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Turtle graphics of morphic sequences

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TURTLE GRAPHICS OF MORPHIC SEQUENCES

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Abstract

The simplest infinite sequences that are not ultimately periodic are pure morphic sequences: fixed points of particular morphisms mapping single symbols to strings of symbols. A basic way to visualize a sequence is by a turtle curve: for every alphabet symbol fix an angle, and then consecutively for all sequence elements draw a unit segment and turn the drawing direction by the corresponding angle. This paper investigates turtle curves of pure morphic sequences. In particular, criteria are given for turtle curves being finite (consisting of finitely many segments), and for being fractal or self-similar: it contains an up-scaled copy of itself. Also space-filling turtle curves are considered, and a turtle curve that is dense in the plane. As a particular result we give an exact relationship between the Koch curve and a turtle curve for the Thue–Morse sequence, where until now for such a result only approximations were known.

Keywords: Turtle Curve; Morphic Sequence; Self-Similar.

1. INTRODUCTION

Infinite sequences, shortly called sequences, are the simplest possible infinite objects. The simplest sequences are periodic, up to an initial part. The next simplest are morphic sequences: fixed points of particular morphisms mapping single symbols to strings of symbols. A typical example is the fixed point starting in 0 of the morphism ϕ defined by $\phi(0) = 01, \phi(1) = 10$, yielding

$$\phi(0) = 01, \quad \phi^2(0) = 0110, \quad \phi^3(0) = 01101001,$$
$$\phi^4(0) = 0110100110010110, \ldots$$

As $\phi^n(0)$ is a prefix of $\phi^{n+1}(0)$ for every n, the limit of this process yields the unique fixed point of ϕ starting in 0: the Thue–Morse sequence.

A sequence can be visualized by a turtle curve: for every alphabet symbol fix an angle. Then the turtle curve is obtained by drawing a unit segment for every sequence element, and adjust the drawing direction by the corresponding angle. In this paper, we investigate turtle curves of morphic sequences. Arbitrarily choosing simple morphisms and angles typically yield turtle curves hardly showing any structure. However, sometimes the turtle curve is finite, that is, it consists of finitely many segments that are drawn over and over again, or shows up fractal or self-similar behavior: it contains an up-scaled copy of itself. When browsing through this paper you will see several examples of turtle curves showing some regular repeating pattern, either finite or indicated by self-similarity. The main goal of this paper is to investigate the criteria for obtaining finite or self-similar turtle curves.

One of the most well-known fractal curves is the Koch curve, going back to Ref. 1, and one of the most well-known morphic sequences is the Thue–Morse sequence as defined above. It was known before that particular variants of turtle curves for the Thue–Morse sequence approximate the Koch curve in the Hausdorff metric, see Ref. 2. In this paper, we go a step further: we show that when connecting particular mid points of segments in a turtle curve for the Thue–Morse sequence, one exactly obtains the Koch turtle curve, rather than only approximating it.

The reason for studying turtle curves of sequences is not only in generating nice pictures. A turtle curve visualizes a sequence, and patterns showing up in the visualization may hint toward properties of the structure of the sequence and help in understanding them.

We restrict to the very simplest version of turtle curves, fully defined by choosing an angle for every alphabet symbol, since the drawing algorithm only draws unit segments in a direction determined by these angles. Obvious generalizations include variants allowing drawing segments of non-fixed lengths, or drawing other shapes rather than segments. Finite initial parts of both morphic sequences and turtle curves can be described by *L-systems*, Ref. 3, in particular *D0L-systems* (deterministic L-systems with 0 context symbols) being a particular kind of context free grammars for which the productions are applied in parallel, and for which drawing instructions are coupled to the terminals. A first approach to draw turtle curves of D0L-systems, including some experimental observations on grid filling and fractal behavior, was presented in Ref. 4.

Other related work includes recurrent sets from Ref. 5. There the sequence generation is similar to L-systems and morphic sequences, but an essential difference is in the way of drawing: there every symbol has a fixed drawing direction, where in our turtle curves the drawing direction is the accumulation of the angles of all symbols inspected before. By extending the alphabet to all drawing directions occurring in the turtle curve, the turtle curve is closely related to a recurrent set over a more complicated sequence over this extended (possibly infinite) alphabet. Another essential difference is that recurrent sets are compact sets obtained as a limit into the small, while our turtle curves are unions of infinitely many unit segments, going to infinity in case of unboundedness.

This paper is organized as follows. In Sec. 2, we introduce morphic sequences; in Sec. 3 we describe turtle curves. In Sec. 4, we give criteria for turtle curves to be finite, illustrated by several examples. In Sec. 5, we introduce self-similarity, and give criteria for self-similarity for point sets of turtle curves, together with a number of examples. In Sec. 6, we give our results relating the Thue–Morse sequence and the Koch turtle curve. Exploiting some results from Sec. 6, in Sec. 7 we give a modified criterion for point sets of turtle curves to be self-similar. In Sec. 8, we give examples of turtle curves that are space-filling in several senses: they meet every grid point exactly once, or every grid segment exactly once, or are dense in the plane. We conclude in Sec. 9.

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2. MORPHIC SEQUENCES

Let A be a finite alphabet. As usual, we write A^* for the set of finite strings over A, ϵ for the empty string and $A^+ = A^* \backslash \{\epsilon\}$. We write $|u|$ for the length of a string u, and $|A|$ for the size of a finite set A.

A *sequence* over A is defined to be a map $\sigma : \mathbf{N} \to A$, where $\mathbf{N}$ consists of the natural numbers $0, 1, 2, \ldots$. We write A^ω for the set of sequences over A.

For $n \in \mathbf{N}$ the string $\sigma{\downarrow_n} = \sigma(0)\sigma(1)\cdots\sigma(n-1) \in A^n$ is called the *prefix of length* n of σ. We write $\mathsf{Pref}(\sigma)$ for the set of all prefixes of the sequence σ.

For $u = u_0 u_2 \cdots u_{n-1} \in A^*$ and $\sigma \in A^\omega$ the sequence $u\sigma$ is defined by $(u\sigma)(i) = u_i$ for $i < n$ and $(u\sigma)(i) = \sigma(n-i)$ for $i \geq n$.

For $n \in \mathbf{N}$ and $\sigma \in A^\omega$ the sequence $\sigma^{(n)}$ is defined by $\sigma^{(n)}(i) = \sigma(i+n)$ for $i \in \mathbf{N}$. So for all $n \in \mathbf{N}$ we have $\sigma = (\sigma{\downarrow_n})\sigma^{(n)}$.

A sequence σ is called *periodic* with period n if $\sigma(i+n) = \sigma(i)$ for all $i \in \mathbf{N}$, or, equivalently, $\sigma = \sigma^{(n)}$. We write $\sigma = u^\omega$ for $u = \sigma{\downarrow_n}$ if σ is periodic with period n. A sequence σ is called *ultimately periodic* if $\sigma^{(m)}$ is periodic for some $m \in \mathbf{N}$.

A *morphism* is a map $\phi : A \to B^*$; we will only consider morphisms $\phi : A \to B^+$ to ensure that infinite sequences will be mapped to infinite sequences. Morphisms are extended to $\phi : A^* \to B^*$ by defining $\phi(a_1 a_2 \cdots a_n) = \phi(a_1)\phi(a_2)\cdots\phi(a_n)$ and to $\phi : A^\omega \to B^\omega$ by defining $\phi(a\sigma) = \phi(a)\phi(\sigma)$.

If $A = B$ and one particular $a \in A$ satisfies $\phi(a) = ax$ for $x \in A^+$, this gives rise to the *pure morphic* sequence (Ref. 6)

$$\phi^\omega(a) = ax\phi(x)\phi^2(x)\phi^3(x)\phi^4(x)\cdots.$$

It is easily shown that this is the only sequence starting in a that is a *fixed point* of ϕ, i.e. $\phi(\sigma) = \sigma$. A *morphic sequence* over A is defined to be a sequence of the form $\tau(\sigma)$ for some pure morphic sequence σ over B and some morphism $\tau : B \to A$. In the literature, the sequence $a, \phi(a), \phi^2(a), \ldots$ is often called a D0L-sequence and the sequence $\tau(a), \tau(\phi(a)), \tau(\phi^2(a)), \ldots$ is then called a CD0L-sequence, see e.g. Ref. 6.

The following simple example shows that not every morphic sequence is purely morphic. Define the sequence $\mathsf{square} = 1100100\cdots$ over $\{0,1\}$ by $\mathsf{square}(n) = 1$ if and only if n is a square. This is not pure morphic over $\{0,1\}$ since $\lim_{n\to\infty} |\mathsf{square}{\downarrow_n}|_1/n = 0$, where $|u|_1$ denotes the number of 1's occurring in u, and this can only be achieved for $\mathsf{square} = \phi^\omega(1)$ if $\phi(0) = 0^k$ for $k > 0$, which yields a contradiction by some case analysis. However, square is morphic since $\mathsf{square} = \tau(\phi^\omega(2))$ for τ, ϕ defined by $\phi(0) = 0$, $\phi(1) = 001$, $\phi(2) = 21$, $\tau(0) = 0$, $\tau(1) = 1$, $\tau(2) = 1$.

Three famous morphic sequences are the Thue–Morse sequence t, the period-doubling sequence pd and the Fibonacci sequence fib. They are defined by $\mathsf{t} = \phi^\omega(0)$ for $\phi(0) = 01$, $\phi(1) = 10$, $\mathsf{pd} = \phi^\omega(0)$ for $\phi(0) = 01$, $\phi(1) = 00$, and $\mathsf{fib} = \phi^\omega(0)$ for $\phi(0) = 01$, $\phi(1) = 0$. In fact up to swapping symbols these are the only three pure morphic sequences over $\{0,1\}$ with $|\phi(a)| \leq 2$ for $a = 0, 1$ that are not ultimately periodic.

3. TURTLE CURVES

Let for every $a \in A$ an angle $\alpha(a) \in \mathbf{R}$ be given. Then for a sequence σ over A its *turtle curve* $C(\sigma, \alpha) \subseteq \mathbf{R}^2$ is described as follows. Start in $(0,0)$ and draw a segment of unit length in the direction $\alpha(\sigma(0))$, by which the current direction is $\alpha(\sigma(0))$. Next for $i = 1, 2, 3, \ldots$ continue by adding $\alpha(\sigma(i))$ to the current direction and draw a segment (starting in the end point of the last drawn segment) in the direction of this current direction. In this paper, we investigate the resulting turtle curves for various sequences and various $\alpha : A \to \mathbf{R}$. Following the above description we now give a formal definition.

Definition 1. Let σ be a sequence over A and $\alpha : A \to \mathbf{R}$.

