

## On representing behaviors in the frequency domain

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**On Representing Behaviors in the  
Frequency Domain**

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## ON REPRESENTING BEHAVIORS IN THE FREQUENCY DOMAIN

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**Abstract.** In this paper we give a fairly complete theory for rational representations of discrete time dynamical systems whose behaviors are assumed to be linear, left-shift invariant and complete subsets of  $\ell_2^+$ . Using the Hilbert space isomorphism between  $\ell_2^+$  and the Hardy space  $\mathcal{H}_2^+$  this leads to frequency domain descriptions of dynamical systems in which system variables are not necessarily partitioned in inputs and outputs. Analytic functions are used to define kernel and image representations of dynamical systems and it is shown that for an important class of discrete time systems rational kernel and rational image representations always exist. We further investigate the concept of state by considering factor spaces of left- and right-shift invariant subspaces of  $\mathcal{H}_2^+$ . It is shown how state space representations are obtained by associating Hankel operators directly with kernel and image representations of dynamical systems.

**Keywords.** realisation theory, linear systems, frequency signal analysis, systems concepts.

### 1. INTRODUCTION

In this paper we focus on the class of discrete time  $\ell_2$ -systems with time set  $T = \mathbb{Z}_+$ . Following the tradition of the behavioral framework (Willems, 1986a; Willems, 1986b; Willems, 1987; Willems, 1991) such a system is specified by a set  $\mathcal{B}$  consisting of (multivariate) square summable trajectories  $w : \mathbb{Z}_+ \rightarrow W$  which are considered to be compatible with the system. We make the

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mathematical assumption that an  $\ell_2$ -system is a *closed* subset  $\mathcal{B}$  of  $\ell_2^+$  where  $\ell_2^+ := \ell_2(\mathbb{Z}_+, W)$  is the set of all trajectories  $w : \mathbb{Z}_+ \rightarrow W$  for which

$$\|w\|_2 := \left\{ \sum_{t=0}^{\infty} \|w(t)\|^2 \right\}^{1/2} < \infty$$

and  $W$  is a normed vector space which in this paper is assumed to be  $W = \mathbb{R}^q$  with  $q$  a positive integer.

Most qualitative properties of dynamical systems are naturally defined in an  $\ell_2$  setting. An  $\ell_2$ -system  $\mathcal{B}$  is called *linear* if  $\mathcal{B}$  is a linear subspace of  $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$  and it is said to be *complete* if  $w \in \mathcal{B}$  whenever for  $w \in \ell_2^+$  the restrictions  $w|_{[t_0, t_1]} \in \mathcal{B}|_{[t_0, t_1]}$  for all intervals  $[t_0, t_1] \subset \mathbb{Z}_+$ . The left-shift on  $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$  is defined as  $(\sigma_L w)(t) = w(t+1)$  and the right-shift on  $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$  is defined as

$$(\sigma_R w)(t) = \begin{cases} 0 & \text{for } t = 0 \\ w(t-1) & \text{for } t \geq 1. \end{cases} \quad (1)$$

An  $\ell_2$  system  $\mathcal{B}$  is called *left-shift invariant* if  $\sigma_L \mathcal{B} \subseteq \mathcal{B}$  and *right-shift invariant* if  $\sigma_R \mathcal{B} \subseteq \mathcal{B}$ .

Since by (1), trajectories in a right-shift invariant  $\ell_2$  system can be preceded by an arbitrary number of zeros it is intuitively clear that in the context of systems defined by difference equations, right-shift invariant  $\ell_2$  systems correspond to systems with “zero initial conditions”. In view of the practical importance of autonomous systems, transient phenomena, non-zero initial conditions, offsets, etc. this makes the class of right-shift invariant  $\ell_2$  systems less suitable for general modeling purposes. In this paper we therefore concentrate on left-shift invariant  $\ell_2$ -systems. More specifically, we define the model class  $\mathbb{B}_2$  as all closed (in the  $\ell_2$  topology) subsets  $\mathcal{B} \subseteq \ell_2(\mathbb{Z}_+, \mathbb{R}^q)$  which are linear, left-shift invariant and complete<sup>3</sup>.

