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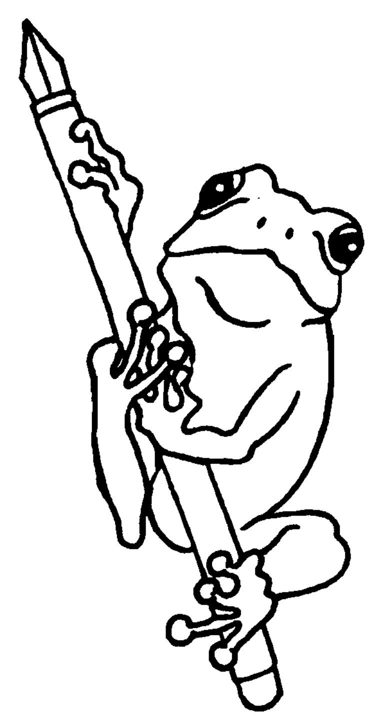

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Abstract

Let $(\mathcal{H}, G, U)$ be an irreducible unitary representation of a homogeneous Lie group G and H a self-adjoint operator on $\mathcal{H}$ associated with a positive Rockland operator. We derive upper and lower bounds on the eigenvalue distribution of H in terms of volume estimates on the coadjoint orbit corresponding to the representation U. Hence we deduce bounds on the partition function $\beta \mapsto \mathrm{Tr}_{\mathcal{H}}(\exp(-\beta H))$. An application is given to the spectrum and eigenfunctions of the general anharmonic oscillator.

1 Introduction

Our purpose is to derive spectral estimates for Rockland operators H in each irreducible unitary representation of a homogeneous group G. These estimates are expressed in terms of the symbol of the differential operator H and are similar in spirit to the estimates of the spectra of quantum-mechanical Hamiltonians in terms of classical phase space integrals (see, for example, [Sim1], [Fef]). The estimates extend recent results of Levy-Bruhl and Nourrigat [LBN] and Levy-Bruhl, Mohamed and Nourrigat [LMN1] for sublaplacians on stratified groups and are related to Weyl's classical results on the asymptotic distribution of eigenvalues of the Laplacian on bounded regions. Our proofs partially rely upon the work of the above authors. Similar results for strongly elliptic operators have also been given by Manchon [Man1], [Man2] although his methods are quite different.

A differential operator H on a homogeneous group G is defined to be a Rockland operator if it is right-invariant, homogeneous and injective in each nontrivial irreducible unitary representation. The theory of Rockland operators began with Rockland's analysis of differential operators on the Heisenberg group [Roc]. Helffer and Nourrigat [HeN1] proved that a Rockland operator on a graded group is hypoelliptic and in addition they derived several inequalities between the norm on the C^n-spaces and the operator norm. Then Miller [Mil] showed that one can replace a graded group by a homogeneous group in the Helffer–Nourrigat theorem. Subsequently, Folland and Stein [FoS] used the proof of an earlier theorem of Nelson and Stinespring [NeS] to deduce that a positive Rockland operator is essentially self-adjoint on the space $C_c^\infty(G)$. Moreover, they established that the closure generates a continuous semigroup with a kernel which is in the Schwartz space over the group. Hence it follows by a general structural theorem for nilpotent groups (see, for example, [CoG] Theorem 4.2.1) that the operators $\exp(-\beta H)$, $\beta > 0$, are trace class in each irreducible unitary representation. Our aim is to estimate the partition functions $\beta \mapsto Z(\beta) = \mathrm{Tr}(\exp(-\beta H))$ for each such representation. This problem is closely related to the estimation of the number $N(\lambda)$ of eigenvalues of H with values less than or equal to λ, because Z is the Abel transform of N.

Throughout the sequel we adopt the notation of [AER], in which we used the general notation of [Rob], but to make this paper more self-contained we repeat the main definitions. Let G be a connected, simply connected, homogeneous group with Lie algebra $\mathfrak{g}$ and let $(\mathcal{X}, G, U)$ denote a strongly continuous, or weakly*, continuous representation of G on the Banach space $\mathcal{X}$ by bounded operators $g \mapsto U(g)$. If $a_i \in \mathfrak{g}$ then $A_i\,(= dU(a_i))$ will denote the generator of the one-parameter subgroup $t \mapsto U(\exp(-ta_i))$ of the representation. Let $(\gamma_t)_{t>0}$ be a family of **dilations** on $\mathfrak{g}$, i.e., a one-parameter group of automorphisms of the form

$$\gamma_t(a_i) = t^{w_i} a_i$$

for some basis $a_1, \ldots, a_d$ of $\mathfrak{g}$ and some positive numbers $w_1, \ldots, w_d$, which we call **weights.** We always assume that the smallest weight is at least one. Let $|||\cdot|||$ be a **homogeneous norm** on $\mathfrak{g}^*$, i.e., a norm such that $|||\gamma_t^*(l)||| = t\,|||l|||$ for all $l \in \mathfrak{g}^*$ and $t > 0$. A homogeneous norm can be constructed as follows. Let $\|\cdot\|$ be the dual norm on $\mathfrak{g}^*$ of a Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$. Define $|||\cdot|||\colon \mathfrak{g}^* \to \mathbf{R}$ by

$$|||l||| = \inf\{\lambda > 0 : \|\gamma_{1/\lambda}^* l\| \leq 1\} \quad .$$

One readily verifies that $|||\cdot|||$ is a homogeneous norm.

Next we introduce a multi-index notation. If $n \in \mathbf{N}_0$ let

$$J_n(d) = \bigoplus_{k=0}^{n} \{1, \ldots, d\}^k$$

and set

$$J(d) = \bigcup_{n=0}^{\infty} J_n(d) \quad .$$

Then if $\alpha = (i_1, \ldots, i_n) \in J(d)$ we denote the **Euclidean length** n of α by $|\alpha|$ and the **weighted length** by

$$\|\alpha\| = \sum_{k=1}^{n} w_{i_k} \quad .$$

If $n \in \mathbf{N}$ we define $\mathcal{X}_n = \mathcal{X}_n(U) = \bigcap_{\alpha \in J_n(d)} D(A^\alpha)$ and

$$\|x\|_n = \max_{\substack{\alpha \in J(d) \\ |\alpha| \leq n}} \|A^\alpha x\| \quad ,$$

where $A^\alpha = A_{i_1} \ldots A_{i_n}$ if $\alpha = (i_1, \ldots, i_n)$. Similarly we define the weighted C^n-spaces

$$\mathcal{X}'_n = \mathcal{X}'_n(U) = \bigcap_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq n}} D(A^\alpha)$$

for all $n \in \mathbf{R}$ with $n > 0$. Now, however, it can happen for a given n that there are no multi-indices α such that $\|\alpha\| = n$. Therefore the corresponding norms and seminorms are given by

$$\|x\|'_n = \|x\|'_{U,n} = \begin{cases} \max\limits_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq n}} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d) \text{ with } \|\alpha\| = n \quad , \\ 0 & \text{otherwise} \quad , \end{cases}$$

$$N'_n(x) = \begin{cases} \max\limits_{\substack{\alpha \in J(d) \\ \|\alpha\| = n}} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d) \text{ with } \|\alpha\| = n \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Note that if $b_1, \ldots, b_d$ is another basis for $\mathfrak{g}$ which satisfies $\gamma_i(b_i) = t^{v_i} b_i$ then the weighted C^n-space with respect to the basis $b_1, \ldots, b_d$ equals the space $\mathcal{X}'_n$, and, if there exists an $\alpha \in J(d)$ with $\|\alpha\| = n$ the norms are also equivalent. Moreover, let $\mathcal{X}_\infty = \mathcal{X}_\infty(U) = \bigcap_{n=1}^{\infty} \mathcal{X}_n$. It follows by a line by line extension of Lemma 2.4 of [ElR1] that the Gårding space, and in particular the space $\mathcal{X}_\infty$, is dense in $\mathcal{X}'_n$ for all $n > 0$. The density is with respect to the weak, or weak*, topology corresponding to the continuity property of the representation. Further we let L denote the left regular representation on $L_2(G)$.