- $\alpha : A \to \mathbf{R}$ is extended to $\alpha : A^* \to \mathbf{R}$ by defining $\alpha(\epsilon) = 0$ and $\alpha(a_1, a_2, \ldots, a_n) = \sum_{i=1}^n \alpha(a_i)$.
- For $u \in A^*$ its *position* $P(u, \alpha) \in \mathbf{R}^2$ is defined inductively by $P(\epsilon, \alpha) = (0,0)$ and
$$P(ua, \alpha) = P(u, \alpha) + (\cos(\alpha(ua)), \sin(\alpha(ua)))$$
for $u \in A^*$ and $a \in A$.
- The *turtle curve point set* $P(\sigma, \alpha)$ is defined by
$$P(\sigma, \alpha) = \bigcup_{u \in \mathsf{Pref}(\sigma)} \{P(u, \alpha)\}.$$
- The *turtle curve* $C(\sigma, \alpha)$ is defined to consist of the union of all subsequent segments between the points in $P(\sigma, \alpha)$:
$$C(\sigma, \alpha) = \bigcup_{ua \in \mathsf{Pref}(\sigma)} \{\lambda P(u, \alpha) + (1-\lambda)P(ua, \alpha) \mid \lambda \in [0,1]\}.$$

Note that this definition of a turtle curve is the very simplest possible one. It allows several obvious extensions, for instance by drawing segments of

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other lengths than only unit length, or even other objects. In Sec. 6, we will use a variant with two angles: one for turning before and one for turning after drawing the segment. But our basic definition only uses one angle per symbol: in every step first turn this angle and then draw the unit segment.

As a first example consider the constant zero sequence $\sigma = 0^\omega$ defined by $\sigma(i) = 0$ for all $i \in \mathbf{N}$, and choose $\alpha(0) = \pi/2$. Then the turtle curve consists of a square, and $P(\sigma, \alpha) = \{(0,0), (0,1), (-1,1), (-1,0)\}$ since $P(\epsilon, \alpha) = (0,0)$, $P(0, \alpha) = (0,1)$, $P(00, \alpha) = (-1,1)$, $P(000, \alpha) = (-1,0)$ and $P(0^i, \alpha) = P(0^{i-4}, \alpha)$ for $i \geq 4$.

For the same sequence $\sigma = 0^\omega$ we obtain

- if $\alpha(0) = 0$ then $P(\sigma, \alpha) = \{(i, 0) \mid i \in \mathbf{N}\}$,
- if $\alpha(0) = \pi$ for $n \geq 3$ then $P(\sigma, \alpha) = \{(0,0), (-1,0)\}$,
- if $\alpha(0) = 2\pi/n$ for $n \geq 3$ then $P(\sigma, \alpha)$ consists of the nodes of a regular n-gon,
- if $\alpha(0) = x\pi$ for an irrational number x, then $P(\sigma, \alpha)$ consists of infinitely many points on a circle.

By definition the distance between two consecutive elements of $P(\sigma, \alpha)$ is 1, so $P(\sigma, \alpha)$ contains at least two points for every σ, α. The above examples already show that it can be finite or infinite; if it is finite then $|P(\sigma, \alpha)|$ may be any number ≥ 2, and if it is infinite then it may be either bounded or unbounded.

More general, the same three types of turtle curves occur for arbitrary periodic sequences $\sigma = u^\omega$. If $\alpha(u) = 0$ and $P(u, \alpha) \neq (0,0)$ then $P(\sigma, \alpha) = S + \mathbf{N} \cdot P(u, \alpha)$ for a finite set $S \subseteq \mathbf{R}^2$, by which $P(\sigma, \alpha)$ is unbounded. Otherwise, if $\alpha(u) = x\pi$ for a rational number x then $P(\sigma, \alpha)$ is finite. In the remaining case $P(\sigma, \alpha)$ is bounded and infinite.

If σ is periodic and $P(\sigma, \alpha)$ is unbounded then it is easily shown that $P(\sigma, \alpha)$ is contained in a strip. For non-periodic sequences this is not the case: Figure 1 shows a fragment of the infinitely spiraling turtle curve obtained by choosing $\alpha(0) = 0$ and $\alpha(1) = \pi/2$ and the sequence square as introduced before.

Earlier results on unboundedness include Ref. 7, in particular proving that $P(\mathsf{fib}, \alpha)$ is unbounded for $\alpha(0) = 0$ and $\alpha(1) = \pi/2$, and even more, contains every grid point in a full quadrant of the plane.

Every turtle curve of any morphic sequence $\tau(\sigma)$ is also a turtle curve of the pure morphic sequence σ by choosing $\alpha(a) = \alpha(\tau(a))$ for every a; this justifies omitting "pure" in the title of this paper.

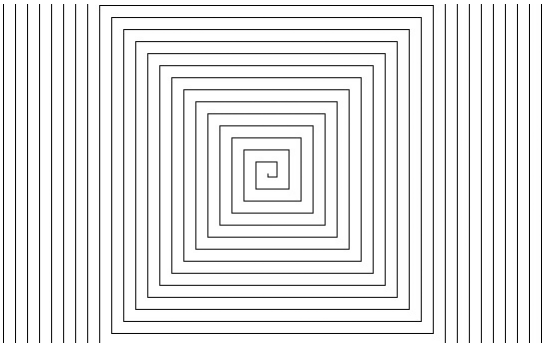

Fig. 1

We end this section by a lemma that we will often use: to obtain $P(uv, \alpha)$ from $P(u, \alpha)$ one adds a rotated version of $P(v, \alpha)$. For an angle θ its rotation $R_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ is defined by $R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. We use $+$ and $-$ for addition and subtraction of vectors in $\mathbf{R}^2$.

Lemma 2. *Let $u, v \in A^*$. Then $P(uv, \alpha) = P(u, \alpha) + R_{\alpha(u)}(P(v, \alpha))$.*

Proof. Induction on $|v|$. For $|v| = 0$ it holds by definition. Using $R_\alpha(\cos\beta, \sin\beta) = (\cos(\alpha+\beta), \sin(\alpha+\beta))$, and using the induction hypothesis on $P(uv, \alpha)$ we obtain

$$\begin{aligned}
P(uva, \alpha) &= P(uv, \alpha) + (\cos(\alpha(uva)), \sin(\alpha(uva))) \\
&= P(u, \alpha) + R_{\alpha(u)}(P(v, \alpha)) \\
&\quad + (\cos(\alpha(uva)), \sin(\alpha(uva))) \\
&= P(u, \alpha) + R_{\alpha(u,\alpha)}(P(v, \alpha)) \\
&\quad + R_{\alpha(u)}(\cos(\alpha(va)), \sin(\alpha(va))) \\
&= P(u, \alpha) + R_{\alpha(u)}(P(v, \alpha) \\
&\quad + (\cos(\alpha(va)), \sin(\alpha(va)))) \\
&= P(u, \alpha) + R_{\alpha(u)}(P(va, \alpha)).
\end{aligned}$$

□

Note that Lemma 2 implies that $P(uv, \alpha) = P(u, \alpha) + P(v, \alpha)$ if $\alpha(u) = 0$ and $P(uv, \alpha) = P(u, \alpha) - P(v, \alpha)$ if $\alpha(u) = \pi$.

4. FINITE TURTLE CURVES

A turtle curve $C(\sigma, \alpha)$ is called *finite* if it consists of finitely many segments; this is equivalent to finiteness of the set $P(\sigma, \alpha)$.

In this section, we give some criteria and examples for finite turtle curves. In Ref. 8, an analysis is given for finiteness of the degenerated class of turtle curves over two symbols only having angles 0 and π, by which the turtle curve is a subset of a line.

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The next theorem provides a fruitful criterion for $P(\sigma, \alpha)$ being finite. For a set $L \subseteq A^+$ of non-empty strings we define L^ω to consist of all sequences that can be written as $u_1u_2u_3\cdots$ for $u_i \in L$ for all i.

Theorem 3. *Let $\sigma \in \{u_1, \ldots, u_n\}^\omega$ for $u_1, \ldots, u_n \in A^+$, $n \geq 2$. Assume that $\alpha(u_i)$ is a multiple of 2π and $P(u_i, \alpha) = (0, 0)$ for $i = 1, \ldots, n$. Then $P(\sigma, \alpha)$ is finite and $|P(\sigma, \alpha)| \leq 1 + \sum_{i=1}^{n}(|u_i| - 1)$.*

Proof. Let $u \in \mathsf{Pref}(\sigma)$. Then $u = v_1v_2\cdots v_k$ for $v_i \in \{u_1, \ldots, u_n\}$ for $i = 0, \ldots, k-1$, and v_k is a prefix of u_i for some $i = 1, \ldots, n$, $v_k \neq u_i$. Then by repeatedly applying Lemma 2 for v_j for $j = 1, 2, \ldots$, satisfying $P(v_j, \alpha) = (0, 0)$ and $\alpha(v_j) = 0$, we obtain $P(u, \alpha) = P(v_k, \alpha)$. As the total number of proper non-empty prefixes of u_i is $|u_i| - 1$ we obtain $|P(\sigma, \alpha)\backslash\{(0, 0)\}| \leq \sum_{i=1}^{n}(|u_i| - 1)$, so $|P(\sigma, \alpha)| \leq 1 + \sum_{i=1}^{n}(|u_i| - 1)$, proving the theorem. □

Theorem 3 admits several variants; for instance by omitting the requirement of all $\alpha(u_i)$ being a multiple of 2π, we can conclude boundedness of the turtle curve, and weakening being a multiple of 2π to being a rational number times π, we can conclude finiteness, but with a higher bound $|P(\sigma, \alpha)|$ depending on the denominators of the rational numbers.

Now we apply Theorem 3 for the Thue–Morse sequence t, exploiting its special structure.

