On the geometry of Hamiltonian systems : lecture notes seminar GISDA - Universidad del Bio Bio

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 17-09
August 2017

On the Geometry of Hamiltonian Systems

Lecture Notes Seminar GISDA
Universidad del Bio Bio

by

J.C. van der Meer

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

Department of Mathematics and
Computer Science
Groene Loper 5, 5612 AZ Eindhoven
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
www.tue.nl

Author
J.C. van der Meer

Date
August 2017

On the Geometry of Hamiltonian Systems

Lecture Notes Seminar GISDA
Universidad del Bio Bio

Where innovation starts

Technische Universiteit
Eindhoven
University of Technology

Table of contents

Title
On the Geometry of Hamiltonian Systems

1 Introduction **3**

2 Geometric reduction **5**

2.1 Some history 5
2.2 Poisson manifolds 5
2.3 Group actions 8
2.4 The Momentum Map 9
2.5 Example: $SO(3)$ 9
2.6 Reduction 10
2.7 Example: Harmonic oscillator [1] 10
2.8 Poisson reduction 11
2.9 Example: Rigid body 11

3 Reduction through invariants **13**

3.1 Invariant theory and reduction 13
3.2 Example: $\mathbb{S}^1$ or $SO(2)$ action [55] 14
3.3 Energy-momentum maps 15
3.4 Example: The harmonic oscillator [77] 16
3.5 Relative equilibria 16
3.6 Toral fibrations 18
3.7 Factorizing $\mathcal{EM}$ through ρ 18
3.8 Example: Hamiltonian Hopf bifurcation 18
3.9 Momentum polytopes 19
3.10 Example: fourfold 1:1 resonance [24] 20

4 Normal forms **21**

4.1 Local Normal Forms 21
4.2 Local Hamiltonian normal forms 23
4.3 Example: saddle-node 23
4.4 Example: Two degrees of freedom, p:q resonance 24

Where innovation starts

Technische Universiteit
Eindhoven
University of Technology

Table of contents

Title
On the Geometry of Hamiltonian Systems

4.5 Constrained dynamics, [56, 19] 24
4.6 Example: geodesic flow 26
4.7 Constrained normalization 26
4.8 Example: Perturbed Kepler problems, [56, 57] 27

5 Hamiltonian Hopf bifurcations **29**
5.1 Introduction 29
5.2 Versal deformation 30
5.3 Bifurcation of periodic orbits 30
5.4 Examples in the literature 31
5.5 Standard form for the HHB 31
5.6 Eigenvalues of the linear system 33
5.7 Geometric Reduction 34
5.8 Nondegenerate Hamiltonian Hopf bifurcations 38
5.9 Lagrange Top 39

1 Introduction

These notes contain the material of a series of four lectures given in the Grupo de Investigación en Sistemas Dinámicos y Aplicaciones (GISDA)of the Departmento de Mathemática at the Universidad del Bio Bio, Chile, in June 2017. It is an extended version of the text of the used slides, to which references are added. Therefore these lecture notes are a sort of compendium of results. For more details and proofs the reader should consult the literature cited.

In this series of lectures the focus is on the geometry of the phase space of Hamiltonian systems with symmetry. The symmetry causes the dynamics of the system to restrict to invariant sets and the aim is to understand how the phase space is organised in invariant sets such as energy manifolds, invariant tori, periodic solutions, stationary points etc. To do so we exploit the Lie algebra structure, or Poisson structure, which is in a natural way present in the theory of Hamiltonian systems. The tools we use will be momentum maps, energy-momentum maps and orbit maps. The orbit map will be used in the reduction of Hamiltonian systems with symmetry. Textbooks in which most of the basic material presented can be found are Abraham and Marsden [1], Marsden and Ratiu [50], the Peyresq lecture notes [12, 41, 75, 73, 77], Van der Meer [55].

In the first lecture (section 2) I will try to give a minimal set of definitions and theorems necessary for the basic classical reduction theorems as formulated bij Marsden and Weinstein [49]. The first section is concluded bij some reduction theorems in the context of Poisson structures [50].

The second lecture (section 3) is devoted to reduction by invariants. It is shown how to construct orbit spaces using a set of invariants for the symmetry. These orbit spaces are foliated with reduced phase spaces. The dynamics on the reduced phase spaces shows how the original phase space is organised. The connection with momentum maps and energy-momentum maps is discussed. As examples the harmonic oscillator (see for instance [77]), the Hamiltonian Hopf bifurcation [55] and the fourfold 1:1 resonance [24] are considered.

The third lecture (section 4) deals with normal forms for Hamiltonian systems and with constrained normalization of constraint Hamiltonian systems. After some general theorems concerning normalization of systems of differential equations, Hamiltonian normalization is discussed. After that the ideas of constrained dynamics and constrained normalization are introduced [56, 19]. This is illustrated by considering perturbed Kepler problems [56, 57].

The fourth lecture (section 5) deals with Hamiltonian Hopf bifurcations. It is shown how normalization, Liapunov-Schmidt reduction and singularity theory lead to a standard form through which this bifurcation can be analyzed. Several methods to prove the presence of Hamiltonian Hopf bifurcation in more complex systems are shown.

Acknowledgement. I would like to thank Francisco Crespo Cutillas for making the visit to the Departmento de Mathemática of the Universidad del Bio Bio, Chile, possible as part of the project Fondecyt No11160224, and the group GISDA for their hospitallity.

2 Geometric reduction

2.1. Some history

Geometric symplectic reduction was possibly first introduced by Reeb [74] who showed that in systems where the phase space has the structure of a circle bundle, the system can be reduced to the base space of the bundle.

Meyer [53], Marsden and Weinstein [49], and Arms, Marsden and Moncrief [2] finally formulated more general theorems for regular reduction.

The more constructive framework for the construction of reduced phase spaces (Constructive Geometric Reduction) was introduced Kummer [47], Cushman and Rod [16], and Van der Meer [55].

The first two papers are on the 1:1 resonance, the third one on the 1:-1 resonance, the latter one being an example of singular reduction.
The singular reduction was then put in a formal framework in by Arms, Cushman and Gotay[3].

Much more on the different kinds of reduction and its history can be found in [52].

Note that the most classical way to perform symplectic reduction, for instance in celestial mechanics, is by choosing clever symplectic coordinate transformations by which the number of equations and variables reduces. Often chosen variables are invariants also appearing in the constructive geometric reduction. A disadvantage is that these transformations are often singular transformations.

2.2. Poisson manifolds

Hamiltonian systems are usually introduced in the context of symplectic geometry. However I think it is more natural to use the context of Poisson geometry, because it emphasizes the Lie algebra structure that is present when dealing with Hamiltonian systems. So we start with introducing Poisson structures.

Let M be a manifold and $\{\ ,\ \}$ a bracket on $C^\infty(M)$ such that
$\{F, G\}$ is real bilinear
$\{F, G\} = -\{G, F\}$ (antisymmetry)
$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$ (Jacobi identity)
$\{FG, H\} = F\{G, H\} + \{F, H\}G$ (Leibnitz identity)

Then $\{\ ,\ \}$ is a **Poisson bracket**, $(M, \{\ ,\ \})$ is a **Poisson manifold**, $(C^\infty(M, \mathbb{R}), \{\ ,\ \})$ is a **Poisson algebra**.

Due to the first three properties $C^\infty(M, \mathbb{R})$ is a Lie algebra. Due to the last property the Poisson bracket is a derivative in each of its components.

Let $(M, \{\ ,\ \})$ be a Poisson manifold and $H \in C^\infty(M)$, then there exists a unique vector field X_H such that $X_H(G) = \{G, H\}$ for all $G \in C^\infty(M)$. We call X_H the **Hamiltonian vector field** with Hamiltonian function H w.r.t. the Poisson structure.

Let $\mathfrak{X}(M)$ denote the space of vector fields on M. Then the map $C^\infty(M) \to \mathfrak{X}(M);\ H \to X_H$ is a Lie algebra morphism, that is, $X_{\{F,G\}} = [X_f, X_G]$.

If $(M, \{\ ,\ \})$ is a Poisson manifold then there exists a contravariant antisymmetric two-tensor $B : T^*M \times T^*M \to \mathbb{R}$ such that $B(z)(dF(z), dG(z)) = \{F, G\}(z)$.

In coordinates $(z_1, \cdots, z_n)$ we have $\{F, G\} = B^{ij} \frac{\partial F}{\partial z^i} \frac{\partial G}{\partial z^j}$, with $B^{ij} = \{z^i, z^j\}$ called the **structure matrix** or just the matrix of the Poisson structure. For the vector field we have $X_H^i = B^{ij} \frac{\partial H}{\partial z^j}$.

If the Poisson structure is nondegenerate, i.e. B^{ij} is invertible, then B^{ij} is the negative inverse of the matrix of a symplectic form. Note that a symplectic form is a covariant antisymmetric two tensor which is nondegenerate.

Proposition 2.2.1 *M is a Poisson manifold with a nondegenerate Poisson bracket if and only if it is a symplectic manifold.*

For $X_f, X_G \in \mathfrak{X}(M)$ $\{F, G\} = \omega(X_F, X_G)$, with ω the symplectic form correponding to the nondegenerate bracket $\{.,.\}$.

On $\mathbb{R}^{2n}$ we have the standard Poisson structure

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right) .$$

Then $X_H(q, p) = \{\begin{pmatrix} q \\ p \end{pmatrix}, H\} = JdH(q, p)$ with $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
We have $\{F, G\} = L_{X_G} F = df \cdot X_G = dF \cdot JdG$, with $L_{X_G} F$ the Lie derivative, or the directional derivative of F in the direction of the Hamiltonian vector field X_H.

A **Poisson map** is a map $\varphi : (M, \{\ ,\ \}_M) \to (N, \{\ ,\ \}_N)$ such that $\{F, H\}_N \circ \varphi = \{F \circ \varphi, H \circ \varphi\}_M$. If $\{\ ,\ \}_M$ and $\{\ ,\ \}_N$ are nondegenerate φ is a symplectic map.

Consider the map

$$\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n; (t, x_0) \to x(t) ,$$

such that $x(t)$ is the solution of X_H, with initial value x_0. The **flow** of the vector field X_H is the map $\varphi_t; \mathbb{R}^n \to \mathbb{R}^n$ given by $\varphi_t(x) = \varphi(t, x)$. The flow of the Hamiltonian vector field X_H is a Poisson map.If φ_t^F is the flow of a Hamiltonian vector field X_F and $\{F, H\} = 0$ then $H \circ \varphi_t^F = H$, i.e. φ_t^F leaves the Hamiltonian invariant and is a symmetry for X_H.

If φ_t is the flow of X_H, then $H \circ \varphi_t = H$, and $\{F, G\} \circ \varphi_t = \{F \circ \varphi_t, G \circ \varphi_t\}$. Differentiation with respect to t then gives

$$\{\{F, G\}, H\} = \{\{F, H\}, G\} + \{F, \{G, H\}\}$$

the Jacobi identity. Furthermore

$$\frac{d}{dt}(F \circ \varphi_t) = \{F \circ \varphi_t, H\} = \{F, H\} \circ \varphi_t \ .$$

A Poisson map leaving the Hamiltonian invariant maps solutions to solutions.

Definition 2.2.2 *Let M be a Poisson manifold, Two points p_1 and p_2 on M are equivalent if they can be connected by a trajectory of a locally Hamiltonian vector field. The corresponding equivalence class is called a* ***symplectic leaf****.*

Theorem 2.2.3 *[Symplectic Stratification Theorem, [50] 10.4.4.] Let M be a finite dimensional Poisson manifold. Then M is a disjoint union of its symplectic leaves. Each leaf is a symplectically immersed Poisson submanifold, and the induced Poisson structure on the leaf is symplectic. The dimension of the leaf through p equals the rank of the Poisson-structure at p.*

An immersion is a map $f : M \to N$ such that $Tf(m) : T_m M \to T_{f(m)} N$ is injective for every $m \in M$. f is only locally a diffeomorphism. $f(M)$ need not be a submanifold.

Recall that Tf is the vector bundle mapping making the following diagram commute

$$\begin{array}{ccc} TU & \xrightarrow{Tf} & TV \\ \Big\downarrow \tau_U & & \Big\downarrow \tau_V \\ U & \xrightarrow{f} & V \end{array}$$

where τ_U and τ_V are the vector bundle projections. More precisely, if $TU = U \times E$, and $TV = V \times F$, then $TF(u, e) = (f(u), DF(u) \cdot e)$.

Theorem 2.2.4 *[Poisson-Darboux Theorem (Lie-Weinstein), [50] 10.4.6.] Let p be a point on a Poisson manifold M. There is a neighborhood U of p and an isomorphism $\varphi = \varphi_S \times \varphi_P : M \to S \times P$, with S symplectic and P Poisson, and the rank of P at $\varphi_P(p)$ is zero. S and P are unique up to local isomorphism. If the rank of the Poisson manifold is constant near p then there are coordinates $(q^1, \cdots, q^k, p_1, \cdots, p_k, y^1, \cdots, y^l)$ near p satisfying $\{q^i, q^j\} = \{p_i, p_j\} = \{q^i, y^j\} = \{p_i, y^j\} = 0$, $\{q^i, p_j\} = \delta^i_j$.*

Let $(M, \{\ ,\ \})$ be a Poisson manifold and $C \in C^\infty(M)$ such that $\{C, F\} = 0$ for all $F \in C^\infty(M)$ then C is called a **Casimir function** for the Poisson structure. C is constant along the flow of all Hamiltonian vector fields, that is, $X_C = 0$. The Casimir functions form the center of the Poisson algebra.

Proposition 2.2.5 *Let $(M, \{\ ,\ \})$ be a Poisson manifold, $C \in C^\infty(M)$ a Casimir, and $S \subset M$ a symplectic leaf, then C is constant on S.*

Remark 2.2.6 Symplectic leaves need not be submanifolds.
Even if all the Casimir functions are constant the Poisson structure can still be degenerate.

2.3. Group actions

Let G be a Lie group. A **Lie group** G is a finite-dimensional smooth manifold such that multiplication and inversion are smooth maps. Let e denote the identity element.
On G consider the left and right translation maps $L_g : G \to G; h \to gh$ and $R_g : G \to G; h \to hg$

$$\begin{array}{ccc} X(h) \in TG & \xrightarrow{T_h L_g} & X(gh) \in TG \\ \Big\downarrow \tau_G & & \Big\downarrow \tau_G \\ h \in G & \xrightarrow{L_g} & gh \in G \end{array}$$

A vector field X on G is left invariant if $L_g^* X = X$, or $T_h L_g X(h) = X(gh)$ for every $h \in G$.
Consider $\mathcal{X}_L(G)$ the space of left-invariant vector fields on G. Then T_eG and $\mathcal{X}_L(G)$ are isomorphic as vector spaces. Thus there exists a Lie bracket on T_eG given by $[\xi, \eta] = [X_\xi, X_\eta](e)$.
T_eG together with this bracket is the **Lie algebra of G**, denoted by $\mathfrak{g}$.

For a manifold M and a Lie group G we define the (left) **action** of G on M as a mapping $\varphi : G \times M \to M$ such that
$\varphi(e, m) = m$ for all $m \in M$ and $\varphi(g, \varphi(h, m)) = \varphi(gh, m)$ for all $g, h \in G$ and $m \in M$.

For every $g \in G$ we have $\varphi_g : m \to \varphi(g, m)$ a diffeomorphism on M. The map $g \to \varphi_g$ is a group homomorphism of G into $Diff(M)$.

The **orbit** of a point $m \in M$ under the action of a Lie group G is $\mathcal{O}(m) = \{\varphi_g(m) | g \in G\} \subset M$.

