A behavioral approach to balanced representations of dynamical systems

Research Report
ISSN 0167-9708
Coden: TEUEDE

Eindhoven
University of Technology
Netherlands

Faculty of Electrical Engineering

A Behavioral Approach to Balanced Representations of Dynamical Systems

by

Siep Weiland

EUT Report 93-E-277
ISBN 90-6144-277-X

August 1993

Eindhoven University of Technology Research Reports

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Faculty of Electrical Engineering
Eindhoven, The Netherlands

ISSN 0167-9708 Coden: TEUEDE

A BEHAVIORAL APPROACH TO BALANCED REPRESENTATIONS OF DYNAMICAL SYSTEMS

by

Siep Weiland

EUT Report 93-E-277
ISBN 90-6144-277-X

Eindhoven
August 1993

CIP-DATA KONINKLIJKE BIBLIOTHEEK, DEN HAAG

Weiland, Siep

A behavioral approach to balanced representations of
dynamical systems / by Siep Weiland. - Eindhoven :
Eindhoven University of Technology, Faculty of Electrical
Engineering. - (EUT report, ISSN 0167-9708 ; 93-E-277)
With ref.
ISBN 90-6144-277-X
NUGI 832
Subject headings: linear systems / systems theory /
balanced representations.

A Behavioral Approach to Balanced Representations of Dynamical Systems

Siep Weiland

Measurement and Control Group
Department of Electrical Engineering
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven,
the Netherlands

Abstract

The behavioral approach to linear systems provides an alternative framework for studying the notion of balanced representations. A new definition for balanced representations is proposed that is one-to-one related to a set of system invariants that is obtained by assuming a specific Hilbert space structure on the system behavior. This notion of balancing is more general than the prevailing notion of balancing in that it is well-defined for non-stable systems, and is independent of a particular (input-output) representation of the system. It is shown that Lyapunov, LQG, and H_∞ balanced representations are obtained as a special case. An application for the problem of model approximation is discussed.

Keywords

Linear systems, balanced representations, model approximation, behavioral theory.

Contents

1 Introduction

This paper addresses the concept of balancing for dynamical systems. Since the introduction in 1978 [9], balanced representations of linear time-invariant systems have proved to be extremely useful in a wide range of applications including model reduction, signal processing, controller design, stochastic realization, system identification and problems related to data reduction. The usual concept of balancing amounts to making a specific choice of coordinates in the state space of a linear time-invariant dynamical system so that the controllability and observability gramians of the system are equal and diagonal [9], [10]. In balanced coordinates, the state of the system is structured in the sense that each state component quantifies to what extent it contributes to the interaction of the system with its environment.

This concept of balanced model representations has led to a straightforward method of model approximation. Without performing further calculations, approximate models may be obtained by discarding those state components of a balanced representation that contribute least to the dynamical relationships between the exogenous variables of the system. See, e.g., [16]. Other applications include the theory of optimal Hankel norm approximations [2], [5], stochastic realization theory [1], and the study of canonical forms [13], [14].

An important drawback of the prevailing concept of balanced representations is that it is only applicable for asymptotically stable systems. Obviously, the stability hypothesis is a very restrictive assumption and prevents applications for many models considered in areas such as controller design, filter design, identification, etc. In the recent past, alternative notions of balancing have been introduced to circumvent this problem. Among these, the most important ones include LQG and H_∞ balanced representations. See, e.g., [3], [4], [15], [12].

In this paper we discuss the concept of balanced representations using the behavioral framework for linear systems as a starting point. We refer to [19], [21] and the references therein for a detailed account on this framework. The main advantage of the approach taken here is that it avoids to study the concept of a balanced state space starting from particular representations or assumptions on representations of dynamical systems. For the class of finite dimensional linear time invariant systems we show that a Hilbert space structure on the exogenous trajectories of a system leads to state space representations in which external characteristics of the system can be naturally reflected by balanced state variables. This leads to an abstract and more general notion of a balanced state space that can be viewed independent of the equations that define a state space representation. Since no reference to a particular state space representation needs to be made this concept applies equally to standard input-state-output systems, systems in descriptor form, driving variable state space representations, etc. Both LQG (or Riccati) balancing, as well as the more recent notion of H_∞ balancing, are obtained as special cases of our setting.

