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by

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1. Introduction.

The cost of uncertainty is, quite generally, defined as the difference between the total cost when we have perfect information regarding the future and act optimally so as to minimize cost given this information, and the total cost when the future is only probabilistic known and we act optimally given this probabilistic information.

In all but the simplest of cases, little empirical knowledge is available about the relation between the cost of uncertainty and the amount of uncertainty present.

In this note we consider the classical production smoothing problem, see [1], [2], [3], [4], and we will show that the cost of uncertainty is proportional to the amount of uncertainty as measured by the standard deviation of the demand per period.

2. Model.

The basic recurrence relation for the production problem considered here is

$$I_t = I_{t-1} + p_t - d_t$$

or: inventory at time t equals inventory at time t-1 plus production during period t minus demand during period t. The inventory and stock-out cost in period t are assumed to be equal to $pI_t^- + hI_t^+$ with $I_t^- = \max(0,- I_t)$ and $I_t^+ = \max(0, + I_t)$. The process can be controlled by changing p_t. For the moment we assume that p_t may also be negative. We come back to that later. The cost of changing the rate of production in period t is equal to $r| p_t - p_{t-1}|$.

To find the cost of uncertainty we have to distinguish the stochastic case (uncertainty) and the deterministic case (certainty, albeit variable certainty).

In the stochastic case the d_t are i.i.d random variables. Let $C^n(I,p)$ be the minimal expected cost in the first n periods for starting state (I,p).

In the deterministic case we suppose that the demand is sampled at t = 0. One can determine for each sample an optimal strategy. For fixed starting state to each sample corresponds a minimal cost for the first n periods. In this way the minimal cost in the first n periods for starting state (I,p) is itself a random variable. Let $B^n(I,p)$ be the expectation of this random variable.

The cost of uncertainty is equal to $C^n(I,p) - B^n(I,p)$.

3. Linearity of the cost of uncertainty.

The process described by the demand distribution (D), starting state the parameters h, p and r. We have to investigate the effect of changes in the demand distribution. Therefore we write $C^n_D(I,p)$ and $B^n_D(I,p)$ to indicate the influence of the demand distribution. The demand distribution for the case where all demands are multiplied by a factor α is denoted by αD, the demand distribution for the case where all demands are an amount t higher is denoted by D + t. The proportionality of the cost of uncertainty to the standard deviation of the demand per period can be shown with aid of the following two propositions.

Proposition 1. $C^n_{\alpha D}(\alpha I,\alpha p) = \alpha C^n_D(I,p)$ and $B^n_{\alpha D}(\alpha I,\alpha p) = \alpha B^n_D(I,p)$

This is easy to see by straightforward dimension analysis. One can change the units in which quality and cost are measured. The new units in which quantity is measured are chosen $1/\alpha$ times the old units, that means that the demand is α times the old demand. If the new units in which cost is measured is also $1/\alpha$ times the old units then the parameters h, p, r do not change.

Proposition 2. $C^n_{D+t}(I,p + t) = C^n_D(I,p)$ and $B^n_{D+t}(I,p + t) = B^n_D(I,p)$

This is easy to see by using in the case of demand distribution D+t the state definition (I,p') where p' = p+t. Then the transition and cost structure for the problem with demand distribution D+t is exactly the same as the transition and cost structure for the problem with demand distribution D and state definition (I,p).

Now let D be the normal distribution $N(\mu,\sigma)$ and D_α the normal distribution $N(\mu,\alpha\sigma)$. Then $D_\alpha = \alpha D-(\alpha-1)\mu$. From proposition 1 and proposition 2 it follows that

$$C_D^n(I,p) = \frac{1}{\alpha} C_{D_\alpha}^n(\alpha I, \alpha p - (\alpha - 1)\mu)$$
$$B_D^n(I,p) = \frac{1}{\alpha} B_{D_\alpha}^n(\alpha I, \alpha p - (\alpha - 1)\mu)$$

That means that for $I = 0$, $p = \mu$ the proportionality of C_D^n, B_D^n, $B_D^n - C_D^n$ to the standard deviation is precise. For a not too small planning horizon n this proportionality holds also for other starting states since the influence of the starting state is small in this case. For the average cost case the results do not depend at all on the starting state. The assumption of normality of the demand distribution is not essential of course. But it is one of the distributions remaining of the same type if multiplied by a constant or translated over a certain distance.

Another assumption is the possibility of negative demand and negative production rates. In reality negative demand and negative production are not possible in general, but if the coefficient of variation (σ/μ) is not too high, say $\leq \frac{1}{2}$, this cannot influence the results too much.

4. Extensions and remarks.

The assumption of symmetric smoothing cost is not essential. The case where the smoothing cost is $r_1(p_t - p_{t-1})^+ + r_2(p_t - p_{t-1})^-$ can be reduced to the symmetric case by adding a cost component $q(p_t - p_{t-1})$. If $q = (r_2 - r_1)/2$ we have as cost for changes in production rate $r|p_t - p_{t-1}|$ with $r = (r_1 + r_2)/2$. The total contribution of the extra component is $q(p_n - p_0)$ which is very small relatively to the total smoothing cost (if n is large enough).
The propositions 1 and 2 can also be used in more general production networks with only one source of uncertainty, e.g. multi-echelon problems and convergent production networks.
The results also hold for the case of a delay in production and for the situation where the case of uncertainty is compared with the case where there is only certainty about the demand in the first k periods ($k = 1,2,\ldots,n$).

The propositions 1 and 2 can also be used to reduce the number of input parameters in this type of production smoothing problem. If we assume normally distributed demand the input parameters are h, p, r, μ, σ. It is always possible of course to choose the units such that h = 1. Together with the propositions 1 and 2 this implies that there are only two input parameters left, e.g. p and r. It is sufficient to analyse the problem for fixed h, μ, σ and varying p and r. The results with respect to the costs for deterministic and stochastic case and the optimal strategy for the stochastic case can be translated to arbitrary settings of h, p, r, μ, σ.

For the analogous continous time case it is possible, by changing also the unit of time, to realize that both h = 1 and r = 1. In this case there is only one input parameter left. The ratio B (.,.)/C (.,.) and the ratio of smoothing costs and inventory and stock out costs only depend on the ratio of h and p.

5. References.

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[3] Sobel, M.J. (1969) "Production smoothing with stochastic demand I: finite horizon case", Management Science 11, 195-207.

[4] Sobel, M.J. (1971) "Production smoothing with stochastic demand II: infinite horizon case", Management Science 17, 724-735.