The importance of this class of models is motivated as follows. Firstly, square summable trajectories are ubiquitous in many physical systems in which dissipativity, power and energy considerations play a natural role. Also, for many problems in robust stabilization and optimal control the square summability assumption of system trajectories is often made implicitly. Secondly, in many modeling problems the partitioning of system variables in inputs and outputs may be unclear or arbitrary. In the model class  $\mathbb{B}_2$  system variables are treated in a symmetric way without distinguishing between inputs and outputs. Thirdly, autonomous systems are naturally included in the model class  $\mathbb{B}_2$ . This in contrast to the class of right-shift invariant linear subsets of  $\ell_2^+$  where

autonomous systems are necessarily trivial. Fourthly, using the Hilbert space isomorphism between  $\ell_2^+$  and the Hardy space  $\mathcal{H}_2^+$  we can interchangeably consider systems in the time domain and in the frequency domain. Specifically, we define for all  $\mathcal{B} \in \mathbb{B}_2$ :

$$\hat{\mathcal{B}} = \{ \hat{w} \in \mathcal{H}_2^+ \mid w \in \mathcal{B} \}$$

where  $\hat{w}(z) := \sum_{t=0}^{\infty} w(t)z^{-t}$  denotes the  $z$ -transform.

In (Willems, 1986a; Willems, 1991) Willems investigated polynomial representations of linear, left-shift invariant subspaces of  $(\mathbb{R}^q)^{\mathbb{Z}_+}$  and  $(\mathbb{R}^q)^{\mathbb{Z}}$  which are closed in the topology of pointwise convergence. In (Heij, 1989) state space representations of  $\ell_2$  systems with doubly infinite time sets are derived, whereas Georgiou and Smith (Georgiou and Smith, 1994) proposed a theory for right-shift invariant  $\ell_2$  systems by taking the  $\ell_2$  graph of an input-output operator as the basic object of study. The role of the  $\ell_2$  graph has been further investigated in (Ober and Sefton, 1991; Sefton and Ober, 1993) in the context of stability and model uncertainty. The class  $\mathbb{B}_2$  of left-shift invariant  $\ell_2$ -systems with one sided time set  $T = \mathbb{Z}_+$  is essentially different than the model classes studied in these works and the representation of systems in this model class is the topic of this paper.

In what follows, we will give characterizations of the model class  $\mathbb{B}_2$  in terms of kernel, image and state space representations. These representations are introduced in the next sections and we address the questions of existence, uniqueness and minimality of analytic functions which represent models in  $\mathbb{B}_2$ .

The following notation will be used. Let  $\mathcal{L}_2$  denote the set of functions  $f : \mathbb{C} \rightarrow \mathbb{C}^q$  which are square integrable on the unit circle.  $\mathcal{H}_2^+$  is the subspace of  $\mathcal{L}_2$  consisting of those functions for which the negative Fourier coefficients are zero.  $\mathcal{H}_2^-$  is the complement of  $\mathcal{H}_2^+$  in  $\mathcal{L}_2$  and consists of all  $\mathcal{L}_2$  functions whose non-negative Fourier coefficients vanish. Let  $\Pi_+$  and  $\Pi_-$  be the canonical projections of  $\mathcal{L}_2$  on  $\mathcal{H}_2^+$  and  $\mathcal{H}_2^-$ , respectively.  $\mathcal{H}_\infty^+$  and  $\mathcal{H}_\infty^-$  denote the Hardy spaces of complex valued functions which are bounded on the unit circle with analytic continuation in  $|z| < 1$  and  $|z| > 1$ , respectively. The prefix  $\mathcal{R}$  will be used to denote rational elements, i.e.  $\mathcal{R}\mathcal{H}_\infty^-, \mathcal{R}\mathcal{H}_\infty^+$ , etc.

