# A multilevel Green function interpolation method to efficiently construct the EFIE MoM-matrix for 2D-periodic PEC structures in 3D space

# A Multilevel Green Function Interpolation Method to Efficiently Construct the EFIE MoM-Matrix for 2D-Periodic PEC Structures in 3D Space

P. Jorna* V. Lancellotti† M.C. van Beurden‡

*Abstract* — **For scattering by perfectly conducting objects in a two-dimensionally periodic setup we employ a surface-integral equation, the Ewald representation of the Green function, and the Method of Moments (MoM). For moderate-size matrices, we observe that the computation time is dominated by the computation of the matrix elements. By employing a multi-level decomposition of the Green function based on Lagrange interpolation on a Chebyshev grid, we demonstrate that the overall computation time can be reduced by 73% compared to the original MoM computation.**

## 1 INTRODUCTION

The two most time consuming tasks in computing the equivalent electric currents for a Surface Integral Equation (SIE) formulation for 2D-periodic PEC structures when using the Method of Moments (MoM) is building the MoM matrix and subsequently solving the linear system of equations. In previous work we have seen that building the MoM matrix is costly in case the 2D Quasi-Periodic Green Function (QPGF) in the double integrals has to be computed time and again, but if we use the Ewald representation of the QPGF to pre-compute the QPGF on a pre-defined grid and use interpolation to compute the QPGF in the double integrals it is shown that for a unit cell that is not too large in terms of wavelength the matrix build time is of the same order of magnitude as for the aperiodic case [1, 2].

By using the combination of tabulation and interpolation of the QPGF we have managed to reduce the computational cost of building the MoM matrix considerably but the cost still increases quadratically with the number of unknowns while the computational cost of applying a direct solver increases cubically with the number of unknowns. A successful strategy to further reduce the computation time, suitable for large scale electromagnetic problems, is to apply fast iterative algorithms that make use of an efficient matrix-vector product. Especially interesting are kernel-independent fast algorithms, such as the Multilevel Green Function Interpolation Method (MLGFIM) [3, 4, 5], that have the potential to be applied in a periodic configuration and that could, in due time, be tailored to a layered-medium Green function.

A disadvantage of these fast algorithms is that iterative solvers are not as powerful as direct solvers in the sense that iterative solvers may exhibit poor convergence or may not converge at all even if the linear system is nonsingular. In particular, the Electric Field Integral Equation (EFIE) formulation is well known for producing a poorly conditioned MoM-matrix. Therefore, it may be necessary to construct a preconditioner that induces fast convergence of the iterative solver. Another drawback of iterative solvers is that the case of multiple right-hand sides often does not benefit from previous computations, whereas for a direct solver only the forward-backward substitution, which is relatively cheap numerically, needs to be repeated after the matrix has been factorized. This is an important observation if one is interested in characterizing the 2D-periodic PEC structure under consideration via a scattering matrix, for which we have to solve each linear system for a large number of right-hand sides.

In previous work we have observed that for moderate-size unit cell problems the bulk of computation time is spent on building the MoM-matrix whereas factorization of the matrix is still relatively cheap. For this class of medium scale electromagnetic problems we present a MLGFIM algorithm that can be used to efficiently construct the EFIE MoM-matrix. By adopting this approach we can keep on using a direct solver to circumvent the convergence problems with an iterative solver and to efficiently compute solutions for multiple right-hand sides.

## 2 MLGFIM-EFIE

In the developed MLGFIM algorithm we use an octree for the hierarchical spatial subdivision of the computational domain. The interaction integrals for Rao-Wilton-Glisson (RWG) basis functions and RWG test functions that are considered to be near

*Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, e-mail: Pieter.Jorna@gmail.com

†ibid. , e-mail: v.lancellotti@tue.nl

‡ibid. , e-mail: m.c.v.beurden@tue.nl

978-1-4673-9811-4/16/$31.00 ©2016 IEEE

to each other are computed using traditional direct integration but all other interaction integrals are computed using low-rank approximations of the QPGF. The required low-rank approximations of the QPGF, for all hierarchical levels, for each combination of a non-empty source box and a non-empty test box that are non-touching near neighbors at that level, are derived in the pre-processing stage. To derive the low-rank approximation of the QPGF we first choose a suitable set of interpolation functions for both the source box and the test box that will allow an efficient and accurate approximation of the QPGF. For a unit cell that is not too large in terms of the wavelength we expect that we can efficiently use polynomial interpolation functions and we therefore use Lagrange interpolation on a Chebyshev grid.