Let $m \in \langle 0, \infty \rangle$ and let $C \colon J(d) \to \mathbf{C}$ be such that $C(\alpha) = 0$ if $\|\alpha\| > m$ and there exists at least one $\alpha \in J(d)$ with $\|\alpha\| = m$ and $C(\alpha) \neq 0$. We call C a **form** of order m. We write $c_\alpha = C(\alpha)$. The **principal part** P of C is the form

$$P(\alpha) = \begin{cases} C(\alpha) & \text{if } \|\alpha\| = m \quad , \\ 0 & \text{if } \|\alpha\| < m \quad . \end{cases}$$

We say that C is **homogeneous** if $C = P$. The **formal adjoint** $C^\dagger$ of C is the function $C^\dagger: J(d) \to \mathbf{C}$ defined by

$$C^\dagger(\alpha) = (-1)^{|\alpha|}\,\overline{C(\alpha_*)} \quad ,$$

where $\alpha_* = (i_n, \ldots, i_1)$ if $\alpha = (i_1, \ldots, i_n)$. We consider the operators

$$dU(C) = \sum_{\alpha \in J(d)} c_\alpha A^\alpha$$

with domain $D(dU(C)) = \mathcal{X}''_m$.

If P is the principal part of a form C we call P a **Rockland form** if the operator $dU(P)$ is injective on the space $\mathcal{X}'_\infty(U)$ for every nontrivial irreducible unitary representation U of G. It follows then from the Helffer–Nourrigat theorem [HeN1] that $dL(P)|_{C_c^\infty(G)}$ is a hypoelliptic operator. In fact the Helffer–Nourrigat theorem is formulated for graded groups. But it follows from Propositions 1.3 and 1.4 of [Mil] that the existence of a Rockland form ensures that the order m of P is an integer multiple of the smallest weight and all weights are rational multiples of this smallest weight. Therefore G is a graded group if one rescales the original weights by a large enough constant. (Actually there is a small gap in the proof of Proposition 1.3 in [Mil] where Miller applies his Lemma 1.2. For the operators that we consider we prove a stronger theorem in the spirit of Proposition 1.3 of [Mil]. This proof requires a lemma, Lemma 2.2, which also fills the gap in [Mil].)

A Rockland form P is called a **positive Rockland form** if $dL(P)$ is symmetric and $(\varphi, dL(P)\varphi) \geq 0$ for all φ in the Schwartz space on G (see [FoS], page 129). Throughout this paper we assume that C is a form of order m and that the principal part P of C is a positive Rockland form. We call $dL(P)$ a **positive Rockland operator**.

We study operators $dU(C)$ where U is a irreducible unitary representation. The irreducible unitary representations of a nilpotent Lie group are described by the Kirillov theory (see [Kir], [CoG], [Puk]). There is a one-to-one correspondence between the orbits in $\mathfrak{g}^*$ under the coadjoint action and the unitary dual of G. For an irreducible unitary representation U we denote by $\mathcal{O}_U$ the corresponding orbit in $\mathfrak{g}^*$ and let μ_U be the canonical invariant measure on $\mathcal{O}_U$ (see [CoG] Section 4.3).

At this point we can state a theorem which indicates the nature of our results.

Theorem 1.1 *Let $(\mathcal{H}, G, U)$ be an irreducible unitary representation of G and C a form of order m whose principal part P is a positive Rockland form. If there exists an $\omega > 0$ such that $dU(C) \geq \omega I$ then there is a $c > 0$ such that*

$$c^{-1}\int_{\mathcal{O}_U} d\mu_U(l)\, e^{-c\beta|||l|||} \leq \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) \leq c\int_{\mathcal{O}_U} d\mu_U(l)\, e^{-c^{-1}\beta|||l|||}$$

for all $\beta > 0$. Moreover, these estimates are valid uniformly for all irreducible unitary representations whenever C is a positive Rockland form.

Note that the condition $dU(C) \geq \omega I$ automatically implies that $dU(C)$ is a positive operator, and hence a self-adjoint operator. The theorem automatically applies to positive Rockland forms because the estimate $dU(C) \geq \omega I$ is a direct consequence of the injectivity hypothesis. The ensuing uniformity over the irreducible representations will be a consequence of the proof. It relies upon a scaling argument. The estimates can be rephrased in terms of Euclidean integrals and symbols of differential operators by using more details of representation theory. These estimates, which will be derived in Section 4, are the direct analogue of the classical phase space estimates for quantum-mechanical partition functions.

The bounds for the partition function given by the theorem can be evaluated in greater detail in particular cases. As an illustration we consider spectral properties of the anharmonic oscillator in Section 5. We establish that the eigenvalues of the operator $P^{2j} + Q^{2k}$ satisfy the bounds $c_0^{-1} n^{2jk/(j+k)} \leq \lambda_n \leq c_0 n^{2jk/(j+k)}$ and the corresponding orthonormal eigenfunctions φ_n can be extended to entire functions on the complex plane satisfying growth bounds

$$|\varphi_n(x+iy)| \leq c^n e^{-a|x|^{(j+k)/j} + b|y|^{(j+k)/j}}$$

for some $a, b, c > 0$, independent of n.

2 Positive Rockland operators

In this section we prove some additional regularity theorems for operators $H = dU(C)$ associated with a (not necessarily unitary) representation $(\mathcal{X}, G, U)$ and a form C whose principal part is a positive Rockland form of order m. In particular we prove that H satisfies a Gårding inequality if U is a unitary representation. Next recall that Theorem 3.4 of [AER] establishes that the closure $\overline{H}$ of H generates a continuous semigroup S which is holomorphic in the right half-plane and has a representation independent kernel K which depends only on the form C.

Proposition 2.1 *Let $(\mathcal{X}, G, U)$ be a (general) continuous representation of G, C a form of order m whose principal part is a positive Rockland form, $H = dU(C)$ and S the semigroup generated by $\overline{H}$.*

I. *If $n \in \mathbf{N}$ and $1 \leq k < mn$ then $D(\overline{H}^n) \subseteq \mathcal{X}_k'$ and there exists $c > 0$ such that*

$$\|x\|_k' \leq \varepsilon^{mn-k} \|\overline{H}^n x\| + c\,\varepsilon^{-k} \|x\|$$

for all $x \in D(\overline{H}^n)$ and $\varepsilon \in \langle 0, 1]$. In particular

$$\mathcal{X}_\infty = \bigcap_{n=1}^{\infty} D(\overline{H}^n)$$

and S maps into the smooth elements, i.e., $S_t\mathcal{X} \subseteq \mathcal{X}_\infty$ for all $t > 0$.

II. *If $k \in \mathbf{N}$ then there exists $c_k > 0$ such that*

$$\|S_t x\|_k' \leq c_k t^{-k/m} \|x\|$$

for all $t \in \langle 0, 1]$ and $x \in \mathcal{X}$.

III. *$\mathcal{X}_\infty$ is a core for $\overline{H}$.*

Proof Let $M, \rho \geq 0$ be such that $\|U(g)\| \leq M e^{\rho |g|'}$ for all $g \in G$, where $|\cdot|'$ is a homogeneous modulus on G. It follows as in Appendix A of [ElR2] that the resolvent kernel $R_\lambda^{(n)}$ defined by

$$R_\lambda^{(n)}(g) = (n-1)!^{-1} \int_0^\infty dt\, e^{-\lambda t} t^{n-1} K_t(g)$$

belongs to $L'^{\rho'}_{1;nm-1}(G)$ and $\|R^{(n)}_\lambda\|^{\rho'}_{1;nm-1} \leq c\lambda^{-(nm-k)/m}$ for all sufficiently large λ. Here K_t is the kernel of S_t (see [AER]) and $L'^{\rho'}_{1;nm-1}(G)$ is the space of weighted C^{nm-1}-vectors with respect to the left regular representation of G in $L_1(G;\, e^{\rho|g|'}dg)$ with norm $\|\cdot\|^{\rho'}_{1;nm-1}$. So $(\lambda I + \overline{H})^{-n} = U(R^{(n)}_\lambda)$ maps $\mathcal{X}$ into $\mathcal{X}'_k$ and

$$\|(\lambda I + \overline{H})^{-n}x\|'_k \leq \|R^{(n)}_\lambda\|^{\rho'}_{1;nm-1}\|x\| \leq c\lambda^{-(nm-k)/m}\|x\|$$

for all $x \in \mathcal{X}$. The proposition now follows as at the end of the proof of Theorem 2.6 of [ElR3]. Note that the constant c depends on the representation U only through the values of M and ρ. □

The next lemma is slightly stronger than Lemma 1.2 of Miller [Mil] and should be used in Proposition 1.3 of [Mil].

