Fast numerical shape optimization of bells using design of experiment and regression techniques

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ABSTRACT

In order to obtain a numerical tool for shape optimization of bells, functions describing the first seven eigenfrequencies of bells were derived. These functions which are a function of the bell geometry, are valid in a wide geometrical region. With the use of regression techniques these functions were fitted on FEM data obtained in a complete factorial design covering the geometrical region. Using these regression functions in an optimization program provided good estimates for bell shapes meeting the required overtone structure of the bell. With little difficulty these estimates could be improved using a FEM-based optimization program.

INTRODUCTION

In the course of several centuries western bell founders have established empirically a number of rules, enabling them to influence the ratios of the eigenfrequencies of a bell and to tune a bell. However, using these rules they have not succeeded in developing completely new, musically interesting bells, such as the major third bell.

A FEM-analysis of bells, combined with an optimization algorithm can be used for designing new bells. This approach was used by Van Asperen [1] in an attempt to find the shape of a major third bell (i.e. a bell who's lowest three eigenfrequency ratios are $1:2:2.5$), starting from the well-known shape of the minor third bell (with eigenfrequency ratios $1:2:2.4$). This approach proved to be time consuming and cumbersome, as the shape converged to a local optimum where the second eigenfrequency was significantly too high. The bell founder then suggested to stretch the bell in order to lower the second eigenfrequency of the bell, while the other important eigenfrequencies remained more or less unchanged. This resulted in a prototype major third bell that approached the required overtone structure, but was still unsatisfactory.

In order to find the shape of a major third bell another approach was used by Maas [2] and Schoofs et al. [3]. The vertical wall section of the bell was described with seven variables. Starting from the existing prototype bell each of these design variables was allowed to shift +5% or -5%. Thus a total of $2^7=128$ different bell shapes were defined. In a FEM-program the first seven important eigenfrequencies and the partial derivatives of the eigenfrequencies with respect to the design variables were calculated for each of the 128 bells. Polynomial functions describing the eigenfrequencies were fitted on the numerical data using regression techniques. These functions were used in a stepwise search and resulted in a design for a major third bell that was cast and could be tuned exactly.

Encouraged by the success of this method in the small geometrical region around the prototype of a major third bell, we decided to derive and use in a similar manner regression functions describing the eigenfrequencies as a function of the bell geometry which are valid in a much wider geometrical region. The geometry of the well-known minor third bell was used as the nominal reference geometry. The number of variables describing the vertical wall section of the bell was increased from 7 to 20 in order to be able to describe more complex bell geometries. A complete factorial design of experiment covering the geometrical region was made for which the required numerical date was gathered. On these data regression functions were fitted. The resulting regression functions were used in an optimization program based on sequential quadratic programming, which led to several possible designs for minor and major third bells.

The work described in this article is the first step in model building and optimization of the complete transient dynamic response of bells.

TONAL STRUCTURE OF BELLS

A partial is the sound field radiated by one of the structural eigenmodes of the bell. The tonal structure of a bell, which is the total sound of the bell, consists of many partials with each its typical frequency, strength and decay time. The most important requirement for the bell to ring pure is that the overtone structure of the lowest (and strongest) partials must be musically well defined like the harmonic or the major third overtone structure [4].

The cent value is a unit frequently used for swinging bells and carillon bells. It is a measure for the ratio of the frequency of a certain partial and the frequency of the partial called nominal, i.e. the 5^{th} eigenfrequency (that corresponds to 2400 cents). The cent value is defined as

$$[\text{cent}] = 1200 \cdot {}^{2}\log(4f_i/f_{nominal}) \qquad (1)$$

A difference of 100 cents between two frequencies corresponds to a distance of half a tone, a difference of 1200 cents denotes a distance of 6 tones, i.e. an octave. (A distance of an octave means that the ratio of the two frequencies equals 0.5 or 2.)
For the overtone structures of the minor third and major third bell the ratios of the lowest important eigenfrequencies and the corresponding cent values are listed in table 1.

The bellfounder is able to tune the first 5 partials of a bell. In a good bell design the eigenfrequencies of the 6^{th} and 7^{th} partial must therefore have approximately the correct values.

