Notes on multilinear algebra

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1. Symmetric functions

The elementary symmetric polynomials in d variables and their generating function are defined by

$$\lambda_k(\underline{x}) := \sum_{j_1<\ldots<j_k} x_{j_1} x_{j_2} \ldots x_{j_k} \qquad \text{for } k \le d, \text{ zero for } k > d\ .$$

$$\lambda_t(\underline{x}) := \sum_{k\ge 0} \lambda_k t^k = \prod_{i=1}^{d} (1 + x_i t)\ .$$

The homogeneous symmetric polynomials and their generating functions are defined by

$$s_k(\underline{x}) := \sum_{j_1\le\ldots\le j_k} x_{j_1} x_{j_2} \ldots x_{j_k} = \sum_{k_1+\ldots+k_d=k} x_1^{k_1} x_2^{k_2} \ldots x_d^{k_d}\ .$$

$$s_t(\underline{x}) := \sum_{k\ge 0} s_k t^k = \prod_{i=1}^{d} \left(\sum_{k\ge 0} x_i^k t^k \right).$$

Theorem 1.1. $\lambda_t(\underline{x})\, s_{-t}(\underline{x}) = 1$. (1)

Proof.

$$\frac{1}{\lambda_{-t}(\underline{x})} = \prod_{i=1}^{d} \frac{1}{1 - x_i t} = \prod_{i=1}^{d} \sum_{k\ge 0} (x_i t)^k = s_t(\underline{x})\ .$$

Formula (1) generalizes in various ways. We have

$$\Lambda_t(V)\, S_{-t}(V) = 1 \tag{2}$$

where

$$\Lambda_t(V) := \sum_{k\geq 0} \Lambda_k(V)\, t^k \quad , \quad S_t(V) := \sum_{k\geq 0} S_k(V)\, t^k ,$$

for the exterior powers $\Lambda_k(V)$ and the symmetric powers $S_k(V)$ of a vector space V, whose definition will be recalled later.

A further generalization of (1) holds in the Grothendieck ring $(R(G),\oplus,\otimes)$ consisting of the G-modules V of a finite group G and, more abstractly, in the theory of λ-rings. Thus (2) follows from (1) by general abstract nonsense.

References

S. Lang (1965), Algebra, page 431.

I.G. Macdonald (1979), Symmetric functions and Hall polynomials, Remark 2.15, page 17.

D. Knutson (1973), λ-rings and the representation theory of the symmetric group, Springer Lect. Notes 308.

T. tom Dieck (1979), Transformation groups and representation theory, Springer Lect. Notes 766.

R.P. Stanley, Combinatorial reciprocity theorems, M.C. tracts 56 (1974), 107 - 118 = Nijenrode Proceedings.
More extended: Advanced Math. 14 (1974), 194 - 253.

2. Tensor algebra

Let V denote a vector space of dim. d, with inner product $(\cdot,\cdot)$. The tensor algebra TV, consisting of all tensors on V, is a graded algebra with

$$TV = \sum_{k=0}^{\infty} T_k V \quad , \quad \dim T_k V = d^k \quad , \quad p(t) = (1 - dt)^{-1} \ .$$

The tensors of the form $\underline{x}_1 \otimes \dots \otimes \underline{x}_k$ form a basis for $T_k V$, and the inner product reads

$$(\underline{x}_1 \otimes \dots \otimes \underline{x}_k , \underline{y}_1 \otimes \dots \otimes \underline{y}_k) = \prod_{i=1}^{k} (\underline{x}_i , \underline{y}_i) \ .$$

The exterior algebra ΛV, consisting of all skew symmetric tensors, is graded with

$$\Lambda V = \sum_{k=0}^{d} \Lambda_k V \quad , \quad \dim \Lambda_k V = \binom{d}{k} \quad , \quad p(t) = (1 + t)^d \ .$$

