Quantifying chaos by the amplitude threshold

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Quantifying chaos by the amplitude threshold

The use of the characteristic frequency in the synchronization method

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ABSTRACT

At the present time the quantification of the chaotic behaviour of nonlinear dynamic systems is being widely studied. A reason for this interest can be the need to have a quantitative method to distinguish chaotic behaviour from noisy or erratic behaviour. Another reason could be to obtain an algorithm to calculate the number of active degrees of freedom needed to describe the behaviour of a dynamical system. Finally a measure of chaos influenced by the parameters of a system could correlate to changes within this system, and might give insight into the transactions from chaotic to periodic behaviour.

The well-known quantifiers for chaotic behaviour; the Lyapunov exponent, the attractor dimension and the Kolmogorov metric entropy all have the disadvantage of being difficult to use on experimental data sets due to the sensibility to noise in the signal or the very large computer times needed to calculate them. In contrast to these methods, a synchronization method as investigated in this report can well be used on time series from experiments.

The synchronization effect is known for dynamic systems executing periodic vibrations. In these periodic systems the effect takes place at a infinitesimal small value of the amplitude of the synchronizing force in the system's fundamental frequency. In chaotic systems the same effect takes place, but the synchronization occurs at a finite value of the amplitude of the external force. The minimal amplitude needed to synchronize is defined to be the amplitude threshold for synchronization. This threshold is an indicator of the chaotic behaviour of a dynamic system. As the relationships to other measures of chaos are known, the synchronization method can be used to estimate the other quantifiers.

A disadvantage of the synchronization method, the fact that the two-dimensional space made up by the amplitude and frequency of the synchronizing force has to be scanned, can be reduced by calculating the characteristic frequency. The characteristic frequency is understood to be the frequency at which the dynamic system will most often oscillate when no synchronizing force is applied to it. From experimental data sets, the characteristic frequency can be estimated by statistical analysis of the Hilbert-transform of the data sets. The characteristic frequency indicates where synchronization might most probably occur in the frequency-space.

Numerical experiments have been performed to verify the synchronization method and the use of the characteristic frequency to accelerate the synchronization procedure. The results show that the characteristic frequency can give a good estimate of the frequency at which the minimal amplitude threshold can be found in the frequency domain.

CONTENTS

LIST OF SYMBOLS

Symbols used in this report

Symbol	Definition	Unit
A	Amplitude	[-]
At	Amplitude threshold	[-]
D	Dimension of an attractor	[-]
f	Frequency	[Hz]
g	Gravity	[m/s2]
λ	Lyapunov exponent	[1/s]
l	length	[m]
Km	Kolmogorov metric entropy	[J/K]
N	Number of samples	[-]
S	Entropy	[J/K]
t	Time	[s]
ω	Angular velocity	[rad/s]

LIST OF FIGURES

1. Introduction

1.1. Quantifying chaos

At the present time the characteristics of nonlinear dynamic systems with chaotic behaviour are being widely studied. One of the points of interest in these studies is the determination of how chaotic a system's chaotic behaviour is.

Three reasons can be stated for this interest. The first reason is the need to have a definitive and quantitative method of recognizing chaos and distinguish chaotic behaviour from noisy or erratic behaviour. Secondly, a quantifier could give an estimate of the number of active degrees of freedom needed to describe the dynamical behaviour of a system. Finally, the changes in the parameters which determine the quantifiers of the system may be correlated to changes in the actual dynamic behaviour of the system. This could give insight into the boundary states where a transaction from periodic to chaotic behaviour takes place.

There exist several quantitative characteristics to describe chaotic oscillations : the Lyapunov exponent, the Kolmogorov-Sinai entropy, and the fractal dimension are the most well-known. These characteristics give a distinct indication of the quantity of chaos. The extraction of these measures of chaos from experimental data is possible but several practical problems can occur when applying these methods on time series such as the need of large amounts of computer time or the uncertainties in the results caused by noise in the time series.

The synchronization amplitude threshold as a measure for chaotic behaviour will be presented in this report. In contrast to the above quantitative characteristics, with this method a measure of chaos can easily be estimated from experimental data. It will be shown numerically that the amplitude threshold is related to the positive Lyapunov exponent and the Kolmogorov entropy.

An important disadvantage of the synchronization method is that the two-dimensional space, spanned by the amplitude and frequency of the external synchronizing force, has to be swept to find the minimal amplitude threshold needed to synchronize the system. A method will be presented to investigate the system without applying the synchronizing force, but by analysing the time series at it's output. From these time series the characteristic frequency of a system can be calculated. This characteristic frequency can help accelerating the search for the amplitude threshold. As the characteristic frequency depends on the parameters of the dynamical system, it can also be used for the purpose of monitoring and diagnostics.

1.2. Aim of this report

The aim of the report is to introduce the amplitude threshold as a measure of chaotic behaviour to an engineer who has a need to quantify a chaotic system from experimental data. The

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theoretical backgrounds of the method as well as the numerical experiments which have been performed will be presented. The advantages and disadvantages of this method over the other methods to measure chaos will be discussed.

This report is not mend to give any insight in chaos itself or the conditions under which chaos can exist but mainly to discuss the possibility to use the synchronization method as a qualifier for chaotic behaviour.

1.3. Structure of this report

In the second chapter of this report several well-know methods to measure chaos will be discussed. The advantages and the disadvantages of these methods will be studied as well as the relationships between them. The synchronization method will be presented as a method that can easily be used on time series.

The synchronization method can provoke some difficulties if no special equipment for sweeping the frequency or amplitude domain is available. In the third chapter the characteristic frequency algorithm to accelerate the synchronization method will be presented. This method can give an indication where to start the synchronization method. The theoretical backgrounds of this method will be explained.

The numerical experiments which have been performed are shown in chapter four. The similarities and differences between the results obtained by numerical experiments and the results expected by theory will be shown and discussed.

Chapter five is the last chapter. In this chapter conclusion will be drawn from the obtained results, as far as possible. Also, the parts of the theory and the experiments which need more attention and investigation are presented.

2. QUALIFIERS FOR CHAOTIC SYSTEMS

The aim of this chapter is to show several well-known methods to measure chaos. It will be explained that these methods are difficult to use when only time series and no system equations are available. In this case the synchronization method can be a helpful tool to measure chaos.

2.1. The Lyapunov exponent

One of the characteristics of chaotic systems is the sensitivity to initial conditions and the consequent divergence of neighbouring trajectories. This divergence is exponential and can be characterized by an estimate of the Lyapunov exponent. The Lyapunov exponent λ is defined for a one-dimensional map as

$$d(t) = d_0 e^{\lambda t} \tag{2.1}$$

where d_0 is a measure for the initial distance between two trajectories and $d(t)$ a measure of the distance between the two trajectories after t units of time. A positive λ would indicate chaotic behaviour as the nearby initial conditions of two trajectories would diverge exponentially in time. The exponent is negative if the function $d(t)$ is attracted to a fixed point, the attractor.

This method can also be used for higher dimensions, in a n-dimensional space we have to consider a spectrum of n Lyapunov exponents. This spectrum is given by

$$\lambda_i = \frac{1}{N} \sum_{j=1}^{N} \lim_{t \to \infty} \frac{1}{t} \ln \frac{d_i(t)}{d_0(t)} \tag{2.2}$$

where N is the number of runs performed to estimate the average Lyapunov exponent as it depends on the initial condition x_i.

In theory it is possible to estimate the Lyapunov exponents from time series using the method proposed by Eckmann and Ruell [1] or, if much noise is included in the signal, the SVD-based method by Darbyshire and Broomhead [2] . In practice however, the amplitude of the noise in the time series can often attain the same magnitude as the thickness of the 'skill' of the attractor. The estimation of the quantifier then becomes a difficult task.

