\mathcal{E}(M^*))}{2} < 1. \tag{4.16}$$

Thus, using the strong Markov property and (4.16), we obtain for any $k \geq 0$,

$$\mathbb{P}_{(0,y)} (\alpha_{2k+1}^* - \alpha_{2k}^* > n(T(M^*) + \log(2M^*B^{-1}))) \leq \left(1 - \frac{\mathbb{P}(\mathcal{E}(M^*))}{2} \right)^n. \tag{4.17}$$

Furthermore, by (4.9) we have constants , for $k \geq 1$ and for all $t \geq t^*$,

$$\mathbb{P}_{(0,y)} (\alpha_{2k}^* - \alpha_{2k-1}^* > t) \leq C_1 \exp(-C_2 t^{1/5}). \tag{4.18}$$

Writing $\alpha_1 = \sum_{j=0}^{2N'} (\alpha_{j+1}^* - \alpha_j^*)$ and using (4.17) and (4.18), we get positive constants C, C', C'' and $t_\alpha^{(2)} > 0$, depending on β, B, M^* , such that for all $t \geq t_\alpha^{(2)}$,

$$\begin{aligned}
\mathbb{P}_{(0,y)} (\alpha_1 > t) & \leq \mathbb{P}(N' > n) + \mathbb{P} \left(\sum_{j=0}^{2n} (\alpha_{j+1}^* - \alpha_j^*) > t \right) \\
& \leq e^{-Cn} + C' n e^{-C(t/n)^{1/5}} \leq C' e^{-C'' t^{1/6}},
\end{aligned}$$

where the last step is obtained by taking $n = \lfloor t^{1/6} \rfloor$. \square

4.2 Up-crossings of Q_2

In this subsection, we will prove tail-asymptotics for the distribution of $\alpha_2 - \alpha_1$ as stated in Lemma 4.2. The proof consists of the following two major parts: (i) First we establish in Lemma 4.8 the tail probability of the hitting time of Q_2 to level $2B$ starting below level B when $Q_1(0)$ is not too small. (ii) Then in Lemma 4.13 we show that at time α_1 , $Q_1(\alpha_1)$ cannot be too small. Lemmas 4.8 and 4.13 are combined to prove Lemma 4.2.

Lemma 4.8. *For any fixed $B > 0$ and $M > 8B + 6\beta$, there exists $c_\alpha^{(2)} > 0$ (depending on M, B, β) such that for all $t \geq 9$,*

$$\sup_{x \in [-M/2, 0], y \in (0, B]} \mathbb{P}_{(x, y)}(\tau_2(2B) > t) \leq \exp(-c_\alpha^{(2)} \sqrt{t}).$$

In order to prove Lemma 4.8, set $M > 0$ to be a fixed large number to be chosen later and $(Q_1(0), Q_2(0)) = (x, y)$ for some $x \in [-M/2, 0], y \in (0, B]$. For $i \geq 1$ define the stopping times

$$\begin{aligned} \tau_{2,2i-1} &:= \inf \left\{ t \geq 0 : Q_2(t) = 2B \quad \text{or} \quad Q_1(t) = -M \right\}, \\ \tau_{2,2i} &:= \inf \left\{ t \geq 0 : Q_2(t) = 2B \quad \text{or} \quad Q_1(t) = -\frac{M}{2} \right\}, \end{aligned}$$

where by convention we take $\tau_{2,0} \equiv 0$. Also define

$$N^* := \inf \left\{ k \geq 0 : Q_2(\tau_{2,2k+1}) = 2B \right\}.$$

Therefore, note that

$$\tau_2(2B) = \sum_{j=1}^{2N^*+1} (\tau_{2,j} - \tau_{2,j-1}). \quad (4.19)$$

The proof of Lemma 4.8 consists of three parts: (i) Lemma 4.9 contains the required probability estimate to analyze the time-interval $\tau_{2,2i-1} - \tau_{2,2i-2}$, (ii) Lemma 4.10 estimates the tail probabilities for the time interval $\tau_{2,2i} - \tau_{2,2i-1}$, and (iii) Lemma 4.12 provides the tail probabilities for the random variable N^* . Lemma 4.11 is used in the proof of Lemma 4.12. Combining Equation (4.19) and Lemmas 4.9, 4.10, and 4.12, we will complete the proof of Lemma 4.8.

Lemma 4.9. *For any fixed $B, M > 0$,*

$$\inf_{\substack{x \in [-M, 0], \\ y \in (0, 2B]}} \mathbb{P}_{(x, y)} \left(\sup_{0 \leq s \leq 1} Q_2(s) > 2B \right) \geq p^{(1)}(M, B) > 0.$$

Proof. Recall that

$$Q_1(t) = Q_1(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-Q_1(s) + Q_2(s)) ds - L(t),$$

where

$$L(t) = \sup_{s \leq t} \left(Q_1(0) + \sqrt{2}W(s) - \beta s + \int_0^s (-Q_1(u) + Q_2(u)) du \right)^+ \geq \sup_{s \leq t} (Q_1(0) + \sqrt{2}W(s) - \beta s)^+.$$

Thus, $\mathbb{P}(L(1) > 4B) \geq \mathbb{P}(\sqrt{2}W(1) > \beta + 4B - Q_1(0))$. Observe that for any $Q_2(0) = y \leq 2B$,

$$\{L(1) > 4B\} \implies \{\sup_{s \leq 1} Q_2(s) > 2B\}$$

To see this, suppose $L(1) > 4B$. If $\sup_{s \leq 1} Q_2(s) \leq 2B$, then

$$Q_2(1) = y + L(1) - \int_0^1 Q_2(s) ds \geq L(1) - 2B > 2B$$

which is a contradiction. Therefore,

$$\begin{aligned} & \inf_{\substack{x \in [-M, 0], \\ y \in (0, 2B]}} \mathbb{P}_{(x,y)} \left(\sup_{0 \leq s \leq 1} Q_2(s) > 2B \right) \geq \inf_{\substack{x \in [-M, 0], \\ y \in (0, 2B]}} \mathbb{P}_{(x,y)} \left(L(1) > 4B \right) \\ & \geq \inf_{\substack{x \in [-M, 0], \\ y \in (0, 2B]}} \mathbb{P}_{(x,y)} \left(\sqrt{2}W(1) > \beta + 4B - x \right) \\ & \geq \mathbb{P} \left(\sqrt{2}W(1) > \beta + 4B + M \right) = p^{(1)}(M, B) > 0. \end{aligned}$$

This completes the proof of Lemma 4.9. \square

Lemma 4.10. For any $j \geq 0$ and any fixed $M \geq 6\beta$, there exists $c_\tau^{(1)} > 0$ such that for all $t \geq 2$,

$$\sup_{\substack{x \in [-M/2, 0], \\ y \in (0, B]}} \mathbb{P}_{(x,y)} \left(\tau_{2,2j+2} - \tau_{2,2j+1} > t \mid N^* > j \right) \leq \exp(-c_\tau^{(1)} t).$$

Proof. Let us denote $Q_1^* = Q_1 + \beta$. Since $N^* > j$, we know $Q_2(\tau_{2,2j+1}) < 2B$. In that case, for $t > \tau_{2,2j+1}$,

$$\begin{aligned} Q_1^*(t) &= Q_1^*(\tau_{2,2j+1}) + \sqrt{2}W(t) + \int_{\tau_{2,2j+1}}^t (-Q_1^*(s) + Q_2(s)) ds \\ &\geq Q_1^*(\tau_{2,2j+1}) + \sqrt{2}W(t) - \int_{\tau_{2,2j+1}}^t Q_1^*(s) ds \\ &= -M + \beta + \sqrt{2}W(t) - \int_{\tau_{2,2j+1}}^t Q_1^*(s) ds. \end{aligned}$$

Thus, we obtain

$$\mathbb{P}_{(x,y)} \left(\tau_{2,2j+2} - \tau_{2,2j+1} > t \mid N^* > j \right) \leq \mathbb{P} \left(\sup_{s \leq t} (\sqrt{2}W(s) - (-M/2 + \beta)s) \leq M/2 \right),$$

since for $t \in (\tau_{2,2j+1}, \tau_{2,2j+2})$, $Q_1^*(s) \leq -M/2 + \beta$. Therefore, as $M \geq 6\beta$, for all $t \geq 2$,

$$\begin{aligned} \mathbb{P}_{(x,y)} \left(\tau_{2,2j+2} - \tau_{2,2j+1} > t \mid N^* > j \right) &\leq \mathbb{P} \left(\sqrt{2}W(t) \leq M/2 - (M/2 - \beta)t \right) \\ &\leq \mathbb{P} \left(\sqrt{2}W(t) \leq -(M/2 - \beta)t/4 \right) \\ &\leq \exp(-c_\tau^{(1)} (M/2 - \beta)^2 t) \leq \exp(-c_\tau^{(1)} t), \end{aligned}$$

where $c_\tau^{(1)}$ does not depend on x, y . \square

Lemma 4.11. For any fixed $B > 0$ and $M > 8B + 2\beta$, there exists $p^{(2)} = p^{(2)}(M, B) > 0$ such that

$$\inf_{\substack{x \in [-M/2, 0], \\ y \in (0, B]}} \mathbb{P}_{(x,y)} \left(\exists t^* \in [0, 1], \text{ such that } \sup_{0 \leq t \leq t^*} Q_2(t) \geq 2B, \inf_{0 \leq t \leq t^*} Q_1(t) > -M \right) \geq p^{(2)}.$$

Proof. For fixed $B > 0$ and $M > 8B + 2\beta$, consider the event

$$\mathcal{E}(\beta, M) := \left\{ \sqrt{2}W(1) > \beta + 4B + \frac{M}{2}, \quad \inf_{t \in [0,1]} \sqrt{2}W(t) > \beta + 4B - \frac{M}{2} \right\}.$$

From the representation (4.2), note that the event $\mathcal{E}(M, B)$ implies the event $\{L(1) > 4B\}$, which in turn implies that there exists $t^* \in [0, 1]$ such that $L(t^*) = 4B$ and $\forall t \leq t^*$,

$$Q_1(t) \geq -\frac{M}{2} + \sqrt{2}W(t) - \beta - 4B > -\frac{M}{2} - \beta - 4B + \left(\beta + 4B - \frac{M}{2}\right) = -M.$$

Therefore, $\inf_{0 \leq t \leq t^*} Q_1(t) > -M$. Furthermore, we claim that $\sup_{0 \leq t \leq t^*} Q_2(t) \geq 2B$. Indeed, if $\sup_{0 \leq t \leq t^*} Q_2(t) < 2B$, then

$$Q_2(t^*) \geq L(t^*) - \int_0^{t^*} Q_2(s) ds > 4B - 2Bt^* \geq 2B,$$

since $0 \leq t^* \leq 1$, which leads to a contradiction. Finally,

$$\begin{aligned} \inf_{\substack{x \in [-M/2, 0], \\ y \in (0, \beta^{-1})}} \mathbb{P}_{(x,y)} \left(\exists t^* \in [0, 1], \text{ such that } \sup_{0 \leq t \leq t^*} Q_2(t) > 2B, \inf_{0 \leq t \leq t^*} Q_1(t) > -M \right) \\ \geq \mathbb{P}(\mathcal{E}(M, B)) > 0. \end{aligned}$$

This completes the proof of the lemma. \square

Lemma 4.12. For any fixed $B > 0$ and $M > 8B + 2\beta$, there exist $c_N^{(2)}, n_N > 0$ such that for all $n \geq n_N$,

$$\sup_{\substack{x \in [-M/2, 0], \\ y \in (0, B]}} \mathbb{P}_{(x,y)}(N^* > n) \leq \exp(-c_N^{(2)} n).$$

Proof. Observe that

$$\mathbb{P}_{(x,y)}(N^* > n) \leq \mathbb{P}_{(x,y)}(Q_1(\tau_{2,2k+1}) = -M \text{ and } Q_2(\tau_{2,2k+1}) < 2B \text{ for all } k \leq n) \leq (1 - p^*)^n,$$

using strong Markov property, where

$$\begin{aligned} p^* &:= \inf_{\substack{x \in [-M/2, 0] \\ y \in (0, B]}} \mathbb{P}_{(x,y)}(Q_2 \text{ hits } 2B \text{ before } Q_1 \text{ hits } -M) \\ &\geq \inf_{\substack{x \in [-M/2, 0] \\ y \in (0, B]}} \mathbb{P}_{(x,y)}(\exists t^* \in [0, 1] \text{ such that } \sup_{0 \leq t \leq t^*} Q_2(t) > 2B, \inf_{0 \leq t \leq t^*} Q_1(t) > -M) \\ &\geq p^{(2)}(M, B) > 0, \end{aligned}$$

by Lemma 4.11, choosing $M > 8B + 2\beta$. \square

Now, we have all the necessary results to prove Lemma 4.8.

