

## Notes on multilinear algebra

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NOTES ON MULTILINEAR ALGEBRA

by

A. Blokhuis and J.J. Seidel

Eindhoven University of Technology

Department of Mathematics and Computing Science

PO Box 513, 5600 MB Eindhoven

The Netherlands

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A. Blokhuis and J.J. Seidel

1. Symmetric functions

The elementary symmetric polynomials in  $d$  variables and their generating function are defined by

$$\lambda_k(\underline{x}) := \sum_{j_1 < \dots < j_k} x_{j_1} x_{j_2} \dots x_{j_k} \quad \text{for } k \leq d, \text{ zero for } k > d .$$

$$\lambda_t(\underline{x}) := \sum_{k \geq 0} \lambda_k t^k = \prod_{i=1}^d (1 + x_i t) .$$

The homogeneous symmetric polynomials and their generating functions are defined by

$$s_k(\underline{x}) := \sum_{j_1 \leq \dots \leq j_k} x_{j_1} x_{j_2} \dots x_{j_k} = \sum_{k_1 + \dots + k_d = k} x_1^{k_1} x_2^{k_2} \dots x_d^{k_d} .$$

$$s_t(\underline{x}) := \sum_{k \geq 0} s_k t^k = \prod_{i=1}^d \left( \sum_{k \geq 0} x_i^k t^k \right) .$$

Theorem 1.1.  $\lambda_t(\underline{x}) s_{-t}(\underline{x}) = 1 .$  (1)

Proof.

$$\frac{1}{\lambda_{-t}(\underline{x})} = \prod_{i=1}^d \frac{1}{1 - x_i t} = \prod_{i=1}^d \sum_{k \geq 0} (x_i t)^k = s_t(\underline{x}) .$$

Formula (1) generalizes in various ways. We have

$$\Lambda_t(V) S_{-t}(V) = 1 \tag{2}$$

where

$$\Lambda_t(V) := \sum_{k \geq 0} \Lambda_k(V) t^k, \quad S_t(V) := \sum_{k \geq 0} S_k(V) t^k,$$

for the exterior powers  $\Lambda_k(V)$  and the symmetric powers  $S_k(V)$  of a vector space  $V$ , whose definition will be recalled later.

A further generalization of (1) holds in the Grothendieck ring  $(R(G), \oplus, \otimes)$  consisting of the  $G$ -modules  $V$  of a finite group  $G$  and, more abstractly, in the theory of  $\lambda$ -rings. Thus (2) follows from (1) by general abstract nonsense.

#### References

S. Lang (1965), Algebra, page 431.

I.G. Macdonald (1979), Symmetric functions and Hall polynomials, Remark 2.15, page 17.

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More extended: Advanced Math. 14 (1974), 194 - 253.

## 2. Tensor algebra

Let  $V$  denote a vector space of dim.  $d$ , with inner product  $(\cdot, \cdot)$ . The tensor algebra  $TV$ , consisting of all tensors on  $V$ , is a graded algebra with

$$TV = \sum_{k=0}^{\infty} T_k V \quad , \quad \dim T_k V = d^k \quad , \quad p(t) = (1 - dt)^{-1} .$$

The tensors of the form  $\underline{x}_1 \otimes \dots \otimes \underline{x}_k$  form a basis for  $T_k V$ , and the inner product reads

$$(\underline{x}_1 \otimes \dots \otimes \underline{x}_k, \underline{y}_1 \otimes \dots \otimes \underline{y}_k) = \prod_{i=1}^k (\underline{x}_i, \underline{y}_i) .$$

The exterior algebra  $\Lambda V$ , consisting of all skew symmetric tensors, is graded with

$$\Lambda V = \sum_{k=0}^d \Lambda_k V \quad , \quad \dim \Lambda_k V = \binom{d}{k} \quad , \quad p(t) = (1 + t)^d .$$