A basis for Λ_k is provided by the skew tensors

$$\underline{x}_1 \wedge \dots \wedge \underline{x}_k := \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} (-1)^{\sigma} \underline{x}_{\sigma 1} \otimes \dots \otimes \underline{x}_{\sigma k} \ ,$$

and the inner product reads

$$(\underline{x}_1 \wedge \dots \wedge \underline{x}_k , \underline{y}_1 \wedge \dots \wedge \underline{y}_k) = \frac{1}{k!} \det[(\underline{x}_i , \underline{y}_j)] \ .$$

The symmetric algebra SV, consisting of all symmetric tensors, is graded with

$$SV = \sum_{k=0}^{\infty} S_k V \quad , \quad \dim S_k V = \binom{d - 1 + k}{k} , \quad p(t) = (1 - t)^{-d} \ .$$

A basis for S_k is provided by the symmetric tensors

$$\underline{x}_1 \vee \ldots \vee \underline{x}_k := \frac{1}{k!} \sum_{\sigma \in \mathcal{T}_k} \underline{x}_{\sigma 1} \otimes \ldots \otimes \underline{x}_{\sigma k} ,$$

and the inner product reads

$$(\underline{x}_1 \vee \ldots \vee \underline{x}_k , \underline{y}_1 \vee \ldots \vee \underline{y}_k) = \frac{1}{k!} \operatorname{per}[(\underline{x}_i , \underline{y}_j)] .$$

Let $\underline{e}_1,\ldots,\underline{e}_d$ denote any orthonormal basis for V. The corresponding orthonormal basis for Λ_k consists of the

$$\binom{d}{k} \text{ elements } \sqrt{k!}\, e^{\underline{k}} ,$$

$$e^{\underline{k}} := \underline{e}_1^{k_1} \wedge \ldots \wedge \underline{e}_d^{k_d} \quad ; \quad k_1 + \ldots + k_d = k \quad ; \quad k_1,\ldots,k_d \in \{0,1\} .$$

The corresponding orthonormal basis for S_k consists of the

$$\binom{d-1+k}{k} \text{ elements } \sqrt{\frac{k!}{k_1! \ldots k_d!}}\, e^{\underline{k}} ,$$

$$e^{\underline{k}} := \underline{e}_1^{k_1} \vee \ldots \vee \underline{e}_d^{k_d} \quad ; \quad k_1 + \ldots + k_d = k \quad ; \quad k_1,\ldots,k_d \in \mathbb{N} ,$$

with $\underline{e}_i^{k_i} := \underline{e}_i \vee \ldots \vee \underline{e}_i$, k_i times. Thus we use the same notation for both cases. It is convenient to abreviate $k_1! \ldots k_d!$ by $\underline{k}!$ and $k_1 + \ldots + k_d$ by $|\underline{k}|$.

References

W.H. Greub, Multilinear Algebra, Springer 1967.

R. Shaw, Linear Algebra and Group Representations I and II, Academic Press 1982.

3. Exterior and symmetric powers of a matrix

Let $A : V \to V$ denote a linear map of V. We define its k-th exterior power by

$$\Lambda_k(A) : \Lambda_k V \to \Lambda_k V : \underline{x}_1 \wedge \ldots \wedge \underline{x}_k \mapsto A\underline{x}_1 \wedge \ldots \wedge A\underline{x}_k ,$$

and its k-th symmetric power by

$$S_k(A) : S_k V \to S_k V : \underline{x}_1 \vee \ldots \vee \underline{x}_k \mapsto A\underline{x}_1 \vee \ldots \vee A\underline{x}_k .$$

We calculate the entries of the power matrices with respect to the standard basis. We use the following notation, which applies for both Λ_k and S_k. For $\underline{k}$ and $\underline{\ell}$ with $|\underline{k}| = |\underline{\ell}| = k$ the matrix $A(\underline{k} \mid \underline{\ell})$ is the $k \times k$ matrix which is obtained from the $d \times d$ matrix A by repeating k_i times row i, and ℓ_j times column j, for $i,j = 1,2,\ldots,d$.