Proof of Lemma 4.8. Recall that $\tau_2(2B) = \sum_{j=1}^{2N^*+1} (\tau_{2,j} - \tau_{2,j-1})$. From Lemma 4.9 observe that for any fixed $M > 0$ and any $x \in [-M/2, 0]$, $y \in (0, B]$,

$$\begin{aligned} \mathbb{P}_{(x,y)}(\tau_{2,1} > n) &= \mathbb{E}_{(x,y)} \left(\mathbb{1}_{[\tau_{2,1} > n-1]} \mathbb{P}_{(Q_1(n-1), Q_2(n-1))}(\tau_{2,1} > 1) \right) \\ &\leq (1 - p^{(1)}(M, B)) \mathbb{P}(\tau_{2,1} > n-1), \end{aligned}$$

Which implies $\mathbb{P}_{(x,y)}(\tau_{2,1} > n) \leq (1 - p^{(1)}(M, B))^n$. Furthermore, following the same argument as above, we can claim that for all $j \geq 1$,

$$\mathbb{P}_{(x,y)}(\tau_{2,2j-1} - \tau_{2,2j-2} \geq n) \leq (1 - p^{(1)}(M, B))^n. \quad (4.20)$$

Therefore for $t \geq 9$, choosing $M > 8B + 6\beta$, we can write for any $x \in [-M/2, 0]$, $y \in (0, B]$,

$$\begin{aligned} \mathbb{P}_{(x,y)}(\tau_2(2B) > t) &\leq \mathbb{P}_{(x,y)}(N^* > n) + \mathbb{P}_{(x,y)} \left(\sum_{j=1}^{2n+1} (\tau_{2,j} - \tau_{2,j-1}) > t \right) \\ &\leq \exp(-c_N^{(2)} n) + (2n+1) \exp(-ct/(2n+1)), \quad \text{Due to Lemmas 4.10 and 4.12, and (4.20)} \\ &\leq c' \sqrt{t} e^{-c\sqrt{t}} \leq e^{c_\alpha^{(2)} \sqrt{t}}, \quad \text{[choosing } n = \lfloor (\sqrt{t} - 1)/2 \rfloor] \end{aligned}$$

where $c_\alpha^{(2)}$ does not depend on (x, y) . □

As mentioned earlier, the next lemma gives a tail estimate on the distribution of $Q_1(\alpha_1)$.

Lemma 4.13. Fix $(Q_1(0), Q_2(0)) = (x, y)$ with $x \leq 0$, $y > 0$. Recall the constant $t_\alpha^{(1)}$ obtained in Lemma 4.1. There exist constants $C_1, C_2 > 0$ possibly depending on (x, y) , B , and β , such that for all $A \geq \max\{8\beta t_\alpha^{(1)}, -4x\}$,

$$\mathbb{P}_{(x,y)}(Q_1(\alpha_1) < -A) \leq C_1 e^{-C_2 A^{1/6}}.$$

Proof. In the proof, C, C' will denote generic positive constants depending on β, x, y whose values change from line to line. Observe that for $t > 0$,

$$Q_1(t) \geq Q_1(0) + \sqrt{2}W(t) - \beta t - L^*(t)$$

where $L^*(t) = \sup_{s \leq t} (Q_1(0) + \sqrt{2}W(s) - \beta s)^+$. Thus, for any $A \geq \max\{8\beta t_\alpha^{(1)}, -4x\}$,

$$\begin{aligned} \mathbb{P}_{(x,y)}(Q_1(\alpha_1) < -A) &\leq \mathbb{P}_{(x,y)}(\alpha_1 > A/(8\beta)) + \mathbb{P}_{(x,y)} \left(\inf_{s \leq A/(8\beta)} Q_1(s) < -A \right) \\ &\leq \mathbb{P}_{(x,y)}(\alpha_1 > A/(8\beta)) + \mathbb{P}_{(x,y)} \left(\inf_{s \leq A/(8\beta)} (Q_1(0) + \sqrt{2}W(s) - \beta s - L^*(s)) < -A \right) \\ &\leq \mathbb{P}_{(x,y)}(\alpha_1 > A/(8\beta)) + \mathbb{P}_{(x,y)}(L^*(A/(8\beta)) > A/2) \\ &\quad + \mathbb{P}_{(x,y)} \left(\inf_{s \leq A/(8\beta)} (\sqrt{2}W(s) - \beta s) < -A/2 - x \right) \quad (4.21) \\ &\leq \mathbb{P}_{(x,y)}(\alpha_1 > A/(8\beta)) + \mathbb{P}_{(x,y)} \left(\sup_{s \leq A/(8\beta)} (\sqrt{2}W(s) - \beta s) > A/2 \right) \\ &\quad + \mathbb{P}_{(x,y)} \left(\inf_{s \leq A/(8\beta)} (\sqrt{2}W(s) - \beta s) < -A/4 \right). \end{aligned}$$

By Lemma 4.1,

$$\mathbb{P}_{(x,y)}(\alpha_1 > A/(8\beta)) \leq Ce^{-C'A^{1/6}}.$$

Using the fact that the scale function (see [17, V.46]) for $\sqrt{2}W(t) - \beta t$ is $s(z) = \exp(\beta z)$,

$$\mathbb{P}_{(x,y)}\left(\sup_{s \leq A/(8\beta)} (\sqrt{2}W(s) - \beta s) > A/2\right) \leq \mathbb{P}_{(x,y)}\left(\sup_{s < \infty} (\sqrt{2}W(s) - \beta s) > A/2\right) = e^{-\beta A/2}.$$

Moreover, by standard estimates on normal distribution functions,

$$\mathbb{P}_{(x,y)}\left(\inf_{s \leq A/(8\beta)} (\sqrt{2}W(s) - \beta s) < -A/4\right) \leq \mathbb{P}_{(x,y)}\left(\inf_{s \leq A/(8\beta)} (\sqrt{2}W(s)) < -A/8\right) \leq Ce^{-C'A}.$$

Using the above bounds in (4.21), we obtain

$$\mathbb{P}_{(x,y)}(Q_1(\alpha_1) < -A) \leq Ce^{-C'A^{1/6}}$$

for any $A \geq \max\{8\beta t_\alpha^{(1)}, -4x\}$, proving the lemma. \square

Proof of Lemma 4.2. In the proof, C, C' will denote generic positive constants depending on β, x, y whose values change from line to line. Fix $M > 8B + 6\beta + 2$. Let $s'_\alpha = \inf\{t \geq \alpha_1 : Q_1(t) = -M/2\}$. Take $t'_\alpha = 4 \max\{9, 8\beta t_\alpha^{(1)}, -4x\}$. Then for $t \geq t'_\alpha$,

$$\begin{aligned} \mathbb{P}_{(x,y)}(\alpha_2 - \alpha_1 > t) &\leq \mathbb{P}_{(x,y)}(Q_1(\alpha_1) < -t/4) + \sup_{u \geq -t/4, v > 0} \mathbb{P}_{(u,v)}(\tau_1(-M/2) > t/4) \\ &\quad + \sup_{y \in (0, B]} \mathbb{P}_{(-M/2, y)}(\tau_2(2B) > t/2) \quad (4.22) \end{aligned}$$

By Lemma 4.13,

$$\mathbb{P}_{(x,y)}(Q_1(\alpha_1) < -t/4) \leq Ce^{-C't^{1/6}}.$$

By computations similar to Lemma 4.10,

$$\begin{aligned} \sup_{u \geq -t/4, v > 0} \mathbb{P}_{(u,v)}(\tau_1(-M/2) > t/4) &\leq \mathbb{P}\left(-\frac{t}{4} + \sqrt{2}W(t/2) + \left(\frac{M}{2} - \beta\right) \frac{t}{2} < -\frac{M}{2}\right) \\ &\leq \mathbb{P}\left(\sqrt{2}W(t/2) < -\frac{t}{4}\right) \leq Ce^{-C't}. \end{aligned}$$

By Lemma 4.8,

$$\sup_{y \in (0, B]} \mathbb{P}_{(-M/2, y)}(\tau_2(2B) > t/2) \leq e^{-C'\sqrt{t}}.$$

Using these bounds in (4.22), we obtain for all $t \geq t'_\alpha$,

$$\mathbb{P}_{(x,y)}(\alpha_2 - \alpha_1 > t) \leq Ce^{-C't^{1/6}}$$

proving the lemma. \square

5 Analysis of fluctuations within a renewal cycle

In this section we prove Theorem 3.4. Specifically, we derive sharp estimates for the fluctuations of excursions of Q_1 and Q_2 between two successive regeneration times defined in (3.1). This will eventually furnish tail estimates for the stationary distribution of Q_1 and Q_2 and the scaling of extrema in large time intervals that are described in Theorems 2.1 and 2.2. First we state and prove Lemmas 5.1 – 5.9, which provide all the necessary results for proving Theorem 3.4 at the end of this section.

Denote the Brownian motion with drift b and its corresponding reflected analogue by

$$\begin{aligned} W^{(b)}(t) &:= \sqrt{2}W(t) + bt, \\ W_R^{(b)}(t) &:= \sqrt{2}W(t) + bt - \sup_{s \leq t} (\sqrt{2}W(s) + bs), \end{aligned}$$

where W denotes the standard Brownian motion. Also, denote the local time of the reflected Brownian motion $W_R^{(b)}$ and its hitting time of level z by $L^{(b)}$ and $\tau^{(b)}(z)$ respectively.