2.2. Attractor dimension

A second measure of chaos is the dimension of the attractor. The characterization of the attractor is to count the minimum number of cubes of linear size ε which are needed to cover the attractor. If the number of cubes is $N(\varepsilon)$, then, as ε is varied, $N(\varepsilon)$ will vary as ε^{-D}, where D is the dimension of the attractor. The dimension D is a quantifier for the chaotic behaviour

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of the system. As the limit of ε tends to zero. D is defined as [3]

$$D = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(\frac{1}{\varepsilon})} \qquad (2.3)$$

The disadvantage of this measure of chaos is that is impractical to implement. The power of the number of cubes is related to the dimension of the attractor. This means that very large amounts of computer memory are required. From experimental data, first the attractor will accurately have to be reconstructed before being able to estimate the attractor dimension.

2.3. Kolmogorov-Sinai entropy

The Kolmogorov-Sinai entropy, also know as the metric entropy, is another qualifier for chaotic behaviour in dynamical systems. This qualifier characterizes the average rate of change of the entropy as the system evolves [4] . The state space is divided into cells. The entropy is defined as

$$S_n = -k \sum_r p_r \ln p_r \qquad (2.4)$$

where p_r is the probability that a trajectory starting from an initial cell is in the rth cell after n units of time and k is Boltzmann's constant. For a periodic dynamical system $p_r = 1$ for one or very few cells in the state space and $p_r = 0$ for all the other cells, because starting from an initial cell it will pass through these few cells regularly and not pass through any other cells in the state space. Thus, $S_n = 0$ for all periodic systems. The K-S entropy after n units of time is defined to be

$$K_n = \frac{1}{\tau}(S_{n+1} - S_n) \qquad (2.5)$$

as it characterizes the rate of change of the entropy. The average K-S entropy over the whole attractor can be defined as

$$K = -\lim_{\tau \to 0}\lim_{L \to 0}\lim_{N \to \infty} \frac{1}{N\tau}[S_N - S_0] \qquad (2.6)$$

As can be seen very large computer times are needed to calculate the K-S or metric entropy from experimental data as the time limit goes to infinity and the limit of cell size in the state space goes to zero. Another disadvantage with experimental data is the difficulty to start in different cells.

2.4. Amplitude threshold for synchronization

2.4.1. Synchronization effect

The synchronization effect is well-known for systems executing periodic vibrations. The synchronization effect is the effect which takes place if a dynamical system is excited by an external harmonic force with a *specific* amplitude and frequency. The system will then oscillate periodically at the same or a multiple of the frequency of the external force. An example is the pendulum shown in figure 2.1. The fundamental frequency of this pendulum is
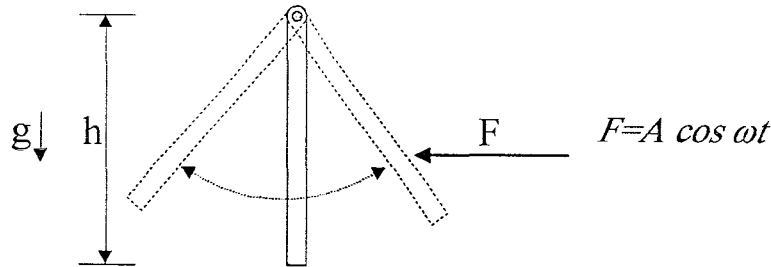
Figure 2.1: Example : The pendulum

$$\omega_e = \sqrt{\frac{g}{h}} \qquad\qquad (2.7)$$

If the pendulum is excited by the external force F at one of the frequencies $\omega = \omega_e, \omega = \frac{1}{2}\omega_e,$ $\omega = \frac{1}{3}\omega_e, \ldots,$ the pendulum will synchronize to this external force and oscillate at the frequency ω_e.

In periodic systems a series of synchronization frequencies exists: the fundamental frequency of the dynamical system, and its subharmonics where synchronization at the fundamental frequency takes place. In periodic systems the synchronization takes place at arbitrarily small values of the amplitude of the external periodic action.

The synchronization has also been observed in nonlinear systems with chaotic behaviour, in which case the synchronization does not begin at an infinitely small amplitude, but synchronization starts at finite values of the amplitude of the external force for all the frequencies at which this phenomenon is observed. The existence of this threshold can be explained by the chaotic nature of the vibrations in these dynamical systems. The synchronization method can be seen as a factor of the ordering in the chaotic motions in nonlinear dynamical systems although these systems tend towards chaotic behaviour.

2.4.2. Amplitude threshold

The amplitude threshold of synchronization is understood to be the minimal value of the amplitude of an external force for which chaotic vibrations are replaced by periodic vibrations, independent of the frequency of the external force acting on the system [7] .

As explained in the subsection above, periodic systems synchronize at an infinitely small amplitude, thus their amplitude threshold equals zero.

2.4.3. Analysing amplitude threshold

To analyse the amplitude threshold the following equation is added to one of the equations of the system

$$A \cos 2\pi f t \qquad\qquad (2.8)$$

where f is the frequency of the external force. For example considering the Lorenz equation
[5]

$$\dot{x} = \sigma(x - y)$$
$$\dot{y} = rx - y - xz \qquad\qquad (2.9)$$
$$\dot{z} = -bx + xy + \boxed{A\cos 2\pi ft}$$

In the two-dimensional space made up by amplitude and frequency of the external force regions can be found where the synchronization effect is observed. This can be done by numerical integration of the system of equations. Figure 2.2a shows an example of obtained results and it also illustrates the definition of the minimal amplitude threshold. In more complex systems several regions of synchronization can be found as synchronization can also be observed at subharmonic vibrations. Figure 2.2b shows a different form of synchronization regions in another system. In domain I the period of the vibrations in the system equals the period of the external force and in domain II the period of vibrations is twice the period of the external force. In the case of multiple regions of synchronization, the amplitude threshold can be defined as [7]

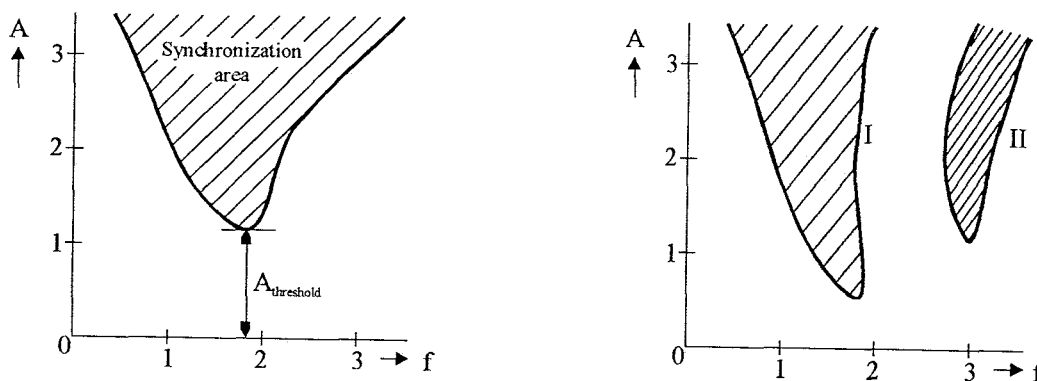$$A_{th} = \min_i A_i \qquad\qquad (2.10)$$

Figure 2.2: Synchronization regions (a) single (b) multiple

The centres of the synchronization domains do not necessarily coincide with the peaks in the power spectrum of the dynamical system. Examples will be shown in chapter 4.

2.5. Relationships between quantifiers for chaos

There exist three different categories of quantifiers for chaotic behaviour, those which depend on the metric properties, those which depend on metric and probabilistic properties of the dynamic system, and those which are defined in terms of dynamical properties of an attractor[13] . The attractor dimension belongs to the first category of measures of chaos which are also called fractal dimensions. The Kolmogorov-Sinai entropy belongs to the second group of quantifiers, also called the dimension of the natural measure. The Lyapunov exponent and the

synchronization threshold depend on the dynamical properties of an attractor. In numerical experiments most quantifiers have shown to relate to each other.