The **isotropy subgroup** (or stabilizer group or symmetry group) of G at $m \in M$ is $G_x = \{g \in G | \varphi_g(x) = x\} \subset G$.

An action is **transitive** if there is only one orbit, that is, for every $m, n \in M$ there is a $g \in G$ such that $g \cdot m = n$.

An action is **effective/faithful** if $\varphi_g = Id_M$ implies $g = e$, that is, $g \to \varphi_g$ is 1-1.

An action is **free** if it has no fixed points, that is, $\varphi_g(m) = m$ implies $g = e$.

Corollary 2.3.1 *An action is free if and only if $G_m = \{e\}$ for all $m \in M$. Every free action is effective.*

An action is **proper** if $\Phi : G \times M \to M \times M; (g, m) \to (m, \varphi(g, m))$ is a proper map. (A map is proper if the inverse images of compact sets are compact)

The inner automorphism associated with $g \in G$ is the map
$I_g : G \to G; h \to ghg^{-1}$, that is, $I_g = R_{g^{-1}} \circ L_g$.

Adjoint action: The adjoint action of G on $\mathfrak{g}$ is given by $Ad : G \times \mathfrak{g} \to \mathfrak{g}; (g, \xi) \to T_e I_g \xi$. that is, $Ad_g = T_e I_g : \mathfrak{g} \to \mathfrak{g}$.

Coadjoint action: The coadjoint action of G on $\mathfrak{g}^*$ is given by $\Phi^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*; (g, \alpha) \to Ad^*_{g^{-1}}\alpha$, with $Ad^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ given by $< Ad^*_g \alpha, \xi > = < \alpha, Ad_g \xi >$. That is, $Ad^*_{g^{-1}} = (T_e(R_g \circ L_{g^{-1}}))^*$.

Consider a manifold M and a Lie group G acting on M. Two points $m, n \in M$ are equivalent if they are in the same G-orbit This is an equivalence relation and we let M/G denote the set of equivalence classes. M/G is called the **orbit space** for the action.

Theorem 2.3.2 *[[1] 4.1.23.] If the action of G on M is proper and free then M/G is a smooth manifold and $\pi : M \to M/G$ is a smooth submersion.*

An submersion is a map $f : M \to N$ such that $Tf(m) : T_m M \to T_{f(m)}N$ is surjective for every $m \in M$.

Consider an action $\varphi : G \times M \to M$. For $\xi \in \mathfrak{g}$ define the map $\varphi^\xi : \mathbb{R} \times M \to M$ by $\varphi^\xi(t, m) = \varphi(\exp t\xi, m)$. φ^ξ is an $\mathbb{R}$-action on M and $\varphi_{\exp t\xi}$ is a flow on M. The corresponding vector field on M is given by

$$\xi_M(m) = \left.\frac{d}{dt}\right|_{t=0} \varphi_{\exp t\xi}(m) \; .$$

It is called the **infinitesimal generator** of the action corresponding to ξ. The **isotropy algebra** $\mathfrak{g}_m = \{\xi \in \mathfrak{g} | \xi_M(m) = 0\}$ is now the Lie algebra of the isotropy group G_m.

2.4. The Momentum Map

Consider a connected symplectic manifold (M, ω) with a symplectic action by a Lie group G given by $\Phi : G \times M \to M$. A **symplectic action** means that Φ_g is symplectic for each $g \in G$. (M, ω, G) is then called a **Hamiltonian G-space**.

Suppose there is a linear map $\mathcal{J} : \mathfrak{g} \to C^\infty(M)$ such that $X_{\mathcal{J}(\xi)} = \xi_M$. Then the mapping $J : M \to \mathfrak{g}^*$ defined by $\mathcal{J}(\xi)(m) =< J(m), \xi >$ is a **momentum mapping** for the action of G.

This can be generalized to Poisson manifolds.

The momentum mapping is Ad^* equivariant if $J(\Phi_g(m)) = Ad^*_{g^{-1}} J(m)$.

Suppose $H : M \to \mathbb{R}$ is invariant under the action Φ, that is, $H(\Phi_g(m)) = H(m)$, for all $m \in M$ and $g \in G$. Thus for all $\xi \in \mathfrak{g}$ we have $H(\Phi_{\exp t\xi}(m)) = H(m)$. By differentiating this expression at $t = 0$ we get

$$0 = dH(m)\xi_M(m) = L_{X_{\mathcal{J}(\xi)}} H = \{H, \mathcal{J}(\xi)\}$$

Thus $\mathcal{J}(\xi)$ is an integral for the Hamiltonian system (M, ω, H).

Theorem 2.4.1 *[Noether, [67], [50]] Let H be a G-invariant Hamiltonian on M with a momentum map J. Then J is conserved on the trajectories of the Hamiltonian vector field X_H.*

2.5. Example: $SO(3)$

Consider $\mathbb{R}^3$ with basis $\{e_1, e_2, e_3\}$, and standard inner product $(\; , \;)$. Consider $SO(3) = \{O \in GL(3, \mathbb{R}) | O^t O = O O^t = I, \; det(O) = 1\}$. Consider the diagonal action on $\mathbb{R}^6$, that is,

$O \cdot (x, y) = (Ox, Oy)$. The Lie algebra is $\mathfrak{so}(3) = \{X \in \mathfrak{gl}(3, \mathbb{R}) | X + X^t = 0\}$. A basis for $\mathfrak{so}(3)$ is $\{E_1, E_2, E_3\}$ with

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The bracket on $\mathfrak{so}(3)$ is the usual commutator $[X, Y] = XY - YX$. We have $[E_1, E_2] = E_3$, $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$.
$(\mathbb{R}^3, \times)$ is a Lie algebra which can be identified with $(\mathfrak{so}(3), [\ ,\ ])$ through $(x_1, x_2, x_3) \to x_1 E_1 + x_2 E_2 + x_3 E_3$.
Consider the momentum mapping $J(x, y) = x \times y$. Then for $\xi \in \mathbb{R}^3$ we have $< J(x, y), \xi >= (\xi, x \times y)$. Thus $X_{\mathcal{J}(\xi)}$ corresponds to rotation about the axis ξ.

2.6. Reduction

Theorem 2.6.1 *[Classical regular reduction theorem (Meyer, Marsden, Weinstein), [1] 4.3.1., 4.3.5.] Let (M, ω, G) be a Hamiltonian G-space with Ad^*-equivariant momentum mapping $J : M \to \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a regular value of J and let the isotropy subgroup G_μ for the coadjoint action on $\mathfrak{g}^*$ act freely and properly on $J^{-1}(\mu)$. Then $M_\mu = J^{-1}(\mu)/G_\mu$ has a unique symplectic form ω_μ making (M_μ, ω_μ) into a symplectic manifold.*
The orbit space $M_\mu = J^{-1}(\mu)/G_\mu$ is the **reduced phase space**.

If $H : M \to \mathbb{R}$ is invariant under the action of Φ then the flow of H leaves $J^{-1}(\mu)$ invariant and commutes with the action of G_μ on $J^{-1}(\mu)$. Thus the flow of H induces a flow on M_μ. This flow is Hamiltonian with Hamiltonian function $H_\mu : M_\mu \to \mathbb{R}$. H_μ is the **reduced Hamiltonian**.

2.7. Example: Harmonic oscillator [1]

Consider $\mathbb{R}^{2n}$ with its canonical symplectic form $\omega = \sum_{i=1}^n dq^i \wedge dp_i$. On this symplectic manifold consider the flow of the vector field X_H with $H(q, p) = \frac{1}{2} \sum_{i=1}^n ((q^i))^2 + p_i^2)$. This flow defines a symplectic $\mathbb{S}^1$-action on $(\mathbb{R}^{2n}, \omega)$. This action is proper and free. The momentum mapping is H itself. Moreover $\frac{1}{2}$ is a regular value for H and $H^{-1}(\frac{1}{2}) = S^{2n-1}$. We now have the reduced phase space

$$H^{-1}(\tfrac{1}{2})/\mathbb{R} = H^{-1}(\tfrac{1}{2})/\mathbb{S}^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$$

2.8. Poisson reduction

Theorem 2.8.1 *[Poisson reduction theorem, [50] 10.5.1.] Consider a Lie Group G acting on a Poisson manifold M by an action Φ such that each Φ_g is Poisson map. Suppose the action is free and proper. Then M/G is a manifold and $\pi : M \to M/G$ is a submersion. Moreover there exists a unique Poisson structure on M/G such that π is a Poisson map. If H is a G-invariant Hamiltonian on M,then there is a function H_G on M/G such that $H = H_G \circ \pi$. Moreover π transforms X_H on M to X_{H_G} on M/G.*

Theorem 2.8.2 *[Lie-Poisson reduction theorem, [50] 13.1.1.] If we identify the set of functions on $\mathfrak{g}^*$ with the set of left-invariant functions on T^*G then the canonical Poisson structure on T^*G induces on $\mathfrak{g}^*$ a Poisson structure given by the* **Lie Poisson bracket** *$\{F, G\} = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle$, with $\frac{\delta F}{\delta \mu}$ the functional derivative.*

$F_L \in C^\infty(T^*G, \mathbb{R})$ is **left-invariant** if $F_L \circ T^*L_g = F_L$. Let $C^\infty(T^*G, \mathbb{R})_L$ denote these functions. For $F : \mathfrak{g}^* \to \mathbb{R}$ and $\alpha_g \in T^*G$ set $F_L(\alpha_g) = F(T_e^*L_g\alpha_g) = (F \circ J_R)(\alpha_g)$, with $J_R : T^*G \to \mathfrak{g}^*; \alpha_g \to T_e^*L_g\alpha_g$ the momentum mapping of the lift of the right translation on G. $F_L = F \circ J_R$ is called the **left-invariant extension** of F from $\mathfrak{g}^*$ to T^*G.

$J_R : (C^\infty(T^*G, \mathbb{R})_L, \{\ ,\ \}_{T^*G}) \to (C^\infty(\mathfrak{g}^*, \mathbb{R})_L, \{\ ,\ \}_{\mathfrak{g}^*})$ is a Poisson isomorphism, with inverse the restriction to $\mathfrak{g}^* = T_e^*G$. We have $\{F, G\}_{\mathfrak{g}^*} = \{F_L, G_L\}_{T^*G}|\mathfrak{g}^*$, and$\{F, G\}_{\mathfrak{g}^*} \circ J_R = \{F_L \circ J_R, G_L \circ J_R\}_{T^*G}$. Note that we have the identification $\mathfrak{g}^* = T^*G/G$.

T^*G is a symplectic manifold and each $\mu \in \mathfrak{g}^*$ is a regular value of momentum mapping J_R. Consequently we have a reduced phase space J_R^{-1}/G_μ. The symplectic form is the Kirillov-Kostant-Souriau symplectic form.

We may identify J_R^{-1}/G_μ with G/G_μ which we may in turn identify with $\mathcal{O}(\mu)$, where $\mathcal{O}(\mu)$ is the coadjoint orbit through μ of the coadjoint action of G on $\mathfrak{g}^*$.

The symplectic form on the coadjoint orbits is the restriction of the Lie-Poisson structure on $\mathfrak{g}^*$.

Coadjoint orbits of finite dimensional Lie groups are even dimensional.

2.9. Example: Rigid body

Consider the rigid body. Its configuration space is $SO(3, \mathbb{R})$. Its phase space $T^*SO(3, \mathbb{R})$. $SO(3, \mathbb{R})$ acts on itself and this action lifts to an action on $T^*SO(3, \mathbb{R})$. Its momentum mapping is $J : T^*SO(3, \mathbb{R}) \to \mathfrak{so}(3, \mathbb{R})^*$. The reduced phase spaces are the co-adjoint orbits of $SO(3, \mathbb{R})$ on $\mathfrak{so}(3, \mathbb{R})^*$ which can be identified with S^2.
This can be shown by identifying $\mathfrak{so}(3, \mathbb{R})^*$ with a Lie subalgebra of the homogeneous quadratic functions on $\mathbb{R}^6$ with the standard Poisson bracket. Let $S_{ij} = x_i y_j - x_j y_i$, then S_{12}, S_{31}, and S_{23} generate a Lie algebra that can be identified with $\mathfrak{so}(3, \mathbb{R})^*$. Furthermore $X = x_1^2 + x_2^2 + x_3^2$, $Y = y_1^2 + y_2^2 + y_3^2$, and $P = x_1y_1 + x_2y_2 + x_3y_3$ are invariant under the diagonal action of $SO(3)$, giving $S_{12}^2 + S_{31}^2 + S_{23}^2 = XY - P^2 = constant$. Which gives a Casimir and defines the

co-adjoint orbit.

3 Reduction through invariants

3.1. Invariant theory and reduction

We will start this section with some theorems about invariants, a minimal generating basis, invariant functions and how invariant may describe the orbit space. The basic result is due to Hilbert.

Theorem 3.1.1 *[Hilbert, [40]] The algebra of polynomials over $\mathbb{C}$ of degree d in n variables which are invariant under $GL(n, \mathbb{C})$, acting by substitution of variables, is finitely generated.*

This was extended to

Theorem 3.1.2 *[Weyl, [85]] The algebra of invariants is finitely generated for any representation of a compact Lie group or a complex semi-simple Lie group.*

Let $\mathbb{R}[x]_G$ denote the space of G-invariant polynomials with coefficients in $\mathbb{R}$.

Corollary 3.1.3 *Consider a compact Lie group G acting linearly on $\mathbb{R}^n$. Then there exist finitely many polynomials $\rho_1, \cdots, \rho_k \in \mathbb{R}[x]_G$ which generate $\mathbb{R}[x]_G$ as an $\mathbb{R}$ algebra. These generators can be chosen to be homogeneous of degree greater then zero. We call $\rho_1, \cdots, \rho_k$ a* **Hilbert basis** *for $\mathbb{R}[x]_G$.*

Theorem 3.1.4 *[Schwarz, [80] Consider a compact Lie group G acting linearly on $\mathbb{R}^n$. Let $\rho_1, \cdots, \rho_k$ be a Hilbert basis for $\mathbb{R}[x]_G$, and let $\rho : \mathbb{R}^n \to \mathbb{R}^k; x \to (\rho_1(x), \cdots, \rho_k(x))$. Then $\rho^* : C^\infty(\mathbb{R}^k, \mathbb{R}) \to C^\infty(\mathbb{R}^n, \mathbb{R})_G$ is surjective, with ρ^* the pull-back of ρ.*

Theorem 3.1.5 *[Poenaru, [71]] The map ρ is proper and separates the orbits of G. Moreover the following diagram commutes, with $\tilde{\rho}$ a homomorphism*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\rho} & \rho(R^n) \\ \pi \searrow & & \swarrow \tilde{\rho} \\ & \mathbb{R}^n/G & \end{array}$$

Thus we can take $\rho(\mathbb{R}^n)$ as a model for the orbit space.

Consider $(\mathbb{R}^{2n}, \omega)$ on which a Lie group G acts linearly and symplectically. Then $(C^\infty(\mathbb{R}^{2n}, \mathbb{R}), \{\ ,\ \})$

is a Poisson algebra. If we consider on $\mathbb{R}^k$ the Poisson structure induced by ρ by taking as structure matrix $W_{ij} = \{\rho_i, \rho_j\}$ then $(C^\infty(\mathbb{R}^k, \mathbb{R}), \{\ ,\ \}_W)$ is a Poisson algebra and ρ a Poisson map. We have a Poisson reduction if we restrict the bracket on $\mathbb{R}^k$ to $\rho(\mathbb{R}^{2n})$.