Lemma 5.1. *There exist positive constants $C_1, C_2 > 0$ that do not depend on β such that*

$$\mathbb{P}_{(0, y + \beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \leq C_1 e^{-C_2 \beta y}$$

for $y \geq \frac{1}{4\beta}$ if $\beta \geq 1$ and $y \geq \frac{64}{\beta} \log \frac{1}{\beta}$ if $\beta < 1$.

Proof. From the evolution equation of Q_1 in (1.1), note that for $y > 0$, $W_R^{(y/2)}$ can be constructed on the same probability space as (Q_1, Q_2) , such that starting from $(Q_1(0), Q_2(0)) = (0, y + \beta)$, almost surely $Q_1(t) \geq W_R^{(y/2)}(t)$ for all $t \leq \tau_2(\frac{y}{2} + \beta)$. The scale function s for $W_R^{(y/2)}(t)$ is obtained by solving the equation $\frac{y}{2}s'(z) + s''(z) = 0$ (see [17, V.46]) and one candidate is

$$s(z) = \frac{2}{y} \left(1 - e^{-yz/2} \right). \quad (5.1)$$

We will estimate the time taken by $W_R^{(y/2)}$ to hit the level $-\beta/2$. Define stopping times for the process $W^{(y/2)}$ as follows: For $i \geq 0$

$$\gamma_{i+1} = \inf \left\{ t \geq \gamma_i : W^{(y/2)}(t) - W^{(y/2)}(\gamma_i) \text{ hits } \beta/4 \text{ or } -\beta/4 \right\},$$

with the convention that $\gamma_0 = 0$. From the explicit form of the scale function s in (5.1), observe that for $i \geq 0$,

$$\mathbb{P} \left(W^{(y/2)}(\gamma_{i+1}) - W^{(y/2)}(\gamma_i) = -\beta/4 \right) = \frac{1 - e^{-\beta y/8}}{e^{\beta y/8} - e^{-\beta y/8}} \leq e^{-\beta y/8}. \quad (5.2)$$

Define

$$\mathcal{N} := \inf \left\{ i \geq 1 : W^{(y/2)}(\gamma_{i+1}) - W^{(y/2)}(\gamma_i) = -\beta/4 \right\}.$$

Then for any $n \geq 1$, by (5.2), $\mathbb{P}(\mathcal{N} \leq n) \leq n e^{-\beta y/8}$. Note that for $t < \gamma_{\mathcal{N}}$, $W_R^{(y/2)}(t) > -\beta/2$. Thus, $\tau^{(y/2)}\left(-\frac{\beta}{2}\right) \geq \gamma_{\mathcal{N}}$. Consequently,

$$L^{(y/2)} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \right) \geq \sup_{t \leq \gamma_{\mathcal{N}}} \left(W^{(y/2)}(t) \right) \geq \mathcal{N} \beta/4.$$

Therefore, for any $n \geq 1$,

$$\mathbb{P} \left(L^{(y/2)} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \right) \leq n\beta \right) \leq \mathbb{P} (\mathcal{N} \leq 4n) \leq 4ne^{-\beta y/8}. \quad (5.3)$$

Further, on the event $\left[\tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right]$, $\tau_1 \left(-\frac{\beta}{2} \right) \geq \tau^{(y/2)} \left(-\frac{\beta}{2} \right)$. Therefore, for $n \geq 1$,

$$\begin{aligned} \mathbb{P}_{(0, y+\beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq n\beta/y, \tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) &\leq \mathbb{P} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \leq n\beta/y \right) \\ &\leq \mathbb{P} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \leq n\beta/y, L^{(y/2)} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \right) > n\beta \right) + \mathbb{P} \left(L^{(y/2)} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \right) \leq n\beta \right). \end{aligned} \quad (5.4)$$

An upper bound for the second probability in the right side of (5.4) has been obtained in (5.3). To estimate the first probability, observe that

$$\begin{aligned} \mathbb{P} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \leq n\beta/y, L^{(y/2)} \left(\tau^{(y/2)} \left(-\frac{\beta}{2} \right) \right) > n\beta \right) \\ &\leq \mathbb{P} \left(\sup_{t \leq n\beta/y} \left(\sqrt{2}W(t) + yt/2 \right) > n\beta \right) \\ &\leq \mathbb{P} \left(\sup_{t \leq n\beta/y} \sqrt{2}W(t) > n\beta/2 \right) \leq \frac{4}{\sqrt{\pi n\beta y}} e^{-n\beta y/16}. \end{aligned} \quad (5.5)$$

Using (5.3) and (5.5) in (5.4), we obtain

$$\mathbb{P}_{(0, y+\beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq n\beta/y, \tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \leq 4ne^{-\beta y/8} + \frac{4}{\sqrt{\pi n\beta y}} e^{-n\beta y/16}, \quad (5.6)$$

where an appropriate choice of $n \geq 1$ (depending on y and β) will be made later. Now, we want to estimate the probability $\mathbb{P}_{(0, y+\beta)} \left(n\beta/y < \tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right)$. Towards this end, recall that $S(t) = Q_1(t) + Q_2(t)$ has the representation

$$S(t) = S(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-Q_1(s)) ds.$$

Thus, for $t \leq \tau_1 \left(-\frac{\beta}{2} \right)$,

$$S(t) \leq S(0) + \sqrt{2}W(t) - \frac{\beta}{2}t.$$

Therefore, if n is chosen such that $y \leq \sqrt{n}\beta/4$,

$$\begin{aligned} \mathbb{P}_{(0, y+\beta)} \left(n\beta/y < \tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \\ &\leq \mathbb{P} \left(y + \beta + \sqrt{2}W(t) - \frac{\beta}{2}t \geq y/2 + \beta/2, \text{ for all } t \leq n\beta/y \right) \\ &\leq \mathbb{P} \left(\sqrt{2}W(n\beta/y) - \frac{n\beta^2}{2y} \geq -y/2 - \beta/2 \right) \\ &\leq \mathbb{P} \left(\sqrt{2}W(n\beta/y) \geq \frac{n\beta^2}{8y} \right) \quad \text{since } y \leq \sqrt{n}\beta/4 \\ &\leq \frac{8\sqrt{y}}{\sqrt{\pi n}\beta^{3/2}} e^{-\frac{n\beta^3}{256y}}. \end{aligned} \quad (5.7)$$

From (5.6) and (5.7) we obtain

$$\mathbb{P}_{(0,y+\beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \leq 4ne^{-\beta y/8} + \frac{4}{\sqrt{\pi n \beta y}} e^{-n\beta y/16} + \frac{8\sqrt{y}}{\sqrt{\pi n \beta^{3/2}}} e^{-\frac{n\beta^3}{256y}}. \quad (5.8)$$

Now, if $\beta \geq 1$, choose $n = 16y^2\beta^2$. Then, clearly $y \leq \sqrt{n}\beta/4$. With this choice of n , the above expression yields the following bound:

$$\mathbb{P}_{(0,y+\beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \leq 64(\beta y)^2 e^{-\beta y/8} + \frac{1}{\sqrt{\pi}(\beta y)^{3/2}} e^{-(\beta y)^3} + \frac{2}{\sqrt{\pi}\beta y} e^{-\frac{\beta y}{16}} \quad (5.9)$$

for $y \geq \frac{1}{4\beta}$ (this ensures $n \geq 1$).

If $\beta < 1$, choose $n = y^4$. Then $y \leq \sqrt{n}\beta/4$ is satisfied if $y \geq 4/\beta$. Some routine calculations reveal that for $y \geq 4/\beta$ the second and third terms appearing on the right side of (5.8) can be estimated by,

$$\frac{4}{\sqrt{\pi n \beta y}} e^{-n\beta y/16} \leq \frac{1}{8\sqrt{\pi}} e^{-(\beta y)^5/16}$$

and

$$\frac{8\sqrt{y}}{\sqrt{\pi n \beta^{3/2}}} e^{-\frac{n\beta^3}{256y}} \leq \frac{1}{\sqrt{\pi}} e^{-(\beta y)^3/256}.$$

To estimate the first term on the right side of (5.8), rewrite it as

$$4ne^{-\beta y/8} = \left[4(\beta y)^4 e^{-(\beta y)/16} \right] \left[\beta^{-4} e^{-(\beta y)/16} \right].$$

Observe that $\beta^{-4} e^{-(\beta y)/16} \leq 1$ for $y \geq \frac{64}{\beta} \log \frac{1}{\beta}$. Therefore, for $\beta < 1$ and $y \geq \frac{64}{\beta} \log \frac{1}{\beta}$, we have the following bound:

$$\mathbb{P}_{(0,y+\beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \leq \frac{1}{8\sqrt{\pi}} e^{-(\beta y)^5/16} + \frac{1}{\sqrt{\pi}} e^{-(\beta y)^3/256} + 4(\beta y)^4 e^{-(\beta y)/16}. \quad (5.10)$$

The lemma follows from (5.9) and (5.10). \square

The above lemma can be used to deduce the following hitting time estimate for Q_2 .

Lemma 5.2. *There exist constants $\tilde{C}_1, \tilde{C}_2 > 0$ that do not depend on β such that*

$$\mathbb{P}_{(0,y+\beta)} \left(\tau_2(2y + \beta) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \leq \tilde{C}_1 e^{-\tilde{C}_2 \beta y}$$

for $y \geq \frac{1}{4\beta}$ if $\beta \geq 1$ and $y \geq \frac{64}{\beta} \log \frac{1}{\beta}$ if $\beta < 1$.

Proof. We can write for any $y > 0$,

$$\begin{aligned} \mathbb{P}_{(0,y+\beta)} \left(\tau_2(2y + \beta) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \\ \leq \mathbb{P}_{(0,y+\beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) + \mathbb{P}_{(0,y+\beta)} \left(\tau_2(2y + \beta) < \tau_1 \left(-\frac{\beta}{2} \right) \right). \end{aligned} \quad (5.11)$$

By Lemma 5.1,

$$\mathbb{P}_{(0,y+\beta)} \left(\tau_1 \left(-\frac{\beta}{2} \right) \leq \tau_2 \left(\frac{y}{2} + \beta \right) \right) \leq C_1 e^{-C_2 \beta y} \quad (5.12)$$

for $y \geq \frac{1}{4\beta}$ if $\beta \geq 1$ and $y \geq \frac{64}{\beta} \log \frac{1}{\beta}$ if $\beta < 1$. To estimate the second probability in (5.11), recall that for $t \leq \tau_1 \left(-\frac{\beta}{2} \right)$, $S(t) = Q_1(t) + Q_2(t)$ satisfies

$$S(t) \leq S(0) + \sqrt{2}W(t) - \frac{\beta}{2}t.$$

Therefore,

$$\mathbb{P}_{(0,y+\beta)} \left(\tau_2(2y + \beta) < \tau_1 \left(-\frac{\beta}{2} \right) \right) \leq \mathbb{P} \left(\sup_{t < \infty} \left(\sqrt{2}W(t) - \frac{\beta}{2}t \right) \geq y \right) = e^{-\frac{\beta y}{2}} \quad (5.13)$$

for $y > 0$. The first inequality above follows from the fact that points of time where Q_2 increases are precisely those where Q_1 equals zero: hence $Q_1(\tau_2(2y + \beta)) = 0$.