It has been proven by Bezaeva et al. [8] that near the boundary to the transition to chaos the curve of the amplitude threshold for synchronization has the same character as the *local* maximal positive Lyapunov exponent $(max(\lambda^+))$. Thus the Lyapunov exponent can be estimated by calculating the amplitude threshold. λ^+ is the Lyapunov exponent calculated by means of the Benettin algorithm

$$\lambda^+ = \lim_{n \to \infty} \frac{1}{n\tau} \sum_{i=1}^{n} \ln \frac{d_i}{d_0} \qquad (2.11)$$

The relationship between the amplitude threshold (A_t) and the Kolmogorov metric entropy (K_m) has been proven numerically by Kuznetov [6] . This relationship is adequately approximated by the formula

$$A_t = C \cdot K_m^{-\alpha} \qquad (2.12)$$

where C and α are constants for a system. Bazeava [8] has proved that α is not a universal number, as was indicated by Kuznetov [6] . For a dynamical system, the α and C parameters are constants for the equations within the system.

The Kolmogorov entropy is assumed to be equal to the sum of the positive Lyapunov exponents.

$$K_m = \sum_{i=1}^{n} \lambda_i \quad \forall \lambda_i > 0 \qquad (2.13)$$

All the qualifiers equal zero for periodic behaviour and are nonnegative, except for the Lyapunov exponent which can be negative. A negative Lyapunov exponent gives an indication of the stability of a periodical system or parameter in a system. The amplitude threshold is only related to the maximum positive Lyapunov function. It can therefor not serve to estimate all Lyapunov exponents or negative exponents.

2.6. Conclusions

The experiments which have been performed previously [6] [8] show that the amplitude threshold, also called the locking threshold, can indeed serve as a convenient and informative characteristic of chaotic motion in dynamic systems. As the relationships to the other well-known measures of chaos are known, it can also be used to estimate these quantifiers from time series.

3. CHARACTERISTIC FREQUENCIES IN DYNAMICAL SYSTEMS

In this chapter an algorithm for using the synchronization method to measure chaos in a more effective way will be presented.

3.1. Introduction

The major disadvantage of the synchronization method in practice is the fact that the two-dimensional space made up by the amplitude A and frequency f of the synchronizing force has to be scanned to determine the regions of synchronization. The application of this method can be complicated if these parameters cannot easily be swept. A tool for solving this problem is the concept of the characteristic frequency where the assumption is made that the dynamical system will resonantly react the most to a periodical external force if the frequency of this external force equals the frequency at which the system oscillates *most often*.[5]

3.2. The analytic signal

To determine the characteristic frequency the instantaneous frequency time series $-f(t)-$ are needed. Gabor [9] introduced a method to calculate the so-called analytic signal by which means these time series can be obtained.

Let $x(t)$ be the time series of one of the degrees of freedom which has been traced when no synchronizing force was applied to the system. $x(t)$ is a real and causal signal so $x(t) \neq 0$ for $t > 0$ and $x(t) = 0$ for $t \leq 0$. This signal can be decomposed into an even component $x_e(t)$ and an odd component $x_o(t)$ [17] . The two signal components must be chosen in such a way that they cancel out for negative times and be identical for positive times. The following applies for this decomposition :

$$x(t) = 2x_e(t) = 2x_o(t), \ t > 0 \qquad (3.1)$$

as $x_e(t) = x_e(-t)$ and $x_0(t) = -x_0(-t)$. Figure 3.1 shows an example of this decomposition.

If $R(\omega)$ is the Fourier transform of the even decomposition and $I(\omega)$ the Fourier transform of the odd part, it follows

$$\begin{aligned} x(t) &= \tfrac{2}{\pi} \int_0^\infty R(\omega)\cos(\omega t)d\omega \\ x(t) &= \tfrac{2}{\pi} \int_0^\infty I(\omega)\sin(\omega t)d\omega \end{aligned} \Rightarrow t > 0 \qquad (3.2)$$
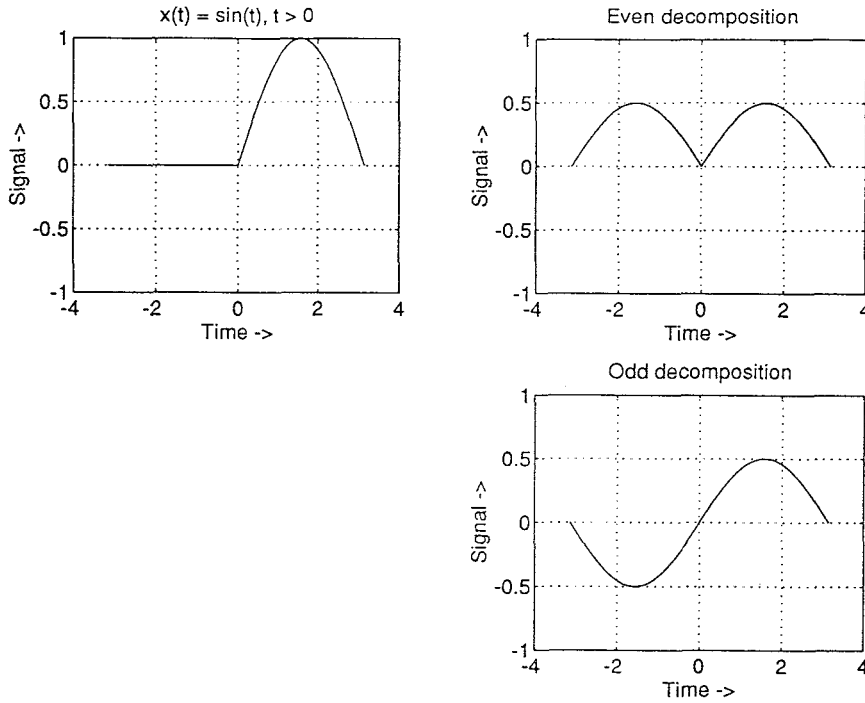
Figure 3.1: Decomposition in even and odd components

From this it can be concluded that $R(\omega)$ and $I(\omega)$, the real and imaginary part of the decomposition, must be related in some way. The Fourier transform of the original causal signal $X(\omega)$ can be calculated by

$$X(\omega) = R(\omega) + jI(\omega) \tag{3.3}$$

The two components are related in the following way (using equation 3.2)

$$x_o(t) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} x_e(\tau) \frac{d\tau}{\tau - t} \tag{3.4}$$

and

$$x_e(t) = -\frac{1}{\pi} \int\limits_{-\infty}^{\infty} x_o(\tau) \frac{d\tau}{\tau - t} \tag{3.5}$$