In general there will be relations and inequalities determining the image of ρ. Therefore $\rho(\mathbb{R}^{2n})$ will in general be a real semi-algebraic subvariety of $\mathbb{R}^k$. A **semi-algebraic subset** of $\mathbb{R}^k$ is a finite union of sets of the form $\{x \in \mathbb{R}^k | R_1(x) = \cdots = R_r = 0\ ,\ \ R_{r+1}(x), \cdots, R_m \geqslant 0\}$. Define $C^\infty(\rho(\mathbb{R}^{2n}), \mathbb{R}) = \{F : \rho(\mathbb{R}^{2n}) \to \mathbb{R} | \rho^*(F) \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})\}$. This is a differential structure on $\rho(\mathbb{R}^{2n})$ and the orbit map is smooth (see [21, 20]).

Let W be a real semi-algebraic variety in $\mathbb{R}^k$. A point $x \in W$ is **nonsingular** if there exists a neighborhood $U \subset W$ of x such that for each $y \in U$ the matrix $\frac{\partial R_i}{\partial x_j}(x)$ has maximal rank. A point $x \in W$ is **singular** if the rank of $\frac{\partial R_i}{\partial x_j}(x)$ is strictly less than the maximal rank.

Let $\mathcal{S}(W)$ denote the set of singular points of W.

Proposition 3.1.6 *$\mathcal{S}(W)$ is a proper semi-algebraic algebraic subvariety of W. $W \setminus \mathcal{S}(W)$ is a non-empty smooth local manifold.*

We may now stratify W in the following way by smooth manifolds M_i. $S_1 = W$, $S_{i+1} = \mathcal{S}(S_i)$, $M_i = S_i \setminus S_{i+1}$. This stratification is called the **Whitney stratification**.

Theorem 3.1.7 *[Bierstone [7]] A stratum of the Whitney stratification of $\rho(\mathbb{R}^{2n}) = \mathbb{R}^{2n}/G$ is the image under the Hilbert map ρ of a connected component of symmetry type K for a compact subgroup K of G.*

For a compact subgroup K of G the set of symmetry type K is defined by
$M_K = \{m \in \mathbb{R}^{2n} | G_m = K\}$.

Theorem 3.1.8 *Let N be a connected component of M_K and let $i_n : N \to M$ be the inclusion map. Then N is a submanifold of $\mathbb{R}^{2n}$ and $\omega_N = i_N^*\omega$ is a symplectic form on N. If $F \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})_G$ then N is an invariant manifold of X_F and $X_F|N = X_{F|N}$.*

If there exists a momentum mapping $J : \mathbb{R}^{2n} \to \mathfrak{g}^*$, then for each $\mu \in \mathfrak{g}^*$ every connected component $M_K^\mu = J^{-1}(\mu) \cap M_K$ is a submanifold of $\mathbb{R}^{2n}$. $\rho(M_K^\mu)$ is a symplectic submanifold of the differential space $(\rho(\mathbb{R}^{2n}), C^\infty(\rho(\mathbb{R}^{2n}), \mathbb{R}))$ with symplectic form ω_H^μ.

The symplectic leaves of $(\rho(\mathbb{R}^{2n}), \{\ ,\ \})$ form a singular foliation. The symplectic leaves are the connected components of $\rho(M_K^\mu)$ as μ runs over $\mathfrak{g}^*$ and K runs over the compact subgroups of G.

Thus the symplectic leaves of the orbit space correspond to the reduced phase spaces.

3.2. Example: $\mathbb{S}^1$ or $SO(2)$ action [55]

Consider on $\mathbb{R}^4$ the $\mathbb{S}^1$-action given by the flow of $L(x, y) = x_1y_2 - x_2y_1$. This flow is $\varphi_t(x, y) = (R_tx, R_ty)$ with $R_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$. A Hilbert basis in terms of invariant polynomials consist of: $L(x, y)$, $X(x, y) = \frac{1}{2}(x_1^2 + x_2^2)$, $Y(x, y) = \frac{1}{2}(y_1^2 + y_2^2)$, $P(x, y) = x_1y_1 + x_2y_2$. With

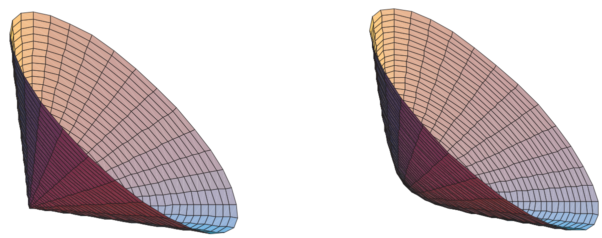

Figure 3.1: Reduced phase spaces for $c = 0$ and $c \neq 0$.

the usual Poisson bracket the polynomials X, Y, and P form a Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. These invariants can be found by determining the kernel of $ad_L = \{L,\ \}$. The orbit map for the $\mathbb{S}^1$-action is $\rho : \mathbb{R}^4 \to \mathbb{R}^4; (x, y) \to (X, Y, P, L)$. Its image is determined by the relation $4XY = L^2 + P^2$ together with $X, Y \geqslant 0$. Consequently the orbit space is half of a solid cone. The reduced phase spaces are obtained by taking $L = c$. See fig 3.1. (Note that $\mathcal{J} : \mathbb{R}^4 \to \mathbb{R}; (x, y) \to S(x, y)$ corresponds to a momentum map for this action). These surfaces correspond to the co-adjoint orbits for the action of $SL(2, \mathbb{R})$ on $\mathfrak{sl}(2, \mathbb{R})^*$. For $c = 0$ the cone consists of two strata, the vertex and the remaining part of the surface.

3.3. Energy-momentum maps

Consider a Hamiltonian system $(\mathbb{R}^{2n}, \omega, H)$, with integrals $F_1, \cdots, F_r$, $r \leqslant n-1$ which are functionally independent of each other and H, that is, $dH \wedge dF_1 \wedge \cdots \wedge dF_r \neq 0$ on an open and dense subset of $\mathbb{R}^{2n}$, then we call the map

$$\mathcal{EM} : \mathbb{R}^{2n} \to \mathbb{R}^{r+1} : x \mapsto (H, F_1(x), \cdots, F_r(x))$$

an **energy-momentum map**. $\mathcal{EM}^{-1}(c)$ is an invariant manifold for X_H.

We have $\{H, F_i\} = \{F_i, F_j\} = 0$ for $i \neq j\ ,\ i = 1, \cdots, r$. When $r = n-1$ the system is called **Liouville integrable**.

The energy-momentum mapping is studied because it gives information about the way the phase space is organized. It gives a fibration in invariant surfaces $\mathcal{EM}^{-1}(c)$ (Smale [81]).

The questions to be answered are:

- What is the topological type of the fibers?
- What does the singularity of the map looks like and what is the type of the singular fibers?
- How do the fibers fit together?
- What is the flow on the fibers?

3.4. Example: The harmonic oscillator [77]

Consider the Hamiltonian system $(T^*\mathbb{R}^2, \omega, H)$, with $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and

$$H : T^*\mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \frac{1}{2}(x_1^2 + y_1^2) + \frac{1}{2}(x_2^2 + y_2^2) \ .$$

This system has integral $L(x, y) = x_1 y_2 - x_2 y_1$, L corresponds to the angular momentum. Because both H and L are integrals of the harmonic oscillator

$$\mathcal{M}_{h,l} = H^{-1}(h) \cap L^{-1}(l) \ .$$

is an invariant manifold for the harmonic oscillator flow.

Define the energy-momentum mapping

$$\mathcal{E}\mathcal{M} : \mathbb{R}^4 \to \mathbb{R}^2 : (x, y) \mapsto (H(x, y), L(x, y)) \ .$$

Then $\mathcal{E}\mathcal{M}^{-1}(h, l) = \mathcal{M}_{h,l}$.
Regular values, that is, dH and dL independent, correspond to T^2. Singular values to S^1 or the origin. The singular fibers correspond to relative equilibria.

Note that $\mathcal{E}\mathcal{M}$ is an energy momentum map, but can also be considered as a $\mathbb{T}^2$ orbit map for the torus group G generated by the flows of X_H and X_L.

$\mathcal{E}\mathcal{M}$ can also be considered as a $\mathbb{T}^2$ momentum mapping.

One could also analyse the problem by doing a reduction with respect to the L symmetry.

By reducing with respect to the H-symmetry one obtains the Hopf fibration.

The $\mathcal{E}\mathcal{M}$ image is a finite intersection of half-planes, that is, a polytope.

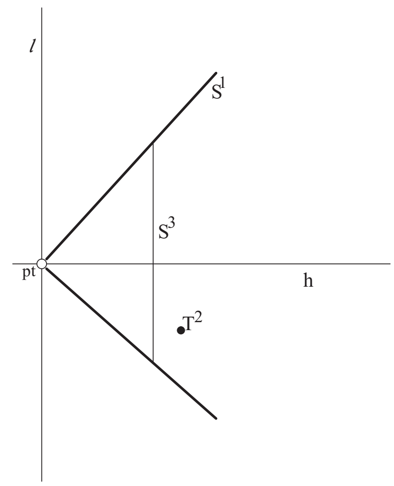

Figure 3.2: EM mapping for the harmonic oscillator

3.5. Relative equilibria

In the example of the Harmonic oscillator a relative equilibrium was a topological S^1 which was an orbit for X_H as well as X_L. More generally (see for instance [12])

Definition 3.5.1 *A point x_e is called a* **relative equilibrium** *if for all t there exists $g_t \in G$ such that $x_e(t) = g_t(x_e)$, where $x_e(t)$ is the dynamic orbit of X_H with $x_e(0) = x_e$. In other words, the trajectory is contained in a single group orbit.*

Proposition 3.5.2 *Let J be a momentum map for the G-action on M and let H be a G-invariant Hamiltonian on M. Let $x_e \in M$ and $\mu = J(x_e)$. The following are equivalent:*

i) x_e *is a relative equilibrium*

ii) the group orbit $G \cdot x_e$ *is invariant under the dynamics*

iii) there is a $\xi \in \mathfrak{g}$ *such that* $x_e(t) = exp(t\xi) \cdot x_e$

iv) there is a $\xi \in \mathfrak{g}$ *such that* x_e *is a critical point of the augmented Hamiltonian:* $H_\xi(x) = H(x) - < J(x), \xi >$.

v) x_e *is a critical point of the restriction of* H *to* $J^{-1}(\mu)$.

vi) the image $\bar{x}_e \in M_\mu$ *of* x_e *is a critical point of the reduced Hamiltonian* H_μ.

3.6. Toral fibrations

When the fibers of the energy momentum mapping are tori, we speak of a toral fibration. When a manifold is fibered with tori in algebraic geometry they speak of a toric manifold.

Theorem 3.6.1 *[Arnol'd, Liouville, [5]] Consider a Hamiltonian system with Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$. Suppose there are n functions $F_1 = H, F_2, \cdots, F_n$ such that $F_1, \cdots, F_n$ are integrals of X_H, the flows of X_{F_i} are complete, $\{F_i, F_j\} = 0$ for all i, j, $dF_1 \wedge \cdots \wedge dF_n \neq 0$ on an open and dense subset of $\mathbb{R}^{2n}$. If the set of regular values $\mathcal{R}$ of the energy-momentum mapping $\mathcal{EM} : \mathbb{R}^{2n} \to \mathbb{R}^n : x \mapsto (F_1(x), \cdots, F_n(x))$ is a nonempty open subset of $\mathbb{R}^n$, and for $c \in \mathcal{R}$, the set $\mathcal{EM}^{-1}(c)$ is compact and connected then $\mathcal{EM}^{-1}(c)$ is an n-torus.*

There are generalizations to the case where:

- The rank is not full (singular fibers). Under certain conditions the singular fibers corresponding to a point of rank $r < n$ correspond to a $\mathbb{T}^r$. (Eliasson [28])
- The system is not completely integrable but has only $k < n$ integrals. Under certain conditions the fiber is a $\mathbb{T}^k$. (Nekhoroshev [66])
- The set $\mathcal{EM}^{-1}(c)$ is not compact. The fibers are $\mathbb{R}^{k-m} \times \mathbb{T}^m$. (Fiorani, Giachetta, Sardanashvilly [30])

3.7. Factorizing $\mathcal{EM}$ through ρ

Consider a Hamiltonian system $(\mathbb{R}^{2n}, \omega, H)$, with integrals $F_1, \cdots, F_r$, $r \leqslant n-1$ and energy-momentum map

$$\mathcal{EM} : \mathbb{R}^{2n} \to \mathbb{R}^{r+1} : x \mapsto (H, F_1(x), \cdots, F_r(x)) \ .$$

The flows of F_i generate a group G which is a group acting symplectically on $\mathbb{R}^{2n}$. When this group is compact and acting linearly it has a orbit mapping $\rho : \mathbb{R}^{2n} \to \mathbb{R}^k; x \to (\rho_1(x), \cdots, \rho_k(x))$. Because H and the F_i are invariant under G there are functions $\bar{H}$ and $\bar{F}_i$ on $\rho(\mathbb{R}^{2n})$ such that, $\bar{H} \circ \rho = H$ and $\bar{F}_i \circ \rho = F_i$. Setting $\overline{\mathcal{EM}}(\rho_1, \cdots, \rho_k) = (\bar{H}, \bar{F}_1, \cdots, \bar{F}_r)$ we have $\overline{\mathcal{EM}} \circ \rho = \mathcal{EM}$. Thus the energy-momentum map factorizes through the orbit map.

Note that in certain situations it is possible to include H and the F_i in the set of generating invariants, which might lose homogeneity. Now the factorization becomes a projection.

3.8. Example: Hamiltonian Hopf bifurcation

Consider $H(x, y) = \frac{1}{2}(x_1^2 + x_2^2) + a(y_1^2 + y_2^2)^2$ with integral $L(x, y) = x_1 y_2 - x_2 y_1$.
The energy-momentum map factorizes through the L orbit map
$\rho : (x, y) \to (X, Y, P, L) \to (H, L)$.
We have $\bar{H}(X, Y, P, L) = X + 4Y^2$ and $\bar{L}(X, Y, P, L) = L$. The fibers of the energy-momentum mapping $\bar{H} \times \bar{L}$ on $\rho(\mathbb{R}^4)$ are just intersections of two surfaces. These are

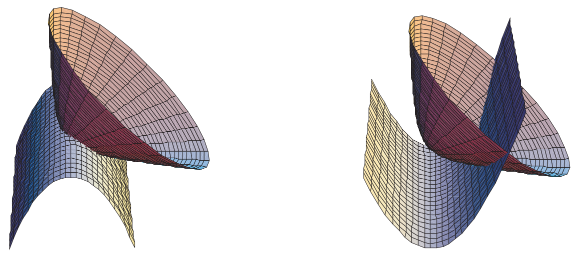

Figure 3.3: Reduced phase spaces and reduced energy surfaces.

the orbits of the reduced vector field. When the surfaces are tangent one obtains a relative equilibrium.

3.9. Momentum polytopes

According to a theorem by Atiyah [6], and Guillimin and Sternberg [32] the image for a momentum map for a torus action is a convex polytope.

Consider $Sp(2, \mathbb{R}^2)$ the group of linear symplectic transformations of the plane. We may identifiy its Lie algebra with the homogeneous quadratic polynomials on $\mathbb{R}^2$. $\frac{1}{2}(p^2 + q^2)$ is the infinitesimal generator for the group of rotations in the (q, p)-plane. Denote this group by $\mathbb{T}^1$ and its Lie algebra by $\mathfrak{t}_1$. Then $\mathfrak{t}_1$ has a generator ξ such that $2\pi\xi = id$ and the ξ corresponds to the function $\frac{1}{2}(p^2 + q^2)$. We may identify $\mathfrak{t}_1$ with $\mathbb{R}$ and $\mathfrak{t}_1^*$ with $\mathbb{R}$ The momentum mapping is given by $J(q, p) = \frac{1}{2}(p^2 + q^2)$ and $< J(q, p), a\xi >= \frac{1}{2}a(p^2 + q^2)$. If we now consider the product $\mathbb{T}^n$ we have momentum mapping $J : \mathbb{R}^2 n \to \mathbb{R}^n, (q, p) \to (\frac{1}{2}(p_1^2 + q_1^2), \cdots, \frac{1}{2}(p_n^2 + q_n^2))$. the image are all the points $(x_1, \cdots, x_n)$ with $x_i \geqslant 0$.