The lemma now follows by using (5.12) and (5.13) in (5.11). \square

The above estimate can be strengthened to the following tail estimate which will be used to study fluctuations of Q_2 between successive regeneration times.

Lemma 5.3. *Recall the constants \tilde{C}_1, \tilde{C}_2 in the statement of Lemma 5.2. There exist constants $C_1^*, C_2^* > 0$ that do not depend on β such that*

$$\mathbb{P}_{(0,y+\beta)} (\tau_2(2y + \beta) \leq \tau_2(y_0 + \beta)) \leq C_1^* e^{-C_2^* \beta y}$$

for all $y \geq y_0$, where $y_0 = \max \left\{ \frac{1}{4\beta}, \frac{\log(4\tilde{C}_1)}{C_2\beta} \right\}$ if $\beta \geq 1$ and $y_0 = \max \left\{ \frac{64}{\beta} \log \frac{1}{\beta}, \frac{\log(4\tilde{C}_1)}{C_2\beta} \right\}$ if $\beta < 1$.

Proof. Define stopping times:

$$T_{2k+1} = \inf \left\{ t \geq T_{2k} : Q_2(t) = 2y + \beta \text{ or } Q_2(t) = \frac{y}{2} + \beta \text{ or } Q_2(t) = y_0 + \beta \right\};$$

$$T_{2k+2} = \inf \{ t \geq T_{2k+1} : Q_2(t) = y + \beta \text{ or } Q_2(t) = y_0 + \beta \},$$

for $k \geq 0$, with the convention that $T_0 = 0$. Let

$$\mathcal{N}^0 = \inf \{ k \geq 1 : Q_2(T_{2k}) = y_0 + \beta \}.$$

Define $\hat{Q}_2(t) = \log_2(Q_2(t) - \beta)$. By Lemma 5.2 and our choice of y_0 , for any $z \geq \log_2(y_0)$,

$$\begin{aligned} \mathbb{P}(\hat{Q}_2 \text{ hits } z+1 \text{ before } z-1 \mid \hat{Q}_2(0) = z, Q_1(0) = 0) \\ = \mathbb{P}_{(0,2^z+\beta)}(Q_2 \text{ hits } 2^{z+1} + \beta \text{ before } 2^{z-1} + \beta) \leq 1/4. \end{aligned}$$

Thus, \hat{Q}_2 starting from any $z \geq \log_2(y_0)$ and observed at the stopping times where the increments are ± 1 until the first time it crosses the level $\log_2(y_0)$ (i.e., strictly less than $\log_2(y_0)$) is stochastically dominated by a random walk $(S_n)_{n \geq 0}$ where

$$\mathbb{P}(S_{n+1} - S_n = 1) = 1 - \mathbb{P}(S_{n+1} - S_n = -1) = 1/4.$$

Therefore,

$$\begin{aligned} \sup_{z \geq \log_2(y_0)} \mathbb{P}(\hat{Q}_2 \text{ hits } z+1 \text{ before it crosses } \log_2(y_0) \mid \hat{Q}_2(0) = z, Q_1(0) = 0) \\ \leq \sup_{z \geq \log_2(y_0)} \mathbb{P}(S_n \text{ hits } z+1 \mid S_0 = z) = p^{(S)} < 1. \end{aligned}$$

which, in turn, implies that for any $y \geq y_0$,

$$\mathbb{P}_{(0, y+\beta)}(\tau_2(2y+\beta) \leq \tau_2(y_0+\beta)) \leq p^{(S)} < 1.$$

Thus, for any $k \geq 1$,

$$\mathbb{P}_{(0, y+\beta)}(\mathcal{N}^0 \geq k+1) \leq (p^{(S)})^k. \quad (5.14)$$

Finally, for any $y \geq y_0$,

$$\begin{aligned} & \mathbb{P}_{(0, y+\beta)}(\tau_2(2y+\beta) \leq \tau_2(y_0+\beta)) \\ &= \mathbb{P}_{(0, y+\beta)}\left(\sup_{0 \leq t \leq T_{2\mathcal{N}^0}} Q_2(t) > 2y+\beta\right) \leq \sum_{k=1}^{\infty} \mathbb{P}_{(0, y+\beta)}\left(\sup_{T_{2k-2} \leq t \leq T_{2k}} Q_2(t) > 2y+\beta, \mathcal{N}^0 \geq k\right) \\ &= \sum_{k=1}^{\infty} \mathbb{E}_{(0, y+\beta)} \mathbb{I}(\mathcal{N}^0 \geq k) \mathbb{P}_{(0, y+\beta)}\left(\tau_2(2y+\beta) \leq \tau_2\left(\frac{y}{2}+\beta\right)\right), \text{ by strong Markov property at } T_{2k-2} \\ &\leq \mathbb{P}_{(0, y+\beta)}\left(\tau_2(2y+\beta) \leq \tau_2\left(\frac{y}{2}+\beta\right)\right) \sum_{k=1}^{\infty} (p^{(S)})^{k-1}, \quad \text{by (5.14)} \\ &\leq (1-p^{(S)})^{-1} \tilde{C}_1 e^{-\tilde{C}_2 \beta y}, \quad \text{by Lemma 5.2,} \end{aligned}$$

which completes the proof of the lemma. \square

The lower bound on the tail probabilities is achieved for all $\beta > 0$ in the following lemma.

Lemma 5.4. *For any $\beta > 0$ and any $B > 0$,*

$$\mathbb{P}_{(0, 2B)}(\tau_2(y) < \tau_2(B)) \geq (1 - e^{-\beta B}) e^{-\beta(y-2B)}$$

for all $y \geq 2B$.

Proof. Note that $Q_2(t) \geq Q_1(t) + Q_2(t) = S(t)$ for all $t \geq 0$. Further, recall that

$$S(t) = S(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-Q_1(s)) ds \geq S(0) + \sqrt{2}W(t) - \beta t, \quad t \geq 0.$$

Therefore, for all $y \geq 2B$,

$$\begin{aligned} \mathbb{P}_{(0, 2B)}(\tau_2(y) < \tau_2(B)) &\geq \mathbb{P}_{(0, 2B)}(S(t) \text{ hits level } y \text{ before level } B) \\ &\geq \mathbb{P}\left(2B + \sqrt{2}W(t) - \beta t \text{ hits level } y \text{ before level } B\right) \\ &= \mathbb{P}\left(\sqrt{2}W(t) - \beta t \text{ hits level } y - 2B \text{ before level } -B\right) \\ &= \frac{1 - e^{-\beta B}}{e^{\beta(y-2B)} - e^{-\beta B}}, \quad \text{by scale function arguments} \\ &\geq (1 - e^{-\beta B}) e^{-\beta(y-2B)}, \end{aligned}$$

proving the lemma. \square

Now, we will study fluctuations of Q_1 within one renewal cycle. Recall $l_0(\beta)$ from (3.3) and the notation

$$\sigma(t) = \inf\{s \geq t : Q_1(s) = 0\}, \quad t \geq 0.$$

Lemma 5.5. *There exist constants $R_1 > 0$ not depending on β and $p^{**}(\beta) \in (0, 1)$ such that for all $R \geq R_1$,*

$$\sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(0,y)}(\tau_1(-\beta) < \tau_2(Rl_0(\beta))) = p^*(\beta, R) \leq p^{**}(\beta). \quad (5.15)$$

Proof. In the proof C, C', C_1, C_2, \dots will denote generic positive constants not depending on β, R whose values might change from line to line. For any $y \geq Rl_0(\beta) - \beta$,

$$\begin{aligned} \mathbb{P}_{(0,y+\beta)}\left(\tau_1(-\beta) < \sigma\left(\tau_2\left(\frac{y}{2} + \beta\right)\right)\right) &\leq \mathbb{P}_{(0,y+\beta)}\left(\tau_1(-\beta/2) \leq \tau_2\left(\frac{y}{2} + \beta\right)\right) \\ &\quad + \mathbb{P}_{(0,y+\beta)}\left(\tau_2\left(\frac{y}{2} + \beta\right) < \tau_1(-\beta/2) < \tau_1(-\beta) < \sigma\left(\tau_2\left(\frac{y}{2} + \beta\right)\right)\right) \\ &\leq \mathbb{P}_{(0,y+\beta)}\left(\tau_1(-\beta/2) \leq \tau_2\left(\frac{y}{2} + \beta\right)\right) + \sup_{x \in [-\beta/2, 0]} \mathbb{P}_{(x, \frac{y}{2} + \beta)}(\tau_1(-\beta) < \tau_1(0)) \end{aligned} \quad (5.16)$$

where the last step is a consequence of the strong Markov property applied at $\tau_2(\frac{y}{2} + \beta)$. From Lemma 5.1, for $R \geq 65$,

$$\mathbb{P}_{(0,y+\beta)}\left(\tau_1(-\beta/2) \leq \tau_2\left(\frac{y}{2} + \beta\right)\right) \leq C_1 e^{-C_2 \beta y}, \quad y \geq Rl_0(\beta) - \beta. \quad (5.17)$$

Now let us take the starting configuration to be $(Q_1(0), Q_2(0)) = (x, \frac{y}{2} + \beta)$ with $y \geq Rl_0(\beta) - \beta$ and $R \geq 5$. In that case, since $(d/dt)Q_2(t) \geq -Q_2(t)$, therefore $Q_2(t) \geq (y/2 + \beta)/2$ for all $t \leq \log 2$. Consequently, for any $t \leq \log 2$,

$$\begin{aligned} Q_1(t) &= Q_1(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-Q_1(s) + Q_2(s))ds - L(t) \\ &\geq x + \sqrt{2}W(t) - \beta t + \int_0^t Q_2(s)ds \\ &\geq x + \sqrt{2}W(t) + (y - 2\beta)t/4 \geq x + \sqrt{2}W(t) + yt/8. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{x \in [-\beta/2, 0]} \mathbb{P}_{(x, \frac{y}{2} + \beta)}(\tau_1(-\beta) < \tau_1(0) \leq \log 2) \\ &\leq \sup_{x \in [-\beta/2, 0]} \mathbb{P}_{(x, \frac{y}{2} + \beta)}\left(x + \sqrt{2}W(t) + yt/8 \text{ hits } -\beta \text{ before } 0\right) \\ &\leq \mathbb{P}\left(\sqrt{2}W(t) + yt/8 \text{ hits } -\beta/2 \text{ before } \beta/2\right) \leq e^{-\beta y/16} \end{aligned} \quad (5.18)$$

where the last step follows from standard scale function arguments. Moreover, for $y \geq Rl_0(\beta) - \beta$

with $R \geq 65$,

$$\begin{aligned}
& \sup_{x \in [-\beta/2, 0]} \mathbb{P}_{(x, \frac{y}{2} + \beta)} (\tau_1(0) > \log 2) \\
& \leq \sup_{x \in [-\beta/2, 0]} \mathbb{P}_{(x, \frac{y}{2} + \beta)} \left(\sup_{t \leq \log 2} (x + \sqrt{2}W(t) + yt/8) < 0 \right) \\
& \leq \mathbb{P} \left(\sup_{t \leq \log 2} (\sqrt{2}W(t) + yt/8) < \beta/2 \right) \leq \mathbb{P} \left(\sqrt{2}W(\log 2) < -y/32 \right) \\
& \leq e^{-y^2/(4(32^2) \log 2)} \leq e^{-\beta y/(64 \log 2)}.
\end{aligned} \tag{5.19}$$