Lemma 2.2 *Let $\mathfrak{g}$ be a homogeneous Lie algebra with dilations $(\gamma_t)_{t>0}$. Then there exist a basis $b_1, \ldots, b_d$ of $\mathfrak{g}$, $v_1, \ldots, v_d \geq 1$ and $d' \in \{1, \ldots, d\}$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \operatorname{span}\{b_{d'+1}, \ldots, b_d\}$, and $\gamma_t(b_i) = t^{v_i} b_i$ for all $i \in \{1, \ldots, d\}$ and all $t > 0$. Moreover, $b_1, \ldots, b_{d'}$ is an algebraic basis for $\mathfrak{g}$.*

Proof There exist $1 \leq u_1 < \ldots < u_n$ and non-trivial subspaces $\mathfrak{g}_{u_1}, \ldots, \mathfrak{g}_{u_n}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_{u_i}$ and $\gamma_t(a) = t^{u_i} a$ for all $i \in \{1, \ldots, n\}$, $t > 0$ and $a \in \mathfrak{g}_{u_i}$. Define $\mathfrak{g}_u = \{0\}$ if $u \notin \{u_1, \ldots, u_n\}$. Then $[\mathfrak{g}_u, \mathfrak{g}_v] \subseteq \mathfrak{g}_{u+v}$ for all $u, v \geq 1$. For all $i \in \{1, \ldots, n\}$ let $b_{i1}, \ldots, b_{id'_i}$ be a basis for

$$\mathfrak{g}_{u_i} \cap \Big(\operatorname{span} \bigcup_{u+v=v_i} [\mathfrak{g}_u, \mathfrak{g}_v]\Big)$$

and let $b_{i\,d'_i+1}, \ldots, b_{id_i} \in \mathfrak{g}_{u_i}$ be such that $b_{i1}, \ldots, b_{id'_i}, b_{i\,d'_i+1}, \ldots, b_{id_i}$ is a basis for $\mathfrak{g}_{u_i}$. Then obviously $[\mathfrak{g}, \mathfrak{g}] \subseteq \operatorname{span}\{b_{ij} : i \in \{1, \ldots, n\},\ j \in \{d'_i + 1, \ldots, d_i\}\}$ since $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_{u_i}$. Let $\mathfrak{h}$ be the Lie algebra generated by $\{b_{ij} : i \in \{1, \ldots, n\},\ j \in \{1, \ldots, d'_i\}\}$. Then it follows by induction on N that $\oplus_{i=1}^N \mathfrak{g}_{u_i} \subseteq \mathfrak{h}$ and hence $\mathfrak{h} = \mathfrak{g}$ and $\{b_{ij} : i \in \{1, \ldots, n\},\ j \in \{1, \ldots, d'_i\}\}$ is an algebraic basis for $\mathfrak{g}$. Now a basis with the required properties is given by the combination $b_{11}, \ldots, b_{1d'_1}, \ldots, b_{n1}, \ldots, b_{nd'_n}, b_{1\,d'_1+1}, \ldots, b_{1d_1}, \ldots, b_{n\,d'_n+1}, \ldots, b_{nd_n}$. □

We will call a basis $b_1, \ldots, b_{d'}, \ldots, b_d$ an **adapted basis** for the homogeneous Lie algebra and $v_1, \ldots, v_d$ the corresponding weights if it satisfies the conclusion of Lemma 2.2. Note that the v_i are a permutation of the w_i.

Example 2.3 Let $\mathfrak{g}$ be the 4-dimensional homogeneous Lie algebra defined by

$$[a_1, a_2] = a_3 + a_4 \quad , \quad \gamma_t(a_1) = ta_1 \quad , \quad \gamma_t(a_2) = ta_2 \quad , \quad \gamma_t(a_3) = t^2 a_3 \quad , \quad \gamma_t(a_4) = t^2 a_4 \quad .$$

Then one can take $b_1 = a_1$, $b_2 = a_2$, $b_3 = a_3$ and $b_4 = a_3 + a_4$.

Lemma 2.4 *Let C be a form of order m whose principal part is a positive Rockland form and let $b_1, \ldots, b_{d'}, \ldots, b_d$ be an adapted basis with weights $v_1, \ldots, v_d$. Then m/v_i is even for all $i \in \{1, \ldots, d'\}$. In particular there exist $\alpha \in J(d)$ such that $\|\alpha\| = m/2$.*

Proof We may assume that C is homogeneous of order m. Fix $j \in \{1, \ldots, d'\}$. Define $U: G \to \mathcal{L}(L_2(\mathbf{R}))$ by

$$\Big(U(\exp(\sum_{n=1}^{d} \xi_n b_n))f\Big)(t) = f(t + \xi_j) \quad .$$

Then it follows from the inclusion $[\mathfrak{g}, \mathfrak{g}] \subseteq \operatorname{span}\{b_{d'+1}, \ldots, b_d\}$ and the Campbell–Baker–Hausdorff formula that U is a unitary representation. Then $B_j = D$, the differentiation operator, and $B_k = 0$ if $k \neq j$. Let $H = dU(C)$ and

$$c = \begin{cases} C(\alpha) & \text{if } m/v_j \in \mathbf{N} \text{ and } \alpha = (j, \ldots, j),\ \|\alpha\| = m, \\ 0 & \text{if } m/v_j \notin \mathbf{N}. \end{cases}$$

Then $H = cD^{m/v_j}$. Since $D(H) = L'_{2;m}(\mathbf{R})$ by Theorem 3.6 of [AER] we have $c \neq 0$, so $m/v_j \in \mathbf{N}$. Moreover, H is self-adjoint since it is symmetric and the generator of a holomorphic semigroup by Theorem 3.4 of [AER]. But as generator of a continuous semigroup it has to be lower semibounded. So cD^{m/v_j} must be lower semibounded and this is only possible if m/v_j is even. □

It is now fairly standard to prove the Gårding inequality. An important tool is that an operator $dU(C)$ is self-adjoint if the representation U is unitary and $C = C^\dagger$.

Theorem 2.5 *Let C be a form of order m whose principal part is a positive Rockland form. Then there exist $p > 0$ and $q \in \mathbf{R}$ such that for each unitary representation U*

$$\operatorname{Re}(dU(C)x, x) \geq p(\|x\|'_{m/2})^2 - q\,\|x\|^2$$

for all $x \in D(dU(C))$. If the form C is homogeneous then there exists $p > 0$, where the value is independent of U, such that

$$\operatorname{Re}(dU(C)x, x) \geq pN'_{m/2}(x)^2$$

for all $x \in D(dU(C))$.

Proof We may assume that the basis $a_1, \ldots, a_{d'}, \ldots, a_d$ is an adapted basis. Let C_1 be the form such that

$$dU(C_1) = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq m/2}} (-1)^{|\alpha|} A^{(\alpha_*, \alpha)}$$

for any representation $(\mathcal{X}, G, U)$. Then the principal part P_1 of C_1 is a positive Rockland form. Indeed, if $(\mathcal{X}, G, U)$ is a nontrivial irreducible unitary representation, $x \in \mathcal{X}_\infty(U)$ and $dU(P_1)x = 0$ then $(dU(P_1)x, x) = 0$ and therefore $A^\alpha x = 0$ for all α with $\|\alpha\| = m/2$. Hence by Lemma 2.4 one deduces that $A_i x = 0$ for all $i \in \{1, \ldots, d'\}$. Since $a_1, \ldots, a_{d'}$ is an algebraic basis this implies that $x = 0$. So

$$\|dU(C_1)^{1/2}x\|^2 = (dU(C_1)x, x) = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq m/2}} \|A^\alpha x\|^2$$

for all $x \in D(dU(C_1))$. Since $\mathcal{X}_\infty$ is a core for $\mathcal{X}'_{m/2}$ and $dU(C_1)$, by Proposition 2.1.III, it is also a core for $dU(C_1)^{1/2}$. It then follows that $D(dU(C_1)^{1/2}) = \mathcal{X}'_{m/2}$ with equivalent norms.