The fact that the image of this momentum mapping for this linear representation of the torus is a convex polytope can be extended to arbitrary linear actions of the torus and by using the Darboux theorem to arbitrary torus actions in the neighborhood of a fixed point. One may then prove the global result

Theorem 3.9.1 *[[33], 32.4] Let M be a compact connected symplectic $2n$ dimensional manifold and G a compact Lie group with Hamiltonian action $G \times M \to M$ and momentum mapping $J : M \to \mathfrak{g}^*$. If $G = \mathbb{T}^n$ then the image of the momentum mapping $J : M \to \mathfrak{t}^*$ is a convex polytope.*

The singularity of the momentum mapping is formed by the vertices, edges and faces of the polytope.

3.10. Example: fourfold 1:1 resonance [24]

Consider the quadratic functions on $\mathbb{R}^8$:

$$
\begin{aligned}
H_2(q,Q) &= \frac{1}{2}(Q_1^2+q_1^2)+\frac{1}{2}(Q_2^2+q_2^2)+\frac{1}{2}(Q_3^2+q_3^2)+\frac{1}{2}Q_4^2+q_4^2)\ ,\\
L_2(q,Q) &= q_1Q_2-Q_1q_2+q_3Q_4-Q_3q_4\ ,\\
L_1(q,Q) &= q_3Q_4-Q_3q_4-q_1Q_2+Q_1q_2\ ,\\
K(q,Q) &= -\frac{1}{2}(Q_1^2+q_1^2)-\frac{1}{2}(Q_2^2+q_2^2)+\frac{1}{2}(Q_3^2+q_3^2)+\frac{1}{2}Q_4^2+q_4^2)\ .
\end{aligned}
$$

The corresponding flows generate a linear torus action with momentum mapping $J_1 : \mathbb{R}^8 \to \mathbb{R}^4; (q,Q) \to (H_2, L_2, L_1, K)$. After reduction with respect to H_2 trough the orbit mapping ρ we get the reduced phase space $\rho(\mathbb{R}^8) \cap \rho(H_2^{-1}(c)) \cong \mathbb{CP}^3$. We may now factorize the momentum mapping $J_2 : \mathbb{R}^8 \to \mathbb{R}^3; (q,Q) \to (L_2, L_1, K)$ through the orbit map to obtain a momentum mapping $\bar{J}_2 : \mathbb{CP}^3 \to \mathbb{R}^3$ which fulfills the conditions of the theorem.

As an image we obtain the Delzant polytope which in this case is a tetrahedron (see fig 3.4).

The fibration of $\bar{J}_2$ is now as follows:

At the vertices the rank of $\bar{J}$ is zero.
The vertices have as fiber a point.

At the edges the rank of $\bar{J}$ is 1.
The points on the edges have as fiber a T^1.

At the faces the rank of $\bar{J}$ is 2
The points an the faces have as fiber a T^2,

In the interior $\bar{J}$ has full rank.
The points in the interior have as fiber a T^3.

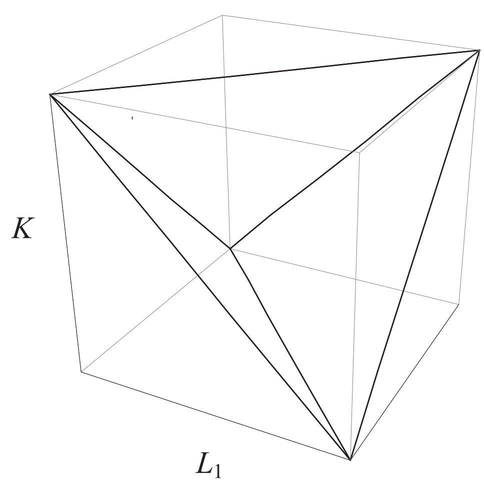

Figure 3.4: Delzant polytope

Next consider $\tilde{J}_2 : \mathbb{CP}^3 \to \mathbb{R}^2$ obtained by factorizing $(q,Q) \to (L_2, L_1)$ through the H_2 orbit mapping. This is a momentum mapping for a $\mathbb{T}^2$-action. dim$(\mathbb{T}^2)$=$\frac{1}{2}$dim$(\mathbb{CP}^3)-1$. We call this a momentum mapping of deficiency one. Projecting the tetrahedron in the K direction we obtain that the critical set is a square with its diagonals.

4 Normal forms

4.1. Local Normal Forms

The text below describes the standard normal form theory and can be found in, for instance, [84].

Consider

$$\dot{x} = Ax + f(x) \ ,$$

where A is a constant $n \times n$ matrix,
and $f(x)$ can be expanded near 0 in homogeneous vector polynomials starting with degree 2. That is, f has a Taylor expansion

$$f(x) = f_2(x) + f_3(x) + \cdots \ ,$$

where $f_m(x)$ is a vector with homogeneous components with terms

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \text{ with } m_1 + m_2 + \cdots + m_n = m \ .$$

of degree m.
Consider near identity transformations of the form

$$x = y + h(y) = y + h_2(y) + h_3(y) + \cdots \ .$$

Such a transformation transforms the equation into

$$\dot{y} = (I + \frac{\partial h}{\partial y})^{-1} \left(Ay + Ah(y) + f(y + h(y))\right) \ .$$

For terms of degree 2 one finds, the **homological equation** for h_2

$$\frac{\partial h_2}{\partial y} Ay - Ah_2(y) = f_2(y) \ ,$$

which can be used to remove as many quadratic terms as possible. One may then proceed with $\frac{\partial h_m}{\partial y} Ay - Ah_m(y) = g_m(y)$, $m > 2$, to remove higher order terms, where g_m depends on all solutions for the lower order terms.
Write

$$L_A(h) = \frac{\partial h}{\partial y} Ay - Ah(y) \ .$$

L_A is a linear operator. Let A be in diagonal form with different eigenvalues λ_i with eigenvectors e_i. Then L_A has eigenvectors $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n} e_i$ with eigenvalues

$$\sum_{j=1}^{n} m_j \lambda_j - \lambda_i \ , \ i = 1, \cdots, n \ .$$

The eigenvalues of A are **resonant** if for $i \in \{1, 2, \cdots, n\}$ one has $\lambda_i = \sum_{j=1}^{n} m_j \lambda_j$, with $m_j \in \{0\} \cup \mathbb{N}$ and $m = m_1 + \cdots + m_n \geqslant 2$.

Thus the eigenvalues tell us which terms are in the image of L_A and which terms are in the kernel. The terms in the image can be removed from the right-hand-side of the equation.

Consider the equation $\dot{x} = Ax + f(x)$, with $f(0) = 0$ and $D_0 f = 0$. Below are some theorems for normal forms for general systems of differential equations.

Theorem 4.1.1 *[Poincaré, [72]] If the eigenvalues of A are nonresonant then the equation $\dot{x} = Ax + f_2(x) + \cdots$ can be transformed into the linear equation $\dot{y} = Ay$ by the formal transformation $x = y + h(y) = y + h_2(y) + \cdots$.*

Theorem 4.1.2 *[Poincaré-Dulac, [26]] The equation $\dot{x} = Ax + f_2(x) + \cdots$ can, by formal transformations, be transformed into the equation $\dot{y} = Ay + r(y)$, such that $r(y)$ only contains resonant term, i.e., consists of monomials that are eigenvectors corresponding to resonant eigenvalues.*

Examples of other normal form theorems are

Theorem 4.1.3 *If the eigenvalues of A are lying either to the right or to the left of the imaginary axis in $\mathbb{C}$, then the equation can be reduced to a polynomial normal form by a formal transformation of variables.*

Theorem 4.1.4 *(Hartman-Grobman) If A is hyperbolic, i.e. A has only eigenvalues with nonzero real part, then there exists a homeomorphism in a neighborhood of the origin locally transforming the equation $\dot{x} = Ax + f(x)$ to $\dot{y} = Ay$.*

We may reformulate the above using the Lie algebra structure of vector fields. Define ad_X by $\mathrm{ad}_X Y = [X, Y]$. This is the adjoint action of the Lie algebra on itself. Consider the equation $\dot{x} = Ax + f(x)$, with $f(0) = 0$ and $D_0 f = 0$. Let $\mathcal{X}^m(\mathbb{R}^n)$ denote the space of polynomial vector fields, homogeneous of degree m. Then $\mathrm{ad}_A : \mathcal{X}^m(\mathbb{R}^n) \to \mathcal{X}^m(\mathbb{R}^n)$ is a linear map. Transforming the equation by $x = y + P(y)$, $P(y) \in \mathcal{X}^k(\mathbb{R}^n)$, one obtains for the kth order terms the homological equation

$$f_k(y) - \mathrm{ad}_A P(y) = 0 \ ,$$

While the terms of order $< k$ are unchanged. Thus in $f(x) = f_2(x) + f_3(x) + \cdots$ one may remove all terms in $\mathrm{Im}\ \mathrm{ad}_A$. The normal form consists of terms in some suitably chosen complement of $\mathrm{Im}\,\mathrm{ad}_A$.

4.2. Local Hamiltonian normal forms

For a given function $H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$, with $H(0) = DH(0) = 0$, consider its power series evaluation in a neighbourhood of the origin

$$H = H_2 + H_3 + H_4 + \cdots .$$

Define ad_H by $\mathrm{ad}_H F = \{H, F\}$. Consider near-identity transformations e^{ad_F}, then $e^{\mathrm{ad}_F} H = H \circ e^{\mathrm{ad}_F}$. Let F_3 be a homogeneous polynomial of degree 3

$$e^{\mathrm{ad}_{F_3}} H = H_2 + H_3 + \{F_3, H_2\} + \text{h.o.t.} .$$

If we can split $H_3 = \tilde{H}_3 + \bar{H}_3$ with $\tilde{H}_3 \in \mathrm{Im}\,(ad_{H_2})$ then we can remove $\tilde{H}_3$ by solving the homological equation $\tilde{H}_3 + \{F_3, H_2\} = 0$, and remove all third order terms in the image of ad_{H_2}, giving

$$H = H_2 + \bar{H}_3 + \check{H}_4 + \text{h.o.t.}$$

Repeat this for the fourth order terms

$$e^{\mathrm{ad}_{F_4}} H = H_2 + \bar{H}_3 + \check{H}_4 + \{F_4, H_2\} + \text{h.o.t.}$$

Up to arbitrary order we may remove all terms in $\mathrm{Im}\,(\mathrm{ad}_{H_2})$.

Thus we can normalize a system if we can solve the homological equation at each order. So the real question is in how to solve these equations.

If the Jordan-Chevalley decomposition of H_2 is $H_2 = S + N$ then we may choose as a complement to $\mathrm{Im}\,(\mathrm{ad}_{H_2})$

$$C = \ker(\mathrm{ad}_S) \cap \ker(\mathrm{ad}_M) ,$$

where M is the homogeneous quadratic polynomial such that the corresponding infinitesimally symplectic matrix $A_M = A_N^T$. here we use the method from [27]. In the next chapter you will find the method using the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra structure from [17].

As a consequence $\{\bar{H}_n, S\} = 0$ for all $n \leqslant 2$. We say H is in **normal form** up to order k, with respect to S.

4.3. Example: saddle-node

Consider the Hamiltonian saddle-node with $\mu = 0$.

$$H(q, p) = \frac{1}{2}p^2 + \frac{1}{3}q^3 + f_{>3}(q, p) .$$

Corresponding equations

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = p + \frac{\partial f}{\partial p}(q, p) , \\ \dot{p} = -\frac{\partial H}{\partial q} = -q^2 - \frac{\partial f}{\partial q}(q, p) . \end{cases}$$

It has normal form up to order k

$$H(q, p) = \frac{1}{2}p^2 + \frac{1}{3}q^3 + a_4 q^4 + a_5 q^5 + \cdots a_k q^k + f_{>k}(q, p) .$$

The terms in the normal form have to commute with q.

4.4. Example: Two degrees of freedom, p:q resonance

Consider a Hamiltonian $H = H_2 + h.o.t.$, with

$$H_2(x, y) = \frac{1}{2}\, p(x_1^2 + y_1^2) + \frac{1}{2}\, q(x_2^2 + y_2^2) \ .$$

Introduce complex conjugate coordinates by $z_j = x_j + iy_j$, $\zeta_j = \bar{z}_j = x_j - iy_j$. Then

$$\mathrm{ad}H_2(z, \zeta) = -i \left(p(z_1 \frac{\partial}{\partial z_1} - \zeta_1 \frac{\partial}{\partial \zeta_1}) + q(z_2 \frac{\partial}{\partial z_2} - \zeta_2 \frac{\partial}{\partial \zeta_2}) \right)$$

This operator has eigenvectors $z_1^{k_1} z_2^{k_2} \zeta_1^{l_1} \zeta_2^{l_2}$ with eigenvalue $-i\,(p(k_1 - l_1) + q(k_2 - l_2))$.

If p and q are non-resonant, that is, $\frac{p}{q} \notin \mathbb{Q}$, then all terms in $\ker \mathrm{ad}H_2$ have $k_i = l_i$ thus the normal form is generated by

$$I_1 = z_1\zeta_1 = x_1^2 + y_1^2 \text{ and } I_2 = z_2\zeta_2 = x_2^2 + y_2^2 \ .$$

The normal form up to order $2k$ is

$$\bar{H} = H_2 + F_2(I_1, I_2) + \cdots F_k(I_1, I_2) + h.o.t \ ,$$

with $F \in \mathbb{R}[I_1, I_2]$. (Birkhof normal form, [8])

When p and q are resonant then the normal form is generated by I_1, I_2 and $R_1 = \frac{1}{2}\,(z_1^q \zeta_2^p + \zeta_1^q z_1^p)$, $R_2 = \frac{1}{2i}(z_1^q \zeta_2^p - \zeta_1^q z_1^p)$. The normal form up to order k is

$$\bar{H} = H_2 + F(I_1, I_2, R_1, R_2) + h.o.t \ ,$$

where $F \in \mathbb{R}[I_1, I_2, R_1, R_2]$ containing only terms of order s, $2 < s \leqslant k$, in (x, y). (Moser, [62]).

This is also called Gustavson normal form [35]. Note that H_2 is an integral for the normalized part.

If we write $B_1 = H_2 = \frac{1}{2}\, I_1 + \frac{1}{2}\, I_2$ and $B_2 = \frac{1}{2}\, I_1 - \frac{1}{2}\, I_2$. Then the normalized terms are in $\mathbb{R}[B_1, B_2, R_1, R_2]$. We have the relation

$$R_1^2 + R_2^2 = \left(\frac{(B_1 + B_2)}{p} \right)^q \left(\frac{(B_1 - B_2)}{p} \right)^p \ .$$

Normal forms are not unique (see [78] and the references therein). One can for instance use the above relation to write the normal form in $\mathbb{R}[B_1, B_2, R_1] + R_2(\mathbb{R}[B_1, B_2, R_1])$. This can be formalized using Groebner basis [79].

4.5. Constrained dynamics, [56, 19]

Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ and standard symplectic form $\omega(x, y) = \sum_{i=1}^n dx_i \wedge dy_i$.