The left- and right-shift operators are defined in the frequency domain as the mappings  $\hat{\sigma}_L, \hat{\sigma}_R : \mathcal{H}_2^+ \rightarrow \mathcal{H}_2^+$  defined, given  $w \in \mathcal{H}_2^+$ , by

$$\begin{aligned} (\hat{\sigma}_L w)(z) &= \Pi_+ z w(z) \\ (\hat{\sigma}_R w)(z) &= z^{-1} w(z) \end{aligned}$$

with  $z \in \mathbb{C}$ . With these definitions an  $\ell_2$  system is left (right) shift invariant if and only if  $\hat{\mathcal{B}}$  is left (right) shift invariant in the sense that  $\hat{\sigma}_L \hat{\mathcal{B}} \subseteq \hat{\mathcal{B}}$  ( $\hat{\sigma}_R \hat{\mathcal{B}} \subseteq \hat{\mathcal{B}}$ ). In the

<sup>3</sup> In fact, completeness in the sense defined above implies closedness in the  $\ell_2$  topology.

sequel we will mainly work in the frequency domain and we will omit the hats  $\hat{\cdot}$  to simplify notation.

## 2. KERNEL REPRESENTATIONS

Let  $\Theta \in \mathcal{H}_\infty^-$  and associate with  $\Theta$  a multiplicative operator  $\Theta : \mathcal{H}_2^+ \rightarrow \mathcal{L}_2$  defined as

$$(\Theta w)(z) := \Theta(z)w(z)$$

with  $z \in \mathbb{C}$ . We will be interested in the subspaces of  $\mathcal{H}_2^+$  defined by the kernel of this operator. To this end, we define two sets

$$\mathcal{B} = \mathcal{B}_{\ker}(\Theta) := \{w \in \mathcal{H}_2^+ \mid \Pi_+ \Theta w = 0\} \quad (2)$$

$$\mathcal{B}^* = \mathcal{B}_{\ker}^*(\Theta) := \{w \in \mathcal{H}_2^+ \mid \Theta w = 0\}. \quad (3)$$

Then it is easily seen that  $\mathcal{B}$  is a linear left-shift invariant and closed subset of  $\mathcal{H}_2^+$  and  $\mathcal{B}^*$  is a linear right-shift invariant and closed subset of  $\mathcal{H}_2^+$ . This means that for all  $\Theta \in \mathcal{RH}_\infty^-$  the set  $\mathcal{B}_{\ker}(\Theta)$  belongs to the model class  $\mathbb{B}_2$ . Note that  $\mathcal{B}_{\ker}(\Theta) = \ker(\Pi_+ \Theta)$  and  $\mathcal{B}_{\ker}^*(\Theta) = \ker \Theta$ . It is thus immediate that  $\mathcal{B}^* \subseteq \mathcal{B}$ . We will see in section 4 that the factor space  $\mathcal{B} \pmod{\mathcal{B}^*}$  plays an important role in the construction of state space representations of models in  $\mathbb{B}_2$ .

Our first main result characterizes the model class  $\mathbb{B}_2$  as precisely those  $\ell_2$ -systems which admit a rational kernel representation of the form (2). Moreover, Theorem 2.1 completely characterizes non-uniqueness of this type of system representations.

*Theorem 2.1.* (1) The following statements are equivalent.

- (a)  $\mathcal{B} \in \mathbb{B}_2$
- (b) there exists  $\Theta \in \mathcal{RH}_\infty^-$  such that  $\hat{\mathcal{B}} = \mathcal{B}_{\ker}(\Theta)$ .
- (2) Let  $\Theta_1, \Theta_2 \in \mathcal{RH}_\infty^-$ . Then
  - (a)  $\mathcal{B}_1 \subseteq \mathcal{B}_2$  if and only if there exists  $U \in \mathcal{RH}_\infty^-$  such that  $\Theta_2 = U\Theta_1$ .
  - (b)  $\mathcal{B}_{\ker}(\Theta_1) = \mathcal{B}_{\ker}(\Theta_2)$  if and only if there exist  $U_1, U_2 \in \mathcal{RH}_\infty^-$  such that  $\Theta_1 = U_1\Theta_2$  and  $\Theta_2 = U_2\Theta_1$ .

*Remark 2.2.* Note that it follows from Theorem 2.1 that if  $\mathcal{B}_{\ker}(\Theta_1) = \mathcal{B}_{\ker}(\Theta_2)$  with  $\Theta_1$  and  $\Theta_2$  both full rank then  $\Theta_1 = U\Theta_2$  for some unit  $U \in \mathcal{RH}_\infty^-$ . In particular, taking  $U$  the left inverse of the outer factor of  $\Theta_2$ , yields that  $\Theta_1$  is co-inner (that is  $\Theta_1\Theta_1^* = I$ ) so that any  $\mathcal{B} \in \mathbb{B}_2$  admits a co-inner kernel representation  $\mathcal{B}_{\ker}(\Theta)$ .