These relationships are known as the *Hilbert-transforms* [17] . The first relationship (3.4) is called the Hilbert-transform, the second equation (3.5) is the inverse Hilbert-transform. Figure 3.2 illustrates the Hilbert transformation of two arbitrarily chosen functions

$$\begin{aligned} &\text{Function A} : x(t) = \sin(t)\\ &\text{Function B} : x(t) = \frac{1}{1 + \left(\frac{t}{t_0}\right)^2} \end{aligned} \tag{3.6}$$
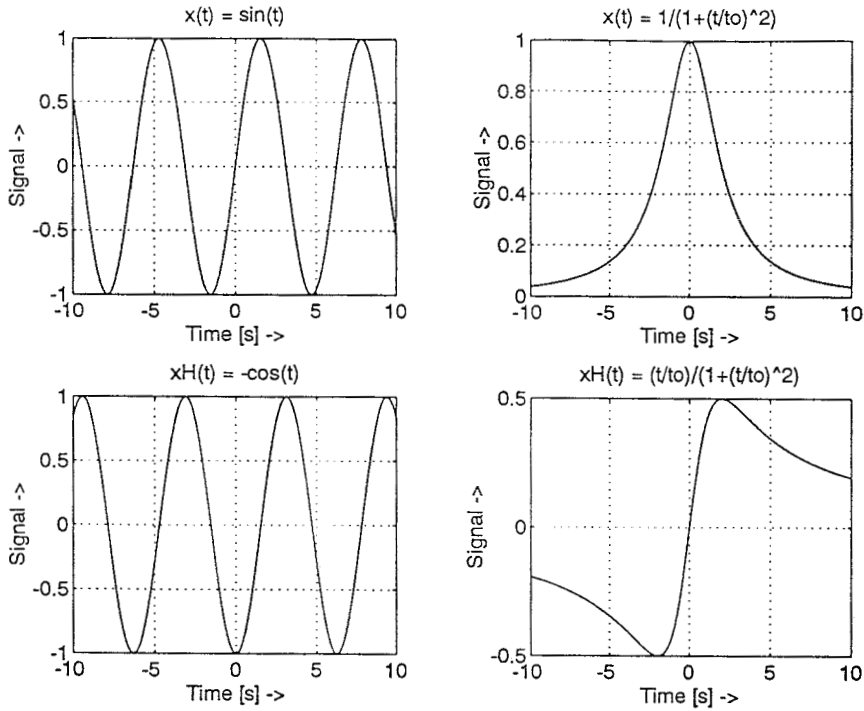
Figure 3.2: Hilbert transforms (a) periodic function (b) non-periodic function

The Hilbert-transforms of the functions A and B are

$$
\begin{aligned}
&\text{Function A}: x^H(t) = -\cos(t) \\
&\text{Function B}: x^H(t) = \frac{(\frac{t}{t_0})}{1+(\frac{t}{t_0})^2}
\end{aligned}
\tag{3.7}
$$

The advantage of the Hilbert transformation is that the two components, the even component and the odd component, have the same spectral components [9] . The analytical signal which is defined as

$$
\Psi(t) = x_e(t) + jx_o(t) = A(t)e^{j\varphi(t)}
\tag{3.8}
$$

also has the same spectral components as the original signal $x(t)$. However, the function $\Psi(t)$ has a real and an imaginary part, which gives the possibility of obtaining phase information. The instantaneous phase can be defined by

$$
\varphi(t) = \tan^{-1} \frac{x_o(t)}{x_e(t)}
\tag{3.9}
$$

and the instantaneous frequency $f(t)$ is the time derivative of the instantaneous phase

$$
f(t) = \frac{d\varphi(t)}{dt} = \frac{\dot{x}_e(t)x_o(t) - x_e(t)\dot{x}_o(t)}{A^2(t)}
\tag{3.10}
$$

So, the Hilbert transformation gives the possibility to obtain phase and frequency informations from the original signal.

This transformation can also be considered to be the convolution of $x(t)$ and $1/\pi t$ as the relationship between the even and the odd components of the signal $x(t)$ is $1/\pi t$. For physically relevant frequencies $\omega > 0$, the Fourier transform of the convolution equals

$$X_o(j\omega) = -jX_e(j\omega) \tag{3.11}$$

so $x_o(t)$ can also be obtained from $x_e(t)$ by a digital filter whose amplitude response equals unity and whose phase response lags $\pi/2$ at all frequencies. The lag of $\frac{\pi}{2}$ at all frequencies and the amplitude response of unity can clearly be seen in figure 3.2a. The original $\sin(t)$-function is transformed to a $-\cos(t)$-function.

In the frequency domain the Hilbert-transform can be described as

> *Suppress the amplitudes belonging to negative frequencies, and multiply the amplitudes of positive frequencies by two* [9] .

This method of calculating the Hilbert-transformation is most used in mathematical software like MATLAB. First the Fast Fourier Transform of the original function is calculated. All the amplitudes for frequencies beneath the folding frequency are multiplied by two, except for the amplitude at frequency 0 [Hz] which is not multiplied. All the amplitudes above the folding frequency are suppressed. Next, the inverse Fast Fourier Transform of this function is calculated. The real part of the inverse Fourier transform equals the original time series $x_e(t)$ and the imaginary part of the inverse Fourier transform equals $x_o(t)$.

3.3. Most occurring frequencies

The next step in this method is to determine the frequency at which the system will oscillate most often when no synchronizing force is being applied to it. The characteristic frequency is the frequency which *occurs most often*, in other words the frequency which takes place more often than other frequencies. It is not the frequency which *will occur most probably*.

The characteristic frequency can be obtained from the distribution function of the instantaneous frequency function $f(t)$. The frequencies with the largest probability is the characteristic frequency. The figures 3.3a, b and c show the times series of a dynamical system with its instantaneous frequency and the relative frequency distribution of the instantaneous frequencies. Again it is stated that the most occurring frequency has no relationship with the peak in the autopower spectrum of the time series. Figure 3.3d shows the autopower spectrum with peak and the location of the characteristic frequency of the system in the spectrum.

Figure 3.4 shows a flowchart which indicates the steps taken to calculate the characteristic frequency starting with time series.

3.4. Conclusion

A disadvantage of the synchronization method, the fact that the two-dimensional space spanned by the A- and f-parameters, must be scanned to obtain the minimal amplitude threshold, can be reduced. By estimating the characteristic frequency from time series, an indication of where the minimal threshold could be found in the frequency domain is obtained. Starting the search for the threshold from this position on might accelerate the synchronization method.
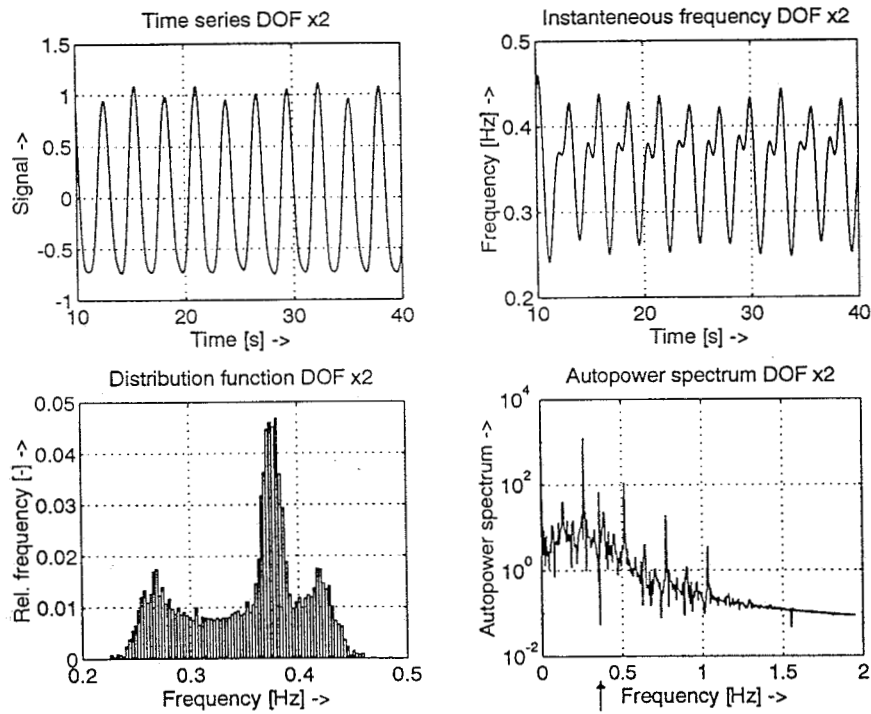
Figure 3.3: Degrees of freedom x1 : (a) Time series (b) Instantaneous frequency (c) Distribution graph (d) Autopower spectrum
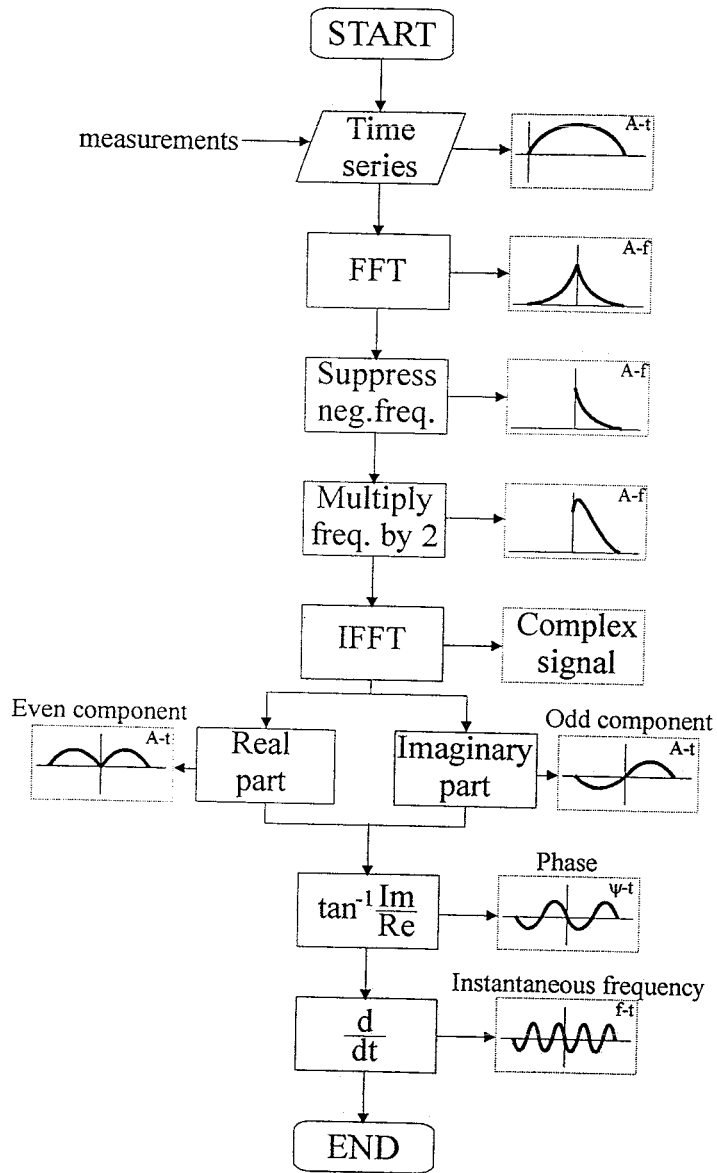
Figure 3.4: Flowchart