Theorem 4. *Let $\alpha(0) + \alpha(1) = k\pi/2^n$ for k odd. Then $P(\mathsf{t}, \alpha)$ is finite and $|P(\mathsf{t}, \alpha)| \leq 2^{n+4}$.*

Proof. Let $\phi(0) = 01, \phi(1) = 10$, so $\mathsf{t} = \phi(\mathsf{t})$. Let $u = \phi^{n+1}(0)$ and $v = \phi^{n+1}(1)$. Since both u and v consists of 2^n 0's and 2^n 1's, we obtain $\alpha(u) = \alpha(v) = k\pi$ and $\alpha(uvv) = 3k\pi$. Observe that $R_{k\pi}(P) = -P$ for every $P \in \mathbf{R}^2$ and k odd. Applying Lemma 2 three times yields

$$P(uvvu, \alpha) = P(uvv, \alpha) - P(u, \alpha),$$
$$P(uvv, \alpha) = P(u, \alpha) - P(vv, \alpha),$$
$$P(vv, \alpha) = P(v, \alpha) - P(v, \alpha) = 0,$$

together yielding $P(uvvu, \alpha) = (0, 0)$, and similarly $P(vuuv, \alpha) = (0, 0)$. We obtain $\alpha(uvvu) = \alpha(vuuv) = 4k\pi$. Since $\mathsf{t} = \phi^{n+3}(\mathsf{t})$, $\phi^{n+3}(0) = uvvu$ and $\phi^{n+3}(1) = vuuv$ we obtain $\mathsf{t} \in \{uvvu, vuuv\}^\omega$. Now by Theorem 3 we conclude that $P(\mathsf{t}, \alpha)$ is finite and $|P(\mathsf{t}, \alpha)| \leq |uvvu| + |vuuv| = 2^{n+4}$. □

The following three pictures show finite turtle curves of t for which Theorem 4 applies. The parameters of Fig. 2 are $\alpha(0) = 0$ and $\alpha(1) = \pi/2$; of Fig. 3 they are $\alpha(0) = \pi/16$ and $\alpha(1) = 3\pi/4$; and of Fig. 4 they are $\alpha(0) = \pi/8$ and $\alpha(1) = 63\pi/64$. The corresponding sets $P(\sigma, \alpha)$ consist of 20, 250 and 1018 points, respectively, all being close to the bounds 32, 256 and 1024 given by Theorem 4.

Next we give a theorem by which finiteness of $P(\sigma, \alpha)$ can be concluded similar to Theorem 3, but with weaker conditions: relaxing the condition on $\alpha(u_1)$ and even removing the condition on $P(u_1, \alpha)$.

Theorem 5. *Let $\sigma \in \{u_1, \ldots, u_n\}^\omega$ for which $\alpha(u_i)$ is a multiple of 2π and $P(u_i, \alpha) = (0, 0)$ for $i = 2, \ldots, n$, and $\alpha(u_1) = q\pi$, where q is rational and not a multiple of 2. Then $P(\sigma, \alpha)$ is finite.*

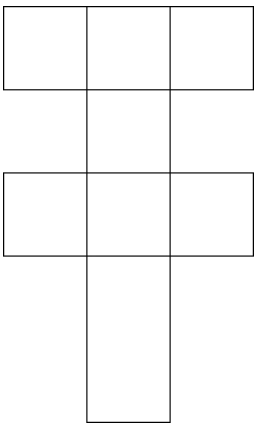

Fig. 2

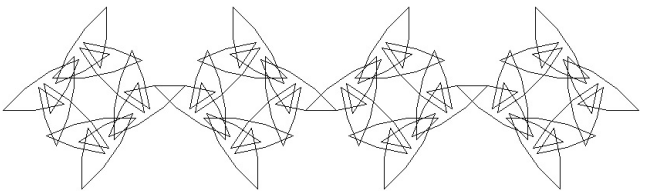

Fig. 3

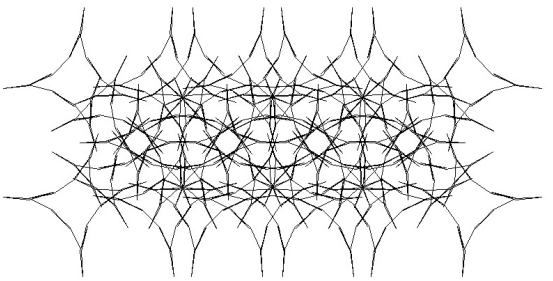

Fig. 4

Proof. Let $P = \{P((u_1^k, \alpha)) \mid k \in \mathbf{N}\}$. From $\alpha(u_1) = q\pi$ and q is rational and not a multiple of 2 it follows that P is finite. Let $u \in \mathsf{Pref}(\sigma)$. Then $u = v_1 v_2 \cdots v_k$ for $v_i \in \{u_1, \ldots, u_n\}$ for $i = 0, \ldots, k-1$, and v_k is a prefix of u_i for some $i = 1, \ldots, n$. Then by repeatedly applying Lemma 2 for v_j for $j = 1, 2, \ldots$, we obtain $P(v_1 v_2 \ldots v_{k-1}, \alpha) \in P$. So $P(u, \alpha) = p + R_{\alpha(u_1)^m}(P(v_k, \alpha))$ for $p \in P$ and $m \in \mathbf{N}$. Since P is finite, v_k is one of the finitely many prefixes of u_i for some i, and $\alpha(u_1)$ is rational, there are only finitely many such points. □

We give a few examples of finite turtle curves of the sequence $\sigma = \phi^\omega(1)$ for ϕ defined by $\phi(0) = 00$ and $\phi(1) = 101$, and $\alpha(0) = k\pi/2^n$ for some k, n. Then $\phi^{n+1}(0) = 0^{2^{n+1}}$, by which the turtle curve of $\phi^{n+1}(0)$ is a regular 2^{n+1}-gon or 2^{n+1}-star, yielding $\alpha(\phi^{n+1}(0)) = 2k\pi$ and $P(\phi^{n+1}(0), \alpha) = (0, 0)$. Further observe that $\sigma \in \{\phi^{n+1}(1), \phi^{n+1}(0)\}^\omega$, by which Theorem 5 applies for proving finiteness of $P(\sigma, \alpha)$ if $\alpha(\phi^{n+1}(1))$ satisfies the rationality condition. Figure 5 is obtained by choosing $\alpha(0) = \pi/16, \alpha(1) = 3\pi/4$, Fig. 6 is obtained by choosing $\alpha(0) = \pi/32, \alpha(1) = -2\pi/3$, Fig. 7 is obtained by choosing $\alpha(0) = \pi/4, \alpha(1) = -17\pi/18$, and Fig. 8 is obtained by choosing $\alpha(0) = 5\pi/16, \alpha(1) = -29\pi/60$.

The next example also applies Theorem 5, but in a more hidden way. Define $\sigma = \phi^\omega(0)$ for ϕ defined by $\phi(0) = 0010$ and $\phi(1) = 1010$; this is the sequence yielding Koch's curve as we will see in Sec. 5. Now we choose $\alpha(0) = 2\pi/5$ and $\alpha(1) = -\pi/5$. Then one checks that $P(\phi(00), \alpha) = P(\phi(10), \alpha) = (0, 0)$, $\alpha(\phi(00)) = 2\pi$ and $\alpha(\phi(10)) = 7\pi/5$. Since $\sigma \in \{\phi(10), \phi(00)\}^\omega$, we obtain by

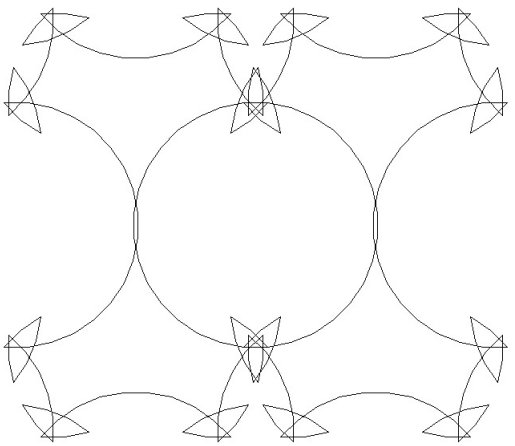

Fig. 5

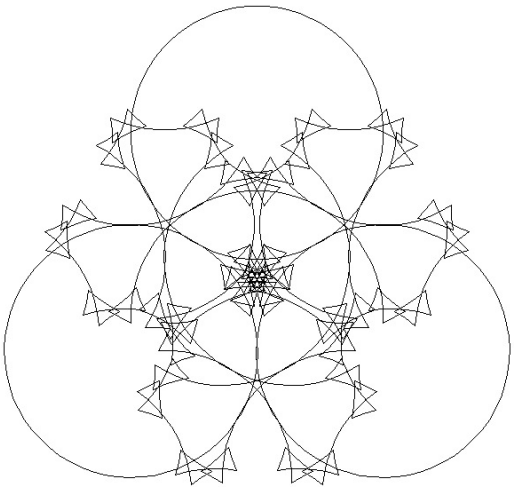

Fig. 6

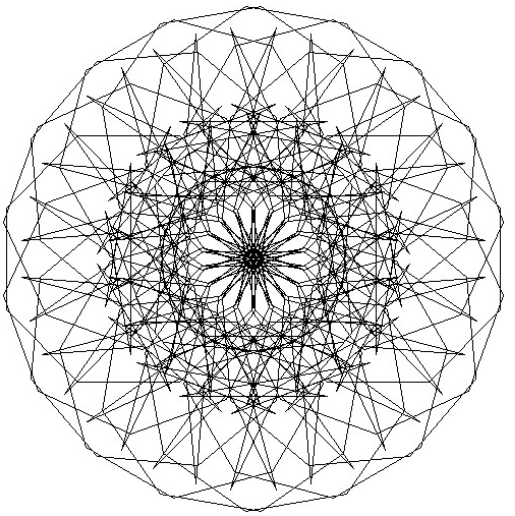

Fig. 7

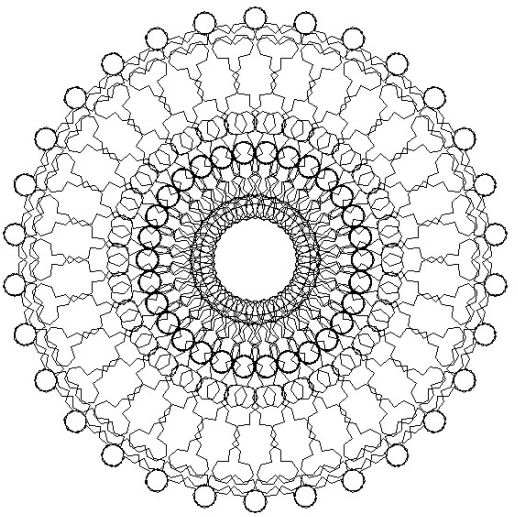

Fig. 8

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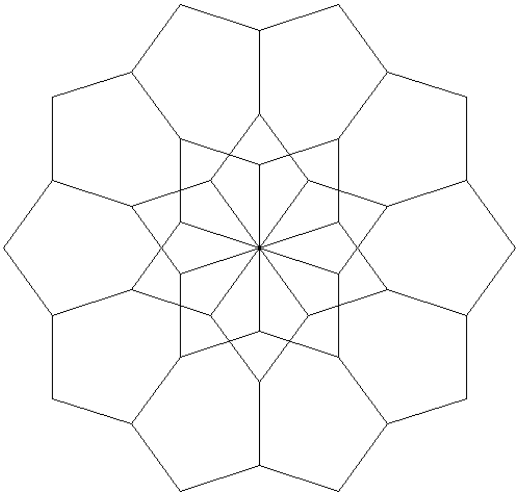

Fig. 9

Theorem 5 that $P(\sigma, \alpha)$ is finite; its turtle curve is shown in Fig. 9.