Theorem 3.1.

$$[\Lambda_k(A)]_{\underline{k},\underline{\ell}} = \det A(\underline{k} \mid \underline{\ell}) \quad ; \quad [S_k(A)]_{\underline{k},\underline{\ell}} = \frac{\operatorname{per} A(\underline{k} \mid \underline{\ell})}{\sqrt{\underline{k}!\,\underline{\ell}!}} .$$

Let A have the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_d$. The eigenvalues of $\Lambda_k(A)$ are the $\binom{d}{k}$ elementary, those of $S_k(A)$ the $\binom{d+k-1}{1}$ homogeneous polynomials of degree k in $\alpha_1, \ldots, \alpha_d$.

Theorem 3.2.

$$\det(I + tA) = \sum_{k=0}^{d} t^k \operatorname{trace} \Lambda_k(A) = \sum_{\underline{k}} t^{|\underline{k}|} \det A(\underline{k} \mid \underline{k})$$

$$\det^{-1}(I - tA) = \sum_{k\geq 0} t^k \operatorname{trace} S_k(A) = \sum_{\underline{k}} t^{|\underline{k}|} \frac{\operatorname{per} A(\underline{k} \mid \underline{k})}{\underline{k}!} .$$

Proof. The first result simply amounts to

$$\det(I + tA) = 1 + t \sum_{i=1}^{d} a_{ii} + t^2 \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} + \ldots + t^d \det A .$$

Both results are easily proved by considering the eigenvalues on both sides, and by using the proof of Theorem 1.1.

For the generating functions

$$\Lambda_t(A) := \sum_{k\geq 0} t^k \Lambda_k(A) \quad , \quad S_t(A) := \sum_{k\geq 0} t^k S_k(A)$$

the theorem implies

$$\det(I + tA) = \operatorname{trace} \Lambda_t(A) = \operatorname{trace}^{-1} S_{-t}(A)$$

and $1 = \lambda_t(\underline{\alpha})\, s_{-t}(\underline{\alpha})$, in agreement with Theorem 1.1.

References

C.C. MacDuffee, The theory of matrices, Chelsea (1946).

M. Marcus, H. Minc, A survey of matrix theory (1964).

H. Minc, Permanents, Addison-Wesley (1978).

H. Minc, Theory of permanents 1978 - 1981, Lin. Multilin. Alg. 12 (1983) 227 - 263.

4. MacMahon's master theorem

Lemma 4.1. The coefficient of $\underline{x}^{\underline{\ell}}$ in $(A\,\underline{x})^{\underline{k}}$ equals

$$\frac{1}{\underline{\ell}!}\ \text{per}\ A(\underline{k}\mid\underline{\ell})\ .$$

Proof. Convince yourself by writing out $A(\underline{k}\mid\underline{\ell})$ in block form.

Theorem 4.2 (MacMahon). The coefficient of $\underline{x}^{\underline{k}}$ in $(A\,\underline{x})^{\underline{k}}$ equals the coefficient of $\underline{x}^{k}$ in

$$1/\det(I - A\,\Delta(\underline{x})),\ \text{where}\ \Delta(\underline{x}) = \text{diag}(x_1,\ldots,x_d)\ .$$

Proof (I.G. Macdonald). Lemma 4.1 and Theorem 3.1 imply that the coefficient of $\underline{x}^{\underline{k}}$ in $(A\,\underline{x})^{\underline{k}}$ equals the $(\underline{k},\underline{k})$-entry in the k-th symmetric power $S_k(A)$, where $k = |\underline{k}|$. Hence it is the coefficient of $\underline{x}^{\underline{k}}$ in trace $S_k(A\,\Delta(\underline{x}))$. But

$$\sum_{k\geq 0} \text{trace}\ S_k(A\,\Delta(\underline{x})) = 1/\det(I - A\,\Delta(\underline{x}))\ .$$

5. Bebiano's formula

Theorem 5.1.

$$\exp(\underline{x}, A\underline{y})t = \sum_{k=0}^{\infty} t^k \sum_{|\underline{k}|=|\underline{\ell}|=k} \frac{\underline{x}^{\underline{k}}}{\underline{k}!} \frac{\underline{y}^{\underline{\ell}}}{\underline{\ell}!} \operatorname{per} A(\underline{k} \mid \underline{\ell}) .$$