The paper is organized as follows. In section 2 we introduce some notation and we briefly review various concepts from the behavioral framework. In section 3 we consider the model class consisting of square integrable trajectories of linear time-invariant finite-dimensional systems. For this class of systems the notion of a balanced state space is defined by considering operators defined on the external behavior of the system. The structure of specific state space representations is analysed in section 4. In particular, in section 4 we derive LQG, H_∞ and Lyapunov balanced representations as a special case. An application to the problem of model approximation is given in section 5.

2 Preliminaries

2.1 Dynamical systems

Following the framework introduced by Willems in [19], [20], a *dynamical system* is a triple $\Sigma = (T, W, \mathcal{B})$ with $T \subseteq \mathcal{R}$ the *time axis*, W the *signal space*, and $\mathcal{B} \subseteq W^T$ the *behavior*, a subset of the family of all trajectories $w : T \to W$. In this paper we will restrict the attention to continuous time systems with time set $T = \mathcal{R}$. For the signal space we take the $q-$variate real vector space $W = \mathcal{R}^q$ with $q > 0$ a fixed number. The system $(\mathcal{R}, \mathcal{R}^q, \mathcal{B})$ is said to be *time-invariant* if $\sigma^t \mathcal{B} = \mathcal{B}$ for all $t \in \mathcal{R}$, where $\sigma^t : W^T \to W^T$ is the $t-$shift $\sigma^t w(t') = w(t + t')$. We call it *linear* if $\mathcal{B}$ is a linear subspace of $(\mathcal{R}^q)^{\mathcal{R}}$.

We will be interested in systems that can be described by a finite number of differential equations. Let $R(s) \in \mathcal{R}^{\cdot \times q}[s]$ be a polynomial matrix with a finite number of rows, q columns and with real coefficients. Consider the behavioral differential equation

$$R(\frac{d}{dt})w = 0. \tag{2.1}$$

This yields the linear time-invariant system $\Sigma = (\mathcal{R}, \mathcal{R}^q, \mathcal{B}(R))$ with

$$\mathcal{B}(R) := \{ \ w : \mathcal{R} \to \mathcal{R}^q \mid w \in \mathcal{L}^{loc} \text{ and (2.1) holds } \}.$$

Here, $\mathcal{L}^{loc}$ is the class of locally integrable vector valued functions and the differential operator $R(\frac{d}{dt})$ is viewed as an operator defined on the space of $q-$dimensional distributions on $\mathcal{R}$. The class of systems which we will study in this paper is given by all such behaviors and will be denoted by $\mathbf{B}$, i.e.,

$$\mathbf{B} := \{\mathcal{B} \mid \exists R \in \mathcal{R}^{\cdot \times q}[s] \text{ such that } \mathcal{B} = \mathcal{B}(R)\}$$

The restrictions $w^- := w|_{(-\infty,0)}$ and $w^+ := w|_{[0,\infty)}$ of a trajectory $w : \mathcal{R} \to \mathcal{R}^q$ are called the *past* and *future* of w respectively. Similarly, $\mathcal{R}^-$ and $\mathcal{R}^+$ will denote the half lines $(-\infty, 0)$ and $[0, \infty)$, respectively. The *past* and *future behavior* of a dynamical system Σ are defined as $\mathcal{B}^- := \mathcal{B}|_{(-\infty,0)}$ and $\mathcal{B}^+ := \mathcal{B}|_{[0,\infty)}$, where $\mathcal{B}$ is the behavior of Σ.

For a trajectory $w \in \mathcal{B}$, we denote by $\mathcal{B}^+(w^-)$ the set of *continuations* of the past w^- of w that belong to $\mathcal{B}$. The set of *antecedents* of the future w^+ in $\mathcal{B}$ will be denoted by $\mathcal{B}^-(w^+)$. Formally,

$$\begin{aligned} \mathcal{B}^+(w^-) &:= \{\tilde{w} \in \mathcal{B}^+ \mid w^- \wedge_0 \tilde{w} \in \mathcal{B}\} \\ \mathcal{B}^-(w^+) &:= \{\tilde{w} \in \mathcal{B}^- \mid \tilde{w} \wedge_0 w^+ \in \mathcal{B}\} \end{aligned}$$