The strategy for building a model for the eigenfrequencies is to calculate eigenfrequencies and partial derivatives of eigenfrequencies with respect to variables describing the bell for many different bells and fit regression functions on the gathered data.

	minor third bell		major third bell	
partial	ratio	[cent]	ratio	[cent]
hum	1	0	1	0
fundamental	2	1200	2	1200
third	2.4	1500	2.5	1600
fifth	3	1900	3	1900
nominal	4	2400	4	2400
twelfth	6	3100	6	3100
double octave	8	3600	8	3600

Table 1 Ideal overtone structure for a minor third and a major third bell, expressed in ratios and cents.

SHAPE DEFINITION

Since a bell is axi-symmetric, ring elements are used in the FEM analysis to calculate the eigenfrequencies and partial derivatives of the eigenfrequencies of the bell. The vertical wall section of the bell is described by 20 design variables, 7 variables for the bell radii and 13 variables for the wall thicknesses. The radii and thicknesses are measured at equidistant points along the bell wall (see figure 1). The 7 radii define points on the reference curve, i.e. the midplane of the bell wall. Splines are used to draw a smooth curve through these radii. The 13 thicknesses are measured perpendicular to this curve, defining points on the inner and outer bell wall. Again splines are used to draw smooth curves. The head and shoulder of the bell are always prescribed to have the shape of the head and shoulder of a minor third bell, their size varying with the 7^{th} radius (see figure 2).

Using the minor third bell as a reference, all the design variables are defined using equations (2) and (3) (see figure 3)

$$x_i = wr_i \cdot \Delta r_i / r_i \qquad i=1,7 \qquad (2)$$

$$t_j = ws_j \cdot \Delta s_j / s_j \qquad j=1,13 \qquad (3)$$

Here r_i and s_j denote the radius i and wall thickness j of the nominal bell, respectively, whereas Δr_i and Δs_j denote the shift of radius i and thickness j measured perpendicularly to the reference curve of the nominal bell, respectively. x_i and t_j denote the design variables for radius i and thickness j, respectively, while wr_i and ws_j denote weighting factors for Δr_i and Δs_j, respectively. The weighting factors are chosen so that a shift of 1 of the design variables corresponds to a shift of 5% for the radii and 10% for the thicknesses.

GEOMETRICAL REGION

First the geometrical region for which the model should be valid is defined. Schoofs [5] made a number of FEM analyses showing some important properties. First, small global geometrical variations in the radii result in less strong nonlinearities in the eigenfrequencies as functions of these radii than large global variations. In order to be able to describe the eigenfrequencies by polynomial functions of low degree, the design variables x_i are constraint to the interval [-3.0, 3.0], which corresponds to variations between -15% and +15% of the nominal radii values.

Local geometrical variations in the radii result in stronger nonlinearities in the eigenfrequencies as functions of these radii than global variations of the same order of magnitude. To avoid strong local geometrical variations additional constraints for the design variables x_i are defined : $|x_i - x_{i-1}| \leq 2.0$

Another property found is that for wall thicknesses varying between 0.7 and 1.4 times the nominal values the eigenfrequencies are approximately linear in the design variables t_j (within 2%). Constraining the design variables t_j to the interval [-3.0, 4.0] it is possible to describe the shift Δf_i of eigenfrequency i due to a shift in t_j as $\Delta f_i = t_j \cdot \partial f_i / \partial t_j$, where $\partial f_i / \partial t_j$ is the partial derivative of eigenfrequency f_i with respect to design variable t_j for a bell with nominal thickness. It should be noted that $\partial f_i / \partial t_j$ is a function of the design variables x_i for the radii.
The geometrical region is thus chosen to be

$$
\begin{aligned}
&x_i \in [-3.0, 3.0] && i=1,7 \\
&t_j \in [-3.0, 4.0] && j=1,13 \\
&|x_i - x_{i-1}| \leq 2.0 && i=2,7
\end{aligned} \tag{4}
$$

The data which will be used to fit the model are calculated in the geometrical region

$$
\begin{aligned}
&x_i \in [-3.0, 3.0] && i=1,7 \\
&t_j = 0.0 && j=1,13 \\
&|x_i - x_{i-1}| \leq 2.0 && i=2,7
\end{aligned} \tag{5}
$$

DESIGN OF EXPERIMENT

For the geometrical region defined by relations (5) a complete factorial design of experiment was made by Schoofs [5]. In a design of experiment a series of experiments is planned for the development of regression models with a minimum of experimental effort and a maximum accuracy of the model predictions [6,7]. Here a complete factorial design of experiment was used. This means that every design variable x_i (i=1,n) is varied on n_i equidistant levels resulting in a total of $n_{total} = n_1 \cdot n_2 \cdot \ldots \cdot n_n$ experiments. These experiments form a n-dimensional grid in the geometrical region.