A basis for  $\Lambda_k$  is provided by the skew tensors

$$\underline{x}_1 \wedge \dots \wedge \underline{x}_k := \frac{1}{k!} \sum_{\sigma \in \sigma_k} (-1)^\sigma \underline{x}_{\sigma 1} \otimes \dots \otimes \underline{x}_{\sigma k} ,$$

and the inner product reads

$$(\underline{x}_1 \wedge \dots \wedge \underline{x}_k, \underline{y}_1 \wedge \dots \wedge \underline{y}_k) = \frac{1}{k!} \det[(\underline{x}_i, \underline{y}_j)] .$$

The symmetric algebra  $SV$ , consisting of all symmetric tensors, is graded with

$$SV = \sum_{k=0}^{\infty} S_k V \quad , \quad \dim S_k V = \binom{d-1+k}{k} \quad , \quad p(t) = (1 - t)^{-d} .$$

A basis for  $S_k$  is provided by the symmetric tensors

$$\underline{x}_1 \vee \dots \vee \underline{x}_k := \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \underline{x}_{\sigma 1} \otimes \dots \otimes \underline{x}_{\sigma k},$$

and the inner product reads

$$(\underline{x}_1 \vee \dots \vee \underline{x}_k, \underline{y}_1 \vee \dots \vee \underline{y}_k) = \frac{1}{k!} \text{per}[(\underline{x}_i, \underline{y}_j)].$$

Let  $\underline{e}_1, \dots, \underline{e}_d$  denote any orthonormal basis for  $V$ . The corresponding orthonormal basis for  $\Lambda_k$  consists of the

$$\binom{d}{k} \text{ elements } \sqrt{k!} e^k,$$

$$e^k := \underline{e}_1^{k_1} \wedge \dots \wedge \underline{e}_d^{k_d} \quad ; \quad k_1 + \dots + k_d = k \quad ; \quad k_1, \dots, k_d \in \{0, 1\}.$$

The corresponding orthonormal basis for  $S_k$  consists of the

$$\binom{d-1+k}{k} \text{ elements } \sqrt{\frac{k!}{k_1! \dots k_d!}} e^k,$$

$$e^k := \underline{e}_1^{k_1} \vee \dots \vee \underline{e}_d^{k_d} \quad ; \quad k_1 + \dots + k_d = k \quad ; \quad k_1, \dots, k_d \in \mathbf{N},$$

with  $\underline{e}_i^{k_i} := \underline{e}_i \vee \dots \vee \underline{e}_i$ ,  $k_i$  times. Thus we use the same notation for both cases. It is convenient to abbreviate  $k_1! \dots k_d!$  by  $\underline{k}!$  and  $k_1 + \dots + k_d$  by  $|\underline{k}|$ .

### References

W.H. Greub, *Multilinear Algebra*, Springer 1967.

R. Shaw, *Linear Algebra and Group Representations I and II*,  
Academic Press 1982.

### 3. Exterior and symmetric powers of a matrix

Let  $A : V \rightarrow V$  denote a linear map of  $V$ . We define its  $k$ -th exterior power by

$$\Lambda_k(A) : \Lambda_k V \rightarrow \Lambda_k V : \underline{x}_1 \wedge \dots \wedge \underline{x}_k \mapsto A\underline{x}_1 \wedge \dots \wedge A\underline{x}_k,$$

and its  $k$ -th symmetric power by

$$S_k(A) : S_k V \rightarrow S_k V : \underline{x}_1 \vee \dots \vee \underline{x}_k \mapsto A\underline{x}_1 \vee \dots \vee A\underline{x}_k.$$

We calculate the entries of the power matrices with respect to the standard basis. We use the following notation, which applies for both  $\Lambda_k$  and  $S_k$ . For  $\underline{k}$  and  $\underline{\ell}$  with  $|\underline{k}| = |\underline{\ell}| = k$  the matrix  $A(\underline{k} | \underline{\ell})$  is the  $k \times k$  matrix which is obtained from the  $d \times d$  matrix  $A$  by repeating  $k_i$  times row  $i$ , and  $\ell_j$  times column  $j$ , for  $i, j = 1, 2, \dots, d$ .