Using (5.17), (5.18) and (5.19) in (5.16), we obtain for $R \geq 65$, there exist positive constants C, C' not depending on β and R such that for all $y \geq Rl_0(\beta) - \beta$

$$\mathbb{P}_{(0, y + \beta)} \left(\tau_1(-\beta) < \sigma \left(\tau_2 \left(\frac{y}{2} + \beta \right) \right) \right) \leq Ce^{-C'\beta y}. \tag{5.20}$$

Now, for any $y \geq Rl_0(\beta) - \beta$, observe that the event $[\tau_1(-\beta/2) \leq \tau_2(Rl_0(\beta))]$ can be written as

$$[\tau_1(-\beta) \leq \tau_2(Rl_0(\beta))] \subseteq \bigcup_{k=1}^{\lfloor \log_2 \left(\frac{y}{Rl_0(\beta) - \beta} \right) + 2 \rfloor} \left[\sigma \left(\tau_2 \left(\frac{y}{2^{k-1}} + \beta \right) \right) < \tau_1(-\beta) < \sigma \left(\tau_2 \left(\frac{y}{2^k} + \beta \right) \right) \right],$$

and therefore,

$$\begin{aligned}
& \mathbb{P}_{(0, y + \beta)} (\tau_1(-\beta) \leq \tau_2(Rl_0(\beta))) \\
& \leq \sum_{k=1}^{\lfloor \log_2 \left(\frac{y}{Rl_0(\beta) - \beta} \right) + 2 \rfloor} \mathbb{P}_{(0, y + \beta)} \left(\sigma \left(\tau_2 \left(\frac{y}{2^{k-1}} + \beta \right) \right) < \tau_1(-\beta) < \sigma \left(\tau_2 \left(\frac{y}{2^k} + \beta \right) \right) \right).
\end{aligned} \tag{5.21}$$

Take any $R \geq 260$. By the strong Markov property, for each $k \leq \lfloor \log_2 \left(\frac{y}{Rl_0(\beta) - \beta} \right) + 2 \rfloor$,

$$\begin{aligned}
& \mathbb{P}_{(0, y + \beta)} \left(\sigma \left(\tau_2 \left(\frac{y}{2^{k-1}} + \beta \right) \right) < \tau_1(-\beta) < \sigma \left(\tau_2 \left(\frac{y}{2^k} + \beta \right) \right) \right) \\
& \leq \sup_{z \in [y/2^k, y/2^{k-1}]} \mathbb{P}_{(0, z + \beta)} \left(\tau_1(-\beta) < \sigma \left(\tau_2 \left(\frac{y}{2^k} + \beta \right) \right) \right) \\
& \leq \sup_{z \in [y/2^k, y/2^{k-1}]} \mathbb{P}_{(0, z + \beta)} \left(\tau_1(-\beta) < \sigma \left(\tau_2 \left(\frac{z}{2} + \beta \right) \right) \right) \leq Ce^{-C'\beta y/2^k},
\end{aligned}$$

where the last inequality follows from (5.20) as for $k \leq \lfloor \log_2 \left(\frac{y}{Rl_0(\beta) - \beta} \right) + 2 \rfloor$, $\frac{y}{2^k} \geq \frac{Rl_0(\beta) - \beta}{4} \geq \frac{R}{4}l_0(\beta) - \beta$ and $\frac{R}{4} \geq 65$.

Writing $p(\beta, R) = C_1 e^{-C_2 \beta (Rl_0(\beta) - \beta)/4}$ and using the above bound in (5.21), we obtain $R_1 > 0$ such that for any $R \geq R_1$ and any $y \geq Rl_0(\beta) - \beta$,

$$\mathbb{P}_{(0, y + \beta)} (\tau_1(-\beta) \leq \tau_2(Rl_0(\beta))) \leq \sum_{k=1}^{\lfloor \log_2 \left(\frac{y}{Rl_0(\beta) - \beta} \right) + 2 \rfloor} C_1 e^{-C_2 \beta y/2^k} \tag{5.22}$$

$$\leq \sum_{k=0}^{\infty} p(\beta, R)^{2^k} \leq \sum_{k=0}^{\infty} p(\beta, R_1)^{2^k} =: p^{**}(\beta) < 1, \tag{5.23}$$

where the second inequality can be seen as follows: For any $y \geq Rl_0(\beta) - \beta$, the last term in the sum in (5.22) is bounded above by $p(\beta, R)$. Also, starting from the last term and counting backwards in k , observe that each next term is the square of the previous term, which provides the 2^k in the exponent of $p(\beta, R)$ in (5.23). Now, it is straightforward to see that for a fixed β the first sum in (5.23) is a decreasing function in R , and is bounded away from 1 for all large enough R . This proves the lemma. \square

Lemma 5.6. *There exists a constant $R_2 > 0$ not depending on β such that for any $R \geq R_2$, there is a constant $C_2(\beta, R) > 0$ (depending on β, R) satisfying*

$$\sup_{z \in [-\beta, 0], y \geq 2Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(-x) < \tau_2(Rl_0(\beta))) \leq C_2(\beta, R)e^{-(x-\beta)^2/2}, \quad \text{for all } x \geq \beta + 1. \quad (5.24)$$

Proof. Take any $R > 0$. Let $(Q_1(0), Q_2(0)) = (z, y)$ where $z \in [-\beta, 0]$ and $y \geq 2Rl_0(\beta)$. Define the stopping times: $\sigma^{(0)} = 0$ and for $k \geq 0$,

$$\begin{aligned} \sigma^{(2k+1)} &= \inf\{t \geq \sigma^{(2k)} : Q_1(t) = -\beta - 1 \text{ or } Q_2(t) \leq Rl_0(\beta)\}, \\ \sigma^{(2k+2)} &= \inf\{t \geq \sigma^{(2k+1)} : Q_1(t) = -\beta \text{ or } Q_2(t) \leq Rl_0(\beta)\}. \end{aligned}$$

Define $\mathcal{N}^\sigma = \inf\{n \geq 1 : Q_2(\sigma^{(n)}) \leq Rl_0(\beta)\}$. Observe that for any $z \in [-\beta, 0]$, by the strong Markov property, we obtain

$$\begin{aligned} &\sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(-\beta - 1) < \tau_2(Rl_0(\beta))) \\ &\leq \sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(0) < \tau_1(-\beta - 1) < \tau_2(Rl_0(\beta))) \\ &\quad + \sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(-\beta - 1) < \tau_1(0) \wedge \tau_2(Rl_0(\beta))) \\ &\leq \sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(0, y)}(\tau_1(-\beta) < \tau_2(Rl_0(\beta))) \\ &\quad + \sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(-\beta - 1) < \tau_1(0) \wedge \tau_2(Rl_0(\beta))). \end{aligned} \quad (5.25)$$

By Lemma 5.5, for large enough R ,

$$\sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(0, y)}(\tau_1(-\beta) < \tau_2(Rl_0(\beta))) \leq p^{**}(\beta) < 1. \quad (5.26)$$

Further, observe that for $t \leq \tau_1(0) \wedge \tau_2(Rl_0(\beta))$,

$$Q_1(t) \geq z + \sqrt{2}W(t) + (Rl_0(\beta) - \beta)t \geq -\beta + \sqrt{2}W(t) + (Rl_0(\beta) - \beta)t.$$

Therefore,

$$\begin{aligned} &\sup_{y \geq Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(-\beta - 1) < \tau_1(0) \wedge \tau_2(Rl_0(\beta))) \\ &\leq \mathbb{P}(-\beta + \sqrt{2}W(t) + (Rl_0(\beta) - \beta)t \text{ hits } -\beta - 1 \text{ before } 0) \leq e^{-(Rl_0(\beta) - \beta)}. \end{aligned} \quad (5.27)$$

Using (5.26) and (5.27) in (5.25), we conclude that there is $R_2 > 0$ such that for all $R \geq R_2$,

$$\sup_{z \in [-\beta, 0], y \geq Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(-\beta - 1) < \tau_2(Rl_0(\beta))) \leq p'(\beta, R) < 1. \quad (5.28)$$

Using (5.28) and the strong Markov property, there exists a constant $C(\beta, R) > 0$ depending on β, R such that

$$\sup_{z \in [-\beta, 0], y \geq 2Rl_0(\beta)} \mathbb{E}_{(z, y)}(\mathcal{N}^\sigma) \leq 2 \sum_{n=0}^{\infty} \mathbb{P}(\mathcal{N}^\sigma > 2n) \leq 2 \sum_{n=0}^{\infty} p'(\beta, R)^n \leq C(\beta, R) < \infty. \quad (5.29)$$

For $(Q_1(0), Q_2(0)) = (-\beta - u, y)$ for any $u \geq 1, y > 0$, by [12, Proposition 2.18], a process Z can be constructed on the same probability space as (Q_1, Q_2) , such that $Q_1(t) + \beta \geq Z(t)$ for $t \leq \tau_1(0)$, where Z is an Ornstein-Uhlenbeck process which solves the SDE:

$$dZ(t) = \sqrt{2}dW(t) - Z(t)dt, \quad Z(0) = -u.$$

The scale function for Z is given by $s_Z(z) = \int_0^z e^{w^2/2} dw$. From this observation and elementary estimates on s_Z , we have for any $x \geq \beta + u$,

$$\begin{aligned} \sup_{y > 0} \mathbb{P}_{(-\beta - u, y)}(\tau_1(-x) < \tau_1(-\beta)) &\leq \mathbb{P}(Z(t) \text{ hits } -x + \beta \text{ before } 0) \\ &= \frac{s_Z(0) - s_Z(-u)}{s_Z(0) - s_Z(-x + \beta)} \leq \sqrt{9\pi/2} e^{u^2/2} e^{-(x-\beta)^2/2}. \end{aligned} \quad (5.30)$$

Finally, using (5.29) and (5.30) along with the strong Markov property, for any $R \geq R_2$ and any $x \geq \beta + 1$,

$$\begin{aligned} \sup_{\substack{z \in [-\beta, 0], \\ y \geq 2Rl_0(\beta)}} \mathbb{P}_{(z, y)}(\tau_1(-x) < \tau_2(Rl_0(\beta))) &= \sup_{\substack{z \in [-\beta, 0], \\ y \geq 2Rl_0(\beta)}} \mathbb{P}_{(z, y)} \left(\inf_{t \leq \sigma^{(\mathcal{N}^\sigma)}} Q_1(t) < -x \right) \\ &\leq \sup_{\substack{z \in [-\beta, 0], \\ y \geq 2Rl_0(\beta)}} \sum_{k=0}^{\infty} \mathbb{P}_{(z, y)} \left(\inf_{t \in [\sigma^{(2k+1)}, \sigma^{(2k+2)}} Q_1(t) < -x, \mathcal{N}^\sigma \geq 2k+2 \right) \\ &\leq \sup_{\substack{z \in [-\beta, 0], \\ y \geq 2Rl_0(\beta)}} \sum_{k=0}^{\infty} \mathbb{E}_{(z, y)} \mathbb{1}_{[\mathcal{N}^\sigma \geq 2k+2]} \sup_{y > 0} \mathbb{P}_{(-\beta - 1, y)}(\tau_1(-x) < \tau_1(-\beta)) \\ &\leq \sup_{\substack{z \in [-\beta, 0], \\ y \geq 2Rl_0(\beta)}} \mathbb{E}_{(z, y)}(\mathcal{N}^\sigma) \sup_{y > 0} \mathbb{P}_{(-\beta - 1, y)}(\tau_1(-x) < \tau_1(-\beta)) \leq C_2(\beta, R) e^{-(x-\beta)^2/2} \end{aligned}$$

where $C_2(\beta, R) > 0$ is a constant depending on β, R . This proves the lemma. \square