We may as well assume that the C is symmetric, i.e., $C = C^\dagger$. Then $dU(C)$ is self-adjoint. Since it is the generator of a semigroup it has to be lower semibounded. Let $\lambda > 0$ be such that $dU(C) + \lambda I \geq 0$. Then both $dU(C) + \lambda I$ and $dU(C_1)$ are positive self-adjoint operators with the same domain (see [AER] Theorem 3.6), so by Kato [Kat] one deduces that $D((dU(C)+\lambda I)^{1/2}) = D(dU(C_1)^{1/2})$, with equivalent norms. Thus there exist $p, q > 0$ such that

$$\|x\|'_{m/2} \leq p\,\|dU(C)^{1/2}x\| + q\,\|x\|$$

for all $x \in \mathcal{X}'_{m/2}$. Then

$$2^{-1}(\|x\|'_{m/2})^2 - q^2\|x\|^2 \leq p^2\|dU(C)^{1/2}x\|^2 = p^2(dU(C)x, x)$$

for all $x \in D(dU(C))$.

The independence of p and q from the representation U follows because the kernels of the semigroups generated by $dU(C)$ and $dU(C_1)$ are independent of U and all constants involved can be expressed in terms of these kernels. Moreover, the Kato theorem involves only a global constant.

If C is homogeneous then one can scale the lower order terms away (see [ElR3] Corollary 3.4). □

Corollary 2.6 *Suppose $(\mathcal{X}, G, U)$ is a unitary representation, C is a form whose principal part is a positive Rockland form of order m. Then*

I. *For all $n \in \mathbf{N}$ and all large $\lambda > 0$ one has*

$$D((dU(C) + \lambda I)^{n/2}) = \mathcal{X}'_{nm/2}$$

with equivalent norms, with factors independent of the representation.

II. *If C is homogeneous then $dU(C)$ is a positive self-adjoint operator. Moreover, for all $n \in \mathbf{N}$ the seminorms $x \mapsto \|dU(C)^{n/2}x\|$ and $N'_{nm/2}$ are equivalent, with factors independent of the representation.*

III. *If $n \in \mathbf{N}$ and $b_1, \ldots, b_{d'}, \ldots, b_d$ is an adapted basis for $\mathfrak{g}$ with weights $v_1, \ldots, v_d$ then*

$$\mathcal{X}'_{nm/2} = \bigcap_{i=1}^{d'} D(B_i^{nm/(2v_i)}) \quad ,$$

where $B_i = dU(b_i)$.

Proof Statement I has been proved for $n = 1$ in the proof of Theorem 2.5. The general case can be dealt with similarly. The second statement follows again by scaling from the first. The proof of the third statement is analogous to the proof of Theorem 5.8.IV in [ElR4]. □

3 Spectral Estimates

In this section we derive some preliminary estimates on the eigenvalue distributions of certain self-adjoint operators. Let H and H_0 be a self-adjoint operators satisfying $H \geq H_0$

in the sense of quadratic forms. If $\exp(-\beta H_0)$ is of trace-class for some $\beta > 0$ it follows that $\exp(-\beta H)$ is also of trace-class and

$$\mathrm{Tr}(e^{-\beta H}) \leq \mathrm{Tr}(e^{-\beta H_0}) \quad .$$

Moreover, if $N(\lambda)$ denotes the number of eigenvalues of H which are less than or equal to λ, counted according to multiplicities, and if $N_0(\lambda)$ is the corresponding measure for H_0 then

$$N(\lambda) \leq N_0(\lambda)$$

for all λ. Both these conclusions are direct consequences of the minimax theorem. Thus the eigenvalue density N of H and the trace of the semigroup generated by H can both be estimated from above by the introduction of a comparator H_0. Similarly they can be estimated from below with the aid of a comparator H_1 satisfying $H \leq H_1$. These various estimates are all closely related and we next give some general results of this nature which will be useful in the sequel.

First we describe two comparison results of Levy-Bruhl and Nourrigat [LBN] which allow the estimation of the eigenvalue density. These results are formulated in terms of a family of 'coherent states'. Thus we assume that $\mathcal{H}$ is a separable Hilbert space, that (X, ρ) is a σ-finite measure space and that there exists a measurable map $x \mapsto \psi_x$ from X into $\mathcal{H}$ satisfying the following properties:

(i) there is a $K > 0$ such that $\|\psi_x\| = K$ for all $x \in X$,

(ii) $\varphi = \int_X d\rho(x)\, (\varphi, \psi_x)\psi_x \quad ,$ for all $\varphi \in \mathcal{H}$, in the weak sense.

The $\{\psi_x : x \in X\}$ are the coherent states.

Proposition 3.1 *Let H be a self-adjoint operator on $\mathcal{H}$ with compact resolvent and D a core of H. Further let h_0 be a positive measurable function over X with the property that $\rho(\{x \in X : h_0(x) \leq \mu\}) < \infty$ for all $\mu > 0$. If*

$$\int_X d\rho(x)\, |(\varphi, \psi_x)|^2\, h_0(x) \leq (\varphi, H\varphi) + \lambda\, \|\varphi\|^2$$

for all $\varphi \in D$ and some $\lambda > 0$ then the eigenvalue density N of H satisfies

$$N(\lambda) \leq 2K^2 \rho(\{x \in X : h_0(x) \leq 4\lambda\}) \quad .$$

This statement is contained in Théorème 1.1 of [LBN] and we refer to this paper for the proof. It is based on the use of approximate spectral projections in the sense of Shubin [Shu]. There is a second complementary result which gives a lower bound on the eigenvalue density.

Proposition 3.2 *Let H be a self-adjoint operator on $\mathcal{H}$ with compact resolvent, D a core of H and h_1 a positive measurable function over X such that $\rho(\{x \in X : h_1(x) \leq \mu\}) < \infty$ for all $\mu > 0$. Assume*

$$(\varphi, H\varphi) \leq \int_X d\rho(x)\, |(\varphi, \psi_x)|^2\, h_1(x)$$

for all $\varphi \in D$.

Then for all $r > 0$, $C > 0$ and $\alpha \geq 0$ there exists an $R > 0$ such that for all $\lambda > 0$ with

$$\int_X d\rho(x)\, |(\psi_x, \psi_y)|\, h(x,y)^{\alpha+1/2} \leq C(h_1(z)/\lambda)^{\alpha} \quad ,$$

where

$$h(x,y) = \max\Big(h_1(x), h_1(y)\Big) \Big/ \min\Big(h_1(x), h_1(y)\Big) \quad ,$$

one has

$$N(R\lambda) \geq 2^{-1} K^2 \rho(\{x \in X : h_1(x) \leq r\lambda\}) \quad .$$

Again this statement is contained in Théorème 1.1 of [LBN] and we refer to this paper for the proof.

Next we examine the relations between estimates on the eigenvalue density N of H and the trace of the semigroup generated by H. We do this in a general measure-theoretic setting. Specifically we compare the properties of two functions related to positive Borel measures μ, ν on $\mathbf{R}_+ = \langle 0, \infty \rangle$. These functions are defined by

$$Z_\mu(\beta) = \int_0^\infty d\mu(x)\, e^{-\beta x}$$

and

$$N_\mu(\lambda) = \int_{\langle 0,\lambda]} d\mu(x)$$

for all $\beta, \lambda > 0$, with similar definitions for Z_ν and N_ν. If the measures are purely atomic these functions are directly comparable with the traces and the eigenvalue densities discussed above. Note that

$$\beta \int_0^\infty d\lambda\, N_\mu(\lambda)\, e^{-\beta\lambda} = \beta \int_0^\infty d\mu(x) \int_x^\infty d\lambda\, e^{-\beta\lambda} = Z_\mu(\beta) \quad .$$

Thus Z_μ is the Abel transform of N_μ. This relationship allows one to relate ordering properties of the Z and N.