Proof. The formula

$$\frac{(\underline{x}, A\underline{y})^k}{k!} = \sum_{|\underline{k}|=|\underline{\ell}|=k} \frac{\underline{x}^{\underline{k}}}{\underline{k}!} \frac{\underline{y}^{\underline{\ell}}}{\underline{\ell}!} \operatorname{per} A(\underline{k} \mid \underline{\ell})$$

is obtained by taking the inner products of the symmetric tensors on the left and on the right hand sides of the following formulae

$$\frac{1}{k!} \underline{x} \vee \ldots \vee \underline{x} = \frac{1}{k!} (x_1 \underline{e}_1 + \ldots + x_d \underline{e}_d) \vee \ldots \vee (x_1 \underline{e}_1 + \ldots + x_d \underline{e}_d) =$$

$$= \sum_{|\underline{k}|=k} \frac{\underline{x}^{\underline{k}}}{\underline{k}!} \underline{e}_1^{k_1} \vee \ldots \vee \underline{e}_d^{k_d} ,$$

$$\frac{1}{k!} A\underline{y} \vee \ldots \vee A\underline{y} = \sum_{|\underline{\ell}|=k} \frac{\underline{y}^{\underline{\ell}}}{\underline{\ell}!} A\underline{e}_1^{\ell_1} \vee \ldots \vee A\underline{e}_d^{\ell_d} .$$

Indeed, we have

$$(\underline{x} \vee \ldots \vee \underline{x}, \underline{z} \vee \ldots \vee \underline{z}) = (\underline{x}, \underline{z})^k = (\underline{x} \otimes \ldots \otimes \underline{x}, \underline{z} \otimes \ldots \otimes \underline{z}) . \quad \square$$

Reference

N. Bebiano, Pac. J. Math. 101 (1982), 1 - 9.

6. Fredholm's determinant

Fredholm's integral equation

$$u(x) = f(x) + \lambda \int_0^1 K(x,t)\, u(t)\, dt$$

is approximated by the set of matrix equations

$$(I - \lambda M_d)\underline{u} = \underline{f} \quad ; \quad d = 1,2,\ldots ,$$

as follows. The interval [0,1] is devided into d equal parts by $0 < \frac{1}{d} < \ldots < \frac{d-1}{d} < 1$, and $\underline{f}$ (= $\underline{f}_d$) has components $f_i = f(\frac{i}{d})$, whereas M_d has entries $M_d(i,j) = d^{-1} K(\frac{i}{d}, \frac{j}{d})$.

Fredholm's determinant is defined as follows:

$$1 - \lambda \int_0^1 K(t,t)dt + \frac{\lambda^2}{2!} \int_0^1 \int_0^1 \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) \\ K(t_2,t_1) & K(t_2,t_2) \end{vmatrix} dt_1\, dt_2 +$$

$$+ \frac{\lambda^3}{3!} \iiint + \ldots .$$

It is the limit, for $d \to \infty$, of

$$\det(I - \lambda M_d) = 1 - \lambda \sum_i M_d(i,i) + \frac{\lambda^2}{2!} \sum_{i,j} \begin{vmatrix} M_d(i,i) & M_d(i,j) \\ M_d(j,i) & M_d(j,j) \end{vmatrix} \ldots +$$

$$+ (-1)^d \lambda^d \times \det M_d .$$

As a consequence of (3) and (4) we have

$$\det(I - \lambda M_d) = \sum_{k\geq 0} (-\lambda)^k \operatorname{trace} \Lambda_k M_d ,$$

$$\det^{-1}(I - \lambda M_d) = \sum_{k\geq 0} \lambda^k \operatorname{trace} S_k M_d .$$

This is in agreement with (2). Thus, Fredholm's equation may be solved in terms of permanents.

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D. Kershaw, Permanents in Fredholm's theory of integral equations, J. Integral Equ. 1 (1979) 281 - 290.