*Definition 2.3.* We call  $\Theta \in \mathcal{H}_\infty^-$  a *kernel representation* of a system  $\mathcal{B}$  if  $\mathcal{B} = \mathcal{B}_{\ker}(\Theta)$ . Such a representation is called *normalized* if in addition  $\Theta$  is co-inner.

*Remark 2.4.* It is shown in (Weiland and Stoorvogel, 1996) that the mapping  $\Theta \mapsto \ker \Pi_+ \Theta$  with  $\Theta \in \mathcal{H}_\infty^-$  in

fact defines a parameterization of the model class of linear, left-shift invariant and closed subsets of  $\mathcal{H}_2^+$ . Theorem 2.1 provides a parameterization of  $\mathbb{B}_2$  by taking rational elements  $\Theta \in \mathcal{H}_\infty^-$  as the domain of this map.

The set  $\mathcal{B}^* = \mathcal{B}_{\ker}^*(\Theta)$  has been studied in (Georgiou and Smith, 1994; Ober and Sefton, 1991) and is easily seen to be a subset of  $\mathcal{B} = \mathcal{B}_{\ker}(\Theta)$ . In fact,  $\mathcal{B}^*$  will be useful for the construction of state space representations of systems  $\mathcal{B} \in \mathbb{B}_2$ . This right shift invariant subspace is characterized as follows (Weiland and Stoorvogel, 1996).

*Theorem 2.5.* Let  $\mathcal{B} = \mathcal{B}_{\ker}(\Theta)$  and  $\mathcal{B}^* = \mathcal{B}_{\ker}^*(\Theta)$ . Then the following statements are equivalent:

- (1)  $\mathcal{B}^* = \{w \in \mathcal{H}_2^+ \mid \hat{\sigma}_R^t w \in \mathcal{B} \text{ for all } t \in \mathbb{Z}_+\}$
- (2)  $\mathcal{B}^*$  is the largest<sup>4</sup> right-shift invariant subspace of  $\mathcal{B}$ .

Theorem 2.5 in fact shows that  $\mathcal{B}^*$  only depends on  $\mathcal{B}$  and is independent of  $\Theta$ .

## 3. IMAGE REPRESENTATIONS

Kernel representations are particularly useful in verifying whether a given trajectory does or does not belong to a system. In this section we wish to parametrize the elements  $\mathcal{B}$  of  $\mathbb{B}_2$  in such a way as to produce the set of compatible trajectories  $w \in \mathcal{B}$ . We will introduce a class of image representations which will parametrize the elements of the model class  $\mathbb{B}_2$  as the images of a map.

Let  $\Psi_a, \Psi_c$  be elements of  $\mathcal{H}_\infty^+$  and associate with  $\Psi_a$  a multiplicative map from  $\mathcal{H}_2^-$  to  $\mathcal{L}_2$  defined by

$$(\Psi_a v)(z) := \Psi_a(z)v(z)$$

and associate with  $\Psi_c$  the map  $\Psi_c : \mathcal{L}_2 \rightarrow \mathcal{L}_2$  defined by the multiplication

$$(\Psi_c v)(z) := \Psi_c(z)v(z)$$

where  $z \in \mathbb{C}$ . Introduce the following sets:

$$\begin{aligned} \mathcal{B} &= \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c) \\ &:= \left\{ \Pi_+ \begin{pmatrix} \Psi_a & \Psi_c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1 \in \mathcal{H}_2^-, v_2 \in \mathcal{L}_2 \right\} \\ \mathcal{B}^* &= \mathcal{B}_{\text{im}}^*(\Psi_c) := \text{im } \Psi_c \Pi_+ = \Psi_c \mathcal{H}_2^+. \end{aligned}$$

Then it is straightforward to see that  $\mathcal{B}$  is a linear left-shift invariant and closed subset of  $\mathcal{H}_2^+$  and  $\mathcal{B}^*$  is a linear right-shift invariant subset of  $\mathcal{H}_2^+$ . In fact,  $\mathcal{B}^* \subseteq \mathcal{B}$  and it is shown in Theorem 3.7 below that  $\mathcal{B}^*$  only depends on  $\mathcal{B}$  and not of a particular representation  $\Psi_c$ .