Lemma 3.3 *Let μ and ν be positive Borel measures on $\mathbf{R}_+$. If $N_\mu(\lambda) \leq N_\nu(\lambda)$ for all $\lambda > 0$ then $Z_\mu(\beta) \leq Z_\nu(\beta)$ for all $\beta > 0$.*

Proof This follows because the Z are the Abel transforms of the N and the measures are positive. □

This argument can also be adapted to give a version of the lemma which relates the small β behaviour of the Z to the large λ behaviour of the N.

Lemma 3.4 *Let μ and ν be non-zero positive Borel measures on $\mathbf{R}_+$. If $Z_\nu(\beta) < \infty$ for all $\beta > 0$, $c \in \mathbf{R}$ and*

$$\lim_{\lambda \to \infty} N_\mu(\lambda)/N_\nu(\lambda) = c$$

then

$$\lim_{\beta \to 0} Z_\mu(\beta)/Z_\nu(\beta) = c \quad .$$

Proof Let $\varepsilon > 0$ and choose $\delta > 0$ such that $(e^\delta - 1)(c+\varepsilon) \le \varepsilon$. There exists $\beta_0 > 0$ such that $N_\mu(\delta/\beta) \le (c+\varepsilon)N_\nu(\delta/\beta)$ for all $\beta \in \langle 0, \beta_0]$. Therefore

$$\begin{aligned}\beta \int_0^{\delta\beta^{-1}} dx\, N_\mu(x)\, e^{-\beta x} &\le N_\mu(\delta/\beta)\, \beta \int_0^{\delta\beta^{-1}} dx\, e^{-\beta x} \\ &\le (c+\varepsilon)(1-e^{-\delta})\, N_\nu(\delta/\beta) \\ &\le (c+\varepsilon)(1-e^{-\delta})\, N_\nu(\delta/\beta)\, e^\delta \beta \int_{\delta\beta^{-1}}^\infty dx\, e^{-\beta x} \\ &\le (e^\delta - 1)(c+\varepsilon)\, \beta \int_{\delta\beta^{-1}}^\infty dx\, N_\nu(x)\, e^{-\beta x} \le \varepsilon\, Z_\nu(\beta)\end{aligned}$$

for all $\beta \in \langle 0, \beta_0]$. Alternatively,

$$\begin{aligned}\beta \int_{\delta\beta^{-1}}^\infty dx\, N_\mu(x)\, e^{-\beta x} &= \int_\delta^\infty dx\, N_\mu(x/\beta)\, e^{-x} \\ &\le (c+\varepsilon) \int_\delta^\infty dx\, N_\nu(x/\beta)\, e^{-x} \\ &= (c+\varepsilon)\beta \int_{\delta\beta^{-1}}^\infty dx\, N_\nu(x)\, e^{-\beta x} \le (c+\varepsilon)\, Z_\nu(\beta) \quad .\end{aligned}$$

Therefore

$$Z_\mu(\beta) \le (c+2\varepsilon)\, Z_\nu(\beta) \tag{1}$$

for all $\beta \in \langle 0, \beta_0]$. Since $Z_\mu(\beta), Z_\nu(\beta) > 0$ for all $\beta > 0$ it follows that $\lim_{\beta\to 0} Z_\mu(\beta)/Z_\nu(\beta) = 0$ if $c = 0$.

If $c \ne 0$ then it follows from (1) that $Z_\mu(\beta) < \infty$ for all $\beta > 0$ and one can interchange μ and ν and deduce that there exists $\beta_1 > 0$ such that

$$Z_\nu(\beta) \le (c^{-1} + 2\varepsilon)\, Z_\mu(\beta)$$

for all $\beta \in \langle 0, \beta_1]$. So $\lim_{\beta\to 0} Z_\mu(\beta)/Z_\nu(\beta) = c$. □

In some situations Lemma 3.4 has a converse. If $d\nu(x) = x^{\alpha-1}dx$ then one can establish that $\lim_{\beta\to 0} Z_\mu(\beta)/Z_\nu(\beta) = c$ implies $\lim_{\lambda\to\infty} N_\mu(\lambda)/N_\nu(\lambda) = c$ (see, for example, [Sim1] pages 107–109). Another special case, with $d\nu(x) = \nu(x)\, dx$ and $\nu(x) \sim x^{-\alpha}(\log x)^{-1}$ as $x \to 0$, occurs in [Sim2].

4 Spectra of positive Rockland operators

Let G be a connected simply connected homogeneous Lie group and C a positive Rockland form of order m. If U is a irreducible unitary representation of G then the operator $dU(C)$ has a discrete spectrum. For $\lambda > 0$ we denote by $N(\lambda, U, C)$ the number of eigenvalues (counted with multiplicity) of the operator $dU(C)$ which are less than or equal to λ.

Since any two homogeneous norms on $\mathfrak{g}^*$ are equivalent we may work with a specific one. For $\lambda > 0$ let

$$N_0(\lambda, U) = \mu_U(\{l \in \mathcal{O}_U : |||l||| \le \lambda\}) \quad .$$

The next theorem extends results of [LMN2], [LBN].

Theorem 4.1 *If C is a positive Rockland form of order m then there exists $c > 0$ such that*

$$c^{-1}N_0(\lambda, U) \le N(\lambda^m, U, C) \le c\, N_0(\lambda, U)$$

uniformly for all $\lambda > 0$ and all irreducible unitary representations U of G.

Proof An irreducible unitary representation is represented in a one dimensional or an infinite dimensional Hilbert space. If the space is infinite dimensional then the representation is unitarily equivalent with a representation U in $L_2(\mathbf{R}^k)$ for some $k \in \mathbf{N}$ such that every infinitesimal generator $dU(a)$ has the following form: there exist polynomials $Y, X_1, \ldots, X_k \colon \mathbf{R}^k \times \mathfrak{g} \to \mathbf{R}$ such that

$$(dU(a)f)(x) = iY(x,a) + \sum_{j=1}^{k} X_j(x,a)\frac{\partial f}{\partial x_j}(x) \qquad (2)$$

for all $a \in \mathfrak{g}$, $f \in \mathcal{S}(\mathbf{R}^k) = D^\infty(U)$ and $x \in \mathbf{R}^k$ (see [CoG] page 125 and Corollary 4.1.2). If the Hilbert space is one dimensional one has to make the obvious changes and $k = 0$. So it suffices to prove the theorem for representations U with infinitesimal generators of the form (2). Therefore, from now on we only consider this type of representation in the proof. Moreover $k \in \mathbf{N}$ will always be such that U is represented in $L_2(\mathbf{R}^k)$.

Next for every $x \in \mathbf{R}^k$ and $\xi \in \mathbf{R}^k$ define the symbol $l^U_{x,\xi} \colon \mathfrak{g} \to \mathbf{R}$ of the partial differential operator $dU(a)$ by

$$l^U_{x,\xi}(a) = Y(x,a) + \sum_{j=1}^{k} X_j(x,a)\,\xi_j \quad .$$

Then

$$(x,\xi) \mapsto l^U_{x,\xi}$$

is a bijection from $\mathbf{R}^k \times \mathbf{R}^k$ onto $\mathcal{O}_U$. A proof can be given along the lines of proofs in [HeN2], e.g., the proof of Proposition VIII.5.1, or [Nou] Theorem 2.13. Moreover,

$$\int_{\mathbf{R}^k \times \mathbf{R}^k} dx\, d\xi\, f(l^U_{x,\xi}) = (2^k k!)^{-1} \int_{\mathcal{O}_U} d\mu_U(l)\, f(l)$$

for every positive measurable function f on $\mathcal{O}_U$. Hence

$$N_0(\lambda, U) = \tau(\{(x,\xi) \in \mathbf{R}^{2k} : |||l^U_{x,\xi}||| \le \lambda\})$$

for all $\lambda > 0$, where τ denotes $(2^k k!)^{-1}$ times the Lebesgue measure on $\mathbf{R}^{2k}$. Let $\|\cdot\|$ be the dual norm on $\mathfrak{g}^*$ of a Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$. For every representation U define $p(x,\xi,U) = \|l^U_{x,\xi}\|$ for all $x, \xi \in \mathbf{R}^k$. Next for all $\lambda > 0$ and all representations U (of the form (2)) define the representation U_λ by $U_\lambda(g) = U(\delta_\lambda(g))$ for all $g \in G$, where δ_λ is the dilation on G obtained via the exponential map. Then $p(x,\xi,U_\lambda) = \|\gamma^*_\lambda l^U_{x,\xi}\|$. Hence if $\mathcal{O}_U \neq \{0\}$ and $c, \lambda > 0$ then $|||l^U_{x,\xi}||| \le c\lambda$ if, and only if, $\|\gamma^*_{1/\lambda}(l^U_{x,\xi})\| \le c$ or, if, and only if, $p(x,\xi,U_{1/\lambda}) \le c$. Here we have chosen for the homogeneous norm on $\mathfrak{g}^*$ the norm defined in the introduction. In particular:

$$N_0(c\lambda, U) = \tau(\{(x,\xi) \in \mathbf{R}^{2k} : p(x,\xi,U_{1/\lambda}) \le c\})$$

for all $c, \lambda > 0$.