Here each design variable x_i was varied at 4 levels, i.e. x_i could take the values {-3.0, -1.0, 1.0, 3.0}. Taking into account the constraints $|x_i - x_{i-1}| \leq 2.0$ this resulted in 1220 different bells to be calculated. A second complete factorial design of experiment was made where each design variable was varied at 3 levels, i.e. x_i could take the values {-2.0, 0.0, 2.0}. This design of experiment contained 577 different bells. For each of the 1797 different bells the first 7 eigenfrequencies and the partial derivatives of these eigenfrequencies with respect to the 20 design variables were calculated using a FEM software package.

FORM OF REGRESSION FUNCTIONS

Schoofs [5] found that the eigenfrequencies are in good approximation linear in the design variables t_j. Therefore the following form for the regression functions has been used

$$
f_i(\underline{x}, \underline{t}) = f_i(\underline{x}) + \sum_{j=1}^{13} t_j \cdot f_{ij}(\underline{x}) \qquad i=1,7 \quad j=1,13 \tag{6}
$$

Here $f_i(\underline{x})$ is a polynomial function describing the i^{th} eigenfrequency as a function of the design variables x_1 up to x_7 at nominal wall thicknesses, $f_{ij}(\underline{x})$ is a polynomial function describing the partial derivative of the i^{th} eigenfrequency with respect to design variable t_j as a function of the design variables x_1 up to x_7 at nominal wall thicknesses, where $\underline{x}$ and $\underline{t}$ are the columns containing the design variables x_i and t_j, respectively. The terms of the polynomial functions $f_i(\underline{x})$ and $f_{ij}(\underline{x})$ are chosen from a complete polynomial function of degree 4 in 7 variables, i.e. a function with 330 terms.

FITTING THE FUNCTIONS

Having defined the regression function and its contributing polynomial functions, the polynomial functions were fitted on the data gathered in both complete factorial designs of experiment using an iterative weighted least squares algorithm [6].

The functions $f_i(\underline{x})$ were fitted on the data of the eigenfrequencies f_i and on the data of the partial derivatives of the eigenfrequencies f_i with respect to all design variables for the radii. Starting with a polynomial function of 109 terms the number of terms was gradually increased until a satisfying predictive accuracy was reached, compromising between the accuracy and the computation time needed. For function $f_i(\underline{x})$ a function consisting of 210 terms sufficed to reach a satisfying predictive accuracy.

The functions $f_{ij}(\underline{x})$ were fitted on the data of the partial derivatives of eigenfrequency f_i with respect to design variable t_j for the j^{th} wall thickness. For the functions $f_{ij}(\underline{x})$ the complete polynomial function of degree 4 in 7 variables has been used in order to reach a satisfying predictive accuracy.

ACCURACY OF THE REGRESSION FUNCTIONS

A first indication of the accuracy of the regression functions is found using 97 bells with nominal wall thicknesses that are not in the designs of experiment used to fit the regression functions. Table 2 shows the mean eigenfrequency, and the mean and the standard deviation of the differences between the eigenfrequencies calculated by the FEM package and the predicted eigenfrequencies for the first 7 eigenfrequencies of a bell, both in Hertz and in cents. The absolute maximum value of the difference is listed in table 2 as well.

partial	mean freq. [Hz]	$\overline{\Delta f}$ [Hz]	σ [Hz]	$\lvert\Delta f\rvert_{max}$ [Hz]	$\overline{\Delta c}$ [cent]	σ [cent]	$\lvert\Delta c\rvert_{max}$ [Hz]
hum	192.1	-1.55	2.07	6.17	-11.71	17.65	51.48
fundamental	360.1	2.29	3.27	10.42	11.96	15.61	49.23
third	436.9	0.38	3.25	9.64	2.97	10.10	42.90
fifth	604.7	-1.11	3.71	12.55	-1.58	14.42	66.92
nominal	735.1	-0.72	3.98	13.21	-	-	-
twelfth	1095.3	0.28	4.49	15.89	2.24	3.10	10.23
double octave	1499.2	-0.06	4.36	11.67	1.77	5.35	15.53

Table 2 Mean frequency, and mean, standard deviations and absolute maximum value of the difference between the predicted and calculated eigenfrequencies for a set of 97 bells with nominal wall thicknesses.