#### Theorem 3.1.

$$[\Lambda_k(A)]_{\underline{k}, \underline{\ell}} = \det A(\underline{k} | \underline{\ell}) \quad ; \quad [S_k(A)]_{\underline{k}, \underline{\ell}} = \frac{\text{per } A(\underline{k} | \underline{\ell})}{\sqrt{k! \ell!}}.$$

Let  $A$  have the eigenvalues  $\alpha_1, \alpha_2, \dots, \alpha_d$ . The eigenvalues of  $\Lambda_k(A)$  are the  $\binom{d}{k}$  elementary, those of  $S_k(A)$  the  $\binom{d+k-1}{1}$  homogeneous polynomials of degree  $k$  in  $\alpha_1, \dots, \alpha_d$ .

#### Theorem 3.2.

$$\det(I + tA) = \sum_{k=0}^d t^k \text{trace } \Lambda_k(A) = \sum_{\underline{k}} t^{|\underline{k}|} \det A(\underline{k} | \underline{k})$$

$$\det^{-1}(I - tA) = \sum_{k \geq 0} t^k \text{trace } S_k(A) = \sum_{\underline{k}} t^{|\underline{k}|} \frac{\text{per } A(\underline{k} | \underline{k})}{\underline{k}!}.$$

Proof. The first result simply amounts to

$$\det(I + tA) = 1 + t \sum_{i=1}^d a_{ii} + t^2 \sum_{i < j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} + \dots + t^d \det A .$$

Both results are easily proved by considering the eigenvalues on both sides, and by using the proof of Theorem 1.1.

For the generating functions

$$\Lambda_t(A) := \sum_{k \geq 0} t^k \Lambda_k(A) \quad , \quad S_t(A) := \sum_{k \geq 0} t^k S_k(A)$$

the theorem implies

$$\det(I + tA) = \text{trace } \Lambda_t(A) = \text{trace}^{-1} S_{-t}(A)$$

and  $1 = \lambda_t(\underline{\alpha}) s_{-t}(\underline{\alpha})$ , in agreement with Theorem 1.1.

### References

- C.C. MacDuffee, The theory of matrices, Chelsea (1946).
- M. Marcus, H. Minc, A survey of matrix theory (1964).
- H. Minc, Permanents, Addison-Wesley (1978).
- H. Minc, Theory of permanents 1978 - 1981, Lin. Multilin. Alg. 12 (1983)  
227 - 263.



4. MacMahon's master theorem

Lemma 4.1. The coefficient of  $\underline{x}^{\underline{\ell}}$  in  $(A\underline{x})^{\underline{k}}$  equals

$$\frac{1}{\underline{\ell}!} \text{ per } A(\underline{k} \mid \underline{\ell}) .$$

Proof. Convince yourself by writing out  $A(\underline{k} \mid \underline{\ell})$  in block form.

Theorem 4.2 (MacMahon). The coefficient of  $\underline{x}^{\underline{k}}$  in  $(A\underline{x})^{\underline{k}}$  equals the coefficient of  $\underline{x}^{\underline{k}}$  in

$$1/\det(I - A\Delta(\underline{x})), \text{ where } \Delta(\underline{x}) = \text{diag}(x_1, \dots, x_d) .$$

Proof (I.G. Macdonald). Lemma 4.1 and Theorem 3.1 imply that the coefficient of  $\underline{x}^{\underline{k}}$  in  $(A\underline{x})^{\underline{k}}$  equals the  $(\underline{k}, \underline{k})$ -entry in the  $\underline{k}$ -th symmetric power  $S_{\underline{k}}(A)$ , where  $\underline{k} = |\underline{k}|$ . Hence it is the coefficient of  $\underline{x}^{\underline{k}}$  in  $\text{trace } S_{\underline{k}}(A\Delta(\underline{x}))$ . But

$$\sum_{\underline{k} \geq 0} \text{trace } S_{\underline{k}}(A\Delta(\underline{x})) = 1/\det(I - A\Delta(\underline{x})) .$$

5. Bebiano's formula

Theorem 5.1.