Many finite turtle curves of the sequence $\sigma = \phi^\omega(0)$ for ϕ defined by $\phi(0) = 011$ and $\phi(1) = 0$, called *rosettes*, are presented in Ref. 9. Finiteness of them can be proved by Theorem 5 as follows. The requirement for the angles in rosettes is $a\alpha(0) + b\alpha(1) = \pi$, up to a multiple of 2π, in which a and b are the numbers of 0s and 1s, respectively, in $\phi^k(0)$ for some k. For instance, for $k = 2$ we have $a = 3$ and $b = 2$. Choosing α in this way yields $\alpha(\phi^k(0)) = \pi$, from which we conclude $\alpha(\phi^k(00)) = 2\pi$ and $P(\phi^k(00), \alpha) = (0, 0)$. Since 1 only occurs in groups of two in $\phi(0)$ and $\phi(1)$, we obtain $\sigma \in \{0, 11\}^\omega$. Since σ is a fixed point of ϕ^{k+1}, we obtain $\sigma \in \{\phi^{k+1}(0), \phi^{k+1}(11)\}^\omega$. Since $\phi^{k+1}(11) = \phi^k(00)$, for applying Theorem 5 the only remaining requirement is a rationality requirement for $\alpha(\phi^{k+1}(0))$, which holds for all rosettes. In Fig. 10, we show a typical rosette from Ref. 9 for $\alpha(0) = 7\pi/9, \alpha(1) = -2\pi/9$, which satisfies the above criteria for $k = 7, a = 85, b = 86$.

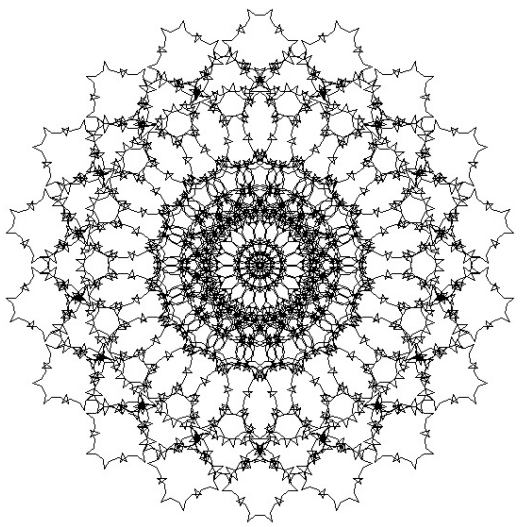

Fig. 10

5. SELF-SIMILAR TURTLE CURVES

Roughly speaking, a set $P \subseteq \mathbf{R}^2$ is called *fractal* or *self-similar* if it contains a copy of itself when zooming in or out. As in our turtle curves all drawing steps have unit length, it is most natural to focus on zooming out, leading to the following definition. As before we use the notation $c(x, y) = (cx, cy)$ for scalar vector multiplication and $R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ for rotation over θ.

Definition 6. A set $P \subseteq \mathbf{R}^2$ is called *self-similar* if there exists $(x_0, y_0) \in \mathbf{R}^2$, a rotation angle θ and a scaling factor $c > 1$ such that

$$(x_0, y_0) + cR_\theta(x, y) \in P$$

for all $(x, y) \in P$.

A turtle curve $C(\sigma, \alpha)$ is called *self-similar* if $P(\sigma, \alpha)$ is self-similar.

A self-similar set containing two distinct elements is always unbounded and hence infinite, since for every $(x, y) \neq (x', y') \in P$ two points $(x_0, y_0) + cR_\theta(x, y)$ and $(x_0, y_0) + cR_\theta(x', y')$ in P can be obtained of which the distance is increased by a factor $c > 1$.

Theorem 7. *Let $\sigma \in A^\omega$ satisfy $\sigma = \phi(\sigma)$ for some $\phi : A \to A^+$ satisfying $\alpha(\phi(a)) = \alpha(a)$ and $P(\phi(a), \alpha) = cR_\theta(P(a, \alpha))$ for all $a \in A$, for some scaling factor $c > 1$ and some rotation angle θ.*

Then $P(\sigma, \alpha)$ is self-similar.

Proof. First we prove the following claim by induction on $|u|$.

Claim: $\alpha(\phi(u)) = \alpha(u)$ and $P(\phi(u), \alpha) = cR_\theta(P(u, \alpha))$ for all $u \in A^+$.

For $|u| = 1$ this is given. For the induction step we prove the claim for ua, assuming that it holds for u. We have

$$\alpha(\phi(ua)) = \alpha(\phi(u)\phi(a)) = \alpha(\phi(u)) + \alpha(\phi(a))$$
$$= \alpha(u) + \alpha(a) = \alpha(ua),$$

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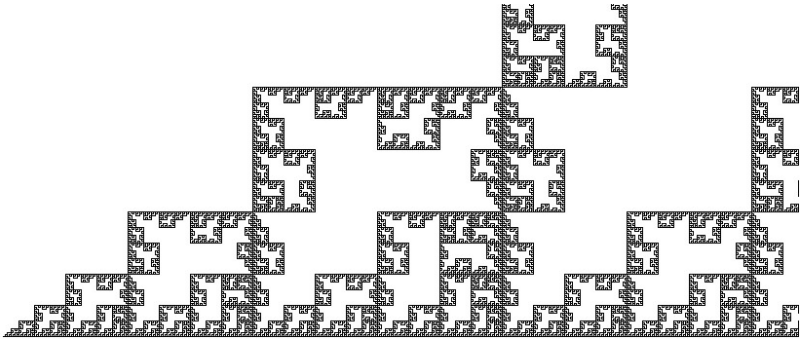

Fig. 11

and

$$
\begin{aligned}
&P(\phi(ua),\alpha)\\
&\quad = P(\phi(u)\phi(a),\alpha)\\
&\quad = P(\phi(u),\alpha) + R_{\alpha(\phi(u))}(P(\phi(a),\alpha))\\
&\qquad \text{(by Lemma 2)}\\
&\quad = P(\phi(u),\alpha) + R_{\alpha(u)}(P(\phi(a),\alpha))\\
&\qquad \text{(induction hypothesis)}\\
&\quad = cR_\theta(P(u,\alpha)) + R_{\alpha(u)}(cR_\theta(P(a,\alpha)))\\
&\qquad \text{(given, induction hypothesis)}\\
&\quad = cR_\theta(P(u,\alpha)) + cR_{\alpha(u)}(R_\theta(P(a,\alpha)))\\
&\quad = cR_\theta(P(u,\alpha)) + cR_\theta(R_{\alpha(u)}(P(a,\alpha)))\\
&\qquad \text{(rotations commute)}\\
&\quad = cR_\theta(P(u,\alpha) + R_{\alpha(u)}(P(a,\alpha)))\\
&\quad = cR_\theta(P(ua,\alpha) \quad \text{by Lemma 2,}
\end{aligned}
$$

proving the claim.

We prove that $P(\sigma,\alpha)$ is self-similar by proving that $cR_\theta(x,y) \in P(\sigma,\alpha)$ for all $(x,y) \in P(\sigma,\alpha)$. Let $(x,y) \in P(\sigma,\alpha)$, then $(x,y) = P(u,\alpha)$ for some prefix u of σ. According to the claim we obtain $cR_\theta(x,y) = cR_\theta(P(u,\alpha)) = P(\phi(u),\alpha)$ Since u is a prefix of σ, we obtain that $\phi(u)$ is a prefix of $\phi(\sigma) = \sigma$, so $P(\phi(u),\alpha) \in P(\sigma,\alpha)$, concluding the proof. □

As the first application of a self-similar turtle curve in Fig. 11 we show a fragment of $C(\sigma,\alpha)$ for $\sigma = \phi^\omega(1)$, for $\phi(0) = 111100$, $\phi(1) = 10$, and $\alpha(0) = 0$, $\alpha(1) = \pi/2$:

Indeed, Theorem 7 applies, since $\alpha(111100) = \alpha(0) = 0$, $\alpha(10) = \alpha(1) = \pi/2$, $P(111100,\alpha) = (2,0) = 2P(1,\alpha)$ and $P(10,\alpha) = (0,2) = 2P(1,\alpha)$, satisfying the requirements for $c = 2$ and $\theta = 0$.

As the second example we consider the well-known Koch curve: start by a single segment, and replace it by the four segments shown in Fig. 12.

The standard Koch curve is obtained by repeating this for every created new segment, where the length of every new segment is one third of the length of its ancestor, and then take the limit. Apart from the scaling, this can be described in turtle graphics by first doing a unit step in the same direction as before, next turn $\pi/3$ to the right, then do a unit step, next turn $2\pi/3$ to the left and do a unit step, and finally turn $\pi/3$ to the right and do a unit step. Taking the limit to the infinite rather than to the finite, this is described as $C(\mathsf{koch},\alpha)$ for $\mathsf{koch} = \phi^\omega(0)$, $\phi(0) = 0010$, $\phi(1) = 1010$, $\alpha(0) = \pi/3$, $\alpha(1) = -2\pi/3$, to which Theorem 7 applies for $c = 3$ and $\theta = 0$. Hence we call $C(\mathsf{koch},\alpha)$ the *Koch turtle curve*. Figure 13 shows a fragment.

By choosing the same sequence, but choosing another value for $\alpha(0)$, and choosing $\alpha(1) = -2\alpha(0)$, still Theorem 7 applies for $\theta = 0$, yielding another scaling factor c. For instance, taking $\alpha(0) = 17\pi/36$ and $\alpha(1) = -17\pi/18$ yields $c = 2 + 2\arccos 17\pi/36$, and Fig. 14 as a fragment of $C(\sigma,\alpha)$, sometimes called *Cesaro fractal*.