From table 2 it can be seen that for the evaluation with 97 bells the predictive accuracy in terms of the 95% confidence interval for each regression function is well within a semi tone, i.e. 50 cents.

To judge the accuracy of the regression functions using bells with non nominal wall thicknesses we chose 400 bells random. This means that every design variable x_i and t_j was randomly chosen from a uniform distribution, taking into account the geometrical region defined by relations (4), and also the constraints

$$|t_j - t_{j-1}| \leq 2.0 \qquad j=2,13 \qquad (7)$$

These constraints are used to insure that the thickness of the bell wall will not vary abruptly. The eigenfrequencies and partial derivatives of the eigenfrequencies with respect to all design variables calculated by the FEM package were compared to the values predicted by the regression functions. In table 3 the mean frequency, and the mean and the standard deviation of the differences between the eigenfrequencies calculated by the FEM package and the predicted eigenfrequencies are listed for the first 7 eigenfrequencies, as well as the absolute maximum value for the differences.

partial	mean freq. [Hz]	$\overline{\Delta f}$ [Hz]	σ [Hz]	$\lvert\Delta f\rvert_{max}$ [Hz]	$\overline{\Delta c}$ [cent]	σ [cent]	$\lvert\Delta c\rvert_{max}$ [cent]
hum	198.8	0.10	2.54	12.90	0.77	26.42	134.21
fundamental	358.3	1.23	3.93	14.58	3.95	25.37	133.39
third	440.0	1.29	4.54	31.76	3.85	17.62	180.90
fifth	623.9	0.58	5.40	23.40	0.59	23.83	149.44
nominal	738.4	0.28	7.33	73.02	-	-	-
twelfth	1093.6	-1.93	16.56	139.03	-3.21	15.00	163.42
double octave	1490.6	-5.50	26.18	266.29	-6.82	22.93	279.28

Table 3 Mean frequency, and mean, standard deviations and absolute maximum of the difference between the predicted and calculated eigenfrequencies for a set of 400 bells with non nominal wall thicknesses.

As can be seen from table 3 the predictive accuracy in terms of the 95% confidence interval of the regression functions for the evaluation with 400 random bells is almost within 50 cents. This is good enough for designing bell shape estimates which can be optimized further with a FEM based optimization program. However, it is not always possible to improve such an estimate within 0 to 20 cents from the target cent values, using the FEM based optimization program.

In figure 4 and 5 the differences between the eigenfrequencies calculated with the FEM package and the predicted eigenfrequencies are displayed in histograms for frequency 1 and frequency 7, respectively. It can be seen that except for some extreme differences the regression functions predicted the eigenfrequencies of the bells reasonably well.

OPTIMIZATION

With the purpose of designing new, musically interesting bells the regression functions are used in an optimization program (OPTIPOL) based on the sequential quadratic programming (SQP) algorithm [8]. In this optimization program the following minimization problem is solved

$$\min_{\underline{x},\underline{t}} F(\underline{x},\underline{t}) \tag{8}$$

$$-3.0 \leq x_i \leq 3.0, \qquad -3.0 \leq t_j \leq 4.0 \qquad i=1,7 \qquad j=1,13$$

$$|x_i - x_{i-1}| \leq 2.0, \qquad |t_j - t_{j-1}| \leq 2.0 \qquad i=2,7 \qquad j=2,13$$

where

$$F(\underline{x},\underline{t}) = \sum_{i=1}^{7} w_i^2[c_i(\underline{x},\underline{t}) - c_i^{opt}]^2 + w_8^2[f\cdot D(\underline{x},\underline{t}) - f\cdot D^{opt}]^2 \tag{9}$$

$$c_i(\underline{x},\underline{t}) = 1200\cdot {}^2\log(4f_i(\underline{x},\underline{t})/f_{nom}(\underline{x},\underline{t})) \tag{10}$$

$$f\cdot D(\underline{x},\underline{t}) = \tfrac{1}{4}\cdot f_{nom}(\underline{x},\underline{t})\cdot 2r_1(1+x_1\cdot wr_1) \tag{11}$$

and

c_i^{opt}, $f\cdot D^{opt}$ optimum value for respectively $c_i(\underline{x},\underline{t})$ and $f\cdot D(\underline{x},\underline{t})$

w_i weighting factors

The $f\cdot D$ value couples the size of a bell, represented by radius 1 (i.e. the radius of the sound bow), to the pitch of the bell partials, represented by frequency f_{nom}. As a rule of thumb the $f\cdot D$ value should be approximately 200 [m/s] for minor third bells and major third bells in order to match size and wall thicknesses of bells [4].