<sup>4</sup> in the sense of subspace inclusions

*Remark 3.1.* The operators  $\Psi_a$  and  $\Psi_c$  constitute a natural decomposition of  $\mathcal{B} = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$  in the sense that  $\mathcal{B} = \mathcal{B}_a + \mathcal{B}_c$  where  $\mathcal{B}_a := \Pi_+ \Psi_a \mathcal{H}_2^-$  and  $\mathcal{B}_c := \Pi_+ \Psi_c \mathcal{L}_2$ . In fact, in such a decomposition  $\mathcal{B}_a$  defines an autonomous  $\ell_2$  system and  $\mathcal{B}_c$  defines a controllable  $\ell_2$  system. We refer to  $\mathcal{B}_a$  and  $\mathcal{B}_c$  as the *autonomous part* and the *controllable part* of  $\mathcal{B}$ . We remark that the autonomous part of an  $\ell_2$  system  $\mathcal{B} \in \mathbb{B}_2$  is in general non-unique.

The main representation result for image representations of systems in  $\mathbb{B}_2$  is as follows.

*Theorem 3.2.*

- (1) The following statements are equivalent.
  - (a)  $\mathcal{B} \in \mathbb{B}_2$
  - (b) There exists  $\Psi_a, \Psi_c \in \mathcal{RH}_\infty$  such that  $\mathcal{B} = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$ .
  - (c) There exists  $\bar{\Psi}_a$  and  $\bar{\Psi}_c$  in  $\mathcal{RH}_\infty^+$  where  $(\bar{\Psi}_a \ \bar{\Psi}_c)$  is square and inner and  $\bar{\Psi}_c$  has no finite or infinite zeros such that  $\mathcal{B} = \mathcal{B}_{\text{im}}(\bar{\Psi}_a, \bar{\Psi}_c)$ .
- (2) With  $\bar{\Psi}_a$  and  $\bar{\Psi}_c$  as defined in statement (c), there holds that  $\mathcal{B} = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$  if and only if there exists  $R \in \mathcal{RH}_\infty^-, T \in \mathcal{RH}_\infty^+$  and  $S \in \mathcal{RL}_\infty$  such that  $R$  and  $T$  have a right-inverse in  $\mathcal{L}_\infty$ ,

$$\begin{aligned}\Psi_a &= \bar{\Psi}_a R + \bar{\Psi}_c S \\ \Psi_c &= \bar{\Psi}_c T\end{aligned}$$

and such that there are no stable pole-zero cancellations between  $\bar{\Psi}_a$  and  $R$  or, in other words, the number of stable poles of  $\bar{\Psi}_a$  equals the number of stable poles of  $\bar{\Psi}_a R$ .

*Remark 3.3.* Together with Theorem 2.1 this result states that systems in the model class  $\mathbb{B}_2$  admit both kernel as well as image representations. We emphasize the difference between these results and the results in e.g. (Willems, 1991) where it is shown that *polynomial* image representations exist only for *controllable* systems.

*Remark 3.4.* Theorem 3.2 shows, among other things, that autonomous left-shift invariant  $\ell_2$ -systems admit analytic image representations. As noted earlier, the model class  $\mathbb{B}_2$  of left-shift invariant linear systems allows for non-trivial autonomous systems. This is in contrast with the class of right-shift invariant  $\ell_2$  systems (or the class of linear (left or right) shift invariant subspaces of  $\ell_2(\mathbb{Z}, \mathbb{R}^q)$ ) in which autonomous systems are necessarily trivial (See (Heij, 1989), (Weiland and Stoorvogel, 1996) for details).

*Remark 3.5.* In words, the last part of Theorem 3.2 states that two image representations define the same  $\ell_2$  system if and only if their image representations have a

common square and inner left factor. With the obvious notation,  $\mathcal{B} = \mathcal{B}_{\text{im}}(\bar{\Psi}_a, \bar{\Psi}_c)$  admits a special decomposition  $\mathcal{B} = \mathcal{B}_a + \mathcal{B}_c$  in which the autonomous part  $\mathcal{B}_a$  is orthogonal to  $\mathcal{B}_c$ . See (Weiland and Stoorvogel, 1996) for details.