Let w be the largest weight. Then

$$\mathcal{X}'_n \subseteq \mathcal{X}_n \subseteq \mathcal{X}'_{nw}$$

for all $n \in \mathbf{N}_0$. Let $n \in \mathbf{N}$ be such that $w \leq nm$. By Corollary 2.6.I there exists $c_1 > 0$ such that

$$\|f\|'_{U,nm} \leq c_1 \|dU(C)^n f\|$$

uniformly for all unitary representations U and $f \in \mathcal{X}_\infty(U)$. We next need the following proposition of Lévy-Bruhl and Nourrigat ([LBN] Proposition 4.1).

Proposition 4.2 *There exists $c > 0$ such that for every unitary representation U of G in $L_2(\mathbf{R}^k)$ there exists a continuous function $(x, \xi) \mapsto \psi_{x,\xi,U}$ from $\mathbf{R}^k \times \mathbf{R}^k$ into $\mathcal{S}(\mathbf{R}^k)$ such that $\|\psi_{x,\xi,U}\|_2 = (2\pi)^{-k/2}$ for all $x, \xi \in \mathbf{R}^k$,*

$$f = \int_{\mathbf{R}^{2k}} dx\, d\xi\, (\psi_{x,\xi,U}, f)\, \psi_{x,\xi,U}$$

for all $f \in L_2(\mathbf{R}^k)$, in the weak sense, and

$$\int_{\mathbf{R}^{2k}} dx\, d\xi\, p(x, \xi, U)^2 |(\psi_{x,\xi,U}, f)|^2 \leq c\, \|f\|^2_{U,1}$$

uniformly for all $f \in \mathcal{X}_\infty(U)$.

Now with $c > 0$ the constant in Proposition 4.2 one has

$$\begin{aligned}\int_{\mathbf{R}^{2k}} dx\, d\xi\, p(x, \xi, U)^2 |(\psi_{x,\xi,U}, f)|^2 &\leq c\|f\|^2_{U,1} \\ &\leq c(\|f\|'_{U,w})^2 \leq c(\|f\|'_{U,nm})^2 \leq cc_1^2 \|dU(C)^n f\|^2\end{aligned}$$

for all U and $f \in \mathcal{S}(\mathbf{R}^k)$. Next let S be the multiplication operator with the function $(x, \xi) \mapsto p(x, \xi, U)^2$ on $L_2(\mathbf{R}^{2k})$. Then the map $T \colon D(dU(C)^n) \to D(S)$ defined by $(Tf)(x, \xi) = (\psi_{x,\xi,U}, f)$ is continuous with norm bounded by cc_1^2. Hence by interpolation ([Kat]) it follows that T is bounded from $D(dU(C)^{1/2})$ into $D(S^{1/(2n)})$ and the norm is bounded by a constant which depends only on cc_1^2. Then with Theorem 2.5 one deduces that there exists a constant $c_2 > 0$ such that

$$\int_{\mathbf{R}^{2k}} dx\, d\xi\, p(x, \xi, U)^{1/n} |(\psi_{x,\xi,U}, f)|^2 \leq c_2\Big((f, dU(C) f) + \|f\|^2\Big)$$

uniformly for all U and $f \in \mathcal{S}(\mathbf{R}^k)$. In particular, for all $\lambda > 0$ and U one obtains the inequalities

$$\begin{aligned}\int_{\mathbf{R}^{2k}} dx\, d\xi\, p(x, \xi, U_{1/\lambda})^{1/n} |(\psi_{x,\xi,U_{1/\lambda}}, f)|^2 &\leq c_2\Big((f, dU_{1/\lambda}(C) f) + \|f\|^2\Big) \\ &= c_2\Big(\lambda^{-m}(f, dU(C) f) + \|f\|^2\Big)\end{aligned}$$

and

$$\int_{\mathbf{R}^{2k}} dx\, d\xi\, c_2^{-1} \lambda^m p(x, \xi, U_{1/\lambda})^{1/n} |(\psi_{x,\xi,U_{1/\lambda}}, f)|^2 \leq \Big((f, dU(C) f) + \lambda^m \|f\|^2\Big) \;\; .$$

Then by Proposition 3.1 one obtains

$$\begin{aligned} N(\lambda^m, U, C) &\leq 2(2\pi)^{-k}\tau(\{(x,\xi) \in \mathbf{R}^{2k} : c_2^{-1}\lambda^m p(x,\xi,U_{1/\lambda})^{1/n} \leq 4\lambda^m\}) \\ &= 2(2\pi)^{-k}\tau(\{(x,\xi) \in \mathbf{R}^{2k} : p(x,\xi,U_{1/\lambda}) \leq (4c_2)^n\}) \\ &= 2(2\pi)^{-k} N_0((4c_2)^n\lambda, U) \end{aligned}$$

for all $\lambda > 0$.

The next proposition gives an inequality in the opposite direction.

Proposition 4.3 *There exists $c > 0$ and, for all $y \in \mathbf{R}$, a $c_y > 0$ such that for every unitary representation U of G in $L_2(\mathbf{R}^k)$ there exists a continuous function $(x,\xi) \mapsto \psi_{x,\xi,U}$ from $\mathbf{R}^k \times \mathbf{R}^k$ into $\mathcal{S}(\mathbf{R}^k)$ with the following properties*

I. $\|\psi_{x,\xi,U}\|_2 = (2\pi)^{-k/2}$ *for all* $x, \xi \in \mathbf{R}^k$,

II. $f = \int_{\mathbf{R}^{2k}} dx\, d\xi\, (\psi_{x,\xi,U}, f)\, \psi_{x,\xi,U}$ *for all* $f \in L_2(\mathbf{R}^k)$, *in the weak sense,*

III. $|(f, dU(C)f)| \leq c\int_{\mathbf{R}^{2k}} dx\, d\xi\, (1 + p(x,\xi,U))^{m+k}|(\psi_{x,\xi,U}, f)|^2$ *for all* $f \in \mathcal{S}(\mathbf{R}^k)$,

IV. $\int_{\mathbf{R}^{2k}} dy\, d\eta\, \Big(h_U(x,\xi,y,\eta)\Big)^y |(\psi_{x,\xi,U}, \psi_{y,\eta,U})| \leq c_y\Big((1 + p(x,\xi,U)\Big)^k$ *uniformly for all* $x, \xi \in \mathbf{R}^k$, *where*

$$h_U(x,\xi,y,\eta) = \frac{1 + \max(p(x,\xi,U), p(y,\eta,U))}{1 + \min(p(x,\xi,U), p(y,\eta,U))} \quad .$$

Proof These properties follows from [LBN] Propositions 6.8 and 6.7. □

In particular, if $\lambda > 0$, U is a representation in $\mathbf{R}^k$ and $f \in \mathcal{S}(\mathbf{R}^k)$ then

$$\lambda^{-m}|(f, dU(C)f)| = |(f, dU_{1/\lambda}(C)f)| \leq c\int_{\mathbf{R}^{2k}} dx\, d\xi\, (1 + p(x,\xi,U_{1/\lambda}))^{m+k}|(\psi_{x,\xi,U_{1/\lambda}}, f)|^2$$

and hence

$$|(f, dU(C)f)| \leq \int_{\mathbf{R}^{2k}} dx\, d\xi\, c\lambda^m(1 + p(x,\xi,U_{1/\lambda}))^{m+k}|(\psi_{x,\xi,U_{1/\lambda}}, f)|^2 \quad .$$