Next, a Green function matrix that relates the interpolation functions in the source box and the test box is formed by sampling the QPGF. Based on the properties of the QPGF it is known that we can find a low-rank approximation of this Green function matrix and we have investigated two techniques (SVD and QR-factorization) to determine such a low-rank approximate matrix. Also computed in the pre-processing stage for each RWG are the integrals of (the divergence of) the RWG with the interpolation functions in the source/test box at the finest level. To approximate for each source RWG the integrals of (the divergence of) the RWG with the interpolation functions in source boxes at a coarser level we apply lower-to-upper-level interpolation in the upward pass. To approximate the interpolation functions in test boxes at coarser levels in terms of the interpolation functions in the test boxes at finer levels we apply upper-to-lower-level interpolation in the downward pass [3, 4].

To efficiently evaluate the interaction integrals for a source RWG and a test RWG, we consecutively perform an upward pass on the integrals of (the divergence of) the source RWG with the interpolation functions in the source box at the finest level, to the level where the source RWG and the test RWG are non-touching near neighbors, perform peer-level Green function interpolation, using the low-rank approximation of the QPGF, perform a downward pass to the finest level and finally multiply with the integrals of (the divergence of) the test RWG with the interpolation functions in the test box at the finest level.

## 3 CONSEQUENCES OF PERIODIC SETUP FOR MLGFIM

In our current implementation, the octree is constructed on the height-limited volume of the unit cell $(-\mathbf{a}_1/2, \mathbf{a}_1/2] \times (-\mathbf{a}_2/2, \mathbf{a}_2/2] \times [z_{\min}, z_{\max}]$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the Bravais lattice vectors of the unit cell and $[z_{\min}, z_{\max}]$ covers at least the height of the entire structure in the $z$-direction, but is otherwise free to choose. As a consequence, the quasi-periodicity of the Green function in the $xy$-plane is also encountered at the discrete level of the octree. This makes the detection of the nearest neighbors in the octree for the periodic case relatively straightforward [5]. It also means that the number of non-touching near neighbors is smaller than in the aperiodic case.

Note that we can immediately determine the finest level, $l_{\max}$, possible in the MLGFIM once the maximum edge length $s_{\max}$ in the mesh and the minimum dimension of the periodic unit cell $a_{\min} = \min\{|\mathbf{a}_1|, |\mathbf{a}_2|\}$ are known, namely

$$3\, s_{\max} < \frac{a_{\min}}{2^{l_{\max}}}, \tag{1}$$

where a separation criterion is used that the minimum dimension of a box is at least three times the maximum edge length in the mesh, to avoid contributions from the singularity of the Green function. Typically, the maximum edge length is given as a fraction of the free-space wavelength $\lambda_0$, that is, $s_{\max} = \lambda_0/N$ while the minimum dimension of the periodic unit cell is given in terms of the free-space wavelength, that is, $a_{\min} = P\lambda_0$, consequently

$$l_{\max} < \log_2(PN) - \log_2(3). \tag{2}$$

A further consequence of the octree spanning the exact cross-section of the unit cell is that the dimensions and shape of the unit cell dictate the cell shape and size at all levels of the octree, as indicated by the above criterion. For example, an elongated unit cell is more restrictive than a square one regarding the deepest level that can be attained for a given mesh for the scattering object inside the unit cell.

## 4 NUMERICAL EXAMPLE

Two identical and mirror-symmetrical PEC structures are put in a unit cell such that the unit cell cannot be reduced further and the distance between the structures in the $\mathbf{a}_2$-direction is given by $d$. In Figure 1, the unit-cell configuration with the two structures with a sidewall angle (SWA) of approximately 84° is shown together with all dimension definitions, which are specified in Table 1 together with the wavelength $\lambda_0$ of the incident plane wave. We use a dense mesh with 4222 triangular elements on each of the structures, that is, there is a total of 8444 triangular elements in the mesh. For this con-

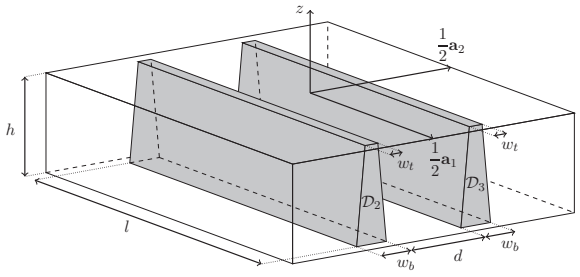


Figure 1: Two PEC copper structures with SWA 84° in the unit cell.