*Definition 3.6.* We call  $\Psi_a, \Psi_c$  an *image representation* of a system  $\mathcal{B}$  if  $\mathcal{B} = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$ . Such a representation is called *normalized* if in addition  $(\Psi_a \ \Psi_c)$  is square and inner and if  $\Psi_c$  has no finite or infinite zeros.

The set  $\mathcal{B}^* = \mathcal{B}_{\text{im}}^*(\Psi_c)$  has similar features as the set  $\mathcal{B}^*$  of Theorem 2.5. Precisely,

*Theorem 3.7.* Let  $\mathcal{B} = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$  and  $\mathcal{B}^* = \mathcal{B}_{\text{im}}^*(\Psi_c)$ . Then the following statements are equivalent:

- (1)  $\mathcal{B}^* = \{w \in \mathcal{H}_2^+ \mid \delta_R^t w \in \mathcal{B} \text{ for all } t \in \mathbb{Z}_+\}$
- (2)  $\mathcal{B}^*$  is the largest right-shift invariant subspace of  $\mathcal{B}$ .

In particular, it follows that the largest right-shift invariant subspace of  $\mathcal{B} = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$  is independent of  $\Psi_a$ .

*Remark 3.8.* It is emphasized that the largest right-shift invariant subspace  $\mathcal{B}^*$  of  $\mathcal{B}$  is a representation independent object which is directly characterized in terms of both the kernel and the image representations of  $\mathcal{B}$ .

#### 4. STATE SPACE REPRESENTATIONS

In this section we will show that state space representations of systems  $\mathcal{B} \in \mathbb{B}_2$  can be constructed directly from the kernel and image representations which have been introduced so far. Such a construction is non-trivial as it amounts to define the state of a system on the basis of a representation of the *external behavior* only. First of all, a state space needs to be defined, and second, its evolution as a function of time needs to be specified. The construction of state space representations will be based on the subsets  $\mathcal{B}$  and  $\mathcal{B}^*$  and will exploit the difference between right- and left shift invariance.

An  $\ell_2$  state space system will be defined as an  $\ell_2$  system for which the signal space is partitioned and for which past and future system trajectories are independent given the current value of the state. Formally,

*Definition 4.1.* Let  $q > 0$  and  $n > 0$  be integers. A (discrete time)  $\ell_2$  state space system is a closed subset  $\mathcal{B}_s$  of  $\ell_2(\mathbb{Z}_+, \mathbb{R}^{q+n})$  with the property that for all  $(w', x'), (w'', x'') \in \mathcal{B}_s$  and  $t_0 \in \mathbb{Z}_+$  there exist  $(w, x) \in \mathcal{B}_s$  which satisfies

$$(w, x) = \begin{cases} (w'(t), x'(t)) & \text{for } t < t_0 \\ (w''(t), x''(t)) & \text{for } t \geq t_0 \end{cases}$$

whenever  $x'(t_0) = x''(t_0)$ .

*Definition 4.2.* An  $\ell_2$  state space system  $\mathcal{B}_s$  is said to represent a system  $\mathcal{B}$  if

$$\mathcal{B} = \{ w \in \ell_2^+ \mid \exists x \in \ell_2^+ \text{ s.t. } (w, x) \in \mathcal{B}_s \}.$$

It is said to be a *minimal representation* of  $\mathcal{B}$  if  $n$  is minimal among all state space representations of  $\mathcal{B}$ .

Let  $\mathcal{B} \in \mathbb{B}_2$  and let  $\mathcal{B}^*$  denote the largest right-shift invariant subspace contained in  $\mathcal{B}$ . We call two trajectories  $w_1, w_2 \in \mathcal{B}$  *right-shift equivalent* if  $w_1 - w_2 \in \mathcal{B}^*$ . Introduce the factor space  $\mathcal{B} \pmod{\mathcal{B}^*}$  which consists of all equivalence classes  $w \pmod{\mathcal{B}^*}$  with  $w \in \mathcal{B}$ . Intuitively, the equivalence class  $x(0) := w \pmod{\mathcal{B}^*}$  can be viewed as the initial state of the system when  $w \in \mathcal{B}$  is observed and the factor space

$$X := \mathcal{B} \pmod{\mathcal{B}^*}$$

can therefore be identified as the *state space* of  $\mathcal{B}$ .