4. NUMERICAL EXPERIMENTS

The synchronization method and especially the characteristic frequency have been tested in numerical experiments. In this chapter the results will be presented.

4.1. The model

The model used to verify the theory of the amplitude threshold and the characteristic frequency is [7]

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2\delta_1 \cdot x_2 + \omega_1^2 \cdot x_1 - b_1 \cdot x_1^3 + d_1 \cdot x_3 + m_1 \cdot x_4 \\
\dot{x}_3 &= -x_3 + x_1 - x_1^2 - x_1^3 \\
\dot{x}_4 &= x_5 \\
\dot{x}_5 &= -2\delta_2 \cdot x_5 - \omega_2^2 \cdot x_4 - b_2 \cdot x_4^3 - d_2 \cdot x_6 + m_2 \cdot x_1 \\
\dot{x}_6 &= -x_6 + x_4 - x_4^2 - x_4^3
\end{aligned}
\tag{4.1}
$$

where $m_1 = 0.65$, $m_2 = 1.4$, $\delta_2 = 0.52$, $\omega_1^2 = 1$, $\omega_2^2 = 1.1$, $b_1 = 17.5$, $b_2 = 17.5$, $d_1 = 20$, $d_2 = 20$, and $\delta_1 = 1.0$

This model with 6 degrees of freedom has be chosen because one of the objectives of the numerical experiments is to find out wether the amplitude threshold differs for the different degrees of freedom. Further more the chaotic behaviour of the system can easily be changed by changing the parameter δ_1. In the article from Dykman [7] it is clearly indicated how this parameter influences the behaviour of the system.

4.2. Creating time series

Time series for the dynamic system above have been created by integration. The discrete time steps were set to 0.05 seconds, and the first 100 seconds of the times series were eliminated to avoid any disturbance due to the initial conditions. Time series over 400 seconds were made. Figures 4.1 (a) and (b) show the first and last degree of freedom of the system as a function of time, $x_1(t)$ and $x_6(t)$, over a period of 20 seconds.

In figure 4.2 (a) $x_1(t)$ as a function of $x_2(t)$ is shown and in figure 4.2 (b) $x_4(t)$ as a function of the degree of freedom $x_6(t)$ is shown. Although they might seem to be periodic, they are *not* periodic.

From the last two figures, a chaotic behaviour between these degrees of freedom can be seen, but these figures are no prove of the fact that the whole systems behaves chaotic. In appendix A the time series of each degree of freedom are included. In appendix B all the frequency autopower spectra of the degrees of freedom x1 to x6 are included. From these
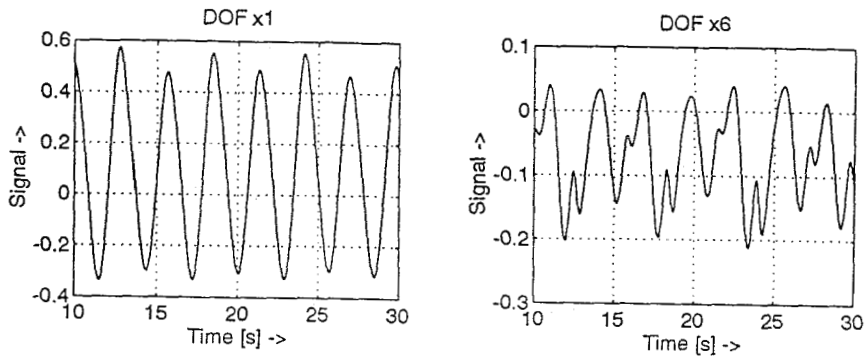
14

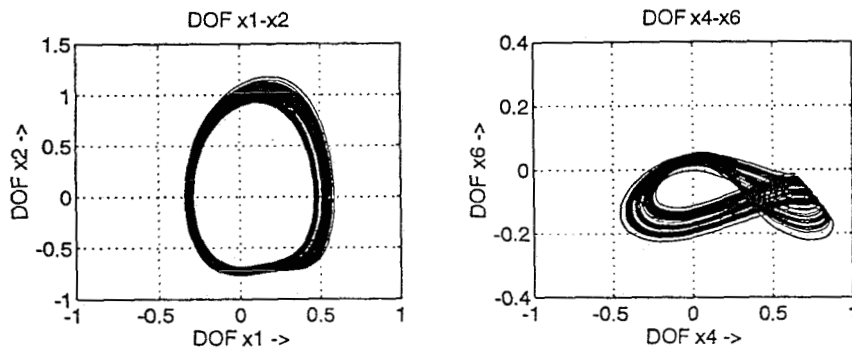Figure 4.1: Time series (a) DOF x1 (b) DOF x6

Figure 4.2: Time series (a) x1-x2 (b) x4-x6

frequency spectra can be concluded that the system behaves chaotically for all six degrees of freedom. Also, all the degrees of freedom x1 to x6 have been plotted by two in all possible combinations, they all show chaotic behaviour.