RESULTS

Having derived a numerical tool for shape optimization of bells we decided to search for the major third bell found by Maas [2] using the regression functions. Starting from several different bell geometries we found four different shapes for a major third bell, including the shape found by Maas.

However, comparing the cents values predicted by the regression functions with the values predicted by the FEM package for the bell shapes found, showed discrepancies of sometimes up to 100 cents. Since for a tunable bell it is necessary for the cent values to be exact within 10 to 20 cent for the first 5 partials, the bell shapes found were improved using the FEM based optimization program. Three of these bell shapes could be improved toward the optimal overtone structure with little difficulty, within 0 to 20 cent from the target values (see figure 6). The 4th bell shape could not be improved within 0 to 20 cent from the target values.

Using the program OPTIPOL alternative shapes for the minor third bell were found as well (see figure 7). An attempt to find a bell with an harmonic overtone structure was not successful. It is very likely that such a harmonic bell does not exist within the geometrical region for which the regression functions are valid.

CONCLUSIONS

A fast numerical tool for shape optimization of bells with respect to the eigenfrequencies was developed, using design of experiment and regression techniques.

Using these regression functions good first estimates of optimal bell designs are found. The estimates could be improved with little difficulty using a FEM based optimization program. Several shapes for a major third bell were found, as well as an alternative shape for the minor third bell. A harmonic bell could not be found in the geometrical region for which the regression functions are valid.

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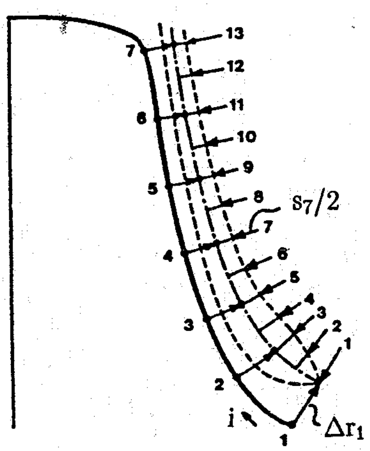


Figure 1 Design variables in the design of experiment

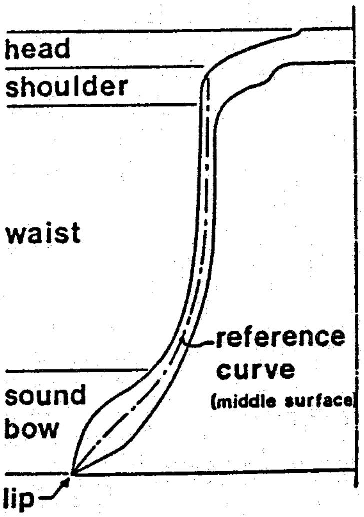


Figure 2 Profile of the well-known minor third bell

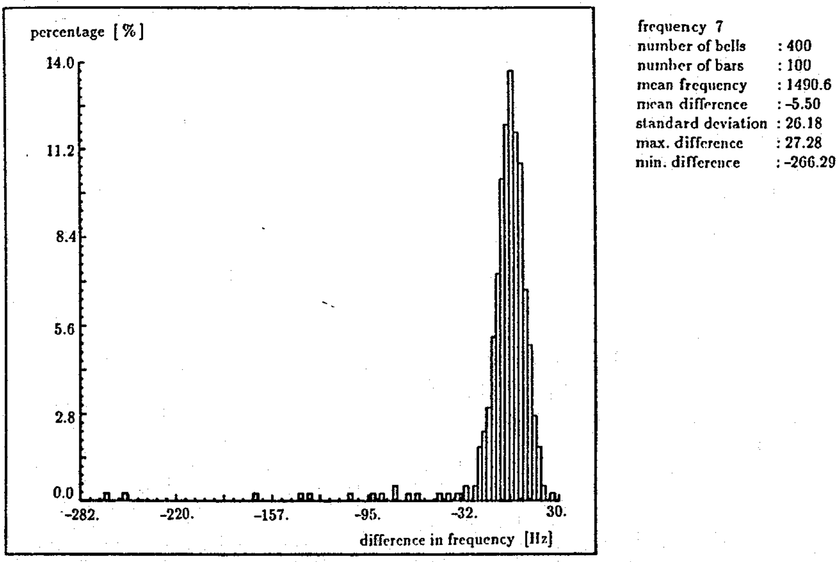


Figure 5 Differences between the eigenfrequencies calculated with the FEM package and the predicted eigenfrequencies for frequency 7, for a set of 400 bells with non nominal wall thicknesses