Fig. 12

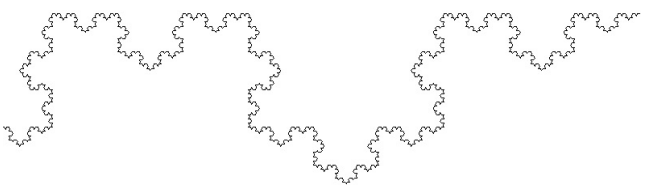

Fig. 13

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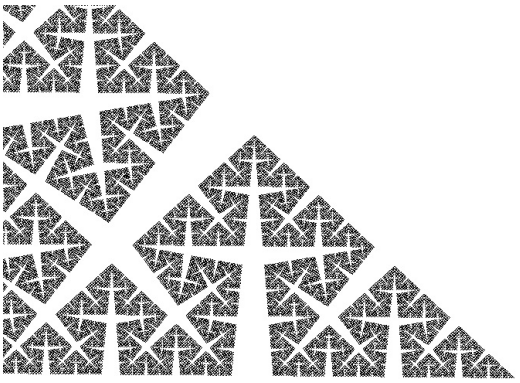

Fig. 14

Theorem 7 also applies for θ being non-zero. As an example consider $\sigma = \phi^\omega(1)$, $\phi(0) = 001$, $\phi(1) = 1010000$, $\alpha(0) = \pi/2$, $\alpha(1) = -\pi/2$. Then $P(0,\alpha) = (0,1)$, $P(1,\alpha) = (0,-1)$, $P(\phi(0),\alpha) = (-1,2)$, $P(\phi(1),\alpha) = (1,-2)$, $\alpha(\phi(0)) = \alpha(0) = \pi/2$ and $\alpha(\phi(1)) = \alpha(1) = -\pi/2$. So Theorem 7 applies for $c = \sqrt{5}$ and $\theta = \arctan 1/2$. Figure 15 shows a fragment in which the starting point $(0,0)$ is on the top left, and in every iteration the picture is simultaneously magnified by a factor $c = \sqrt{5}$ and rotated by an angle $\theta = \arctan 1/2$.

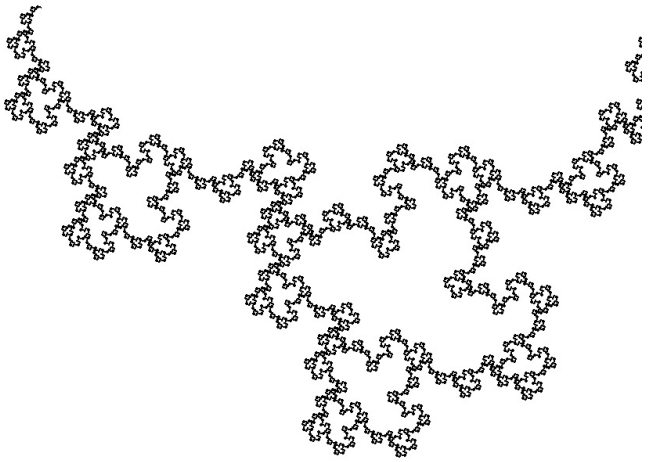

Fig. 15

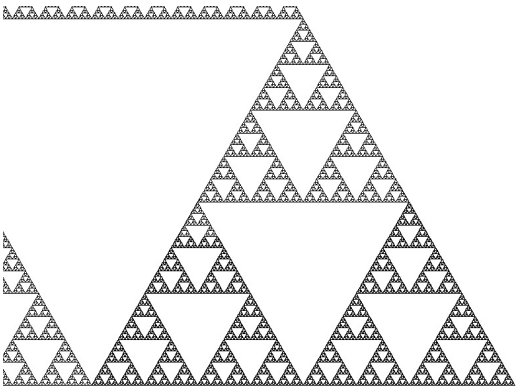

Fig. 16

In Fig. 16, we see a fragment of the well-known Sierpiński triangle, obtained as $C(\sigma,\alpha)$ for $\sigma = \phi^\omega(1)$, $\phi(0) = 00$, $\phi(1) = 11110$, $\alpha(0) = 0$, $\alpha(1) = 2\pi/3$, for which Theorem 7 applies for $c = 2$ and $\theta = 0$.

6. THUE–MORSE MEETS KOCH

In earlier work, e.g. Refs. 2 and 10, it has been shown that particular turtle curves for the Thue–Morse sequence t converge to the Koch curve in the Hausdorff metric. Related work includes Ref. 11, where it is investigated why and to which extent the sum of binary digits of multiples of 3 is more often even than odd. It is well known that the sum of binary digits of n is even if and only if $\mathsf{t}(n) = 0$. Essentially, Ref. 11 considers the turtle curve of t in which for both 0 and 1 the direction turns $\pi/3$, but the unit segment is only drawn at every 1, and for every 0 nothing is drawn. Its analysis is based on "the classical fractal scheme", essentially being the Koch curve.

But in all this work the focus is on approximation and convergence. Here we go a step further: we show that the point set of the Koch curve is a subset of a turtle curve for the Thue–Morse sequence. More precisely, by choosing $\delta(0) = \pi/3$ and $\delta(1) = \pi$, we prove that the points $(P(u,\delta) + P(ua,\delta))/2 \in C(\mathsf{t},\delta)$ for ua being prefixes of t of length divisible by 4, exactly form the point set $P(\mathsf{koch},\beta)$ of the Koch turtle curve, defined by $\mathsf{koch} = \psi^\omega(0)$, $\psi(0) = 0010$, $\psi(1) = 1010$, $\beta(0) = \pi/3$, $\beta(1) = -2\pi/3$, up to some scaling and rotation.

In Fig. 17, we show a fragment of $C(\mathsf{t},\delta)$ from which the relationship with the Koch curve clearly appears.

In order to give the proof, first we extend the notion of turtle curve to two angles α,α' for every symbol, rather than only one single angle α. Until now steps in a turtle curve when reading a symbol a were defined by first turning around the angle $\alpha(a)$

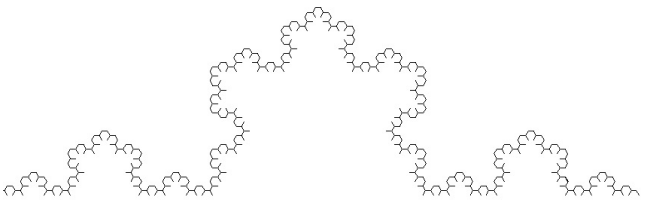

Fig. 17

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and then drawing a unit segment. In the extended variant, this is replaced by: first turn around the angle $\alpha(a)$, then draw a unit segment, and then turn around the angle $\alpha'(a)$. More precisely, for $u \in A^*$ its position $P(u,\alpha,\alpha') \in \mathbf{R}^2$ and angle $\overline{\alpha}(u)$ is defined inductively by $P(\epsilon,\alpha,\alpha') = (0,0)$, $\overline{\alpha}(\epsilon) = 0$, and

$$P(ua,\alpha,\alpha') = P(u,\alpha,\alpha') + (\cos(\overline{\alpha}(u)+\alpha(a)), \sin(\overline{\alpha}(u)+\alpha(a))),$$
$$\overline{\alpha}(ua) = \overline{\alpha}(u) + \alpha(a) + \alpha'(a),$$

for $u \in A^*$ and $a \in A$. Now for a sequence σ over A the set $P(\sigma,\alpha,\alpha')$ is defined by

$$P(\sigma,\alpha,\alpha') = \bigcup_{u \in \mathsf{Pref}(\sigma)} \{P(u,\alpha,\alpha')\}.$$

For $\alpha'(a) = 0$ for all $a \in A$ this definition coincides with the earlier definition of $P(\sigma,\alpha)$. Conversely, the following lemma shows that the point set $P(\sigma,\alpha,\alpha')$ of a turtle curve in this extended set coincides with the point set of a turtle curve in the basic setting. For a sequence σ over A define the sequence $\mathsf{pair}(\sigma)$ over $A \times A$ by $\mathsf{pair}(\sigma)(i) = (\sigma(i),\sigma(i+1))$ for $i \geq 0$.

Lemma 8. *Let σ be a sequence over A and let $\alpha,\alpha' : A \to \mathbf{R}$. Then $P(\sigma{\downarrow_{k+1}},\alpha,\alpha') = (\cos(\alpha(\sigma(0)),\sin(\alpha(\sigma(0))) + R_{\alpha(\sigma(0))}(P(\mathsf{pair}(\sigma){\downarrow_k},\gamma)))$ for all $k \geq 0$, for γ defined by $\gamma(a,b) = \alpha'(a) + \alpha(b)$ for $a,b \in A$.*

Proof. Induction on k. For $k = 0$ it holds by definition, and the induction steps follows since the drawing instructions coincide. □

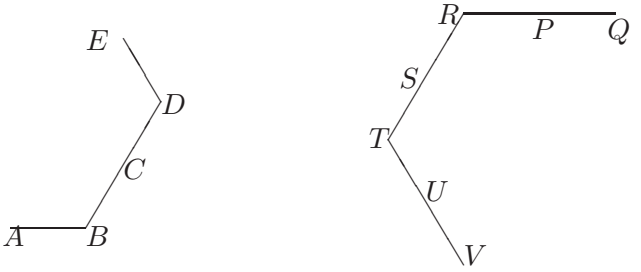
The points $(P(u,\delta) + P(ua,\delta))/2 \in C(\mathsf{t},\delta)$ represent midpoints between two consecutive points in the turtle curve point set $P(\mathsf{t},\delta)$. Up to scaling and translation, they can also be obtained as points in $P(\mathsf{t}^+,\delta)$, where $\mathsf{t}^+ = 20212120\cdots$ is defined by $\mathsf{t}^+(2i) = 2$ and $\mathsf{t}^+(2i+1) = \mathsf{t}(i)$ for $i \geq 0$, and for which the definition of δ is extended to $\{0,1,2\}$ by defining $\delta(2) = 0$. Next define $\alpha(0) = \alpha'(0) = \pi/3$ and $\alpha(1) = \alpha'(1) = -2\pi/3$. The next lemma is the key to the relation between Koch and Thue–Morse.

Lemma 9. *Let $k \geq 0$, and let $u = \mathsf{t}^+{\downarrow_{8k}}$ and let $v = \mathsf{t}{\downarrow_k}$. Then $P(u,\delta) = 3P(v,\alpha,\alpha')$.*

Proof. We extend the claim to be proved by $\delta(u) = \overline{\alpha}(v)$ and do this by induction on k. For $k = 0$ we have $P(u,\delta) = (0,0) = 3P(v,\alpha,\beta)$ and $\delta(u) = 0 = \overline{\alpha}(v)$.

For the induction step assume $\delta(u) = \overline{\alpha}(v)$ and $P(u,\delta) = 3P(v,\alpha,\alpha')$. We extend v by one element $a \in \{0,1\}$.

In case of $a = 0$ we have to prove $\delta(u20212120) = \overline{\alpha}(v0)$ and $P(u20212120,\delta) = 3P(v0,\alpha,\alpha')$. Since $\delta(u) = \overline{\alpha}(v)$ and $\delta(20212120) = 2\pi/3 = \alpha(0) + \alpha'(0)$ we conclude $\delta(u20212120) = \overline{\alpha}(v0)$. Let $A = P(u,\delta)$, and we turn the picture in such a way that the direction $\delta(u) = \overline{\alpha}(v)$ is horizontal from left to right. Then starting in A, the eight symbols 20212120 cause the turtle to move to B,C,D,C,B,C,D,E, successively. The total effect is a move from A to E, which is also obtained by moving 3 unit steps in the direction of $\alpha(0) = \pi/3$. So $P(u20212120,\delta) = 3P(v0,\alpha,\alpha')$.