For $m < n$ let $M_1, ..., M_{2m} \in C^\infty(\mathbb{R}^{2n})$ be such that $dM_1, ..., dM_{2m}$ are independent on $M = \{(x, y) \in \mathbb{R}^{2n} | M_1(x, y) = M_2(x, y) = ... = M_{2m}(x, y) = 0\}$, that is, M is a smoothly embedded submanifold of $\mathbb{R}^{2n}$. Furthermore suppose that the matrix $C = (c_{ij}) = (\{M_i, M_j\})$ is nonsingular at every point of M. Then M is a symplectic manifold with symplectic form $\omega|M$, the restriction of the symplectic form ω to M. We say that M is a **co-symplectic manifold** of $\mathbb{R}^{2n}$.

For $H \in C^\infty(\mathbb{R}^{2n})$ the restriction of the Hamiltonian vector field X_H to M need not be tangential to M. However we can construct a vector field tangential to M by considering $X_{H|M}$ on $(M, \omega|M)$, where $H|M$ is the restriction of H to M.

We call $X_{H|M}$ the **constrained Hamiltonian vector field** corresponding to H. $X_{H|M}$ is the image of the projection of X_H on TM with respect to the splitting of $T\mathbb{R}^{2n}$ into TM and its ω-orthogonal complement.

Let $\mathcal{I}$ be the ideal of $C^\infty(\mathbb{R}^{2n})$ generated by $M_1, ..., M_{2m}$, that is, $\mathcal{I}$ is the ideal of functions vanishing on M. Let L_H denote the derivative defined by $L_H = \{., H\}$, where $\{., .\}$ is the Poisson bracket on $\mathbb{R}^{2n}$.

Lemma 4.5.1 *The following statements are equivalent:*

(i) $X_{H|M} = X_H$ *on* M.

(ii) $\{H, M_j\} \in \mathcal{I}$, *for* $j = 1, ..., 2m$.

(iii) $(\exp L_H)(\mathcal{I}) \subset \mathcal{I}$.

(iv) M is an invariant manifold of X_H.

(v) X_H *is tangent to M at each point of M.*

Let $H \in C^\infty(\mathbb{R}^{2n})$. When X_H is not tangent to M we can construct a function $\hat{H}$ such that $\hat{H}|M = H|M$, $X_{\hat{H}}$ is tangent to M, and $X_{\hat{H}}|M = X_{H|M}$.

Lemma 4.5.2 *If* $\hat{H} = H + \sum_{i=1}^{2m} \alpha_i M_i$, *with* $\alpha_i = \sum_{j=1}^{2m} c_i^{-1} j\{H, M_j\}$, *then* $X_{H|M} = X_{\hat{H}}$ *on* *M.*

Note that here the α_i are chosen such that $\{\hat{H}, M_i\} = 0$, however, we only need $\{\hat{H}, M_i\} \in \mathcal{I}$. Consequently the choice of the α_i may be modified.

Note that $\hat{H}$ need not be a smooth function on all of $\mathbb{R}^{2n}$. In fact $\hat{H}$ is first constructed on M and then extended to some open neighborhood of M in $\mathbb{R}^{2n}$.

The Poisson bracket $\{., .\}^M$ on $(M, \omega|M)$ can be computed in terms of the Poisson bracket on $\mathbb{R}^{2n}$ by using the Dirac bracket

$$[H, G] = \{H, G\} - \sum_{i,j=1}^{2m} \{H, M_i\} c^{ij} \{M_j, G\} .$$

Lemma 4.5.3 $\{H|M, G|M\}^M = \{H, G\} - \sum_{i,j=1}^{2m} \{H, M_i\} c^{ij} \{M_j, G\}$ *on* M, *where the right hand side is calculated for any smooth extension of* $H|M$ *and* $G|M$ *to an open neighborhood of* M *in* $\mathbb{R}^{2n}$.

Lemma 4.5.4 *If $X_H|M = X_H$ on M then $\{H|M, G|M\}^M = \{H, G\}$ on M for all $G \in C^\infty(\mathbb{R}^{2n})$.*

Note that $(C^\infty(\mathbb{R}^{2n}), \{\ ,\ \})$ as well as $(C^\infty(M), \{\ ,\ \}^M)$ are Poisson algebra's.

We may call two function $F, G \in C^\infty(\mathbb{R}^{2n})$ equivalent if $F|M = G|M$, or, in other words, if $F - G \in \mathcal{I}$. To compute a constrained bracket one may choose any two representatives for the two functions within their equivalence classes.

4.6. Example: geodesic flow

Consider a free particle moving in $\mathbb{R}^4$. This gives a Hamiltonian system on $T^*\mathbb{R}^4$. Using coordinates (q, p) the Hamiltonian is given by $G(q, p) = \frac{1}{2}|p|^2$. The geodesic flow on $S^3 \subset \mathbb{R}^4$ is obtained by constraining this vector field to TS^3 given by the constraints $M_1(q, p) = |q|^2 - 1 = 0$, and $M_2(q, p) = \langle q, p\rangle = 0$. $c^{-1} = \frac{1}{2|x|^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus TS^3 is a co-symplectic manifold. The constrained Hamiltonian is $\hat{G} = G + \sum_{i,j=1}^{2}\{G, M_i\}c_{ij}M_i$. However, we will pick another representative. Note that when $G = F \mod \mathcal{I}$ then $\hat{G} = \hat{F} \mod \mathcal{I}$ and
$X_{\hat{G}} = X_{\hat{F}} = X_G|M$. Therefore take $F = |q|^2 G$. Then

$$\hat{F} = \frac{1}{2}(|q|^2|p|^2 - \langle q, p\rangle^2)\ .$$

Thus we may replace G by $\hat{F}$ on M.

4.7. Constrained normalization

Consider a Hamiltonian system with Hamiltonian H on $\mathbb{R}^{2n}$ which is constrained to some manifold M as above. We want to put $H|M$ into normal form up to some order. We may do this by mappings $e^{\mathrm{ad}(F_k)}$ on $\mathbb{R}^{2n}$ but these mappings should leave M invariant. Suppose that X_{H_2} has a periodic flow that leaves M invariant.

Theorem 4.7.1 *If H is in normal form up to order k with respect to H_2, then $H|M$ is in normal form up to order k with respect to $H_2|M$.*

Definition 4.7.2 *$e^{\mathrm{ad}(F)}H$ is in normal form up to order k with respect to H_2 modulo $\mathcal{I}$ if*

(i) $\{F, M_j\} \in \mathcal{I}$ for all $j = 1, \cdots, 2m$,

(ii) All terms in $e^{\mathrm{ad}(F)}H$ of order $\leqslant k$ are in $(\ker \mathrm{ad}(H_2)) + \mathcal{I}$.

Consequently if H is in normal form up to order k with respect to H_2 modulo $\mathcal{I}$ then $H|M$ is in normal form up to order k with respect to $H_2|M$.

We may now normalize H by mappings $e^{\mathrm{ad}(F_k)}$, but at each step we take $\hat{F}_k = F_k + \sum_{i=1}^{2m}\alpha_i M_i$, with $\alpha_i = \sum_{j=1}^{2m} c^{ij}\{F_k, M_j\}$. Now F_k is obtained by solving the homological equation $\tilde{H}_k +$

$\{F_k, H_2\} = 0$, and it follows that $\{\hat{F}_k, H_2\} = -\tilde{H}_k + I$, with $I = \{H_2, \sum_{i=1}^{2m} \alpha_i M_i\} \in \mathcal{I}$. Splitting $H_k = (\bar{H}_K + I) + (\tilde{H}_k - I)$ we get $H_k + \{\hat{F}_k, H_2\} = \bar{H}_k + I$. This way we obtain a normal form up to order k with respect to H_2 modulo $\mathcal{I}$

4.8. Example: Perturbed Kepler problems, [56, 57]

We start with regulariztion of the Kepler problem with negative energy using Moser's regularization [63].

Consider the Kepler Hamiltonian

$$\tilde{K}(q, p) = \frac{1}{2}\langle p, p\rangle - \frac{1}{|q|} .$$

Suppose that $\tilde{K} = E < 0$ and consider

$$\hat{K}(q, p) = \frac{|q|}{\sqrt{-2E}}(\tilde{K}(q, p) + |E|) + \frac{\mu}{\sqrt{-2E}} = \frac{1}{2\sqrt{-2E}}|q|(|p|^2 + 2|E|) .$$

Then the Hamiltonian vector field for $\hat{K}$ is

$$\frac{dq}{ds} = \frac{1}{\sqrt{-2E}}|q|\frac{\partial \tilde{K}}{\partial p} , \ \frac{dp}{ds} = -\frac{1}{\sqrt{-2E}}|q|\frac{\partial \tilde{K}}{\partial q} .$$

After re-scaling time according to $\frac{ds}{dt} = \frac{\sqrt{-2E}}{|q|}$ this becomes the Kepler vector field.

The Hamiltonian vector field corresponding to $\hat{K}$ with energy $\frac{\mu}{\sqrt{-2E}}$ is a re-parametrzation of the Kepler vector field with Hamiltonian $\tilde{K}$ with energy E. Without loss of generality we may assume that $\mu = 1$ by choosing appropriate units, while scaling q by $2|E|$ and p by $\frac{1}{\sqrt{2|E|}}$ allows us to restrict to $E = -\frac{1}{2}$. Thus we may set

$$\hat{K}(q, p) = \frac{1}{2}|q|(|p|^2 + 1) .$$

Consider Moser's map based on the stereographic projection

$$\mu : T_pS^3 \subset \mathbb{R}^8 \to T^*\mathbb{R}^3 \subset \mathbb{R}^6; (u, v) \to (p, q) ,$$

with

$$p_k = \frac{u_{k+1}}{1 + u_1} , \ q_k = v_{k+1}(1 - u_1) + v_1 u_{k+1} , \ k = 1, 2, 3. .$$

The inverse is given by

$$u_{k+1} = \frac{2p_k}{1 + |p|^2} , \ u_1 = \frac{|x|^2 - 1}{|x|^2 + 1} ,$$
$$v_{k+1} = \frac{1}{2}(|p|^2 + 1)q_k - \langle p, q\rangle p_k , \ v_1 = \langle p, q\rangle , \ k = 1, 2, 3 ,$$

with $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathbb{R}^3$.
μ is a symplectic map when restricting the standard symplectic form to T_pS^3.

We have $|v| = \hat{K} = \frac{1}{2}|q|(|p|^2+1)$.

In the Kepler system we have to exclude the collision orbits corresponding to $|q| = 0$. So we should consider only $T_p^+S^3$, that is $|u| = 1$, $\langle u, v\rangle = 0$, $|v| \neq 0$, $u \neq (1, 0, 0, 0)$.

The regularized flow is now obtained by including $|v| = 0$. The Kepler flow corresponds to the constrained flow of $|v|$ on T_pS^3. As Hamiltonian we may choose

$$H_0(u, v) = \sqrt{|u|^2|v|^2 - \langle u, v\rangle^2}\ .$$

The regularized Kepler flow is a re-parametrization of the geodesic flow.

Consider perturbed Kepler system $K(q, p) + \epsilon K_1(q, p) + \epsilon^2 K_2(q, p) + \cdots$. Using Moser's map this transforms to $H_0(u, v) + \epsilon H_1(u, v) + \epsilon^2 H_2(u, v) + \cdots$. Where H_i, $i > 0$ are functions on $\mathbb{R}^8$.

The invariants for the periodoc H_0-flow are

$$|u|^2\ ,\ |v|^2\ ,\ \langle u, v\rangle\ ,\ S_{ij} = u_i v_j - u_j v_i\ ,\ i, j = 1, \cdots, 4\ ,\ i \neq j\ .$$

$|u|^2$, $|v|^2$, $\langle u, v\rangle$ span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, the S_{ij} span a Lie algebra isomorphic to $\mathfrak{so}(4)$.

In the relation $|u|^2|v|^2 - \langle u, v\rangle^2 = S_{12}^2 + S_{13}^2 + S_{14}^2 + S_{23}^2 + S_{24}^2 + S_{34}^2$ the left hand side is the Casimir for $\mathfrak{sl}(2,\mathbb{R})$, while the right hand side is the Casimir for $\mathfrak{so}(4)$.

Consequently

$$\rho : T_pS^3 \to (S_{12}, S_{13}, S_{14}, S_{23}, S_{24}, S_{34})$$

is an orbit map for the H_0-flow. The reduced phase spaces are given by

$$\sum_{1\leqslant i,j\leqslant 4} S_{ij}^2 = H_0^2\ ,\ S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0\ .$$

and can be shown to be isomorphic to $S^2 \times S^2$.

Note that $S_{ij}|T_pS^3 \circ \mu^{-1}$ give the integrals for the Kepler problem spanning the Lie algebra of its symmetry group $SO(4)$.

By the constrained normal form algorithm we may now normalize

$$H_0(u, v) + \epsilon H_1(u, v) + \epsilon^2 H_2(u, v) + \cdots$$

such that the normalized H_i become functions in the S_{ij}.

5 Hamiltonian Hopf bifurcations

5.1. Introduction

Consider a C^∞ function $H : \mathbb{R}^4 \to \mathbb{R}; (x, y) \to H(x, y)$.
The system of ordinary differential equations

$$\dot{x} = -\frac{\partial H}{\partial y} \, , \dot{y} = \frac{\partial H}{\partial x} \, ,$$

is the Hamiltonian system with Hamiltonian function H.
With $z = (x, y)$ the system can be written as

$$\dot{z} = J dH(z) = X_H(z) \text{ with } J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \, .$$

X_H is called the Hamltonian vector field.

Assume

$$H(0) = dH(0) = 0 \, .$$

That is, $(0, 0)$ is a stationary point and the powerseries expansion of H starts with quadratic terms

$$H(x, y) = H_2(x, y) + H_3(x, y) + \cdots \, .$$

The linearized vector field corresponds to H_2, i.e.

$$X_{H_2}(z) = Az \, .$$

Define a Poisson bracket by

$$\begin{aligned} \{F, G\} &= \frac{\partial F}{\partial x_1}\frac{\partial G}{\partial y_1} + \frac{\partial F}{\partial x_2}\frac{\partial G}{\partial y_2} - \frac{\partial F}{\partial y_1}\frac{\partial G}{\partial x_1} - \frac{\partial F}{\partial y_2}\frac{\partial G}{\partial x_2} \\ &=< dF, J^{-1} dG > \, . \end{aligned}$$

C^∞ with the bracket $\{., .\}$ is a Poisson algebra, and $\mathrm{ad}_F = \{F, .\}$ is a derivative. Note the Hamiltonian equations corresponding to H can also be written as

$$\dot{z} = \{H, z\} \, .$$

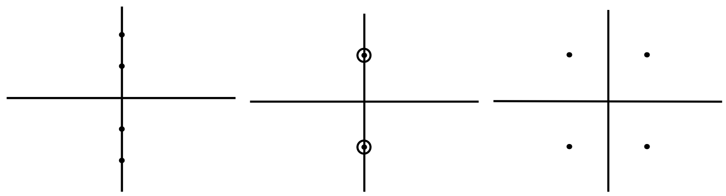

Figure 5.1: Eigenvalues for $\nu > 0$, $\nu = 0$, and $\nu < 0$.

Consider a Hamiltonian system such that the linearized system

$$\dot{z} = Az$$

has purely imaginary eigenvalues, and is not diagonalizable.
The normal form of corresponding quadratic Hamiltonian, with eigenvalues $\pm i\alpha, \pm i\alpha$, is

$$H_2(x, y) = \alpha(x_2 y_1 - x_1 y_2) + \tfrac{1}{2}(x_1^2 + x_2^2) .$$

H_2 has a nontrivial Jordan-Chevalley decomposition in a semisimple part $S = \alpha(x_2 y_1 - x_1 y_2)$ and a nilpotent part $X = \frac{1}{2}(x_1^2 + x_2^2)$, $\{X, S\} = 0$.