| parameter | dimension |
|---|---|
| $\|\mathbf{a}_1\|$ | 1.000 |
| $\|\mathbf{a}_2\|$ | 0.800 |
| $d$ | 0.296 |
| $l$ | 0.920 |
| $h$ | 0.240 |
| $w_b$ | 0.840 |
| $w_t$ | 0.360 |
| $\lambda_0$ | 0.850 |

Table 1: Dimensions of the scattering setup as defined in Figure 1.

figuration $a_{\min} = \min\{1.000, 0.800\} \approx \lambda_0$, whereas $s_{\max} \approx \lambda_0/35$ applies, hence if we want to invoke the MLGFIM, the rule of thumb, given in (2), suggests the finest level $l_{\max}$ we can use has to satisfy

$$l_{\max} \quad < \quad \log_2(35) - \log_2(3) \approx 3.5.$$

Hence, we can apply the MLGFIM on a single level $l = 2$ with $l_{\max} = 2$ or on two levels $l = 2$ and $l = 3$ with $l_{\max} = 3$. We may increase the computational domain in the $z$-direction and take its height equal to $d_{z,0} = \max\{a_{\min}, z_{\max} - z_{\min}\}$ so that $d_{z,0} = 0.800$. Thus, the boxes at level $l = 2$ have dimensions $d_{a_1,2} = 0.250$, $d_{a_2,2} = 0.200$ and $d_{z,2} = 0.200$ and the boxes at level $l = 3$ have dimensions $d_{a_1,3} = 0.125$, $d_{a_2,3} = 0.100$ and $d_{z,3} = 0.100$.

To determine the order of the interpolation polynomials in each box at each level in the MLGFIM, we compute the reflection coefficients for three angles of incidence, $\theta = 0°$, $\theta = 30°$ and $\theta = 60°$, while the azimuth angle is fixed at $\phi = 45°$. We compute the reflection coefficients using three different methods to build the MoM-matrix, namely, the MLGFIM on a single level, the MLGFIM on two levels, and the standard-MoM approach. We use the reflection coefficients computed with the MoM-matrix that is built with the standard-MoM approach as the reference. In Table 2 we provide the overview of timings for the MLGFIM at level-two only, with $5 \times 5 \times 5$ interpolation points in each of the level two boxes, and in Table 3 we provide the comparison between the standard MoM and the MLGFIM with $5 \times 5 \times 5$ interpolation points in each of the level-two boxes, $4 \times 4 \times 4$ interpolation points in each of the level-three boxes. These settings are sufficient for a root-mean-square error of $10^{-3}$ or less in the reflection coefficients. In both cases a low-rank approximation of the QPGF was obtained via the $QR$-factorization with a relative error criterion of $10^{-4}$. During the tabulation phase, the $QR$ factorization was found to be faster than the SVD, while both exhibit similar low-rank performance. The CPU times for the standard MoM computation are of course unchanged for both cases and many of the entries in the first column of the two tables are therefore the same, but now that we use either one or two levels in the MLGFIM the definition of the touching near-neighbor interactions at the finest level has changed and as a consequence the entries in the third and fourth row have changed. If we compare the entries in Tables 2 and 3, we can see that the majority of time in the standard MoM in Table 3 was spent on computing the non-touching near-neighbor interactions ($\mathcal{L}_{1,\Lambda}$ nnn), whereas in Table 2 the majority of time was spent on computing the touching near-neighbor interactions ($\mathcal{L}_{1,\Lambda}$ tnn). Consequently, the room for improvement in the combined level-two and level-three octree is larger, as only non-touching near-neighbor interactions are amenable to speedup with the low-rank decomposition of the Green function.