It then follows from Proposition 3.2, applied with $y = 2^{-1} + k(m+k)^{-1}$ that there exists $c_2 > 0$ such that

$$\int_{\mathbf{R}^{2k}} dy\, d\eta \left(\frac{1 + \max(p(x,\xi,U_{1/\lambda}), p(y,\eta,U_{1/\lambda}))}{1 + \min(p(x,\xi,U_{1/\lambda}), p(y,\eta,U_{1/\lambda}))}\right)^{\frac{k}{m+k}+\frac{1}{2}} |(\psi_{x,\xi,U_{1/\lambda}}, \psi_{y,\eta,U_{1/\lambda}})|$$

$$\leq c_2\left(\frac{c\lambda^m(1 + p(x,\xi,U_{1/\lambda}))^{m+k}}{\lambda^m}\right)^{\frac{k}{m+k}}$$

uniformly for all U, all $x, \xi \in \mathbf{R}^k$ and $\lambda > 0$. Therefore, by Proposition 3.2 one deduces that there exists $c_3 > 0$ such that

$$\begin{aligned} N(c_3\lambda^m, U, C) &\geq 2^{-1}(2\pi)^{-k}\tau(\{(x,\xi) \in \mathbf{R}^{2k} : c\lambda^m(1 + p(x,\xi,U_{1/\lambda}))^{m+k} \leq c2^{m+k}\lambda^m\}) \\ &= 2^{-1}(2\pi)^{-k}\tau(\{(x,\xi) \in \mathbf{R}^{2k} : p(x,\xi,U_{1/\lambda}) \leq 1\}) \\ &= 2^{-1}(2\pi)^{-k}N_0(\lambda, U) \end{aligned}$$

for all $\lambda > 0$. This completes the proof of the theorem. □

Corollary 4.4 *Let $(\mathcal{H}, G, U)$ be an irreducible unitary representation of G and C a form of order m whose principal part P is a positive Rockland form. If there exists an $\omega > 0$ such that $dU(C) \geq \omega I$ then there is a $c > 0$ such that*

$$c^{-1}N_0(c^{-1}\lambda, U) \leq N(\lambda^m, U, C) \leq cN_0(c\lambda, U)$$

for all $\lambda > 0$.

Proof Since P is a Rockland form the operator $dU(P)$ is strictly positive and there exists an $\omega_0 > 0$ such that $\|x\|^2 \leq \omega_0(dU(P)x, x)$ for all $x \in \mathcal{X}_\infty$. Now $2P - C$ is a form whose principal part is a positive Rockland form. So by Theorem 2.5 there exists a $q > 0$ such that

$$(dU(2P - C)x, x) \geq -q\,\|x\|^2$$

for all $x \in \mathcal{X}_\infty$. Hence

$$(dU(C)x, x) \leq 2(dU(P)x, x) + q\,\|x\|^2 \leq (2 + q\omega_0)\,(dU(P)x, x)$$

for all $x \in \mathcal{X}_\infty$. By assumption one similarly has

$$(dU(P)x, x) \leq 2(dU(C)x, x) + q'\|x\|^2 \leq (2 + q'\omega^{-1})\,(dU(C)x, x)$$

for some $q' > 0$, uniformly for all $x \in \mathcal{X}_\infty$. Then by the minimax theorem one obtains

$$N((2 + q\omega_0)^{-1}\lambda, U, P) \leq N(\lambda, U, C) \leq N((2 + q'\omega^{-1})\lambda, U, P)$$

for all $\lambda > 0$. Now the corollary follows from Theorem 4.1. □

These estimates for the eigenvalue density can be converted into estimates on the partition function of $H = dU(C)$ by the observation of Lemma 3.3. First for $\beta > 0$, $m > 1$ and U an irreducible unitary representation of G define

$$Z_0(\beta, m, U) = \int_{\mathcal{O}_U} d\mu_U(l)\, e^{-\beta |||l|||^m} = 2^k k! \int_{\mathbf{R}^k \times \mathbf{R}^k} dx d\xi\, e^{-\beta |||l^U_{x,\xi}|||} \quad .$$

Corollary 4.5 *Let $(\mathcal{H}, G, U)$ be an irreducible unitary representation of G and C a form of order m whose principal part P is a positive Rockland form. If there exists an $\omega > 0$ such that $H = dU(C) \geq \omega I$ then there is a $c > 0$ such that*

$$c^{-1}Z_0(c\beta, m, U) \leq \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) \leq cZ_0(c^{-1}\beta, m, U)$$

uniformly for all $\beta > 0$. Moreover, these estimates are valid, uniformly for all irreducible unitary representations whenever C is a positive Rockland form.

Proof We shall prove that

$$Z_0(\beta, m, U) = \beta \int_0^\infty d\lambda\, N_0(\lambda^{1/m}, U)\, e^{-\beta\lambda}$$

for all $\beta > 0$ and irreducible unitary representations U. The equality follows from Fubini's theorem since

$$\begin{aligned}\beta \int_0^\infty d\lambda\, N_0(\lambda^{1/m}, U)\, e^{-\beta\lambda} &= \beta \int_0^\infty d\lambda \int_{\mathcal{O}_U} d\mu_U(l)\, 1_{\{l \in \mathcal{O}_U : |||l||| \leq \lambda^{1/m}\}}(l)\, e^{-\beta\lambda} \\ &= \beta \int_{\mathcal{O}_U} d\mu_U(l) \int_{|||l|||^m}^\infty d\lambda\, e^{-\beta\lambda} \\ &= \int_{\mathcal{O}_U} d\mu_U(l)\, e^{-\beta |||l|||^m} = Z_0(\beta, m, U) \quad .\end{aligned}$$

Now the corollary easily follows from Corollary 4.4 and the fact that $\beta \mapsto \mathrm{Tr}_{\mathcal{H}}(\exp(-\beta H))$ is the Abel transform of N. See Lemma 3.3. □

The estimates of Corollary 4.5 are potentially useful for calculating the behaviour of the partition function for small β. In the next section this will be achieved for the simplest example, the anharmonic oscillator on the Heisenberg group. The general case appears more intractable and requires understanding the behaviour of $\lambda \mapsto N_0(\lambda, U)$ for large λ.

The large β behaviour of the partition function is much easier to establish. If λ_1 is the smallest eigenvalue of H then $\lim_{\beta\to\infty} e^{\beta\lambda_1} \mathrm{Tr}_{\mathcal{H}}(\exp(-\beta H)) = n_1$, where n_1 is the multiplicity of the eigenvalue λ_1. It is an interesting question whether $n_1 = 1$ for homogeneous C. For non-homogeneous C the spectrum of H need not be simple and in general $n_1 > 1$. An example is the operator

$$(P^2 + Q^2)^2 - 4(P^2 + Q^2) + 3I \quad ,$$

where P and Q are the usual self-adjoint operators in $L_2(\mathbf{R})$, see Section 5.

The simplicity of the lowest eigenvalue is often established by positivity arguments based on some variation of the Perron–Frobenius theorem. But this type of reasoning is in general not applicable in the present setting. Although the semigroup $t \mapsto \exp(-tH)$ has a kernel K it is usually not positive. It follows from [Rob], Chapter III, Section 5, that the semigroup kernel K^G on the group corresponding to the form C is positive if and only if C is second-order, in the unweighted sense, with real coefficients and with the principal coefficients satisfying an ellipticity condition. But positivity of K^G does not imply positivity of the kernel K on $\mathbf{R}^k \times \mathbf{R}^k$ corresponding to the semigroup generated by $H = dU(C)$ even for homogeneous, real, second-order C. An example is given by the operator $(P + Q)^2 + Q^2$, which is a sublaplacian for the algebraic basis $a_1 + a_2, a_2$ of the Heisenberg algebra, but as a second-order operator on $L_2(\mathbf{R})$ it is not even real.

5 The anharmonic oscillator

As an application we consider the general anharmonic oscillator.