The theory of normal forms for lineair Hamiltonian systems (infinitesimal symplectic matrices) was developped by Williamson (1936) [86], Burgoyne and Cushman (1974) [13], Melbourne and Dellnitz (1993) [60].

5.2. Versal deformation

If we consider an homogeneous quadratic Hamiltonian, and consider it as a point in the space of all quadratic Hamiltonians, we might wonder if, applying diffeomorphisms, we might cover a complete neighbourhood of the point. In general this will not be possible and we will need versal deformations or versal unfoldings.

H_2 has a **versal deformation** or **versal unfolding**

$$H_2(x, y; \delta, \nu) = (\alpha + \delta)(x_2 y_1 - x_1 y_2) + \tfrac{1}{2}(x_1^2 + x_2^2) + \tfrac{1}{2}\nu(y_1^2 + y_2^2) .$$

The eigenvalue behaviour is given in figure 5.1. Versal deformations are studied by Arnold [4], Galin [31], Hoveijn [43], Dellnitz, Melbourne, and Marsden [22], Van der Meer [58].

5.3. Bifurcation of periodic orbits

Consider a nonlinear Hamiltonian system such that the linear system has Hamiltonian H_2 as above.We want to study the bifurcation of relative equilibria of such a system.
Steps in attacking the bifurcation problem:

- Put the nonlinear Hamiltonian function into normal form up to sufficiently high order. (introducing additional symmetry)

- Show by a Liapunov-Schmidt like procedure that the set of periodic solutions of the original system and the normalized system are diffeomorphic.
- Determine, using singularity theory, a sufficient jet to describe the bifurcation. (Standard form)
- Analyse the standard form by geometrical reduction.

One of the first known examples of such a bifurcation is in the planar restricted problem of three bodies. The bifurcation value of the parameter is, in the restricted three body problem also known as Routh critical mass ratio (Routh (1875), [76]). The bifurcation was studied by Brown (1911), [11], Krein (1955), [46], Meyer and Schmidt (1971), [61], Sokol'skij (1974), [82], and Van der Meer (1985), [55].

5.4. Examples in the literature

- Restricted three body problem, Van der Meer, 1985, [55].
- Lagrange top, Cushman and Van der Meer, 1990, [18].
- Spinning orthogonal planar pendulum, Bridges, 1990, [9]
- Double spherical pendulum, Marsden and Scheurle, 1993, [51]
- Water wave problem, Bridges, 1992, [10], Dias and Iooss, 1996, [23].
- Hydrodynamical stability problems (reversible), Iooss and Pérouème, 1993, [44]
- Torsional buckling, Champneys and Thompson, 1996, [14]
- Buckling of a strut on a elastic foundation, Champneys, 1998, [15].
- Ferroelectric liquid cristals, Pitanga, Ribeiro Filho, and Mundim, 1996, [70].
- Rotating H_3^+, Kozin, Roberts, and Tennyson, 1999, [45].
- Rydberg electron in rotating electric field. Lahiri and Roy, 2001, [48].
- The 3-D Hénon-Heiles family. Ferrer, Hanßmann, Palacián and Yanguas, 2002. [29], Hanßmann, Van der Meer, 2002, 2005, [36, 38].
- Gyrostat in an incompressible ideal fluid, Guirao, J.L. and Vera-López, J.A., 2012, [34].

5.5. Standard form for the HHB

The complement C to im(ad$_{H_2}$) (recall $H_2 = S + X$) is

$$C = \ker(\mathrm{ad}_S) \cap \ker(\mathrm{ad}_Y) \text{ with } Y = \tfrac{1}{2}(y_1^2 + y_2^2)\ .$$

A theorem of Jacobson-Morosov says: there exist $Y = \frac{1}{2}(y_1^2 + y_2^2)$ and $P = x_1 y_1 + x_2 y_2$ such that

$$\begin{aligned}
&\{S, X\} = \{S, Y\} = \{S, P\} = 0 \ , \\
&\{X, Y\} = P \ , \ \{X, P\} = 2X \ , \ \{Y, P\} = -2Y \ .
\end{aligned}$$

That is, X, Y, and P span a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Normal form of the Hamiltonian up to order k

$$\begin{aligned}
H &= \alpha S + X + aY^2 + bYS + cS^2 + \cdots + H_k(S, Y) + R_{>k}(x, y) \\
&= H_{NF} + R_{>k}(x, y)
\end{aligned}$$

The following theorem is a Liapunov-Schmidt reduction theorem [55, 83]. It relates an arbitrary system in HHB to a system with a Hamiltonian which is fully in normal and has a diffeomorphic set of X_S relative equilibria.

It is followed by a theorem based on algberaic geometric singularity theory. This theory shows how to find a simple as possible representation within a set of maps haven diffeomorphc singular sets. In our case we want to find a simple as possible form for the energy momentum map of the HHB in such a way that its singular set is only changed up to diffeomorphism. This singular set corresponds to the stationary points and the X_S relative equilibria.

In these theorems transformations need not be symplectic. We are only focussing on the geometry of the singular set of the energy-momentum map.

Theorem 5.5.1 *There exists for each $k \geq 2$ a C^{k+1}-mapping $\widetilde{H} : U \times \Lambda \to \mathbb{R}$, and a C^k-mapping $\Psi : U \times \Lambda \to V$, with the following properties:*

(i) *$\widetilde{H}(0, \lambda) = 0$ and $D_u \widetilde{H}(0, \lambda) = 0$ for all $\lambda \in \Lambda$, and $\widetilde{H}(u, \lambda) = H_{NF}(u, \lambda) + O(\|u\|^{k+2})$ as $u \to 0$;*

(ii) *$\Psi(0, \lambda) = 0$ for all $\lambda \in \Lambda$, and $D_u \Psi(0, \lambda_0) \cdot u = u, \forall u \in U$;*

(iii) *$\widetilde{H}$ and Ψ are C^∞ on $(U - \{0\}) \times \Lambda$;*

(iv) *$\{\widetilde{H}_\lambda, \widetilde{S}\}_U = 0, \quad \forall \lambda \in \Lambda$;*

(v) *$\tilde{x} : \mathbb{R} \to V$ is a T-periodic solution of $X_H(x, \lambda)$ if and only if*

$$\tilde{x}(t) = \Psi(\tilde{u}(t), \lambda), \quad \forall t \in \mathbb{R},$$

where $\tilde{u} : \mathbb{R} \to U$ is a sufficiently small T-periodic solution of the reduced Hamiltonian system

$$\dot{u} = X_{\widetilde{H}}(u, \lambda).$$

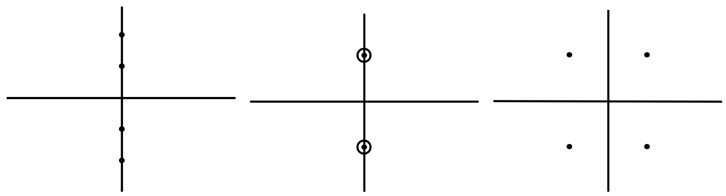

Figure 5.2: Eigenvalues for $\nu > 0$, $\nu = 0$, and $\nu < 0$. $H_2 = \alpha S + X + \nu Y$.

Theorem 5.5.2 *Let ϱ denote the S^1–action given by the flow of X_S. Consider a ϱ–invariant mapping*

$$(\tilde{H}, S) : \mathbb{R}^4 \longrightarrow \mathbb{R}^2; (S, X, Y, P) \mapsto (\tilde{H}(S, X, Y, P), S) \text{ with}$$

$$\tilde{H}(S, X, Y, P) \;=\; S + X + aY^2 + bSY + cS^2 + \text{h.o.t.}$$

where $a \neq 0$. Then there exists a ϱ–equivariant origin preserving diffeomorphism φ on $\mathbb{R}^4$ and a origin preserving diffeomorphism ψ on $\mathbb{R}^2$ such that

$$\psi \circ (\tilde{H}, S) \circ \varphi \;=\; (\hat{G}, S)$$

with

$$\hat{G}(S, X, Y, P) \;=\; X + aY^2 \;.$$

Moreover, the universal unfolding of $(\hat{G}, S)$ is (G, S) with

$$G(S, X, Y, P; \nu) \;=\; X + \nu Y + aY^2 \;.$$

and every deformation $\tilde{H}(S, X, Y, P; \mu)$ of $\tilde{H}(S, X, Y, P)$ is right-left equivalent to $G(S, X, Y, P; \nu)$.

Standard form (singularity theoretic normal form)

$$G_\nu = X + \nu Y + aY^2 \;.$$

5.6. Eigenvalues of the linear system

In figure 5.2 and figure 5.3 we see the eigenvalue behavior for the Hamiltonian Hopf bifurcation. Note that in the case of zero eigenvalues the S-symmetry is crucial.

Note that, provided an S^1-invariant setting, with the S^1-action given by the flow of X_S, also the eigenvalue behavior as given in figure 5.3 gives an hamiltonian Hopf bifurcation.

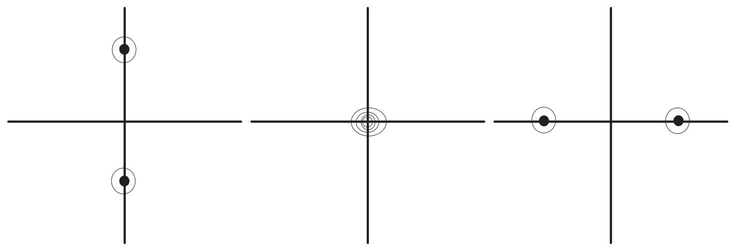

Figure 5.3: Eigenvalues for $\nu > 0$, $\nu = 0$, and $\nu < 0$. $H_2 = X + \nu Y$.

5.7. Geometric Reduction

Consider the symplectic S^1-action induced by the flow of X_S. The invariants for this action are S, X, Y, and P. They span a Lie algebra isomorphic to $\mathbb{R} \times \mathfrak{sl}(2, \mathbb{R})$)
We have the following relation among the invariants

$$4XY = S^2 + P^2 \ .$$

The orbit mapping for S^1-action is

$$\rho : (x, y) \to (X, Y, P, S) \ .$$

The orbit space has a Poisson structure given by

$$\{F, G\} = < dF, WdG > \ .$$

with

$$W = \begin{pmatrix} 0 & -P & -2X & 0 \\ P & 0 & 2Y & 0 \\ 2X & -2Y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

The reduced (symplectic) phase spaces

$$\rho(S^{-1}(s)) \ .$$

are given in (X, Y, P)-space by the equation

$$C(X, Y, P; s) = 4XY - P^2 - s^2 = 0 \ .$$

The Poisson structure on the image is given by $\{F, G\} = < \nabla C, \nabla G \times \nabla F >$. Brackets of this form can be considered as Nambu-brackets [65]. It is the Casimir C that determines the Poisson structure.

Tangent points of $G_\nu = X + \nu Y + aY^2$ with the reduced phase space give periodic orbits. Tangent points are always in the plane $P = 0$. How the energy surfaces and the reduced phase spaces intersect can be found in figures 5.4, 5.5, 5.6.

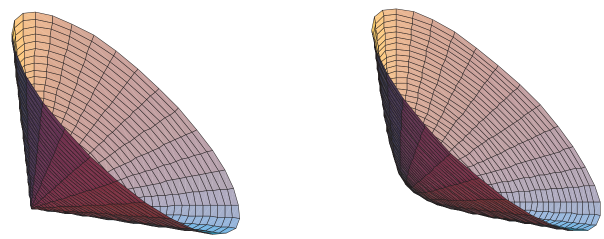

Figure 5.4: Reduced phase spaces for $s = 0$ and $s \neq 0$.

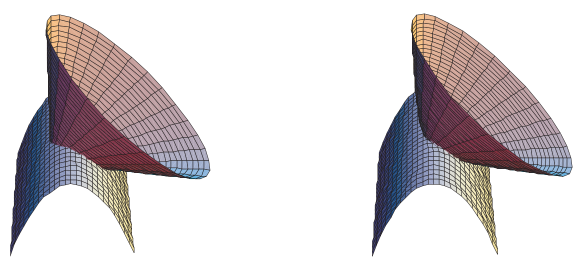

Figure 5.5: Intersection of energy surface and reduced phase space, $a > 0$

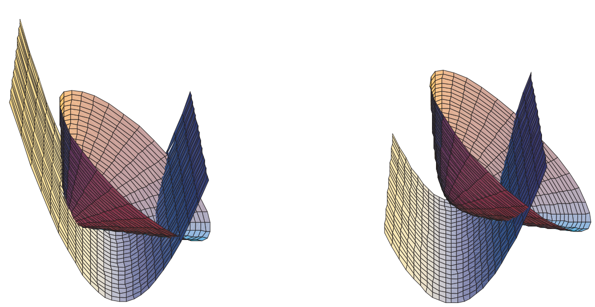

Figure 5.6: Intersection of energy surface and reduced phase space, $a < 0$

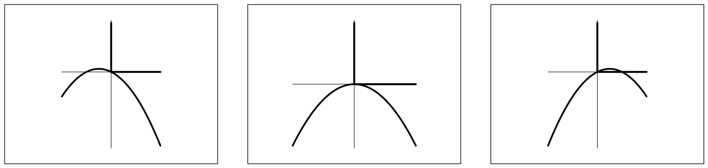

Figure 5.7: $a > 0$, G_ν at the cone for $\nu > 0$, $\nu = 0$, and $\nu < 0$.

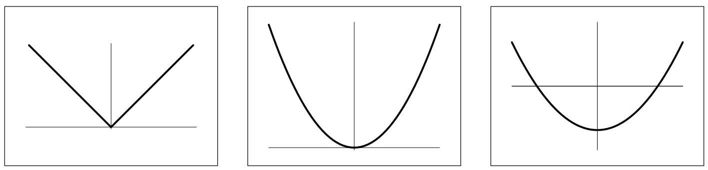

Figure 5.8: $a > 0$, bifurcation in (S, G)-plane for $\nu > 0$, $\nu = 0$, and $\nu < 0$.

The bifurcation is completely determined by the tangency of the energy surface to the vertex of the cone at $\nu = 0$. For $a > 0$ we have supercritical bifurcation, for $a < 0$ we have subcritical bifurcation. Of course the derivative at the origin should change sign as a function of ν in order to have a genuine bifurcation. When $a = 0$ we say that the bifurcation is degenerate. The essence of the tangency is captured in figures 5.7, and 5.9. The corresponding bifurcation is illustrated in 5.8, 5.10 where the curve of relative equilibria is given in the image of the energy momentum map.

The image of the reduction map is the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The map

$$\psi : \mathbb{R}^3 \to \mathbb{R}^3; (X, Y, P) \to (G, Y, P)$$

is a diffeomorphism. The energy-momentum map can now easily be factorized through $\psi \circ \rho$. It changes C into

$$\tilde{C}(G, Y, P; s, \nu) = 4YG - 4\nu Y^2 - 4aY^3 - P^2 - s^2 \ ,$$

which gives the energy-momentum representation of the Poisson structure [59]. Note that in the energy-momentum representation the bifurcation equation, which also describes the singularity of the energy-momentum map, is identified with the singularity of the Poisson structure.

In figure 5.11 the reduced phase spaces are given after the transformation that makes the vertical axis the G-axis.

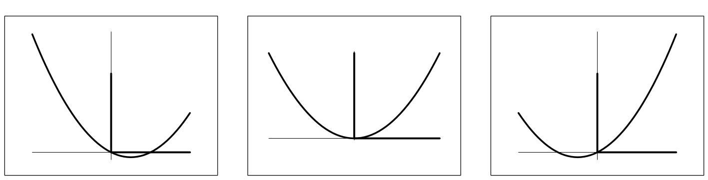

Figure 5.9: $a < 0$, G_ν at the cone for $\nu > 0$, $\nu = 0$, and $\nu < 0$.