In case of $a = 1$ we have to prove $\delta(u21202021) = \overline{\alpha}(v1)$ and $P(u21202021,\delta) = 3P(v1,\alpha,\alpha')$. The former holds since $\delta(u) = \overline{\alpha}(v)$ and $\delta(21202021) = 2\pi/3 = \alpha(1)+\alpha'(1)$, up to 2π. Let $P = P(u,\delta)$, and we turn the picture in such a way that the direction $\delta(u) = \overline{\alpha}(v)$ is horizontal from left to right. Then starting in P, the eight symbols 21202021 cause the turtle to move to Q,P,R,S,T,U,V,U, successively. The total effect is a move from P to U, which is also obtained by moving 3 unit steps in the direction of $\alpha(1) = -2\pi/3$. So $P(u21202021,\delta) = 3P(v1,\alpha,\alpha')$, concluding the proof. □

Recall that the period-doubling sequence pd is defined by $\mathsf{pd} = \phi^\omega(0)$ for $\phi(0) = 01$, $\phi(1) = 00$. The next lemma states that it is obtained from koch by only removing the first element, and is closely related to t.

Lemma 10. *$\mathsf{pd}(i) = \mathsf{koch}(i+1) = \tau(\mathsf{pair}(\mathsf{t})(i))$ for all $i \in \mathsf{N}$, for τ defined by $\tau(0,0) = \tau(1,1) = 1$, $\tau(0,1) = \tau(1,0) = 0$.*

Proof. We apply induction on i; we have $\mathsf{pd}(0) = 0 = \mathsf{koch}(1) = \tau(\mathsf{pair}(\mathsf{t})(0))$. Recall that $\mathsf{koch} = \psi^\omega(0)$, $\psi(0) = 0010$, $\psi(1) = 1010$, and observe $\mathsf{pd} = (\phi^2)^\omega(0)$ where $\phi^2(0) = 0100$ and $\phi^2(1) = 0101$.

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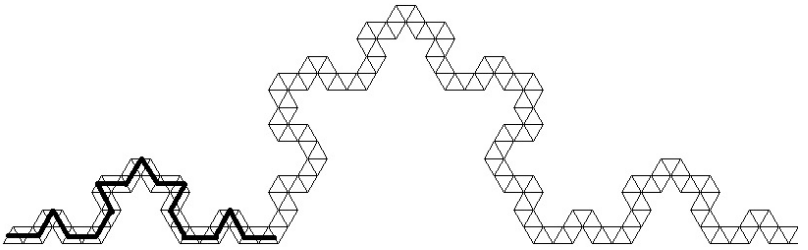

Fig. 18

Since $\mathsf{pd} = \phi^2(\mathsf{pd})$ and $\mathsf{koch} = \psi(\mathsf{koch})$, from this we conclude $\mathsf{pd}(i) = 0 = \mathsf{koch}(i+1)$ for i even, and $\mathsf{pd}(i) = 1 = \mathsf{koch}(i+1)$ for $i \equiv 1 \bmod 4$, and $\mathsf{koch}(4j) = \mathsf{koch}(j)$ and $\mathsf{pd}(4j+3) = \mathsf{pd}(j)$ for $j \geq 0$. So for $i \not\equiv 3 \bmod 4$ the claim follows directly, and for $i = 4j+3$ we conclude $\mathsf{pd}(i) = \mathsf{pd}(4j+3) = \mathsf{pd}(j) = \mathsf{koch}(j+1) = \mathsf{koch}(4j+4) = \mathsf{koch}(i+1)$ by the induction hypothesis.

Since $\mathsf{t} \in \{0110, 1001\}^\omega$ we obtain $\tau(\mathsf{pair}(\mathsf{t})(i)) = 0 = \mathsf{pd}(i)$ for i even, and $\tau(\mathsf{pair}(\mathsf{t})(i)) = 1 = \mathsf{pd}(i)$ for $i \equiv 1 \bmod 4$, and $\mathsf{t}(4k+3) = \mathsf{t}(4k)$ for all k. Combined with $\mathsf{t}(2k) = \mathsf{t}(k)$ for all k, for $i = 4j+3$ we obtain

$$\begin{aligned}\tau(\mathsf{pair}(\mathsf{t})(i)) &= \tau(\mathsf{pair}(\mathsf{t})(4j+3))\\ &= \tau(\mathsf{t}(4j+3), \mathsf{t}(4j+4))\\ &= \tau(\mathsf{t}(4j), \mathsf{t}(4j+4))\\ &= \tau(\mathsf{t}(j), \mathsf{t}(j+1))\\ &= \tau(\mathsf{pair}(\mathsf{t})(j))\\ &= \mathsf{pd}(j) = \mathsf{pd}(4j+3) = \mathsf{pd}(i),\end{aligned}$$

using the induction hypothesis, concluding the proof. □

As a side remark, $\mathsf{pd}(i) = \mathsf{koch}(i+1)$ follows from the more general statement that can be proved in a similar way: if $\psi(0) = 0u$, $\psi(1) = 1u$, $\phi(0) = u0$, $\phi(1) = u1$ for any u starting with 0, then $\phi^\omega(0)(i) = \psi^\omega(0)(i+1)$ for all $i \in \mathbf{N}$.

Now we arrive at the main theorem.

Theorem 11. *Let $\delta(0) = \pi/3$ and $\delta(1) = \pi$, then connecting the points $(P(\mathsf{t}{\downarrow_{4k+3}}, \delta) + P(\mathsf{t}{\downarrow_{4k+4}}, \delta))/2$ on $C(\mathsf{t}, \delta)$ for $k = 0, 1, 2, \ldots$ exactly yields the Koch turtle curve, scaled up by a factor 3/2.*

Proof. Up to a translation, these points are exactly the points $\frac{1}{2}P(\mathsf{t}^+{\downarrow_{8k}}, \delta)$ for $k = 1, 2, \ldots$. According to Lemma 9, these are equal to $\frac{3}{2}P(\mathsf{t}{\downarrow_k}, \alpha, \alpha')$. According to Lemma 8, after removing the first one, up to a rotation and translation these are equal to $\frac{3}{2}P(\mathsf{pair}(\mathsf{t}){\downarrow_k}, \gamma)$, for γ defined by $\gamma(a,b) = \alpha'(a) + \alpha(b)$ for $a, b \in \{0,1\}$. Since $\alpha(0) = \alpha'(0) = \pi/3$ and $\alpha(1) = \alpha'(1) = -2\pi/3$, we obtain (up to 2π) $\gamma(0,0) = \gamma(1,1) = 2\pi/3$ and $\gamma(0,1) = \gamma(1,0) = -\pi/3$. So these points are equal to $\frac{3}{2}P(\tau(\mathsf{pair}(\mathsf{t}){\downarrow_k}), \beta)$ for $\beta(0) = -\pi/3$ and $\beta(1) = 2\pi/3$. Since the sequences koch and $\tau(\mathsf{pair}(\mathsf{t}))$ coincide by Lemma 10 up to removing a first element, these points are exactly the points on (a mirrored version of) the Koch turtle curve, multiplied by 3/2. □

In Theorem 11, we worked out the relationship of the Koch turtle curve with the turtle curve $C(\mathsf{t}, \alpha)$ for the particular angles $\alpha(0) = \pi/6$ and $\alpha(1) = \pi$. A similar relationship appears for other choices. Figure 18 shows a fragment of $C(\mathsf{t}, \alpha)$ for $\alpha(0) = 2\pi/3$ and $\alpha(1) = \pi$, showing in bold an initial part of the underlying Koch curve, again scaled up by a factor 3/2.

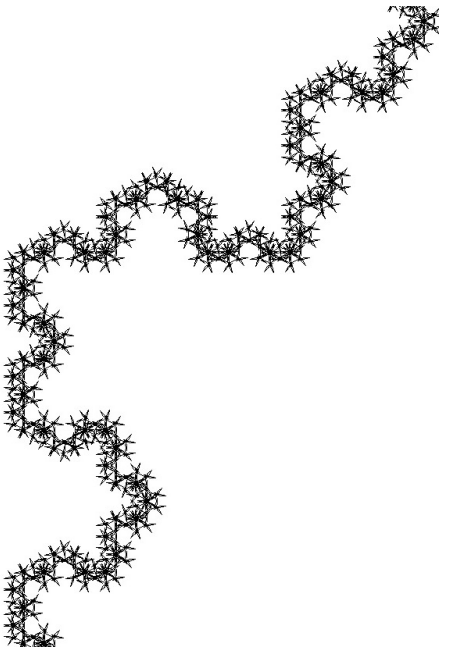

Fig. 19

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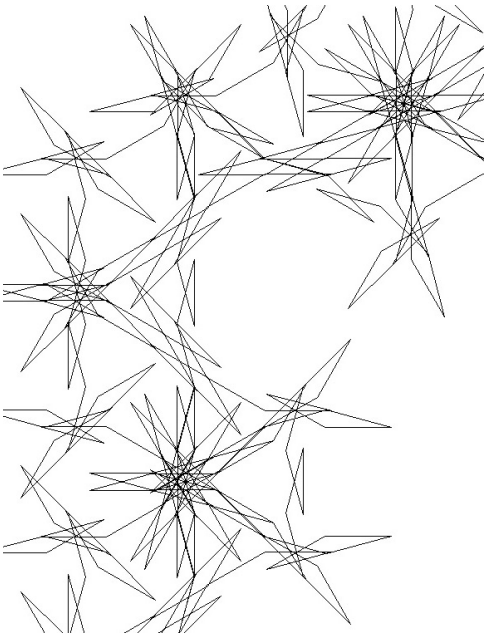

Fig. 20

Experiments show that $C(\mathsf{t}, \alpha)$ converges to the Koch curve if $\alpha(0) + \alpha(1) = k\pi/(3 * 2^n)$ for k not divisible by $2, 3$, where in Theorem 4 we showed that without the 3 in the denominator a finite turtle curve is obtained. For instance, by choosing $\alpha(0) = 11\pi/12$ and $\alpha(1) = \pi/6$ we obtain the infinite Koch like turtle curve of which a fragment is shown in Fig. 19, and from which a detail is shown in Fig. 20.