Let G be the simply connected Heisenberg group, U the standard irreducible unitary representation of G in $L_2(\mathbf{R})$ and a_1, a_2, a_3 a basis in the Lie algebra $\mathfrak{g}$ of G such that $[a_1, a_2] = a_3$, $A_1 = -iP$, $A_2 = iQ$ and $A_3 = iI$, where P and Q are the self-adjoint operators in $L_2(\mathbf{R})$ such that $(Pf)(x) = if'(x)$ and $(Qf)(x) = xf(x)$ for all $f \in C_c^\infty(\mathbf{R})$ and $x \in \mathbf{R}$. Then

$$l^U_{x,\xi}(a_1) = \xi \quad , \quad l^U_{x,\xi}(a_2) = x \quad , \quad l^U_{x,\xi}(a_3) = 1 \quad .$$

Fix $j, k \in \mathbf{N}$. There are dilations $(\gamma_t)_{t>0}$ on $\mathfrak{g}$ such that $\gamma_t(a_1) = t^k a_1$, $\gamma_t(a_2) = t^j a_2$ and $\gamma_t(a_3) = t^{j+k} a_3$ for all $t > 0$. Let C be the positive Rockland form of order $m = 2jk$ such that

$$H = dU(C) = (-1)^j A_1^{2j} + (-1)^k A_2^{2k} = P^{2j} + Q^{2k} \quad .$$

Define the modulus $|||\cdot|||_1 \colon \mathfrak{g}^* \to [0, \infty\rangle$ by

$$|||l|||_1 = \Big(l(a_1)^{2j} + l(a_2)^{2k} + l(a_3)^{2jk/(j+k)} \Big)^{1/(2jk)} \quad .$$

Then there exists a $c > 0$ such that

$$c^{-1} |||l|||_1 \leq |||l||| \leq c |||l|||_1$$

for all $l \in \mathfrak{g}^*$, so in Theorem 4.1 we may as well replace $|||\cdot|||$ by $|||\cdot|||_1$. Now $|||l^U_{x,\xi}|||_1^m = \xi^{2j} + x^{2k} + 1$. Then an elementary estimate shows that there exist $\Lambda > 0$ and $c > 0$ such that

$$c^{-1} \lambda^{1/\sigma} \leq \tau(\{(x, \xi) \in \mathbf{R}^2 : |||l^U_{x,\xi}|||_1 \leq \lambda\}) \leq c \lambda^{1/\sigma}$$

for all $\lambda \geq \Lambda$, where τ now denotes the Lebesgue measure on $\mathbf{R}$ and $\sigma = 2jk/(j + k)$. So if $N(\lambda)$ denotes the number of eigenvalues of H which are less than or equal to λ one deduces that there exist $\Lambda > 0$ and $c > 0$ such that

$$c^{-1} \lambda^{1/\sigma} \leq N(\lambda) \leq c \lambda^{1/\sigma}$$

for all $\lambda \geq \Lambda$ and then, by increasing the value of c, for all $\lambda \geq \lambda_1$ where λ_1 is the strictly positive lowest eigenvalue of H. Next let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of H, repeated according to multiplicity. Then for all small $\varepsilon > 0$ and $n \in \mathbf{N}$ with $\lambda_n > \Lambda$ one has

$$c^{-1} (\lambda_n - \varepsilon)^{1/\sigma} \leq N(\lambda_n - \varepsilon) \leq n \leq N(\lambda_n) \leq c \lambda_n^{1/\sigma} \quad ,$$

so

$$(c^{-1} n)^\sigma \leq \lambda_n \leq (c n)^\sigma$$

for all $\lambda_n \geq \Lambda$. By increasing c one concludes that

$$c^{-1} n^\sigma \leq \lambda_n \leq c n^\sigma$$

for all $n \in \mathbf{N}$. This proves a conjecture in [ElR4] page 40.

In [ElR4] we proved that the eigenfunctions φ_n corresponding to H with eigenvalue λ_n belong to the Gel'fand–Shilov space $S^{k/(j+k)}_{j/(j+k)}$, which consists of all infinitely differentiable functions φ on $\mathbf{R}$ for which there exist $a, b, c > 0$ (depending on φ) such that

$$|x^r \varphi^{(s)}(x)| \leq c\, a^r\, b^s\, r!^{j/(j+k)}\, s!^{k/(j+k)}$$

for all $r, s \in \mathbf{N}_0$ and $x \in \mathbf{R}$, or, equivalently, which consists of all functions φ on $\mathbf{R}$ which can be extended to entire functions (also denoted by φ) into the complex plane satisfying the growth bounds

$$|\varphi(x + iy)| \leq c e^{-a|x|^{(j+k)/j} + b|y|^{(j+k)/j}}$$

for all $x, y \in \mathbf{R}$, for some constants $a, b, c > 0$, depending on φ, or, equivalently,

$$|\varphi(z)| \leq c e^{b|z|^{(j+k)/j}} \quad , \quad |\varphi(x)| \leq c e^{-a|x|^{(j+k)/j}}$$

for all $z \in \mathbf{C}$ and $x \in \mathbf{R}$ (see [ElR4] Section 7).

For the eigenfunctions $\varphi_1, \varphi_2, \ldots$ we next show that the constants a, b, c do not behave wildly if n varies. For the harmonic oscillator $P^2 + Q^2$ this was proved before by [Zha], equation (2).

Theorem 5.1 *Let $j, k \in \mathbf{N}$ and let $\lambda_1 \leq \lambda_2 \leq \ldots$ denote the eigenvalues of the operator $P^{2j} + Q^{2k}$, repeated according to multiplicity, with $\varphi_1, \varphi_2, \ldots$ a corresponding orthonormal basis of eigenfunctions. Then there exists $C > 0$ such that*

$$C^{-1} n^{2jk/(j+k)} \leq \lambda_n \leq C n^{2jk/(j+k)}$$

for all $n \in \mathbf{N}$. Moreover, there exist $a, b, c > 0$ such that

$$|x^r \varphi_n^{(s)}(x)| \leq c^n a^r b^s r!^{j/(j+k)} s!^{k/(j+k)}$$

uniformly for all $n \in \mathbf{N}$, $r, s \in \mathbf{N}_0$ and $x \in \mathbf{R}$.

Equivalently, each φ_n can be extended to an entire function and there exist $a, b, c > 0$ such that

$$|\varphi_n(x + iy)| \leq c^n e^{-a|x|^{(j+k)/j} + b|y|^{(j+k)/j}}$$

uniformly for all $x, y \in \mathbf{R}$ and $n \in \mathbf{N}$, or, equivalently,

$$|\varphi_n(z)| \leq c^n e^{b|z|^{(j+k)/j}} \quad , \quad |\varphi_n(x)| \leq c^n e^{-a|x|^{(j+k)/j}}$$

uniformly for all $z \in \mathbf{C}$, $x \in \mathbf{R}$ and $n \in \mathbf{N}$.

Proof We have already proved the existence of the constant C for the eigenvalue estimates. Then for all $n \in \mathbf{N}$ one has

$$\|H^l \varphi_n\|_2 = \lambda_n^l \leq (C n^{2jk/(j+k)})^l \leq (e^{2jk/(j+k)})^n \, C^l \, l!^{2jk/(j+k)}$$

for all $l \in \mathbf{N}_0$. If one now traces all the constants in [ElR4], [Wlo] §29.5 and [GeS] Section IV.3.3 one obtains the uniform estimates of the theorem. □

Finally the partition function $Z(\beta) = \mathrm{Tr}(\exp(-\beta H))$ can be bounded above and below by use of the eigenvalue estimates. One has

$$c^{-1} \beta \int_{\lambda_1}^{\infty} d\lambda \, \lambda^{\sigma} e^{-\beta\lambda} \leq \mathrm{Tr}(e^{-\beta H}) \leq c\beta \int_{\lambda_1}^{\infty} d\lambda \, \lambda^{\sigma} e^{-\beta\lambda}$$

where again $\sigma = 2jk/(j+k)$. Alternatively one obtains bounds

$$c_{\sigma}^{-1} \beta^{-\sigma} e^{-\beta\lambda_1} \leq \mathrm{Tr}(e^{-\beta H}) \leq c_{\sigma} \min(\beta^{-\sigma}, e^{-\beta\lambda_1})$$

by straightforward estimations of the integral.

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