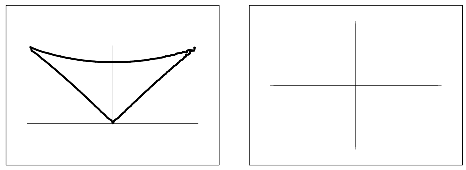

Figure 5.10: $a < 0$, bifurcation in (S, G)-plane for $\nu > 0$, and $\nu \leq 0$.

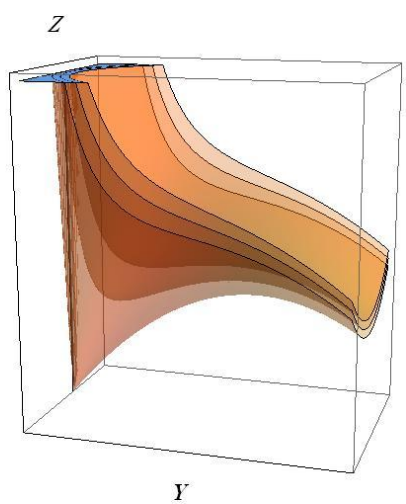

Figure 5.11: Reduced phase space for different values of s in energy-representation.

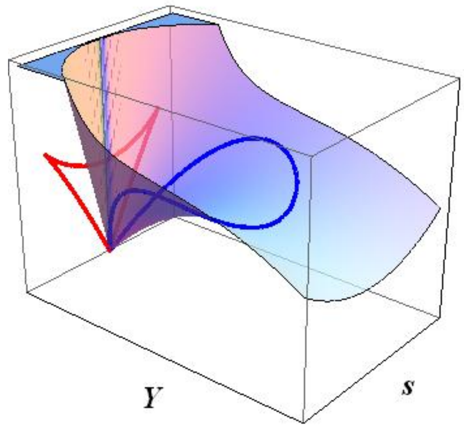

Figure 5.12: Projection of $\tilde{C}(G, Y, 0; S, \nu)$ on the (S, G)-plane for $\nu > 0$ and $a < 0$.

In figure 5.12 one can see how the family of periodic solutions in the (S, G) parameter plane is obtained by projection of $\tilde{C}(G, Y, 0; S, \nu) = 0$ surface on the (S, G)-plane.

5.8. Nondegenerate Hamiltonian Hopf bifurcations

How to determine (nondegenerate) Hamiltonian Hopf bifurcations in three degree of freedom systems

- Flattening the Poisson structure.
  Perform a first reduction. One obtains a reduced system with a nonlinear Poisson structure. The aim is to obtain a system with standard symplectic form. Linearize the Poisson structure or symplectic structure within the local chart on the first reduced phase space. (see for instance [25].)

- Singularity theoretic equivalence.
  Perform a double reduction. Choose coordinates such that the S^1–action on the second reduced phase space is standard. Apply singularity theory to the energy-momentum mapping.(see [37])

- The geometric method.
  Perform a double reduction. Perform an analysis of the relative position of the reduced phase space and the energy level sets of the one–degree–of–freedom system (see figures 5.7, and 5.9). The second reduction is not regular and leads to a conical singularity.(see [36].)

To apply the geometric method certain hypotheses should be checked.

H1. The stratification of the orbit space into reduced phase spaces should locally be equivalent to the standard case.
At the critical value of the integral one locally has a cone deforming into a hyperboloid

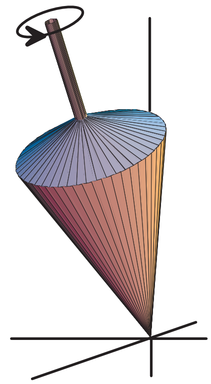

Figure 5.13: Heavy symmetric rigid body with one point fixed.

when the value of the integral is varied, locally the invariants form a Lie algebra isomorphic to $sl(2, \mathbb{R})$.

H2. As the bifurcation parameter varies, the energy level set through the vertex of the cone should change "with non–zero speed" from passing outside the cone to intersecting the interior of the cone.
Ensures that the unfolding of the linear part is (uni)versal.

H3. The energy level set should have second order contact at the vertex of the cone at the bifurcation value of the parameter.
Non–degeneracy condition.

5.9. Lagrange Top

The Lagrange top problem is completely integrable on its phase space $T^*SO(3)$ with canonical symplectic form. The two symmetries are a right S^1–action corresponding to rotation about the axis of symmetry of the body and a left S^1–action corresponding to rotation about the vertical axis in space. Reduction with respect to the left S^1–action gives the Euler–Poisson equations

$$\dot{x} = x \times \frac{\partial H}{\partial y}\,, \quad \dot{y} = x \times \frac{\partial H}{\partial x} + y \times \frac{\partial H}{\partial y}\,.$$

With Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{\gamma}{2} y_3^2 + x_3\,.$$

We will first perform reduction like in [18]. The reduced phase space is a 4–dimensional smooth submanifold

$$T_a S^2 = \left\{ (x, y) \in \mathbb{R}^6 \mid C_1 = 1\,,\ C_2 = a \right\} \subseteq \mathbb{R}^6\,.$$

with

$$C_1 = x_1^2 + x_2^2 + x_3^2 \ ,$$
$$C_2 = x_1 y_1 + x_2 y_2 + x_3 y_3 \ .$$

The equilibrium point $(x, y) = (0, 0, 1, 0, 0, a)$ is the upright standing position of the top, which becomes gyroscopically stabilized ("sleeping") if the top spins sufficiently fast.

The reduced system still has the integral of motion $L(x, y) = y_3$, which is the angular momentum corresponding to rotation about the symmetry axis of the top.

The invariants for the right S^1-action are

$$\begin{aligned} \pi_1 &= x_1^2 + x_2^2 \ , & \pi_3 &= x_1 y_1 + x_2 y_2 \ , & \pi_5 &= x_3 \ , \\ \pi_2 &= y_1^2 + y_2^2 \ , & \pi_4 &= x_1 y_2 - x_2 y_1 \ , & \pi_6 &= y_3 \end{aligned}$$

with relations

$$\pi_3^2 + \pi_4^2 = \pi_1 \pi_2 \ , \quad \pi_1 \geq 0 \ , \quad \pi_2 \geq 0 \ .$$

Furthermore

$$C_1 = \pi_1 + \pi_5^2 = 1 \ , \quad C_2 = \pi_3 + \pi_5 \pi_6 = a \ , \quad L = \pi_6 = b$$

Twice reduced phase space is given by

$$V_a^b = \left\{ (\pi_2, \pi_4, \pi_5) \in \mathbb{R}^3 \ \mid \ R_a^b(\pi) = 0 \ , \ \pi_2 \geq 0 \ , \ |\pi_5| \leq 1 \right\}$$

with

$$R_a^b(\pi) = \pi_4^2 + (a - b\pi_5)^2 - (1 - \pi_5^2)\pi_2 \ .$$

After skipping constant terms the reduced Hamiltonian on V_a^b is

$$\mathcal{H} = \frac{1}{2}\pi_2 + \pi_5$$

The twice reduced phase space is smooth for $|a| \neq |b|$ and degenerates into a cone at $a = b$ (and also at $a = -b$, but this "hanging top" is always stable). Figure 5.14 shows that one can show the presence of a Hamiltonian Hopf bifurcation using the geometric method [38].

The map $\psi : \mathbb{R}^3 \to \mathbb{R}^3; (\pi_5, \pi_4, \pi_2) \to (\pi_5, \pi_4, \mathcal{H})$ is a diffeomorphism. It changes R_a^b into

$$\tilde{R}_a^b(\pi_5, \pi_4, \mathcal{H}; a, b) = -\pi_4^2 - (a - b\pi_5)^2 + 2(\mathcal{H} - \pi_5)(1 - \pi_5^2) \ .$$

which gives an energy-momentum representation for the Poisson structure [59]. The following theorems are also taken from [59].

Theorem 5.9.1 *The Poisson structure given by R_a^b is at $(\pi_5, \pi_4, \pi_2) = (1, 0, 0)$ locally of type $\mathfrak{sl}(2)$.*

Proof : Set $a = N + \frac{1}{2}S$ and $b = N - \frac{1}{2}S$, replacing the integrals for the right and left

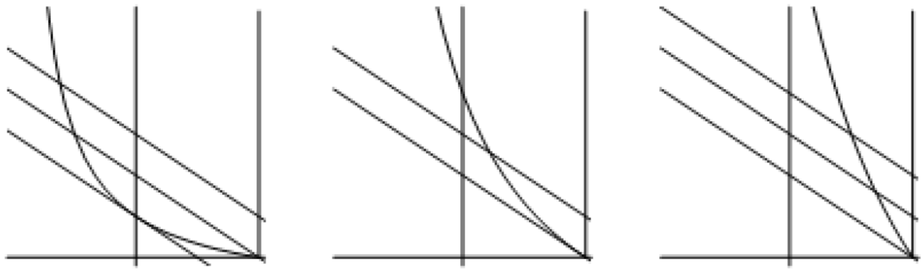

Figure 5.14: Relative position of $\{\mathcal{H} = h\}$ and V_a^a within $\{\pi_4 = 0\}$ for $|a| < 2$, $|a| = 2$ and $|a| > 2$. The horizontal axis is π_5 and the vertical axis is $\frac{1}{3}\pi_2$.

action by N and S. Furthermore apply the translation $\pi_5 = 1 - \tilde{\pi}_5$, $0 \leqslant \tilde{\pi}_5 \leqslant 2$, then R_a^b becomes

$$-\pi_4^2 - S^2 + 2\tilde{\pi}_5\pi_2 - 2NS\tilde{\pi}_5 + S^2\tilde{\pi}_5 - \pi_2\tilde{\pi}_5^2 - N^2\tilde{\pi}_5^2 + NS\tilde{\pi}_5^2 - \frac{1}{4}S^2\tilde{\pi}_5^2 \, , \qquad 0 \leqslant \tilde{\pi}_5 \leqslant 2 \, , \ \pi_2 \geqslant 0 \, ,$$

which is at zero locally equivalent to $-\pi_4^2 - S^2 + 2\tilde{\pi}_5\pi_2$. **q.e.d.**

Theorem 5.9.2 *The Poisson structure given by $\tilde{R}_a^b(\pi_5, \pi_4, \mathcal{H}; a, b)$ is at $(\pi_5, \pi_4, \pi_2) = (1, 0, 0)$ locally projection equivalent to $\tilde{C}(H, Y, Z; s, \nu)$.*

Here projection equivalent means that the projections on the energy-momentum plane (like in figure 5.12) are diffeomorphic.

Proof : Set $a = N + \frac{1}{2}\tilde{S}$ and $b = N - \frac{1}{2}\tilde{S}$. Furthermore apply $\pi_5 = 1 - \tilde{\pi}_5$, $0 \leqslant \tilde{\pi}_5 \leqslant 2$ and $\mathcal{H} = \tilde{H} + \frac{1}{2}NS$. Then $\tilde{R}_a^b$ transforms to

$$-\tilde{S}^2 - \pi_4^2 + 4\tilde{H}\tilde{\pi}_5 + \tilde{S}^2\tilde{\pi}_5 + 4\tilde{\pi}_5^2 - 2\tilde{H}\tilde{\pi}_5^2 - \frac{1}{4}\tilde{S}^2\tilde{\pi}_5^2 - 2\tilde{\pi}_5^3 \, .$$

Finally transform $S = \tilde{S} - \frac{1}{2}\tilde{S}\tilde{\sigma}_1$, and $H = \tilde{H} - \frac{1}{2}\tilde{H}\tilde{\pi}_5$. Setting $\pi_4 = Z$, $\tilde{\pi}_5 = Y$, and $1 - \frac{1}{4}N^2 = \nu$ finally gives

$$4YH - 4\nu Y^2 - 2Y^3 - Z^2 - S^2 \, .$$

q.e.d.

Bibliography

[1] Abraham, R, and Marsden, J.E. : *Foundations of Mechanics*, Benjamin/Cummings publishing Company, (1978).

[2] Arms, J.H., Marsden, J.E., and Moncrief, V. : *Symmetry and bifurcations of momentum mappings*, Commun. Math. Phys. **78**, (1981), pp 455–478.

[3] Arms, J.H., Cushman, R.H. and Gotay, M.J.: *A Universal Reduction Procedure for Hamiltonian Group Actions*, in: The Geometry of Hamiltonian Systems, Berkeley 1989, ed. T.S. Ratiu, MSRI series **22**, Springer Verlag,(1991), pp 26–38.

[4] Arnold, V.I.: *On matrices depending on parameters*, Russian Math.Surv. **26**, (1971), pp 46–60.

[5] Arnold, V.I. : *Mathematical methods of classical mechanics*, GTM **60**, Springer-Verlag, (1978).

[6] Atiyah, M.F. : *Convexity and commuting Hamiltonians*, Bull. London Math.Soc. **14**, (1982), pp 1–15.

[7] Bierstone, E.: *The structure of orbit spaces and the singularities of equivariant mappings*, Monografias de Matematica **35**, Inst. de Matematica Pura et Aplicada, Rio de Janeiro, (1980).

[8] Birkhoff, G.D.: *Dynamical systems*, A.M.S. collected papers IX, (1927).

[9] Bridges, T.J. : *Bifurcation of periodic solutions near a collission of eigenvalues of opposite signature*, Math.Proc.Cambr.Soc. 108, (1993), pp 575–601.

[10] , Bridges, T.J.: *Spatial Hamiltonian structure, energy flux and the water-wave problem*, Proc.Roy.Soc.Lond.Ser.A. **439**, pp 297–315.

[11] Brown, E.W. : *On the oscillating orbits about the triangular equilibrium points in the problem of three bodies*, Mon.Not.Roy.Astron.Soc. **71**, (1911), pp 492–502.

[12] Buono, P.-L., Laurant-Polz, F., and Montaldi, J.A.: *Symmetric Hamiltonian Bifurcations (J.A. Montaldi)*, In: Geometric Mechanics and Symmetry: the Peyresq lectures, Editors J. Montaldi and T.S. Ratiu. London Math. Soc. Lect. Note Series **306**, Cambridge Univ. Press, 2005, pp 357-402.

[13] Burgoyne, N. and Cushman, R.: *Normal forms for real linear Hamiltonian systems with purely imaginary eigenvalues*,Cel.Mech. **8**, (1974), pp 435–443.

[14] Champneys, A.R. and Thompson, J.M.T. : *A multiplicity of localized buckling modes for twisted rod equations*, Proc.Roy.Soc.Lond.-A. **452**, (1996), pp 2467–2491.

[15] Champneys, A.R. : *Homoclinic orbits in reversible sytems and their applications in mechanics, fluids and optics*, Physica D **112**, (1998), pp158–186.

[16] Cushman, R.H. and Rod, D.L.: *Reduction of the semisimple 1:1 resonance*, Physica D **6**, (1982), pp 105–112.

[17] Cushman, R.,and Sanders, J.: *Nilpotent normal forms and representation theory of* $sl_2(R)$ M. Golubitsky, J. Guckenheimer (Eds.), Multiparamter Bifurcation Theory, Contemporary Mathematics, **56**, Amer. Math. Soc, Providence (1987), pp. 353-371.

[18] Cushman, R. and Van der Meer, J.C. : *The Hamiltonian Hopf bifurcation in the Lagrange top*. In : *Géométrie Symplectique et Mécanique, La Grande Motte 1988* (ed. C. Albert) LNM **1416**, Springer Verlag, Berlin etc. (1990), pp 26 – 38.

[19] Cushman, R.H. and Bates, L.M. : *Global Aspects of Classical Integrable Systems*, Birkhäuser Verlag, (1997).