$$\exp(\underline{x}, A \underline{y})^t = \sum_{k=0}^{\infty} t^k \sum_{|\underline{k}|=|\underline{\ell}|=k} \frac{\underline{x}^{\underline{k}}}{\underline{k}!} \frac{\underline{y}^{\underline{\ell}}}{\underline{\ell}!} \text{ per } A(\underline{k} | \underline{\ell}) .$$

Proof. The formula

$$\frac{(\underline{x}, A \underline{y})^k}{k!} = \sum_{|\underline{k}|=|\underline{\ell}|=k} \frac{\underline{x}^{\underline{k}}}{\underline{k}!} \frac{\underline{y}^{\underline{\ell}}}{\underline{\ell}!} \text{ per } A(\underline{k} | \underline{\ell})$$

is obtained by taking the inner products of the symmetric tensors on the left and on the right hand sides of the following formulae

$$\begin{aligned} \frac{1}{k!} \underline{x} \vee \dots \vee \underline{x} &= \frac{1}{k!} (x_1 \underline{e}_1 + \dots + x_d \underline{e}_d) \vee \dots \vee (x_1 \underline{e}_1 + \dots + x_d \underline{e}_d) = \\ &= \sum_{|\underline{k}|=k} \frac{\underline{x}^{\underline{k}}}{\underline{k}!} \underline{e}_1^{k_1} \vee \dots \vee \underline{e}_d^{k_d} , \\ \frac{1}{\ell!} A \underline{y} \vee \dots \vee A \underline{y} &= \sum_{|\underline{\ell}|=\ell} \frac{\underline{y}^{\underline{\ell}}}{\underline{\ell}!} A \underline{e}_1^{\ell_1} \vee \dots \vee A \underline{e}_d^{\ell_d} . \end{aligned}$$

Indeed, we have

$$(\underline{x} \vee \dots \vee \underline{x}, \underline{z} \vee \dots \vee \underline{z}) = (\underline{x}, \underline{z})^k = (\underline{x} \otimes \dots \otimes \underline{x}, \underline{z} \otimes \dots \otimes \underline{z}) . \quad \square$$

Reference

N. Bebiano, Pac. J. Math. 101 (1982), 1-9.

6. Fredholm's determinant

Fredholm's integral equation

$$u(x) = f(x) + \lambda \int_0^1 K(x,t) u(t) dt$$

is approximated by the set of matrix equations

$$(I - \lambda M_d) \underline{u} = \underline{f} \quad ; \quad d = 1, 2, \dots ,$$

as follows. The interval [0,1] is divided into d equal parts by

$0 < \frac{1}{d} < \dots < \frac{d-1}{d} < 1$ , and  $\underline{f}$  ( $= \underline{f}_d$ ) has components  $f_i = f(\frac{i}{d})$ , whereas  $M_d$  has entries  $M_d(i,j) = d^{-1} K(\frac{i}{d}, \frac{j}{d})$ .

Fredholm's determinant is defined as follows:

$$1 - \lambda \int_0^1 K(t,t) dt + \frac{\lambda^2}{2!} \int_0^1 \int_0^1 \begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) \\ K(t_2, t_1) & K(t_2, t_2) \end{vmatrix} dt_1 dt_2 +$$

$$+ \frac{\lambda^3}{3!} \iiint + \dots .$$

It is the limit, for  $d \rightarrow \infty$ , of

$$\det(I - \lambda M_d) = 1 - \lambda \sum_i M_d(i,i) + \frac{\lambda^2}{2!} \sum_{i,j} \begin{vmatrix} M_d(i,i) & M_d(i,j) \\ M_d(j,i) & M_d(j,j) \end{vmatrix} \dots +$$

$$+ (-1)^d \lambda^d \times \det M_d .$$

As a consequence of (3) and (4) we have

$$\det(I - \lambda M_d) = \sum_{k \geq 0} (-\lambda)^k \text{trace } \Lambda_k M_d ,$$

$$\det^{-1}(I - \lambda M_d) = \sum_{k \geq 0} \lambda^k \text{trace } S_k M_d .$$

This is in agreement with (2). Thus, Fredholm's equation may be solved in terms of permanents.

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