4.3. Time series analysis

The next step in the experiment was the analysis of each time series. From each time series the Hilbert transformation was calculated using a digital filter with amplitude response unity and a constant phase lag of $\frac{\pi}{2}$ as explained in section 3.2. The instantaneous frequency was obtained from the results of the Hilbert transformation by deriving the phase information in these calculations. The figures 4.3 (a) and 4.3 (b) show the instantaneous frequency $f(t)$ for the time series of the degrees of freedom x_1 and x_6. The instantaneous frequencies of the other
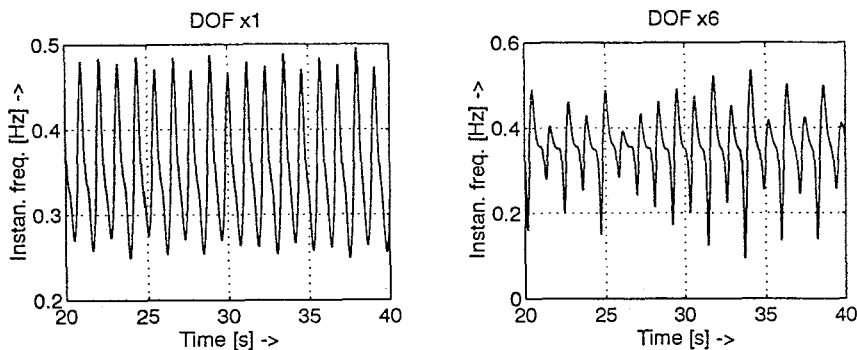
Figure 4.3: Instantanious frequency (a) DOF x1 (b) DOF x6

time series are included in appendix C.

A statistical analysis of the instantaneous frequency gives the frequency which *occurs the most* (not the most probable). The relative frequency distributions of the degrees of freedom x_1 to x_6 are shown on the next page in figure 4.4. It can clearly be seen that the frequency distributions are completely different for the six degrees of freedom. The characteristic frequencies differ from 0.29 up to 0.41 Hz. The table 4.1 shows the characteristic frequencies obtained by numerical simulations.

As stated in chapter 2, the characteristic frequency does not necessarily coincide with the peaks in the power spectrum of the time series. In the figure 4.5 can be seen that the characteristic frequency does not coincide. The first degree of freedom apparently oscillates often at the frequency of 0.285 Hz, but the amplitude at this frequency is relatively small compared to other frequencies.
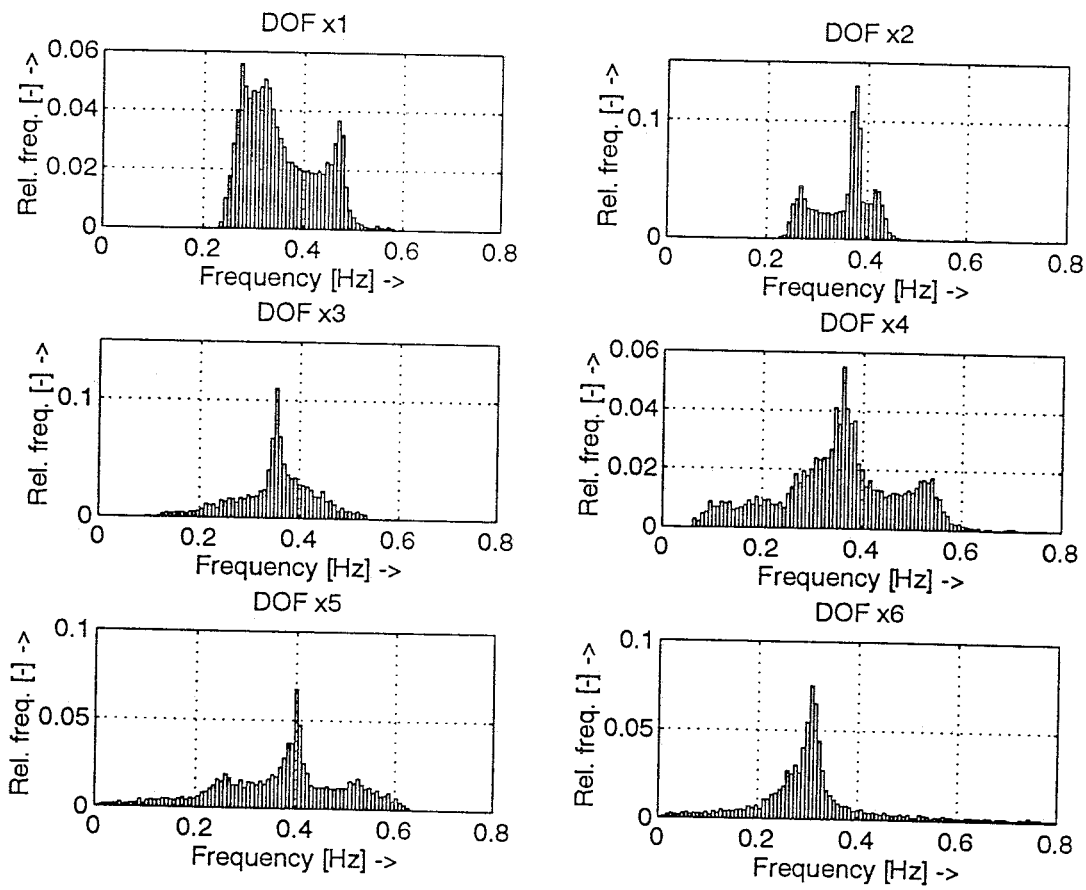
Figure 4.4: Frequency distributions

Analysis numerical experiments - Instanteneous frequency					
DOF	**Charact. freq.**	**Expectation**	**Min**	**Max**	**Peak spectrum**
[-]	[Hz]	[Hz]	[Hz]	[Hz]	[Hz]
x1	0.285	0.355	0.236	1.836	0.218
x2	0.312	0.356	0.227	3.436	0.203
x3	0.350	0.354	0.094	4.449	0.225
x4	0.362	0.355	0.062	1.867	0.213
x5	0.408	0.355	0.006	4.327	0.216
x6	0.319	0.316	0.019	4.125	0.206

Table 4.1: Analysis numerical experiments - Instanteneous frequency
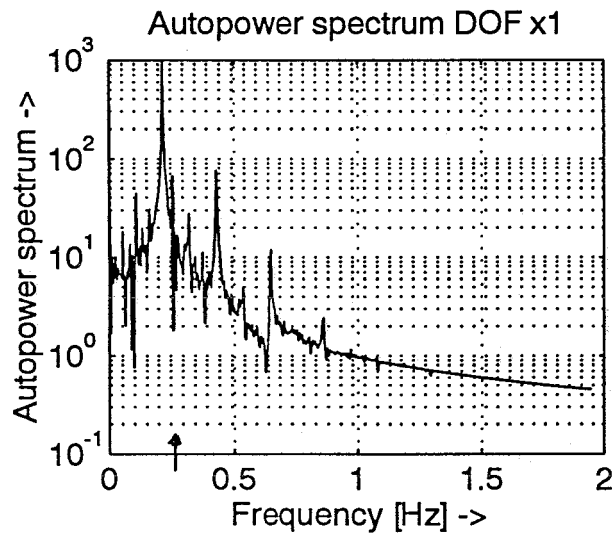
Figure 4.5: Power spectrum and location characteristic frequency