[20] Cushman, R., and Sniaticky, J. : *lectures on reduction theory for Hamiltonian systems*, Utrecht spring summer school on Lie groups, (2005),

`http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.524.564&rep=rep1&type=pdf}`

[21] Cushman, R., and Sniaticky, J. : *Differential structure of orbit spaces*, Canad. J. Math., **53**, (2001), pp 715 – 755. Errratum: Canad. J. Math., **55**, (2003), pp 247.

[22] Dellnitz, M, Melbourne, I, and Marsden, E.M.: *Generic bifurcation of Hamiltonian vectr fields with symmetry*, Nonlinearity **5**, (1992), pp 979–996.

[23] Dias, F. and Iooss,G. : *Capillary-gravity interfacial waves in infinite depth*, Eur.J.Mech.-B-Fluids **15**, (1996), pp 367–393.

[24] Diaz, G., Egea, J., Ferrer,S., Meer, J.C. van der, Vera, J.A.: *Relative equilibria and bifurcations in the generalized van der Waals 4D oscillator*. Physica D **239**, (2010), pp 1610–1625.

[25] Dufour, J.P.,Zung, N.T.: *Poisson Structures and Their Normal forms*, Birkhauser-Verlag, (2005).

[26] Dulac, M.H. : *Solution d'un système d'équations différentielles dans le voisinage de valuer singulières*, Bull.Soc. Math. France **40**, (1912), pp 324–383.

[27] Elphick, C., Tirapegui, E.,Brachet, M.E., Coullet, P., Iooss, G.: *A simple global characterization for normal forms of singular vector fields* Physica D, **29** (1987), pp. 95-127.

[28] Eliasson, H. : *Hamiltonian systems with Poisson commuting integrals*, Thesis University of Stockholm, (1984).

[29] Ferrer, S., Hanßmann, H., Palacián, J. and Yanguas, P. : *On Perturbed Oscillators in 1-1-1 Resonance : The Case of Axially Symmetric Cubic Potentials*, J. of Geometry and Physics **40**, (2002), pp 320–369.

[30] Fiorani,E., Giachetta,G., Sardanashvily, G.: *The Liouville-Arnold-Nekhoroshev theorem for non-compact invariant manifolds*, J. Phys. A **36**, (2003), pp L101– L107.

[31] Galin, D.M.: *Versal deformations of linear Hamiltonian systems*, A.M.S. Trans. **118**, (1982), pp 1–12. (Trudy Sem. Petrovsk. 1, 63-74, 1975.).

[32] Guillemin, V., and Sternberg, S. : *Convexity properties of the moment mapping*, Invent. Math. **67**, (1982), pp 491–513.

[33] Guillemin, V., and Sternberg, S. : *Symplectic techniques in physics*, Cambridge Univ. Press, (1984).

[34] Guirao, J.L. and Vera-López, J.A. : *Equilibria, stability and Hamiltonian Hopf bifurcation of a gyrostat in an incompressible ideal fluid*, Physica D, Vol. **241**, (2012), pp. 1648 - 1654.

[35] Gustavson, F.G.: *On constructing formal integrals of a Hamiltonian system near an equilibrium point*. Astron. J. **71**, (1966), pp 670-686.

[36] Hanßmann, H. and Van der Meer, J.C. : *On the Hamiltonian Hopf bifurcations in the $3D$ Hénon–Heiles family*, J. Dynamics Diff. Eq. **14**, (2002), pp 675–695.

[37] Hanßmann, H. and Van der Meer, J.C. : *Algebraic methods for determining Hamiltonian Hopf bifurcations in three-degree-of-freedom systems*, Journal of Dynamics and Differential Equations, **17**, (2005), pp 455–474.

[38] Hanßmann, H. and Van der Meer, J.C. : *On non-degenerate Hamiltonian Hopf bifurcations in 3DOF systems*. In: *EQUADIFF 2003, Proceedings of the International Conference on Differential Equations*, Hasselt, Belgium 22 - 26 July 2003, eds: F. Dumortier , H. Broer , J. Mawhin , A. Vanderbauwhede, S. Verduyn Lunel. World Scientific, Singapore, (2005), pp 476–481.

[39] Hénon, M. and Heiles, C.:*The applicability of the third integral of motion: some numerical experiments*, Astron. J. **69**, (1976), pp 73–79.

[40] Hilbert, D.: *Ueber die Theorie der algebraischen Formen*, Math. Ann., **36** , (1890).

[41] Holm, D.D.: *The Euler-Poincaré variational framework for modeling fluid dynamics(D.D. Holm)*; In: Geometric Mechanics and Symmetry: the Peyresq lectures, Editors J. Montaldi and T.S. Ratiu. London Math. Soc. Lect. Note Series **306**, Cambridge Univ. Press, (2005), pp 157-210.

[42] Hopf, E.: *Abzweigungen einer periodischen Lösung von einer stationären Lösung eines differential-systems*; Ber.Math.-Phys.Kl.Sächs Acad. Wiss. Leipzig **94**, (1942), pp 1-22,and **95**, (1942), pp 3-22.

[43] Hoveijn, I.: 1996, *Versal deformations and normal forms for reversible and Hamiltonian linear sytems*, J. Dif. Eq. **126**, (1996), pp 408-442.

[44] Iooss and Pérouème, : *Perturbed homoclinic solutions in reversible 1:1 resonance vector fields*, J.Diff.Eq. **102**, (1993), pp 62-88.

[45] Kozin, I.N., Roberts, R.M., and Tennyson, J. : *Symmetry and structure of rotating* $H_3^{[}+]$, J.Chem.Phys. **111**, (1999), pp 140–150.

[46] Krein, M..: *Foundations of the theory of* λ*-zones of stability of a canonical system of linear differential equations with periodic coefficients*, Am.Math.Soc.Trans. Ser. 2 **120**, (1955), pp 1–70.

[47] Kummer, M.: *On the construction of the reduced phase space of a Hamiltonian systen with symmetry*, Indiana Univ. math. J. **30**, (1981), pp 281–291.

[48] Lahiri, A. and Roy, M.S. : *The Hamiltonian Hopf bifurcation: an elementary perturbative approach*, Int.J.Non-lin.Mech. **36**, (2001), pp 787–802.

[49] Marsden, J.E. and Weinstein, A.: *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5**, (1974), pp 121–130.

[50] Marsden, J.E. and Ratiu,T.S.: *Introduction to Mechanics and Symmetry*, TAM series **17**, Springer-Verlag, (1999).

[51] Marsden, J.E. and Scheurle, J.: *Lagrangian reduction and the double sperical pendulum*, Zeitschr.Angew.Math.Phys. **44**, (1993), pp 17–43.

[52] Marsden, J.E., Misiolek, G., Ortega, J.P., Perlmutter, M., Ratiu, T.S.: *Hamiltonian reduction by stages*. Lect. Notes in Math. **1913**, Springer-Verlag, (2007)

[53] Meyer, K.R.: *Symmetries and integrals in mechanics*, in: Dynamical Systems, ed. M.M. Peixoto, Academic Press, (1973), pp 259–272.

[54] Meyer, K.R.: *Normal forms for Hamiltonian systems*, In: Dynamical system- Warwick, Lect. Notes Math. **468**, pp , Springer-Verlag, Berlin, 1974.

[55] van der Meer, J.C.: *The Hamiltonian Hopf bifurcation*, LNM 1160, Springer Verlag, (1985).

[56] van der Meer, J.C., Cushman, R.: *Constrained normalization of Hamiltonian systems and perturbed Keplerian motion*. Zeitschrift fÃ$\frac{1}{4}$r Angewandte Mathematik und Physik **37**, (1986), 402-424. Corrections, ZAMP **37**, (1986), pp 931.

[57] van der Meer, J.C.: *Integrability and reduction of normalized perturbed Keplerian systems*. RANA-report 88-15, (1988).

[58] van der Meer: *One-parameter versal deformations of symmetric Hamiltonian systems in 1:1 resonance*, International Journal of Pure and Applied Mathematics **53** (4), (2009), pp 547-561.

[59] van der Meer, J.C.: *Singularities of Poisson structures and Hamiltonian bifurcations*. CASA report 1005, TU/e, 2010.

[60] Melbourne, I. and Dellnitz, M.:*Normal forms for linear Hamiltonian vector fields commuting with the action of a compact Lie group.*, Proc.Cambr.Phil.Soc. **114**, (1993), pp 235–268.

[61] Meyer, K.R., and Schmidt, D.: *Periodic orbits near L_4 for mass ratio's near the critical mass ration of Routh*, Cel.Mech **4**, (1971), pp 99–109.

[62] Moser, J. : *New aspects in the theory of stability of Hamiltonian systems*, Comm. Pure Appl. Math. **11**, (1958), pp 81–114.

[63] Moser, J. : *Regularization of Kepler's Problem and the Averaging Method on a Manifold*, Comm. Pure Appl. Math. **23**, (1970), pp 609 – 636.

[64] Moser, J.: *Periodic orbits near an equilibrium and a theorem of Alan Weinstein*, Comm. Pure Appl. Math. **29**, (1976), pp 727-747.

[65] Nambu, Y. : *Generalized Hamiltonian dynamics*, Physical Review D **7**, (1973), pp 2405-2412.

[66] Nekhoroshev, N.N. : *The Poincaré-Lyapunov-Liouville-Arnol/d Theorem*, Funct. Anal. Appl. **28**, (1994), pp 128–129.

[67] Noether, E. : *Invariante Variationsprobleme*, Nachrichten on der KÃ¶niglichen Gesellschaft der Wissenschaften zu G|"ottingen, Mahtematisch-Physikalische Klasse, (1918), pp 235–257.

[68] Ortega, J.-P., and Ratiu, T.S. : *Momentum maps and Hamiltonian reduction*, Birkhäuser, Boston, (2004).

[69] Palacian, J., and Yanguas, P.: Reduction of polynomial Hamiltonians by the construction of formal integrals; Nonlinearity **13**, (2000), pp 1021-1054.

[70] Pitanga, P., Mundim, K.C., Ribeiro Filho, A.: *Stable states in ferroelectric liquid crystals*, Physica status solidi B **197**, (1996), pp 517–524.

[71] Poénaru, V. : *Singularités C^∞ en présence de symétrie*, LNM 510, Springer-Verlag, (1976).

[72] Poincaré, H. : *Thèse*, (1879), Also Oevres I, 59-129, Gauthiers-Villars, (1928).

[73] Ratiu,T.S., Sousa Dias, E., Sbano, L., Terra, G., Tudoran, R.: *A Crash Course in Geometric Mechanics (T.S. Ratiu)*; In: Geometric Mechanics and Symmetry: the Peyresq lectures, Editors J. Montaldi and T.S. Ratiu. London Math. Soc. Lect. Note Series **306**, Cambridge Univ. Press, 2005, pp 23-156.

[74] Reeb, G.: *Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques*, Mémoires de l'Académie Royale de Belgique, **27**:9, 1952.

[75] Rink, B. and Tuwankotta, T.: Stability In Hamiltonian Systems: Applications to the restricted three-body problem (K.R. Meyer); In: Geometric Mechanics and Symmetry: the Peyresq lectures, Editors J. Montaldi and T.S. Ratiu. London Math. Soc. Lect. Note Series **306**, Cambridge Univ. Press, 2005, pp 1-22.

[76] Routh, E.J. : *On Laplace's three particles, with a supplement on the stability of steady motion*, Proc.Lond.Math.Soc. **6**, (1875), pp86–97.

[77] Sadovskií,D. and Efstathiou, K.: *No polar coordinates (R.H. Cushman)*; In: Geometric Mechanics and Symmetry: the Peyresq lectures, Editors J. Montaldi and T.S. Ratiu. London Math. Soc. Lect. Note Series **306**, Cambridge Univ. Press, (2005), pp 211-302.

[78] Sanders, J.A.; Meer, J.C. van der: *Unique normal form of the Hamiltonian 1:2 resonance*. In: Geometry and analysis in nonlinear dynamics, Proceedings of the Conference on Chaotic Dynamics and Bifurcations, Groningen, March 1989, eds. H.W. Broer and F. Takens, Pitman Research Notes in Mathematics Series 222, Longman Scientific & Technical, Harlow, (1992), pp 56-69.

[79] Sanders, J.A., Verhulst, F., and Murdock, J.: Averaging Methods in Nonlinear Dynamical Systems, 2nd ed. Appl. Math. Sc. **59**, Springer-Verlag, New-York, 2007.

[80] Schwarz, G. : *Smooth function invariant under the action of a compact Lie group*, Topology **14**, (1975), pp 63–68.

[81] Smale. S. : *Topology and Mechanics*, Inventiones Math. **10**, (1970), pp 305–331 ; **11**, (1970),pp 45–64.

[82] Sokol'skij, A.G. : *On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies*, J. Appl. Math. Mech. **38**, (1974), pp 791–799.

[83] Vanderbauwhede, A.; Meer, J.C. van der: *A general reduction method for periodic solutions near equilibria in Hamiltonian systems*, in: Normal Forms and Homoclinic Chaos, eds. W.F. Langford and W. Nagata, Fields Institute Communications, vol. 4, AMS, Providemce, R.I., (1995), pp 273-294.

[84] Verhulst, F.:*Nonlinear Differential Equations and Dynamical Systems*, UTX, 2nd ed., Springer-Verlag, (1996).

[85] Weyl, H.: *The classical groups, their invariants and representations* , Princeton Univ. Press, (1946).

[86] Williamson, J. : *On the algebraic problem concerning the normal forms of linear dynamical systems*, Am.J.Math.**58**, (1936),pp 141–163.

Index

adjoint action, 8

Casimir function, 7
co-symplectic manifold, 25
coadjoint action, 8
constrained Hamiltonian vector field, 25

effective action, 8
energy-momentum map, 15

faithful action, 8
flow, 6
free action, 8

Hamiltonian G-space, 9
Hamiltonian normal form, 23
Hamiltonian vector field, 6
homological equation, 21

infinitesimal generator, 9
integrable system, 15
isotropy algebra, 9
isotropy subgroup, 8

Kirillov-Kostant-Souriau symplectic form, 11

left invariant, 11
left invariant extension, 11
Lie algebra, 8
Lie group, 8
Lie Poisson bracket, 11
Lie-Poisson reduction, 11
Liouville integrable, 15

momentum mapping, 9

orbit, 8
orbit space, 9

Poisson algebra, 6
Poisson bracket, 6
Poisson manifold, 6
Poisson map, 6
Poisson reduction, 11
proper map, 8

reduced Hamiltonian, 10
reduced phase space, 10
relative equilibrium, 16
resonant eigenvalues, 22

semi-algebraic subset, 14
structure matrix, 6
symplectic action, 9
symplectic leaf, 7

transitive action, 8

versal deformation, 30
versal unfolding, 30

Whitney stratification, 14

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
17-05	B.S. van Lith J.H.M. ten Thije Boonkkamp W.L. IJzerman	Full linear multistep methods as root-finders	February ‘17
17-06	J.H.M. ten Thije Boonkkamp M.J.H. Anthonissen R.J. Kwant	A two-dimensional complete flux scheme in local flow adapted coordinates	March ‘17
17-07	C. Filosa J.H.M. ten Thije Boonkkamp W.L. Ijzerman	Phase space ray tracing for a two-dimensional parabolic reflector	June ‘17
17-08	J.H.M. ten Thije Boonkkamp M.J.H. Anthonissen	Compact high order complete flux schemes	June ‘17
17-09	J.C. van der Meer	On the Geometry of Hamiltonian Systems Lecture Notes Seminar GISDA Universidad del Bio Bio	August ‘17

Ontwerp: de Tantes,
Tobias Baanders, CWI