Suppose that

$$\mathcal{B} = \mathcal{B}_{\ker}(\Theta) = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$$

define kernel and image representations of  $\mathcal{B}$ . We associate with  $\Theta$  and the pair  $(\Psi_a, \Psi_c)$  the *state spaces*

$$\begin{aligned} X_{\ker} &= \Pi_- \Theta \mathcal{H}_2^+ \\ X_{\text{im}} &= \Pi_+ (\Psi_a \ \Psi_c) \mathcal{H}_2^- \end{aligned}$$

*Remark 4.3.* Note that  $X_{\ker}$  is a subset of the infinite dimensional space  $\mathcal{H}_2^-$  and  $X_{\text{im}}$  a subset of the infinite dimensional space  $\mathcal{H}_2^+$ . Their dimensions, however, are *finite* if and only if  $\Theta$  and  $(\Psi_a \ \Psi_c)$  are rational operators. In this case,  $\dim X_{\ker}$  equals the McMillan degree of  $\Theta$  and  $\dim X_{\text{im}}$  equals the McMillan degree of  $(\Psi_a \ \Psi_c)$ .

Next, let  $w \in \mathcal{B}$  and suppose that  $v_1 \in \mathcal{H}_2^-$  and  $v_2 \in \mathcal{L}_2$  are such that

$$w = \Pi_+ \Psi_a v_1 + \Pi_+ \Psi_c v_2.$$

We define for  $t \in \mathbb{Z}_+$  the state trajectories  $x_{\ker} : \mathbb{Z}_+ \rightarrow X_{\ker}$  and  $x_{\text{im}} : \mathbb{Z}_+ \rightarrow X_{\text{im}}$  as

$$x_{\ker}(t) := \Pi_- \Theta \bar{\sigma}_L^t \hat{w} \quad (4)$$

$$x_{\text{im}}(t) := \Pi_+ \Psi_a \bar{\sigma}_L^t v_1 + \Pi_+ \Psi_c \Pi_- v_2 \quad (5)$$

where  $\bar{\sigma}_L$  is the left shift operator on  $\mathcal{L}_2$  defined as  $(\bar{\sigma}_L v)(z) := zv(z)$  with  $z \in \mathbb{C}$ .

It can be shown that, in the sense of definition 4.1, the sets

$$\mathcal{B}_s^{\ker} := \{(w, x) \in \ell_2^+ \mid w \in \mathcal{B}_{\ker}(\Theta); x = x_{\ker}\}$$

$$\mathcal{B}_s^{\text{im}} := \{(w, x) \in \ell_2^+ \mid w \in \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c); x = x_{\text{im}}\}$$

define  $\ell_2$  state space systems.<sup>5</sup>

Clearly, this result is of little practical interest as it does not provide an iterative way to compute state trajectories of  $\ell_2$  systems. Therefore, consider the equations

$$x(t+1) = Ax(t) + Bw(t) \quad (6)$$

$$0 = Cx(t) + Dw(t) \quad (7)$$

and associate with (6) the *output nulling behavior*

$$\mathcal{B}_{\text{on}} := \{(w, x) \in \ell_2^+ \mid (6) \text{ holds}\}. \quad (8)$$

Similarly, associate with the equations

$$x(t+1) = Ax(t) + Bv(t) \quad (9)$$

$$w(t) = Cx(t) + Dv(t) \quad (10)$$

the *driving variable behavior*

$$\mathcal{B}_{\text{dv}} := \{(w, x) \in \ell_2^+ \mid \exists v \in \ell_2^+ \text{ such that (9) holds}\}. \quad (11)$$

The matrices  $A, B, C, D$  are assumed to be compatible with the indicated partitionings. It is easy to see that  $\mathcal{B}_{\text{on}}$  and  $\mathcal{B}_{\text{dv}}$  define  $\ell_2$  state space systems in the sense of definition 4.1 and we will refer to these sets as *output nulling* and *driving variable state space systems*, respectively.