Lemma 5.7. *For any $R > 1$ and any $x \geq 18Rl_0(\beta)$, there exists a constant $C_3(\beta, R) > 0$ (depending on β, R) such that*

$$\sup_{z \in [-9Rl_0(\beta), 0], y \leq 2Rl_0(\beta)} \mathbb{P}_{(z, y)}(\tau_1(-x) < \tau_2(2Rl_0(\beta))) \leq C_3(\beta, R) e^{-(x-\beta)^2/2}.$$

Proof. Fix any $R > 1$, $Q_1(0) = z \geq -9Rl_0(\beta)$ and $Q_2(0) = y \leq 2Rl_0(\beta)$. Define the stopping times: $\gamma^{(0)} = 0$ and for $k \geq 0$,

$$\begin{aligned} \gamma^{(2k+1)} &= \inf\{t \geq \gamma^{(2k)} : Q_1(t) = -18Rl_0(\beta) \text{ or } Q_2(t) = 2Rl_0(\beta)\}, \\ \gamma^{(2k+2)} &= \inf\{t \geq \gamma^{(2k+1)} : Q_1(t) = -9Rl_0(\beta) \text{ or } Q_2(t) = 2Rl_0(\beta)\}. \end{aligned}$$

Define $\mathcal{N}^\gamma = \inf\{n \geq 1 : Q_2(\gamma^{(n)}) = 2Rl_0(\beta)\}$. Taking $B = 2Rl_0(\beta)$ and $M = 18Rl_0(\beta)$ in Lemma 4.11, we know there exists $q(\beta, R)$ such that

$$\begin{aligned} & \inf_{z \in [-9Rl_0(\beta), 0], y \leq 2Rl_0(\beta)} \mathbb{P}_{(z,y)}(\tau_2(2Rl_0(\beta)) < \tau_1(-18Rl_0(\beta))) \\ & \geq \inf_{z \in [-9Rl_0(\beta), 0], y \leq 2Rl_0(\beta)} \mathbb{P}_{(z,y)}(\tau_2(4Rl_0(\beta)) < \tau_1(-18Rl_0(\beta))) \geq q(\beta, R) > 0. \end{aligned} \quad (5.31)$$

Using (5.31) and the strong Markov property, there exists a constant $C(\beta, R) > 0$ depending on β, R such that

$$\begin{aligned} \sup_{z \in [-9Rl_0(\beta), 0], y \leq 2Rl_0(\beta)} \mathbb{E}_{(z,y)}(\mathcal{N}^\gamma) & \leq 2 \sum_{n=0}^{\infty} \mathbb{P}(\mathcal{N}^\gamma > 2n) \\ & \leq 2 \sum_{n=0}^{\infty} (1 - q(\beta, R))^n \leq C(\beta, R) < \infty. \end{aligned} \quad (5.32)$$

Using (5.32) and (5.30) along with the strong Markov property, we obtain for any $x \geq 18Rl_0(\beta)$,

$$\begin{aligned} & \sup_{\substack{z \in [-9Rl_0(\beta), 0], \\ y \leq 2Rl_0(\beta)}} \mathbb{P}_{(z,y)}(\tau_1(-x) < \tau_2(2Rl_0(\beta))) \\ & = \sup_{\substack{z \in [-9Rl_0(\beta), 0], \\ y \leq 2Rl_0(\beta)}} \mathbb{P}_{(z,y)}\left(\inf_{t \leq \gamma^{(\mathcal{N}^\gamma)}} Q_1(t) < -x\right) \\ & \leq \sup_{\substack{z \in [-9Rl_0(\beta), 0], \\ y \leq 2Rl_0(\beta)}} \sum_{k=0}^{\infty} \mathbb{P}_{(z,y)}\left(\inf_{t \in [\gamma^{(2k+1)}, \gamma^{(2k+2)}} Q_1(t) < -x, \mathcal{N}^\gamma \geq 2k+2\right) \\ & \leq \sup_{\substack{z \in [-9Rl_0(\beta), 0], \\ y \leq 2Rl_0(\beta)}} \sum_{k=0}^{\infty} \mathbb{E}_{(z,y)} \mathbb{1}_{[\mathcal{N}^\gamma \geq 2k+2]} \sup_{y>0} \mathbb{P}_{(-18Rl_0(\beta), y)}(\tau_1(-x) < \tau_1(-\beta)) \\ & \leq \sup_{\substack{z \in [-9Rl_0(\beta), 0], \\ y \leq 2Rl_0(\beta)}} \mathbb{E}_{(z,y)}(\mathcal{N}^\gamma) \sup_{y>0} \mathbb{P}_{(-18Rl_0(\beta), y)}(\tau_1(-x) < \tau_1(-\beta)) \\ & \leq C_3(\beta, R) e^{-(x-\beta)^2/2} \end{aligned}$$

for some constant $C_3(\beta, R) > 0$ depending on β, R . This proves the lemma. \square

Now, we are in a position to give an upper bound to the fluctuations of Q_1 between two successive regeneration times Ξ_k and Ξ_{k+1} , $k \geq 0$, defined in (3.1) taking $B = Rl_0(\beta)$ for sufficiently large fixed R .

Lemma 5.8. *Fix any $R \geq \max\{2, R_1, R_2\}$, where R_1 and R_2 are obtained from Lemmas 5.5 and 5.6 respectively. Let $(Q_1(0), Q_2(0)) = (0, 2Rl_0(\beta))$ and take $B = Rl_0(\beta)$ in (3.1). There exists a constant $C^*(\beta, R) > 0$ depending on β, R such that for any $x \geq 18Rl_0(\beta)$,*

$$\mathbb{P}_{(0, 2Rl_0(\beta))}\left(\inf_{t \leq \Xi_0} Q_1(t) < -x\right) \leq C^*(\beta, R) e^{-(x-2\beta)^2/8}.$$

Proof. Choose and fix $R \geq \max\{2, R_1, R_2\}$. Define

$$\Xi^* = \inf \{t \geq \tau_2(\mathbf{R}l_0(\beta)) : Q_1(t) \geq -\beta - 1\}.$$

Then for any $x \geq 2(\beta + 1)$, by Lemma 5.6 and (5.30) along with the strong Markov property,

$$\begin{aligned} & \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{\tau_2(\mathbf{R}l_0(\beta)) \leq t \leq \Xi^*} Q_1(t) < -x \right) \\ & \leq \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{\tau_2(\mathbf{R}l_0(\beta)) \leq t \leq \Xi^*} Q_1(t) < -x, Q_1(\tau_2(\mathbf{R}l_0(\beta))) \geq -x/2 \right) \\ & \quad + \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} (\tau_1(-x/2) < \tau_2(\mathbf{R}l_0(\beta))) \quad (5.33) \\ & \leq \sup_{u \in [1, \frac{x}{2} - \beta]} \mathbb{P}_{(-\beta - u, \mathbf{R}l_0(\beta))} (\tau_1(-x) < \tau_1(-\beta)) + \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} (\tau_1(-x/2) < \tau_2(\mathbf{R}l_0(\beta))) \\ & \leq \sqrt{9\pi/2} e^{(\frac{x}{2} - \beta)^2/2} e^{-(x - \beta)^2/2} + C_2(\beta, R) e^{-(\frac{x}{2} - \beta)^2/2} \\ & \leq (\sqrt{9\pi/2} + C_2(\beta, R)) e^{-(x - 2\beta)^2/8}. \end{aligned}$$

Therefore, for any $x \geq 18\mathbf{R}l_0(\beta)$, using (5.33) along with Lemmas 5.6 and 5.7,

$$\begin{aligned} & \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{t \leq \Xi_0} Q_1(t) < -x \right) \leq \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{t \leq \tau_2(\mathbf{R}l_0(\beta))} Q_1(t) < -x \right) \\ & \quad + \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{\tau_2(\mathbf{R}l_0(\beta)) \leq t \leq \Xi^*} Q_1(t) < -x \right) + \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{\Xi^* \leq t \leq \Xi_0} Q_1(t) < -x \right) \\ & \leq \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} (\tau_1(-x) < \tau_2(\mathbf{R}l_0(\beta))) + \mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{\tau_2(\mathbf{R}l_0(\beta)) \leq t \leq \Xi^*} Q_1(t) < -x \right) \\ & \quad + \sup_{z \in [-9\mathbf{R}l_0(\beta), 0], y \leq 2\mathbf{R}l_0(\beta)} \mathbb{P}_{(z, y)} (\tau_1(-x) < \tau_2(2\mathbf{R}l_0(\beta))) \\ & \leq C_2(\beta, R) e^{-(x - \beta)^2/2} + (C_2(\beta, R) + \sqrt{9\pi/2}) e^{-(x - 2\beta)^2/8} + C_3(\beta, R) e^{-(x - \beta)^2/2} \\ & \leq C^*(\beta, R) e^{-(x - 2\beta)^2/8} \end{aligned}$$

which proves the lemma. \square

Now, we prove a lower bound for the fluctuation of Q_1 .

Lemma 5.9. *Let $(Q_1(0), Q_2(0)) = (0, 2\mathbf{R}l_0(\beta))$ and take $B = \mathbf{R}l_0(\beta)$ in (3.1). There exist constants $R^{**} > 0$ not depending on β such that for any $R \geq R^{**}$ and any $x \geq \beta$,*

$$\mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} \left(\inf_{t \leq \Xi_0} Q_1(t) < -x \right) \geq C^{**}(\beta, R) e^{-x^2},$$

where the positive constant $C^{**}(\beta, R)$ depends on both β and R .