7. MORE SELF-SIMILAR TURTLE CURVES

In this section, we prove a variant of Theorem 7 for proving turtle curves to be self-similar, yielding a number of surprising examples.

Theorem 12. *Let $\sigma \in A^\omega$ satisfy $\sigma = \phi(\sigma)$ for $\phi : A \to A^+$, and there is $(x, y) \in \mathbf{R}^2$ with $(x + 1)^2 + y^2 > 1$ such that for all $a \in A$ the following holds*:

- $\alpha(\phi(a)) = \alpha(a)$, *and*
- $P(u_a, \alpha) = (x, y)$ *for u_a obtained from $\phi(a)$ by removing its last element.*

Then $P(\sigma, \alpha)$ is self-similar.

Proof. We use turtle curves with two angles as introduced in Sec. 6, in which $P(u, \alpha, 0) = P(u, \alpha)$ is the point obtained after proceeding u where for every symbol first a turn is made and then a unit segment is drawn, while $P(u, 0, \alpha)$ is the point obtained after proceeding u where for every symbol first a unit segment is drawn and then a turn is made.

Choose $c > 1$ and θ such that $cR_\theta(1, 0) = (x + 1, y)$; this is possible due to $(x + 1)^2 + y^2 > 1$. In computing $P(\phi(a), 0, \alpha)$ first the segment from $(0, 0)$ to $(1, 0)$ is drawn, followed by unit segments in the directions $\alpha(u)$ for the consecutive prefixes u of $\phi(a)$. So

$$\begin{aligned} P(\phi(a), 0, \alpha) &= (1, 0) + P(u_a, \alpha) = (x + 1, y) \\ &= cR_\theta(1, 0) = cR_\theta P(a, 0, \alpha), \end{aligned}$$

very similar to the condition of Theorem 7. Using this property, the following claim is proved in exactly the same way as the claim in the proof of Theorem 7, where instead of Lemma 2 its variant $P(uv, 0, \alpha) = P(u, \alpha) + R_{\alpha(u)}(P(v, 0, \alpha))$ is used, allowing a similar proof as for Lemma 2.

Claim: $\alpha(\phi(u)) = \alpha(u)$ and $P(\phi(u), 0, \alpha) = cR_\theta(P(u, 0, \alpha))$ for all $u \in A^+$.

We prove that $P(\sigma, \alpha)$ is self-similar by proving that $cR_\theta(1, 0) - (1, 0) + cR_\theta(x, y) \in P(\sigma, \alpha)$ for all $(x, y) \in P(\sigma, \alpha)$. Let $(x, y) \in P(\sigma, \alpha)$, then $(x, y) = P(u, \alpha)$ for some prefix u of σ. By the construction of P we have $P(ua, 0, \alpha) = (1, 0) + P(u, \alpha)$ for every $u \in A^*, a \in A$. So $(x + 1, y) \in P(\sigma, 0, \alpha)$. Using the claim we conclude $cR_\theta(x + 1, y) = P(v, 0, \alpha)$ for some non-empty prefix v of σ. Again using $P(ua, 0, \alpha) = (1, 0) + P(u, \alpha)$, we conclude that $cR_\theta(1, 0) - (1, 0) + cR_\theta(x, y) = cR_\theta(x + 1, y) - (1, 0) \in P(\sigma, \alpha)$, concluding the proof. □

In case $\phi(0) = u0$, $\phi(1) = u1$, for some string u starting in 0, for $\sigma = \phi^\omega(0)$ Theorem 12 applies if $\alpha(u) = 0$ and $P(u, \alpha) = (x, y)$ satisfies $(x + 1)^2 + y^2 > 1$. This is the case for $u = 00110$ and $\alpha(0) = 7\pi/18$, $\alpha(1) = -7\pi/12$, yielding a self-similar turtle curve with an initial fragment starting at the bottom left is shown in Fig. 21.

From Lemma 10, we know that the period-doubling sequence pd is obtained from koch by removing the first element. Since koch yields a self-similar curve for $\alpha(1) = -2\alpha(0)$ for $\arccos(\alpha(0)) > -1/2$, we expect the same for pd. This can be proved directly by Theorem 12: pd is a fixed point of ϕ defined by $\phi(0) = 01$, $\phi(1) = 00$, so also of ϕ^2 satisfying $\phi^2(0) = 0100$, $\phi^2(1) = 0101$, being of the above pattern for which the requirements of Theorem 12 are easily checked.

In case $\alpha(a) = 2\pi/k$ for some $a \in A$, $k > 1$, we may plug in a^k in the definition of $\phi(b)$ for

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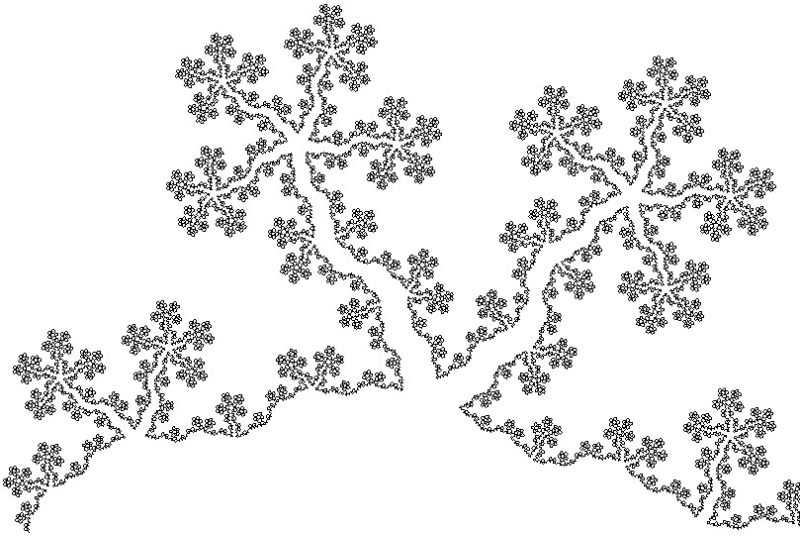

Fig. 21

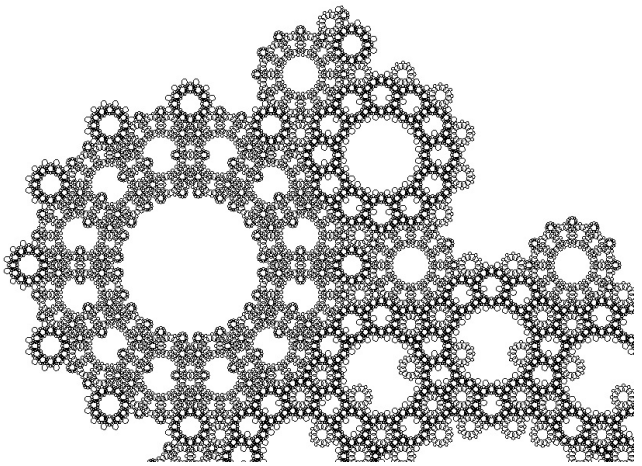

Fig. 22

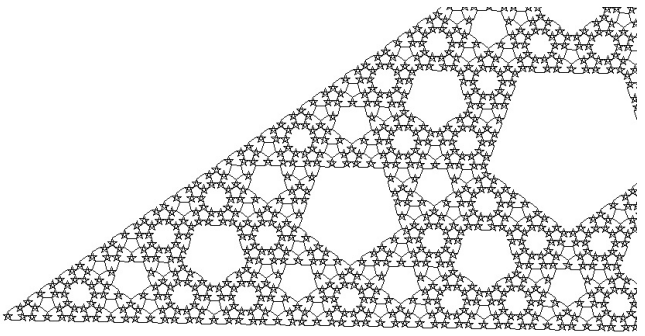

Fig. 23

any $b \in A$, without affecting the requirements for Theorem 12, since in the turtle a^k draws a regular k-gon, ending in the same point and direction where started. Applying this for pd for $k = 6$ by choosing $\phi(0) = 01111111 = 01^7$, $\phi(1) = 00$, $\alpha(0) = \pi/12$, $\alpha(1) = -\pi/6$ yields a self-similar turtle curve for $\phi^\omega(0)$ of which Fig. 22 shows the initial fragment, starting at the top.

Figure 23 shows an initial part of $C(\phi^\omega(0), \alpha)$ for ϕ, α defined by $\phi(0) = 0111111 = 01^6$, $\phi(1) = 00$, $\alpha(0) = 2\pi/5$, $\alpha(1) = -4\pi/5$. Again this is a self-similar turtle curve since $\phi^\omega(0)$ is a fixed point of ϕ^2, and ϕ^2, α satisfy the requirements of Theorem 12.

8. SPACE-FILLING CURVES

A turtle curve can be space-filling in several senses:

- It contains every edge in a grid (or half or one quarter of it) at least once or exactly once.
- It contains every node in a grid (or half or one quarter of it) at least once or exactly once.
- It is dense in the whole plane.

In this section, we prove that these three phenomena all occur for turtle curves of morphic sequences. First we focus on the rectangular grid of which the nodes are (x, y) for integers x, y, and the edges are of the shape $\{\lambda(x, y) + (1 - \lambda)(x + 1, y) \mid \lambda \in [0, 1]\}$, or $\{\lambda(x, y)+(1-\lambda)(x, y+1) \mid \lambda \in [0, 1]\}$, for integers x, y. A result from Ref. 7 states that $C(\mathsf{fib}, \alpha)$ for $\alpha(0) = 0$ and $\alpha(1) = \pi/2$ contains every edge in a full quadrant of this grid. This is "at least once" in the above setting; the next theorem states "exactly once".

Theorem 13. *Let* $\alpha(0) = \pi/2$ *and* $\alpha(1) = -\pi/2$. *Define* $\phi(0) = 010001110$ *and* $\phi(1) = 110001110$.

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Then $C(\phi^\omega(0),\alpha)$ contains every edge in the quarter plane $\{(x,y)\,|\,x<y\wedge x>-y\}$ exactly once.

Proof. For a string v write $\mathsf{Pref}(v)$ for the set of strings u for which w exists such that $uw=v$, and

$$C(v,\alpha)=\bigcup_{ua\in\mathsf{Pref}(v)}\{\lambda P(u,\alpha)+(1-\lambda)P(ua,\alpha)\,|\,\lambda\in[0,1]\}.$$

It is easily proved by induction on n that $C(\phi^n(0),\alpha)$ consists of all 9^n grid edge in the square bounded by the four points $(0,0)$, $(3^n/2,3^n/2)$, $(0,3^n)$, $(-3^n/2,3^n/2)$, all occurring exactly once. So $C(\phi^\omega(0),\alpha)$ contains every edge in the quarter plane $\{(x,y)\,|\,x<y\wedge x>-y\}$ exactly once. □

Showing the turtle curve indicated in Theorem 13 only shows the filled grid quarter; more information is obtained by Fig. 24 showing $C(\phi^3(0),\alpha)$ in which the corners are rounded off. The starting point $(0,0)$ is at the bottom.