The next theorem is the main result of this section and provides explicit expressions for the state space matrices  $A, B, C$  and  $D$  which define output nulling and driving variable state space representations of a given system  $\mathcal{B} \in \mathbb{B}_2$ . To state the result we introduce some more notation. Let  $P$  be the mapping from the signal space  $\mathbb{R}^q$  to  $\mathcal{H}_2^+$  defined by

$$(Pw)(z) := w$$

for all  $z \in \mathbb{C}$ . Further, let  $\Pi_0$  denote the map  $\mathcal{H}_2^+ \rightarrow \mathbb{R}^q$  which assigns the Fourier coefficient with index 0 to elements in  $\mathcal{H}_2^+$ , i.e.,

$$\Pi_0(w(z)) := \lim_{|z| \rightarrow \infty} w(z).$$

Finally, let  $\Pi_{X_{\ker}}$  be the orthogonal projection on  $X_{\ker}$ .

*Theorem 4.4.* Let  $\mathcal{B} \in \mathbb{B}_2$  and suppose that

$$\mathcal{B} = \mathcal{B}_{\ker}(\Theta) = \mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$$

define kernel and image representations of  $\mathcal{B}$ .

(1) Let

$$A := \Pi_{X_{\ker}} \bar{\sigma}_L$$

$$B := -\Pi_{X_{\ker}} \Pi_- \bar{\sigma}_L \Theta P$$

$$C := (I - \Pi_{X_{\ker}}) \bar{\sigma}_L$$

$$D := -(I - \Pi_{X_{\ker}}) \Pi_- \bar{\sigma}_L \Theta P.$$

<sup>5</sup> Here, all signals need to be interpreted in the time domain by taking inverse  $z$ -transforms.

Then  $\mathcal{B}_{\text{on}}(A, B, C, D)$  defines an output-nulling state space representation of  $\mathcal{B}$ . Moreover, if  $\mathcal{B}_{\text{ker}}(\Theta)$  is a normalized kernel representation of  $\mathcal{B}$  then this state space representation is minimal.

(2) Let

$$\begin{aligned} A &:= \hat{\sigma}_L \\ B &:= \Pi_+ \Psi_c \bar{\sigma}_L P \\ C &:= \Pi_0 \\ D &:= \Pi_0 \Psi_c P. \end{aligned}$$

Then  $\mathcal{B}_{\text{dv}}(A, B, C, D)$  defines a driving-variable state space representation of  $\mathcal{B}$ . Moreover, if  $\mathcal{B}_{\text{im}}(\Psi_a, \Psi_c)$  is a normalized image representation of  $\mathcal{B}$  then this state space representation is minimal.

The proof of this result can be found in (Weiland and Stoorvogel, 1996) and is based on the fact that for every  $(w, x) \in \mathcal{B}_{\text{on}}(A, B, C, D)$  the state trajectory  $x(\cdot)$  coincides with the trajectory  $x_{\text{ker}}(\cdot)$  as defined in (4). Similarly, for every  $(w, x) \in \mathcal{B}_{\text{im}}(A, B, C, D)$  the state trajectory  $x(\cdot)$  coincides with the trajectory  $x_{\text{im}}(\cdot)$  defined in (5) where  $v_1 \in \mathcal{H}_2^-$  is any element for which  $x(0) = x_{\text{im}}(0)$  and  $v_2$  is the Laplace transform of  $v$  in (11).

*Remark 4.5.* Note that the matrices  $A, B, C, D$  defining the driving variable state space representation are independent of  $\Psi_a$ . This is in accordance with the intuitive idea that the autonomous subset  $\Pi_+ \Psi_a \mathcal{H}_2^-$  is only depending on initial conditions of the state.

## 5. CONCLUSION

In this paper we investigated the class of discrete time left-shift invariant linear  $\ell_2$ -systems with time set  $T = \mathbb{Z}_+$ . It is shown that systems in this class admit both kernel representations as well as image representations in terms of multivariable  $\mathcal{H}_\infty$  functions. The set of  $\mathcal{H}_\infty$  functions that represent the same  $\ell_2$  system has been characterized completely for both the kernel as well as the image representations of  $\ell_2$ -systems. It is shown how state space representations can be obtained directly from kernel and image representations, by associating a Hankel operator with the  $\mathcal{H}_\infty$  functions which define the representation of the external behavior.

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