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4.4. Synchronization

The last step in the experiments is the search for the boundaries of the transition to chaos. Two searches have been performed. The first search was a 20 x 20 matrix in the A-f state space around the characteristic frequency found by the time series analysis. The frequency range was chosen from 0 to 1 [Hz] and the amplitude range from 0 to 1 [-]. At certain points in the A-f space, the synchronization effect was very clear, as well in the plots of the degrees of freedom as well as in the frequency spectra of the degrees of freedom. The figures 4.6 (a) and (b) show a plot of two degrees of freedom and the power spectrum of a degree of freedom in a point in a chaotic region. Figures 4.7 (a) and (b) illustrate the plot of two degrees of freedom and
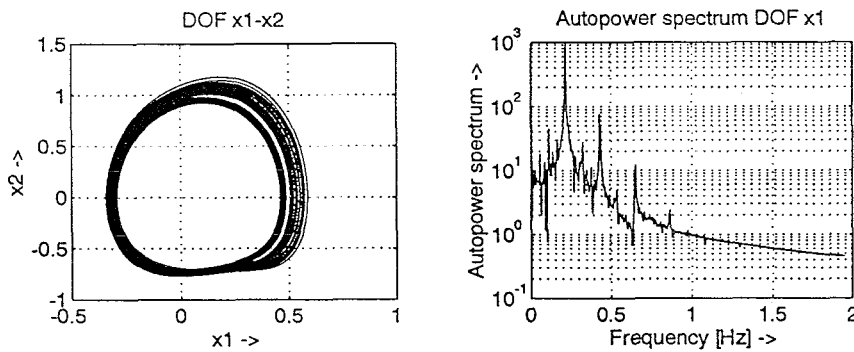
Figure 4.6: Chaotic region (a) Plot DOF x1-x2 (b) Autopower spectrum

the autopower spectrum in the synchronization region An interesting phenomenon that occurs
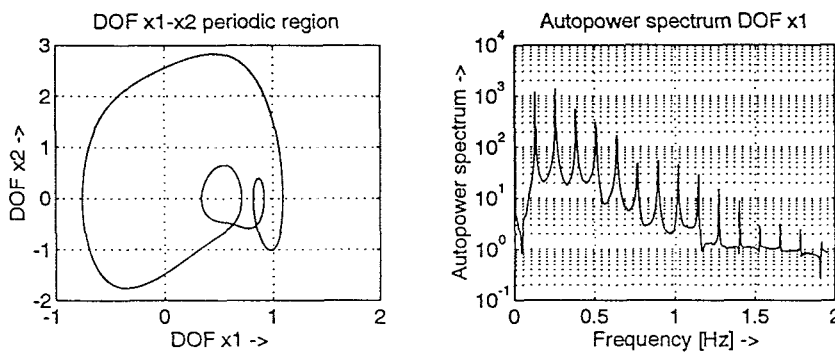
Figure 4.7: Periodic region (a) Plot DOF x1-x2 (b) Autopower spectrum

when applying the synchronization is the fact that there exist multiple synchronization regions. Figure 4.8 shows the same degrees of freedom as in figure 4.7 but in a different synchronization region.
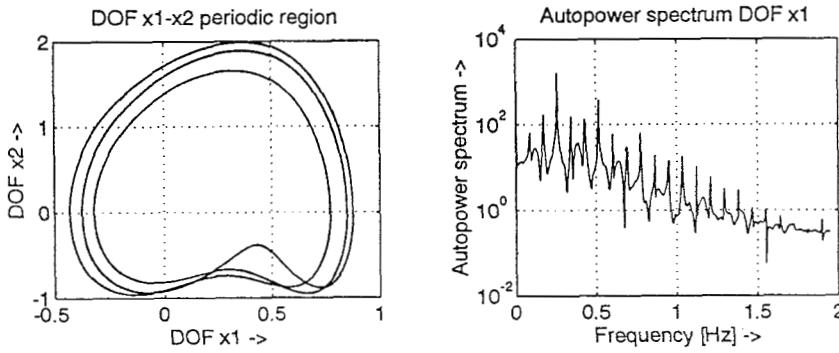

Figure 4.8: Synchronization in different region

This difference is due to the supharmonic synchronization of the degrees of freedom in other regions. From the results of the 400 simulations in the A-f space the map shown in figure 4.9 has been constructed.
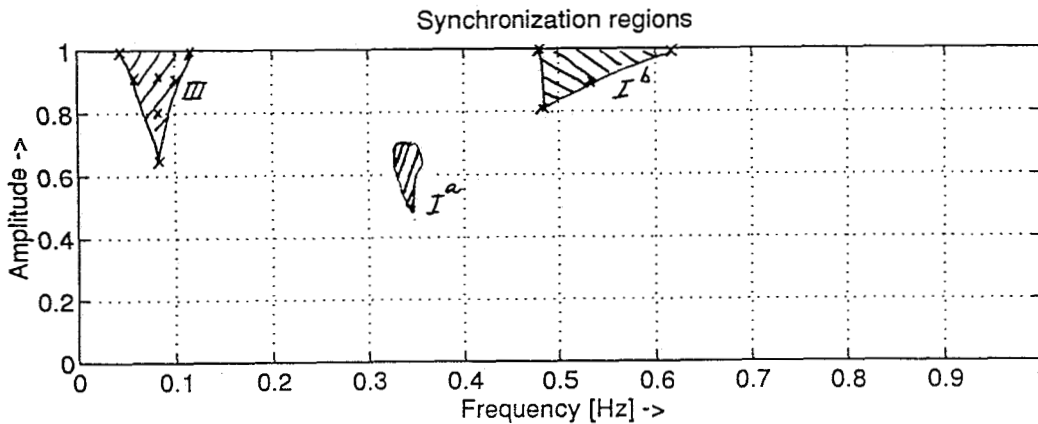

Figure 4.9: A-f synchronization regions

After this general view of the synchronization regions, another series of 400 simulations was performed. As the amplitude threshold was estimated to occur at 0.341 [Hz] (see table 4.1), the frequency range was reduced to from 0.32 [Hz] to 0.37 [Hz]. The range was chosen to be ±10% of the characteristic frequency as this approximate error was found by Rosenblum [5] . The amplitude range was chosen from 0 to 1 [-]. The results are shown in the figure 4.10

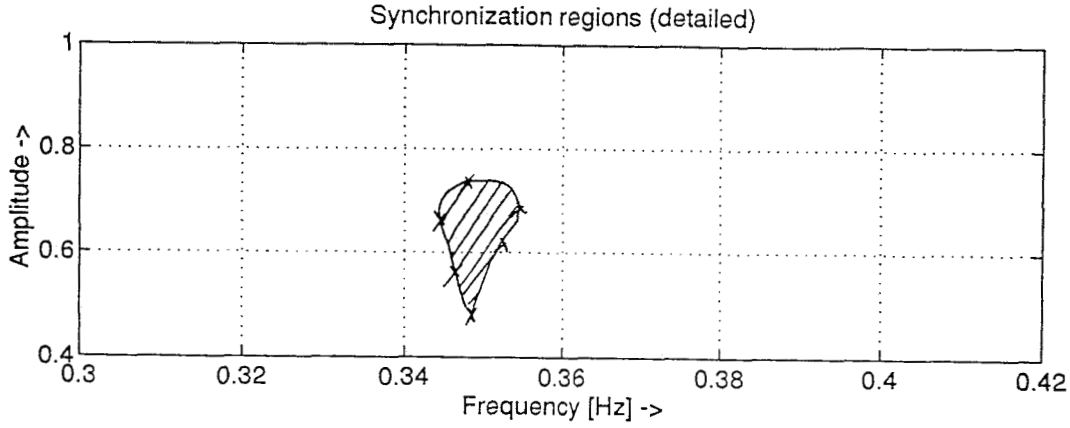which gives more detail as figure 4.9.

Figure 4.10: A-f synchronization regions (detailed)

The synchronization threshold for the second equation in this system was already calculated by Dykman [7] , the results found during his experiments are presented in figure 4.10. The
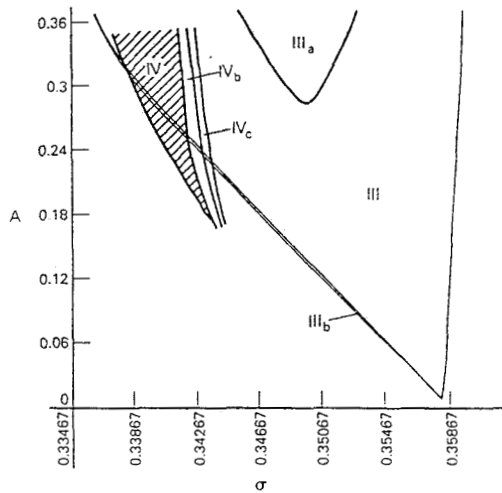

Figure 4.11: Synchronization regions second equation

characteristic frequency equals 0.312 [Hz]. It can be seen in figure 4.11 that it almost coincides with the results from Dykman if an uncertainty of 10% is taken into account.