In this example, every grid edge in the quarter plane is passed exactly once, so all grid points are passed exactly twice, except for the points on the border that are passed once. A similar curve, but then filling 1/8 of the plane, is obtained by a sequence obtained from the Rudin–Shapiro sequence, see Ref. 6.

More interesting and slightly harder is the second variant: all grid points are passed exactly once. In the literature and wikipedia, many examples of such curves are given, including the Hilbert curve, the Peano curve and the Moore curve. Typically, they can be presented as L-systems, but not in our more basic format of turtle curves. To obtain a turtle curve passing every grid point exactly once we consider the turtle curve from Theorem 13, but instead of drawing the full edges, only the midpoints of the edges are drawn, and every two consecutive midpoints are connected by a segment. Since all angles in the original turtle curve from Theorem 13 are $\pi/2$ or $-\pi/2$, this yields a rectangular grid again, but then turned over $\pi/4$. The resulting curve can be obtained as the turtle curve of the sequence $\mathsf{pair}(\phi^\omega(0))$, where ϕ is from Theorem 13 and pair is defined by $\mathsf{pair}(\sigma)(i)=(\sigma(i),\sigma(i+1))$ for $i\geq 0$, just like in Sec. 6. The angles in this turtle curve are as follows: $\alpha(0,0)=\pi/2$, $\alpha(0,1)=\alpha(1,0)=0$, $\alpha(1,1)=-\pi/2$. Next we observe that this sequence $\mathsf{pair}(\phi^\omega(0))$ is pure morphic: writing a,b,c,d for $(0,0),(0,1),(1,0),(1,1)$, respectively, it is easily proved to be a fixed point of ψ defined by

$$\psi(a)=bcaabddca,\quad \psi(b)=bcaabddcb,$$
$$\psi(c)=dcaabddca,\quad \psi(d)=dcaabddcb.$$

Hence $\mathsf{pair}(\phi^\omega(0))=\psi^\omega(b)$. This is the proof sketch of the following theorem.

Theorem 14. *Let $\alpha(a)=\pi/2$, $\alpha(b)=\alpha(c)=0$ and $\alpha(d)=-\pi/2$. Define ψ as above. Then $P(\psi^\omega(b),\alpha)$ contains every grid point in the quarter plane $\{(x,y)\,|\,x\geq 0\wedge y\geq 0\}$ exactly once.*

An initial part of the corresponding turtle curve $C(\psi^\omega(b),\alpha)$ starting at the bottom left is given in Fig. 25.

This section is concluded by the observation that a particular turtle curve of a morphic sequence is

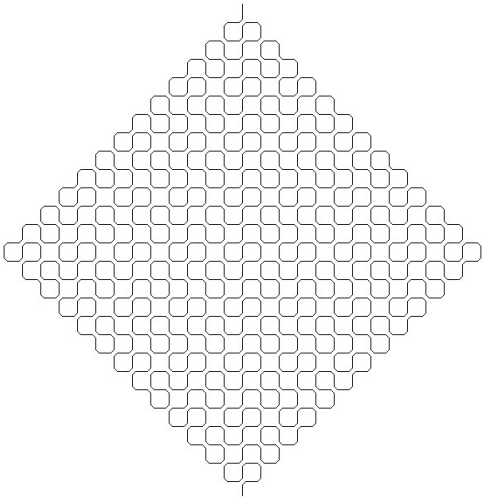

Fig. 24

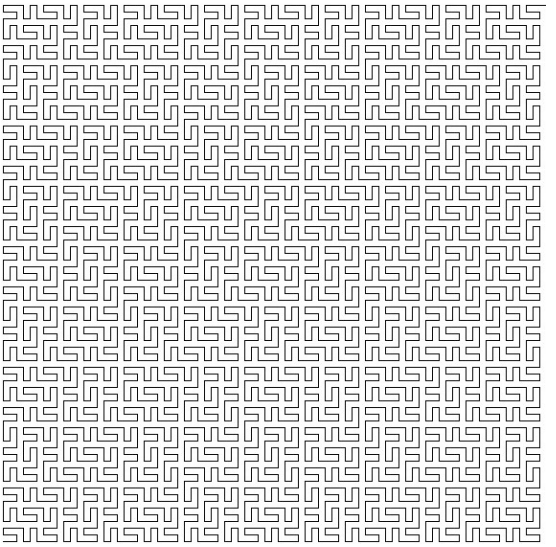

Fig. 25

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dense in the full plane, that is, for every point $p \in \mathbf{R}^2$ and every $\epsilon > 0$ the point set of the turtle curve contains a point that is closer than ϵ to p.

The example and the idea of the proof was suggested by Tonny Hurkens.

Theorem 15. *Let $\sigma = \phi^\omega(0)$ for ϕ defined by*

$$\phi(0) = 011111, \quad \phi(1) = 100000,$$

and let $\alpha(0) = 2\pi/5$ and $\alpha(1) = -2\pi/5$. Then $P(\sigma, \alpha)$ is dense in $\mathbf{R}^2$.

Write $p_k = (\cos(2\pi i/5), \sin(2\pi i/5))$ for $k = 0, 1, 2, 3, 4$ and let P be the set of points that can be written as $\sum_{i=0}^{4} a_i p_i$ for integers a_0, a_1, a_2, a_3, a_4. It is well known that P is dense in $\mathbf{R}^2$; we will prove that $P(\sigma, \alpha) = P$. Since $\alpha(0) = 2\pi/5$ and $\alpha(1) = -2\pi/5$ we obtain $P(u, \alpha) \in P$ for all $u \in \{0, 1\}^*$, so $P(\sigma, \alpha) \subseteq P$. For the converse we first observe that due to $p_0 + p_1 + p_2 + p_3 + p_4 = (0, 0)$ we may restrict to $a_i \geq 0$:

$$P = \left\{ \sum_{i=0}^{4} a_i p_i \,|\, a_0, a_1, a_2, a_3, a_4 \in \mathbf{N} \right\}.$$

For the remaining proof obligation $P \subseteq P(\sigma, \alpha)$ we have to prove that for every $a_0, a_1, a_2, a_3, a_4 \in \mathbf{N}$ the point $\sum_{i=0}^{4} a_i p_i$ is in $P(\sigma, \alpha)$. This follows by induction on $a_0 + a_1 + a_2 + a_3 + a_4$ using the following lemma.

Lemma 16. *Let $p \in P(\sigma, \alpha)$. Then $p + p_k \in P(\sigma, \alpha)$ for $k = 0, 1, 2, 3, 4$.*

Proof. Write $P(i) = P(\sigma{\downarrow_i}, \alpha)$ and $\alpha(i) = \alpha(\sigma{\downarrow_i})$ for $i \in \mathbf{N}$. Then $p \in P(\sigma, \alpha)$ means that $p = P(n)$ for some $n \in \mathbf{N}$. Since $\alpha(n)$ is a multiple of $2\pi/5$ we conclude that $P(n + 1) = p + p_i$ for some $i \in \{0, 1, 2, 3, 4\}$. So for this particular $k = i$ we have $p + p_k \in P(\sigma, \alpha)$, we have to prove that this also holds for the other k. Since $P(\sigma, \alpha)$ is self-similar with scaling factor 1 and rotation angle 0 by Theorem 7, and $|\phi(u)| = 6|u|$ for all u, we obtain $P(6n) = P(n)$ and $\alpha(6n) = \alpha(n)$. If the last element of $\sigma{\downarrow_n}$ is 1 we replace n by $6n$, keeping the same $P(n)$ and $\alpha(n)$, but replacing the last element by 0. So we assume that the last element of $\sigma{\downarrow_n}$ is 0. Then $\phi(\sigma{\downarrow_n})$ ends in 011111, by which the last 5 steps in its turtle curve is a regular pentagon. Hence $P(6n - 5) = P(6n) = P(n)$ and $\alpha(6n - 5) = \alpha(6n) = \alpha(n)$. Doing the same observation on the 1-step from $P(6n - 4)$ to $P(6n - 5)$ we obtain $\phi(\sigma{\downarrow_{6n-4}})$ ending in 100,000 by which the last 5 steps in its turtle curve is a regular pentagon in the other direction. This yields $P(6(6n-4)-6) = P(n)$ and $\alpha(6(6n - 4) - 6) = \alpha(n) + 6\pi/5$. Hence $P(6(6n - 4) - 5) = p + p_k \in P(\sigma, \alpha)$ for $k = (i + 3) \mod 5$. This argument is repeated until for all five values of k the point $p + p_k$ has been proved to be in $P(\sigma, \alpha)$.

This concludes the proof of both the lemma and Theorem 15. □

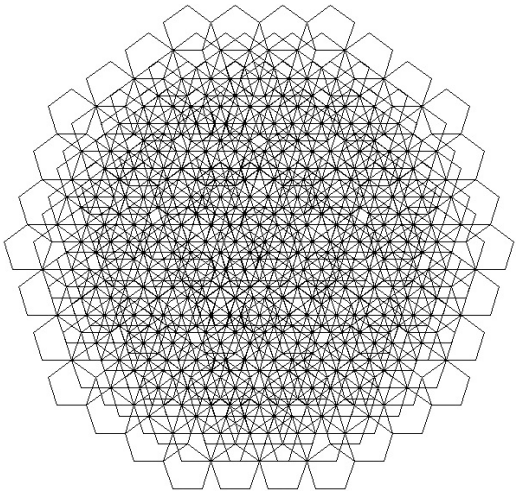

Fig. 26

Showing a fragment of the full turtle curve indicated in Theorem 15 would give a fully black picture. Instead in Fig. 26 we show $C(\phi^7(0), \alpha)$.

9. CONCLUSIONS

Ultimately periodic sequences yield boring turtle curves. Morphic sequences still have very simple definitions, but typically are not ultimately periodic. When randomly generating morphic sequences and arbitrarily choosing angles, the resulting turtle curves typically show up a mess in which no structure is recognized. In this paper, we developed criteria for morphic sequences and angles yielding turtle curves with a special structure. One special structure is finiteness, meaning that only finitely many distinct segments are drawn. Since the sequence is infinite, these finitely many segments are drawn over and over again. A second special structure is self-similarity: the set of end points of the segments contain an up-scaled copy of itself. Surprisingly, both for finiteness and self-similarity the resulting turtle curves typically look nice and well-structured, for which we gave several examples. For all our criteria we gave rigorous proofs. We also gave examples of space-filling turtle curves and dense turtle curves.

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