Proof. Using $y = 2\mathbf{R}l_0(\beta) - \beta$ in Lemma 5.1, we observe that there exists $R^{**} > 0$ such that for all $R \geq R^{**}$, there is a constant $q_1(\beta, R) > 0$ (depending on β, R) for which

$$\mathbb{P}_{(0, 2\mathbf{R}l_0(\beta))} (\tau_1(-\beta/2) > \tau_2(\mathbf{R}l_0(\beta) + \beta/2)) \geq q_1(\beta, R) > 0. \quad (5.34)$$

Recall $S(t) = Q_1(t) + Q_2(t)$. Recall that $Q_1(t) \leq S(t) \leq Q_2(t)$ for every t , and when $Q_1(0) \in [0, \beta/2]$,

$$S(t) = S(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-Q_1(s)) ds \leq S(0) + \sqrt{2}W(t) - \frac{\beta}{2}t$$

for $t \leq \tau_1(-\beta/2)$. Moreover, observe that if $Q_2(0) \leq 2Rl_0(\beta)$, then $Q_1(\tau_2(2Rl_0(\beta))) = 0$ and consequently, $S(\tau_2(2Rl_0(\beta))) = Q_2(\tau_2(2Rl_0(\beta))) = 2Rl_0(\beta)$. Thus,

$$\begin{aligned}
& \sup_{z \in [-\beta/2, 0]} \mathbb{P}_{(z, Rl_0(\beta) + \beta/2)} (\tau_2(2Rl_0(\beta)) < \tau_1(-\beta/2)) \\
& \leq \sup_{z \in [-\beta/2, 0]} \mathbb{P}_{(z, Rl_0(\beta) + \beta/2)} (S(t) \text{ hits } 2Rl_0(\beta) \text{ before } -\beta/2) \\
& \leq \sup_{z \in [-\beta/2, 0]} \mathbb{P}_{(z, Rl_0(\beta) + \beta/2)} \left(z + Rl_0(\beta) + \beta/2 + \sqrt{2}W(t) - \frac{\beta}{2}t \text{ hits } 2Rl_0(\beta) \text{ before } -\beta/2 \right) \\
& \leq \mathbb{P} \left(\sqrt{2}W(t) - \frac{\beta}{2}t \text{ hits } Rl_0(\beta) - \beta/2 \right) \leq e^{-\beta(Rl_0(\beta) - \beta/2)/2} =: 1 - q_2(\beta, R) < 1.
\end{aligned} \tag{5.35}$$

For $y \leq 2Rl_0(\beta)$ and $(Q_1(0), Q_2(0)) = (-\beta/2, y)$, by [12, Proposition 2.18], a process U can be constructed on the same probability space as (Q_1, Q_2) such that almost surely $Q_1(t) + \beta \leq U(t)$ for all $t \leq \tau_1(0)$, where U is an Ornstein-Uhlenbeck process which solves the SDE:

$$dU(t) = \sqrt{2}dW(t) + (2Rl_0(\beta) - U(t))dt, \quad U(0) = \beta/2.$$

The scale function for U is given by $s_U(u) = \int_0^u e^{(w - 2Rl_0(\beta))^2/2} dw$. Therefore, by elementary estimates on s_U , there exists a constant $C(\beta, R) > 0$ (depending on β, R) such that for any $x \geq \beta$,

$$\begin{aligned}
\inf_{y \leq 2Rl_0(\beta)} \mathbb{P}_{(-\beta/2, y)} (\tau_1(-x) < \tau_1(0)) & \geq \mathbb{P}(U(t) \text{ hits } -(x - \beta) \text{ before } \beta) \\
& = \frac{s_U(\beta) - s_U(\beta/2)}{s_U(\beta) - s_U(-(x - \beta))} \geq C(\beta, R)e^{-x^2}. \tag{5.36}
\end{aligned}$$

Recall the notation $\sigma(t) = \inf\{s \geq t : Q_1(s) = 0\}$ and define the stopping time

$$\sigma_R = \inf\{t > \tau_2(Rl_0(\beta) + \beta/2) : Q_2(t) = 2Rl_0(\beta)\}.$$

From (5.34), (5.35) and (5.36) and the strong Markov property, for any $R \geq R^{**}$ and any $x \geq \beta$,

$$\begin{aligned}
& \mathbb{P}_{(0, 2Rl_0(\beta))} \left(\inf_{t \leq \Xi_0} Q_1(t) < -x \right) \\
& \geq \mathbb{P}_{(0, 2Rl_0(\beta))} (\tau_2(Rl_0(\beta) + \beta/2) < \tau_1(-\beta/2) < \sigma_R, \tau_1(-x) \in (\tau_1(-\beta/2), \sigma(\tau_1(-\beta/2)))) \\
& \geq \mathbb{P}_{(0, 2Rl_0(\beta))} (\tau_1(-\beta/2) > \tau_2(Rl_0(\beta) + \beta/2)) \times \inf_{z \in [-\beta/2, 0]} \mathbb{P}_{(z, Rl_0(\beta) + \beta/2)} (\tau_1(-\beta/2) < \tau_2(2Rl_0(\beta))) \\
& \quad \times \inf_{y \leq 2Rl_0(\beta)} \mathbb{P}_{(-\beta/2, y)} (\tau_1(-x) < \tau_1(0)) \\
& \geq q_1(\beta, R)q_2(\beta, R)C(\beta, R)e^{-x^2}.
\end{aligned}$$

This proves the lemma. □

Proof of Theorem 3.4. Fix any

$$R_0 \geq 4 \max\{64, \log(4\tilde{C}_1)/\tilde{C}_2, R_1, R_2, R^{**}\}, \tag{5.37}$$

where where R_1, R_2 and R^{**} are obtained from Lemmas 5.5, 5.6 and 5.9 respectively and \tilde{C}_1, \tilde{C}_2 are the constants defined in the statement of Lemma 5.2. Choose $B = R_0l_0(\beta)$ in (3.1).

To prove (i), note that y_0 defined in Lemma 5.3 satisfies $y_0 + \beta < R_0 l_0(\beta)$ for our specific choice of R_0 . Therefore, taking $z = \frac{y-\beta}{2}$ in place of y in Lemma 5.3 and applying the strong Markov property at $\tau_2(z + \beta)$, we have for any $y \geq 4R_0 l_0(\beta)$,

$$\begin{aligned} \mathbb{P}_{(0, 2R_0 l_0(\beta))}(\tau_2(y) \leq \Xi_0) &= \mathbb{P}_{(0, 2R_0 l_0(\beta))}(\tau_2(y) \leq \tau_2(R_0 l_0(\beta))) \\ &\leq \mathbb{P}_{(0, z+\beta)}(\tau_2(2z + \beta) \leq \tau_2(R_0 l_0(\beta))) \leq \mathbb{P}_{(0, z+\beta)}(\tau_2(2z + \beta) \leq \tau_2(y_0 + \beta)) \leq C_1^* e^{-C_2^* \beta z}. \end{aligned}$$

Part (ii) follows from Lemma 5.4 by taking $B = R_0 l_0(\beta)$. Parts (iii) and (iv) are direct consequences of Lemmas 5.8 and 5.9 respectively. \square

6 Proofs of the main results

Proof of Theorem 2.1. We will show that the tail bounds stated in the theorem hold with $C_R(\beta) = 18R_0 l_0(\beta)$ and $D_R(\beta) = 4R_0 l_0(\beta)$, where R_0 is defined in (5.37) and $l_0(\beta)$ was defined in (3.3). Taking $B = R_0 l_0(\beta)$ in Theorem 3.3, note that for any $x \geq 0, y > 0$,

$$\begin{aligned} \pi(Q_1(\infty) < -x) &= \frac{\mathbb{E}_{(0, 2R_0 l_0(\beta))} \left(\int_0^{\Xi_0} \mathbb{1}_{[Q_1(s) < -x]} ds \right)}{\mathbb{E}_{(0, 2R_0 l_0(\beta))}(\Xi_0)}, \\ \pi(Q_2(\infty) > y) &= \frac{\mathbb{E}_{(0, 2R_0 l_0(\beta))} \left(\int_0^{\Xi_0} \mathbb{1}_{[Q_2(s) > y]} ds \right)}{\mathbb{E}_{(0, 2R_0 l_0(\beta))}(\Xi_0)}. \end{aligned} \quad (6.1)$$

To prove the theorem, we only need to estimate the numerators in the above representation. By the Cauchy-Schwarz inequality, for $x \geq 18R_0 l_0(\beta)$,

$$\begin{aligned} \mathbb{E}_{(0, 2R_0 l_0(\beta))} \left(\int_0^{\Xi_0} \mathbb{1}_{[Q_1(s) < -x]} ds \right) &\leq \mathbb{E}_{(0, 2R_0 l_0(\beta))}(\mathbb{1}_{[\tau_1(-x)] < \Xi_0} (\Xi_0 - \tau_1(-x))) \\ &\leq \sqrt{\mathbb{P}_{(0, 2R_0 l_0(\beta))}(\tau_1(-x) < \Xi_0)} \sqrt{\mathbb{E}_{(0, 2R_0 l_0(\beta))}(\Xi_0)^2} \leq \sqrt{C^*(\beta)} e^{-(x-2\beta)^2/16} \sqrt{\mathbb{E}_{(0, 2R_0 l_0(\beta))}(\Xi_0^2)}, \end{aligned}$$

where the last inequality is a consequence of Part (iii) of Theorem 3.4. By Proposition 3.2, $\mathbb{E}_{(0, 2R_0 l_0(\beta))}(\Xi_0^2) < \infty$. Now, using this in the above bound, we obtain the upper bound on $\pi(Q_1(\infty) < -x)$ claimed in the theorem. The upper bound for $\pi(Q_2(\infty) > y)$ is obtained similarly using part (i) of Theorem 3.4.

To obtain the lower bound on $\pi(Q_1(\infty) < -x)$, we proceed along the same line of arguments as in the proof of Lemma 5.9. Recall the stopping time

$$\sigma_R = \inf\{t > \tau_2(Rl_0(\beta) + \beta/2) : Q_2(t) = 2Rl_0(\beta)\}.$$

Observe that for $x \geq \beta$,

$$\begin{aligned}
& \mathbb{E}_{(0, 2R_0 l_0(\beta))} \left(\int_0^{\bar{x}_0} \mathbb{1}_{[Q_1(s) < -x]} ds \right) \\
& \geq \mathbb{E}_{(0, 2R_0 l_0(\beta))} \left(\mathbb{1}_{[\tau_2(R_0 l_0(\beta) + \beta/2) < \tau_1(-\beta/2) < \sigma_{R_0}] } \int_{\tau_1(-\beta/2)}^{\sigma(\tau_1(-\beta/2))} \mathbb{1}_{[Q_1(s) < -x]} ds \right) \\
& \geq \mathbb{P}_{(0, 2R_0 l_0(\beta))} (\tau_2(R_0 l_0(\beta) + \beta/2) < \tau_1(-\beta/2)) \times \inf_{y \leq 2R_0 l_0(\beta)} \mathbb{E}_{(-\beta/2, y)} \left(\int_0^{\tau_1(0)} \mathbb{1}_{[Q_1(s) < -x]} ds \right) \\
& \quad \times \inf_{z \in [-\beta/2, 0]} \mathbb{P}_{(z, R_0 l_0(\beta) + \beta/2)} (\tau_1(-\beta/2) < \tau_2(2R_0 l_0(\beta))) \\
& \geq q_1(\beta, R_0) q_2(\beta, R_0) \inf_{y \leq 2R_0 l_0(\beta)} \mathbb{E}_{(-\beta/2, y)} \left(\int_0^{\tau_1(0)} \mathbb{1}_{[Q_1(s) < -x]} ds \right)
\end{aligned}$$

where $q_1(\beta, R_0) > 0$, $q_2(\beta, R_0) > 0$ are obtained in (5.34) and (5.35) respectively with R_0 in place of R .