4.5. Conclusion

The numerical experiments show that the method of calculating the characteristic frequency can be very useful if the synchronization method is used. Although the characteristic frequency ,
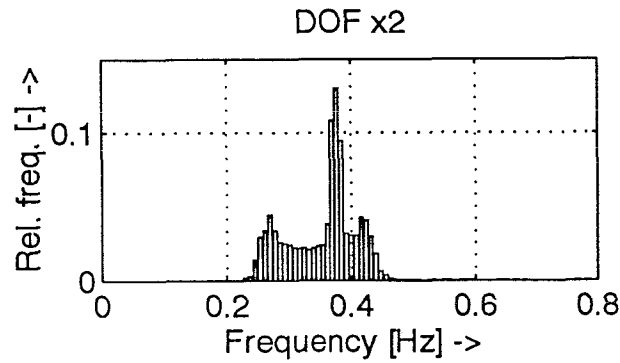
Figure 4.12: Characteristic frequency second parameter

is not the exact frequency at which the amplitude threshold has been found, it gives a good indication with an estimated error of about 10%.

On the other hand the results which where done during this project as well as the results which were obtained by Rosenblum [5] show that if there is a clear peak visible in the autopower spectrum, this peak is located very close to the characteristic frequency. The peak in the autopower spectrum also seem to be a good estimate of the location where the minimal threshold could be found.

5. CONCLUSIONS

The characteristic frequency has numerically proven to be a useful tool when the synchronization method is used to estimate the quantity of chaotic behaviour in a dynamical system. Although the characteristic frequency does not always indicate the exact frequency at which the amplitude threshold will be found, it is a good and helpful estimate.

The amplitude threshold is related to other measures of chaotic behaviour. As the amplitude threshold can be estimated by time series from experiments in contrast to the other qualifiers, this method of quantifying can be useful. How ever, still large computer times are needed to obtain an accurate estimate of the quantity of chaos.

Too few experiments have been performed to make any conclusion on the advantages over the other methods for quantifying chaos. As the method seem to be interesting, more experiments and comparisons with other quantifiers might be worthwhile.

6. RECOMMENDATIONS

The following recommendations can be made based on the study of the backgrounds of the theory and the experiments that have been performed :

- a study on the possibilities to estimate a *characteristic amplitude.* Now it is still necessary to estimate the position of the amplitude threshold in the A-space by trial and error. An indication at what amplitude level the amplitude threshold can be found would be very interesting to reduce more the computer times needed,

- an algorithm that tracks the boundary between chaotic and periodic behaviour once a point on it has been found. It is not interesting to keep trying large array in the A-v space as only the boundaries are interesting,

- more experiments as to estimate the error between the estimated characteristic frequency and the actual frequency at which the amplitude threshold has been found,

- a study on the errors which are made by choosing a certain class-size when estimating the probability distribution of the characteristic frequency,

- a number of experiments to calculate the computer times needed to estimate the quantity of chaos by means of the synchronization method and another method for example the method proposed by Darbyshire[2] . From these experiments the real advantage of the amplitude threshold could be obtained,

- and as a last point but quite important, a simple experiment to test the synchronization method in practice.
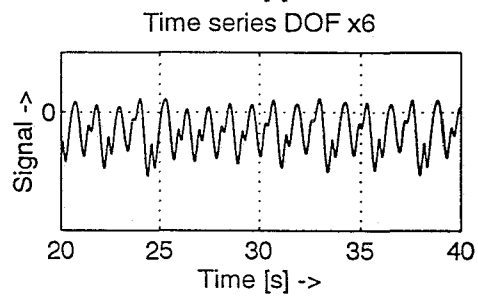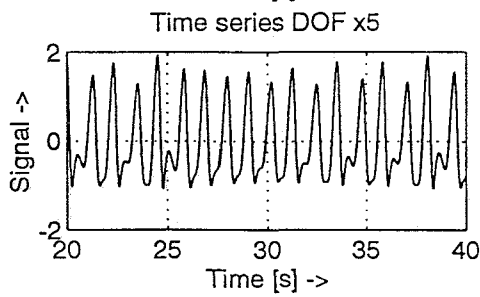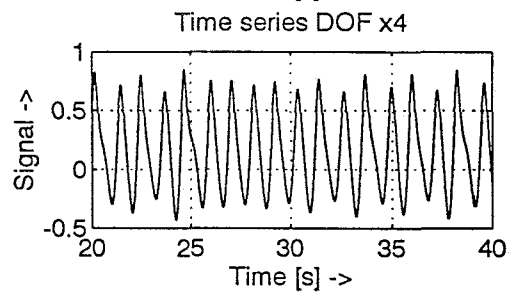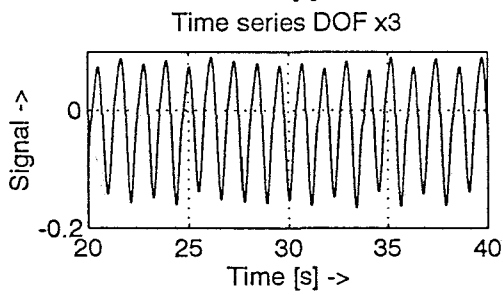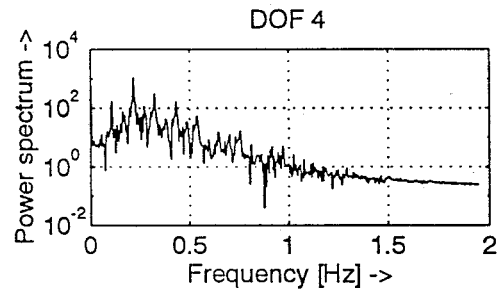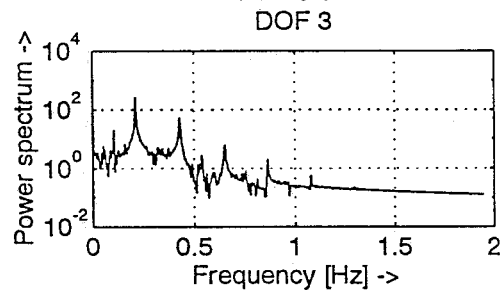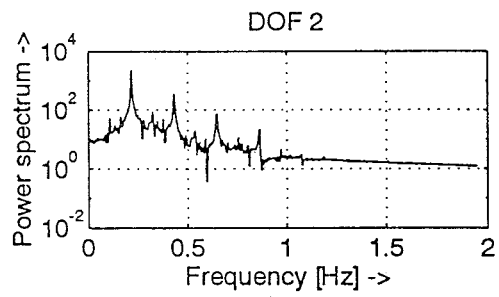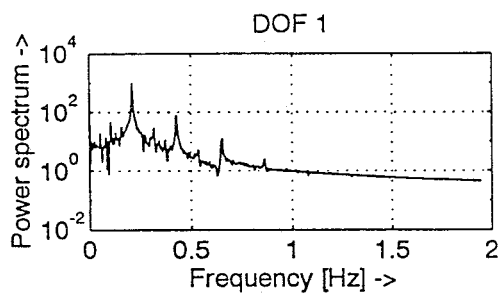
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A. Time series

Time series DOF x1

Time series DOF x2

Time series DOF x3

Time series DOF x4

Time series DOF x5

Time series DOF x6

B. FREQUENCY SPECTRA

DOF 1

DOF 2

DOF 3

DOF 4

DOF 5

DOF 6

C. INSTANTANEOUS FREQUENCIES

DOF x1

DOF x2

DOF x3

DOF x4

DOF x5

DOF x6