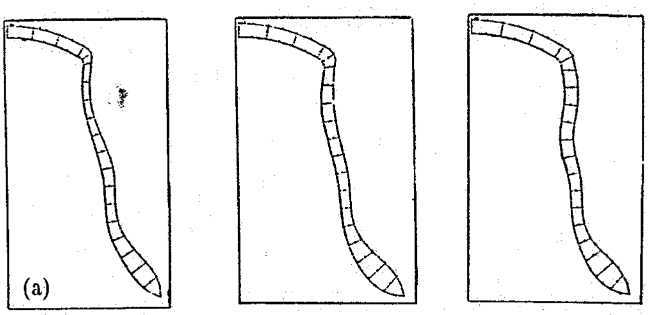


Figure 6 3 possible profiles for a major third bell, including the profile found by Maas (a) (Patented by the Royal Eijsbouts Bellfoundry, Asten, the Netherlands)

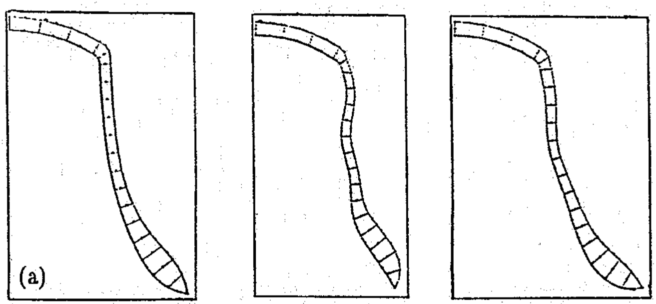


Figure 7 Profile of the minor third bell (a) and two alternative profiles of the minor third bell

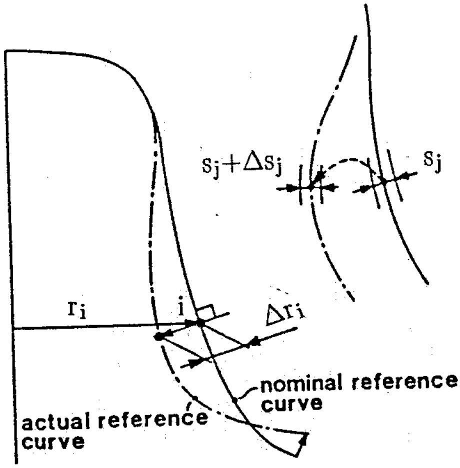
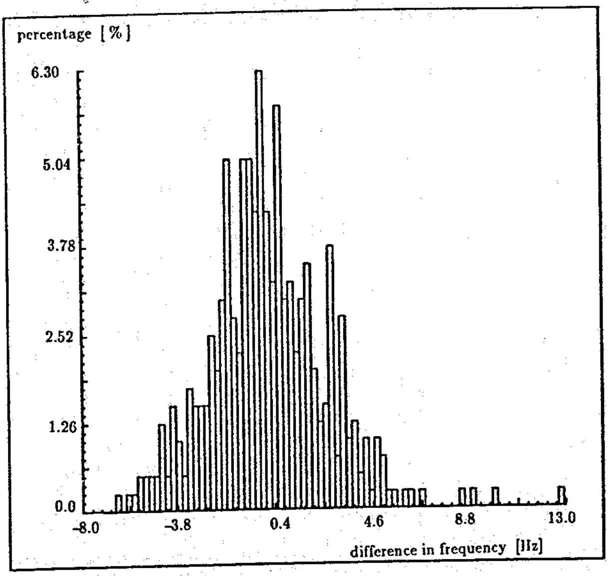


Figure 3 Nominal and actual reference curve and definition of Δr_i, s_j and $s_j+\Delta s_j$



frequency 1	
number of bells	: 400
number of bars	: 80
mean frequency	: 198.8
mean difference	: 0.10
standard deviation	: 2.53
max. difference	: 12.90
min. difference	: -6.53

Figure 4 Differences between the eigenfrequencies calculated with the FEM package and the predicted eigenfrequencies for frequency 1, for a set of 400 bells with non nominal wall thicknesses