Recall that for $y \leq 2R_0 l_0(\beta)$ and $(Q_1(0), Q_2(0)) = (-\beta/2, y)$, by [12, Proposition 2.18], a process $U_{\beta/2}$ can be constructed on the same probability space as (Q_1, Q_2) , such that $Q_1(t) + \beta \leq U_{\beta/2}(t)$ for $t \leq \tau_1(0)$, where U_z is an Ornstein-Uhlenbeck process which solves the SDE:

$$dU_z(t) = \sqrt{2}dW(t) + (2Rl_0(\beta) - U(t))dt, \quad U_z(0) = z.$$

where the scale function for U_z is given by $s_U(u) = \int_0^u e^{(w-2Rl_0(\beta))^2/2} dw$.

Define $\tau_z^U(w) = \inf\{t \geq 0 : U_z(t) = w\}$ and write the law of U_z and the corresponding expectation as \mathbb{P}_z^U and \mathbb{E}_z^U respectively. Then, for $x \geq \beta$,

$$\begin{aligned}
& \inf_{y \leq 2R_0 l_0(\beta)} \mathbb{E}_{(-\beta/2, y)} \left(\int_0^{\tau_1(0)} \mathbb{1}_{[Q_1(s) < -x]} ds \right) \geq \mathbb{E}_{\beta/2}^U \left(\int_0^{\tau_{\beta/2}^U(\beta)} \mathbb{1}_{[U_{\beta/2}(s) < -x + \beta]} ds \right) \\
& \geq \mathbb{P}_{\beta/2}^U \left(\tau_{\beta/2}^U(-2x + \beta) < \tau_{\beta/2}^U(\beta) \right) \mathbb{E}_{-2x + \beta}^U \left(\tau_{-2x + \beta}^U(-x + \beta) \right), \quad \text{by strong Markov property,} \\
& = \frac{s_U(\beta) - s_U(\beta/2)}{s_U(\beta) - s_U(-2x + \beta)} \mathbb{E}_{-2x + \beta}^U \left(\tau_{-2x + \beta}^U(-x + \beta) \right) \geq C(\beta) e^{-4x^2} \mathbb{E}_{-2x + \beta}^U \left(\tau_{-2x + \beta}^U(-x + \beta) \right)
\end{aligned} \tag{6.2}$$

where $C(\beta)$ is a positive constant that only depends on β . Now, from the Doob representation of Ornstein-Uhlenbeck process,

$$U_{-2x + \beta}(t) = (-2x + \beta)e^{-t} + 2R_0 l_0(\beta)(1 - e^{-t}) + e^{-t} \widetilde{W}(e^{2t} - 1)$$

for a standard Brownian motion \widetilde{W} . Therefore, taking $T = \log(5/4)$, for $x \geq 4R_0 l_0(\beta)$,

$$\begin{aligned}
& \mathbb{P}_{-2x+\beta}^{\mathbb{U}} \left(\tau_{-2x+\beta}^{\mathbb{U}}(-x+\beta) \leq T \right) \\
& \leq \mathbb{P} \left(\sup_{t \leq T} \left((-2x+\beta)e^{-t} + 2R_0 l_0(\beta)(1-e^{-t}) + e^{-t} \widetilde{W}(e^{2t}-1) \right) > -x+\beta \right) \\
& \leq \mathbb{P} \left((-2x+\beta)e^{-T} + 2R_0 l_0(\beta)(1-e^{-T}) + \sup_{t \leq T} \left(\widetilde{W}(e^{2t}-1) \right) > -x+\beta \right) \\
& \leq \mathbb{P} \left(\sup_{t \leq T} \left(\widetilde{W}(e^{2t}-1) \right) > x/2 \right) \quad \text{by our choice of } T \\
& = \mathbb{P} \left(\sup_{t \leq 1} \widetilde{W}(t) > \frac{x}{2\sqrt{\exp(2T)-1}} \right) \quad \text{by Brownian scaling} \leq \frac{4\sqrt{\exp(2T)-1}}{\sqrt{2\pi x}} < \frac{1}{2}.
\end{aligned}$$

Thus,

$$\mathbb{E}_{-2x+\beta}^{\mathbb{U}} \left(\tau_{-2x+\beta}^{\mathbb{U}}(-x+\beta) \right) = \int_0^\infty \mathbb{P}_{-2x+\beta}^{\mathbb{U}} \left(\tau_{-2x+\beta}^{\mathbb{U}}(-x+\beta) > t \right) dt \geq \frac{1}{2} \log(5/4).$$

Using this in (6.2) gives us the lower bound on $\pi(Q_1(\infty) < -x)$ claimed in the theorem.

Finally, we prove the lower bound on $\pi(Q_2(\infty) > y)$. Note that by the strong Markov property, for any $y \geq R_0 l_0(\beta)$,

$$\begin{aligned}
\mathbb{E}_{(0,2R_0 l_0(\beta))} \left(\int_0^{\Xi_0} \mathbb{1}_{[Q_2(s) > y]} ds \right) & \geq \mathbb{P}_{(0,2R_0 l_0(\beta))} \left(\tau_2(2y) \leq \Xi_0 \right) \times \mathbb{E}_{(0,2y)} \left(\tau_2(y) \right) \\
& \geq (1 - e^{-\beta R_0 l_0(\beta)}) e^{-\beta(2y-2R_0 l_0(\beta))} \mathbb{E}_{(0,2y)} \left(\tau_2(y) \right)
\end{aligned} \tag{6.3}$$

where the last step follows from Part (ii) of Theorem 3.4. Recall that

$$Q_2(t) \geq S(t) \geq S(0) + \sqrt{2}W(t) - \beta t, \quad t \geq 0,$$

where $S(t) = Q_1(t) + Q_2(t)$. Therefore, starting with $(Q_1(0), Q_2(0)) = (0, 2y)$, the hitting time of level y of Q_2 is stochastically bounded below by the hitting time of y by $S(0) + \sqrt{2}W(t) - \beta t$. Denoting the latter hitting time by $\tau^S(y)$, we obtain $\mathbb{E}_{(0,2y)} \left(\tau_2(y) \right) \geq \mathbb{E}_{(0,2y)} \left(\tau^S(y) \right)$. For $y \geq R_0 l_0(\beta)$,

$$\begin{aligned}
\mathbb{P}_{(0,2y)} \left(\tau^S(y) \leq \frac{y}{2\beta} \right) & = \mathbb{P} \left(\inf_{t \leq \frac{y}{2\beta}} \left(2y + \sqrt{2}W(t) - \beta t \right) < y \right) \leq \mathbb{P} \left(\inf_{t \leq \frac{y}{2\beta}} \left(\sqrt{2}W(t) \right) < -y/2 \right) \\
& = \mathbb{P} \left(\inf_{t \leq 1} \left(W(t) \right) < -\sqrt{\beta y}/2 \right) \leq \frac{4}{\sqrt{2\pi\beta y}} \leq \frac{4}{\sqrt{2\pi R_0}} < \frac{1}{2},
\end{aligned}$$

for our choice of R_0 . This gives

$$\mathbb{E}_{(0,2y)} \left(\tau^S(y) \right) = \int_0^\infty \mathbb{P}_{(0,2y)} \left(\tau^S(y) > t \right) dt \geq \frac{y}{4\beta}.$$

Using this in (6.3) gives us the lower bound on $\pi(Q_2(\infty) > y)$ claimed in the theorem. \square

Proof of Theorem 2.2. Below we provide the proof of the fluctuation result for Q_2 . The proof for Q_1 follows using analogous arguments.

Take \mathcal{C}^* in the theorem to be the positive constant C_2^* not depending on β that was obtained in Part (i) of Theorem 3.4. Fix $\epsilon \in (0, 1/2)$. Fix any starting point $(Q_1(0), Q_2(0)) = (x, y)$. Then by Parts (i) and (ii) of Theorem 3.4, we obtain constants $D_1(\beta)$ and $D_2(\beta)$ and an integer $N(\beta) > 0$ depending only on β and an such that for all $n \geq N(\beta)$,

$$\begin{aligned} \mathbb{P}_{(x,y)} \left(\sup_{t \in [\Xi_n, \Xi_{n+1}]} Q_2(t) > \frac{2(1+\epsilon) \log n}{C_2^* \beta} \right) &\leq \frac{D_1(\beta)}{n^{1+\epsilon}}, \\ \mathbb{P}_{(x,y)} \left(\sup_{t \in [\Xi_n, \Xi_{n+1}]} Q_2(t) > \frac{(1-\epsilon) \log n}{\beta} \right) &\geq \frac{D_2(\beta)}{n^{1-\epsilon}}. \end{aligned}$$

Therefore, by the Borel-Cantelli Lemma,

$$\frac{1-\epsilon}{\beta} \leq \limsup_{n \rightarrow \infty} \frac{\sup_{t \in [\Xi_n, \Xi_{n+1}]} Q_2(t)}{\log n} \leq \frac{2(1+\epsilon)}{C_2^* \beta}, \quad \text{a.s.} \quad (6.4)$$

By Proposition 3.2, $\mathbb{E}_{(0,2R_0, l_0(\beta))}(\Xi_0) < \infty$ and as $\{\Xi_{n+1} - \Xi_n\}_{n \geq 0}$ are i.i.d., therefore by the Strong Law of Large Numbers,

$$\lim_{n \rightarrow \infty} \frac{\Xi_n}{n} \rightarrow \mathbb{E}_{(0,2R_0, l_0(\beta))}(\Xi_0), \quad \text{a.s.} \quad (6.5)$$

From the lower bound in (6.4), with probability one, there exists a subsequence $\{n_k\} \subseteq \{n\}$ and $t_{n_k} \in [\Xi_{n_k}, \Xi_{n_k+1}]$ such that

$$Q_2(t_{n_k}) \geq (1-2\epsilon) \frac{\log n_k}{\beta}$$

for all sufficiently large k . Moreover, by (6.5), almost surely,

$$\log t_{n_k} \leq \log \Xi_{n_k+1} = \log \left(\frac{\Xi_{n_k+1}}{n_k+1} \right) + \log(n_k+1) \leq (1+\epsilon) \log n_k$$

for all sufficiently large k . Therefore, almost surely, for all sufficiently large k ,

$$\frac{Q_2(t_{n_k})}{\log t_{n_k}} \geq \frac{1-2\epsilon}{(1+\epsilon)\beta}.$$

Since this holds for every $\epsilon \in (0, 1/2)$, we obtain

$$\limsup_{t \rightarrow \infty} \frac{Q_2(t)}{\log t} \geq \frac{1}{\beta}, \quad \text{a.s.}$$

From the upper bound in (6.4) and (6.5), we obtain n_0 such that for all $n \geq n_0$

$$\frac{\sup_{t \in [\Xi_n, \Xi_{n+1}]} Q_2(t)}{\log n} \leq \frac{2(1+\epsilon)}{C_2^* \beta}, \quad \text{and} \quad \log t \geq (1-\epsilon) \log n.$$

Therefore,

$$\frac{Q_2(t)}{\log t} \leq \frac{2(1+\epsilon)}{(1-\epsilon)C_2^* \beta}, \quad \text{for all } t \geq \Xi_{n_0}$$

and hence,

$$\limsup_{t \rightarrow \infty} \frac{Q_2(t)}{\log t} \leq \frac{2}{C_2^* \beta}, \quad \text{a.s.}$$

The fluctuation result for Q_1 is obtained similarly using Parts (iii) and (iv) of Theorem 3.4. \square

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