With an improvement factor in the CPU time that is even slightly higher than the improvement factor in the CPU time measured for the single-level MLGFIM over the standard MoM for the non-touching near-neighbor interactions, the reduction in CPU time that we obtain by computing the non-touching near-neighbor interactions with the two-level MLGFIM instead of the standard MoM now really makes a big difference. While the standard MoM requires more than 30 minutes to compute the non-touching near-neighbor interactions, the two-level MLGFIM is finished in approximately one and a half minute. More precisely, by using the two-level MLGFIM the CPU time that is needed to compute the non-touching near-neighbor interactions is only 5.4 % of the CPU-time that is needed with the standard MoM. With this reduction in computation time for the RWGs that are considered to be sufficiently separated the total build time when using the two-level MLGFIM is only 21.6 % of the total build time with the standard MoM. The total CPU-time to solve the scattering problem

| CPU Time | MoM (in sec.) | MLGFIM (in sec.) | Improvement factor | Percentage MLGFIM vs. MoM |
|---|---|---|---|---|
| Tabulation QPGF | 6.0 | 9.8 | 0.61 | 164.0 |
| $\mathcal{L}_{1,\Lambda}$ tnn | 1334.3 | 1334.3 | | |
| $\mathcal{L}_{1,\Lambda}$ nnn | 843.4 | 53.5 | 15.73 | 6.4 |
| Build | 2183.6 | 1397.8 | 1.56 | 64.0 |
| Factorize | 137.1 | 137.5 | | |
| Total | 2321.4 | 1535.9 | 1.51 | 66.2 |

Table 2: Comparison CPU time tasks in MoM EFIE with and without MLGFIM. The MLGFIM makes use of a single level, $5 \times 5 \times 5$ interpolation points in each box, and $QR$-factorization with a relative error criterion of $10^{-4}$.

| CPU Time | MoM (in sec.) | MLGFIM (in sec.) | Improvement factor | Percentage MLGFIM vs. MoM |
|---|---|---|---|---|
| Tabulation QPGF | 6.0 | 9.5 | 0.63 | 159.8 |
| $\mathcal{L}_{1,\Lambda}$ tnn | 363.7 | 363.7 | | |
| $\mathcal{L}_{1,\Lambda}$ nnn | 1813.9 | 97.0 | 18.69 | 5.4 |
| Build | 2183.6 | 470.5 | 4.64 | 21.6 |
| Factorize | 137.1 | 136.7 | | |
| Total | 2321.4 | 607.8 | 3.82 | 26.2 |

Table 3: Comparison CPU time tasks in MoM EFIE with and without MLGFIM. The MLGFIM makes use of two levels, $5 \times 5 \times 5$ interpolation points in each box at level $l = 2$, $4 \times 4 \times 4$ interpolation points in each box at level $l = 3$, and $QR$-factorization with a relative error criterion of $10^{-4}$.

when using the two-level MLGFIM is only 26.2 % of the total CPU-time with the standard MoM, versus 66.2 % for the one-level MLGFIM. By applying the two-level MLGFIM to build the MoM matrix we have managed to reduce the total computation time by 73 %.

## 5 CONCLUSIONS

We have demonstrated the efficiency of the MLGFIM for building the full MoM matrix of a periodic problem with 12666 unknowns. We show that computing the non-touching near neighbor contributions in the EFIE MoM-matrix using the two-level MLGFIM only takes 5.4% of the CPU time it takes to compute the matrix entries with standard MoM. We also show that the two-level MLGFIM reduces the required total CPU time to build the EFIE MoM-matrix by more than 73 %.

## References


[1] P. Jorna, V. Lancellotti, and M.C. van Beurden, "SIE approach to scattered field computation for 2D periodic diffraction gratings in 3D space consisting of high permittivity dielectric materials and plasmonic scatterers", Proceedings ICEAA, pp. 143-146, 2014.

[2] P. Jorna, V. Lancellotti, and M.C. van Beurden, "Computational Aspects of 2D-Quasi-Periodic-Green-Function Computations for Scattering by Dielectric Objects via Surface Integral Equations", PIER B, Vol. 63, pp. 49-66, 2015.

[3] Hao Gang Wang, Chi Hou Chan, and Leung Tsang, "A new multilevel Green's function interpolation method for large-scale low-frequency EM simulations", IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 24(9), pp. 1427-1443, September 2005.

[4] Hao Gang Wang and Chi Hou Chan, "The implementation of multilevel Green's function interpolation method for full-wave electromagnetic problems", IEEE Transactions on Antennas and Propagation, 55(5), pp. 1348-1358, May 2007.

[5] Yan Shi and Chi Hou Chan, "Multilevel Green's function interpolation method for analysis of 3-D frequency selective structures using volume/surface integral equation", Journal of the Optical Society of America A, Optics and Image Science, 27(2), pp. 308-318, February 2010.