Here, $\wedge_t$ denotes the concatenation product

$$(w_1 \wedge_t w_2)(t') := \begin{cases} w_1(t') & \text{if } t' < t \\ w_2(t') & \text{if } t' \geq t \end{cases}.$$

Hence, $\mathcal{B}^+(w^-)$ consists of all futures which are compatible with the past of w, while $\mathcal{B}^-(w^+)$ consists of all past trajectories which are compatible with the future of w. Finally, a time-invariant behavior $\mathcal{B}$ is called *controllable* if for all $w^- \in \mathcal{B}^-$ and $w^+ \in \mathcal{B}^+$ there exists a $\tilde{w} \in \mathcal{B}$ and $T \geq 0$ such that $\tilde{w}^- = w^-$ and $\tilde{w}(t) = w^+(t-T), t \geq T$.

2.2 State space systems

State space systems will play an important role in the sequel. We will view a state space system as a special case of a system with *latent variables*. As opposed to external (or manifest) variables, latent variables should be viewed as internal (or auxiliary) quantities that serve to provide an implicit description of a system. We formalize this as follows. A quadruple $\Sigma_l = (T, W, L, \mathcal{B}_l)$, with T, W as before, L a set of *latent variables* and $\mathcal{B}_l \subseteq (W \times L)^T$ is called a *dynamical system with latent variables.* If Σ_l is such a system, then the system $\Sigma = (T, W, \mathcal{B})$ with

$$\mathcal{B} := \{w \in W^T \mid \exists l \in L^T \text{ such that } (w, l) \in \mathcal{B}_l\}$$

is said to be *induced* by Σ_l. Consider a time-invariant latent variable system Σ_l and consider the set

$$\mathcal{B}(l_0) := \{w \in W^T \mid \exists l \in L^T, l(0) = l_0, (w, l) \in \mathcal{B}_l\}.$$

Then clearly $\mathcal{B}(l_0) \subseteq \mathcal{B}(l_0)^- \wedge_0 \mathcal{B}(l_0)^+$. In case equality holds we have the property that a trajectory $w := w^- \wedge_0 w^+ \in \mathcal{B}(l_0)$ whenever $w^- \in \mathcal{B}(l_0)^-$ and $w^+ \in \mathcal{B}(l_0)^+$, i.e., the variable l_0 will *split* the trajectories w^- and w^+. A system with this property is called a *splitting variable system.* This is of course closely related to the intuitive notion of *state.* Let $\Sigma_s = (T, W, X, \mathcal{B}_s)$ be a dynamical system with latent variables. We will call Σ_s a *state space system* with *state space* X if the following implication is satisfied

$$\{(w_1, x_1), (w_2, x_2) \in \mathcal{B}_s, t \in T, x_1(t) = x_2(t)\} \Longrightarrow \{(w_1, x_1) \wedge_t (w_2, x_2) \in \mathcal{B}_s\} \tag{2.2}$$

A state space system $\Sigma_s = (T, W, X, \mathcal{B}_s)$ is said to *represent* a system $\Sigma = (T, W, \mathcal{B})$, if Σ is induced by Σ_s. In that case, we call Σ_s a *state space representation* of Σ. Hence, any splitting variable system that satisfies (2.2) will be viewed as a state space system.

The state x in a time-invariant state space behavior $\mathcal{B}_s$ is called *past induced* (*future induced*) if there exists a partial map $f_- : W^{T^-} \to X$ (resp. $f_+ : W^{T^+} \to X$) such that for any $(w, x) \in \mathcal{B}_s$ the restriction w^- is in the domain of f_- and $x(0) = f_-(w^-)$ (resp. $w^+ \in \mathrm{Dom}(f_+)$ while $x(0) = f_+(w^+)$). In fact, Theorem 3.2 shows that each system $\mathcal{B} \in \mathbf{B}$ admits a state space representation which is both past and future induced. That is, it has the property that

$$\{ \ w \in \mathcal{B} \ \} \Longleftrightarrow \{ \ f_-(w^-) = f_+(w^+) \ \}$$

We emphasize that in our definition of a state space system, no reference to specific equations is made. In fact, this level of generality turns out to be a useful starting point to define a concept of a balanced state space. It will be shown that systems in the model class $\mathbf{B}$ admit a wide variety of linear time-invariant state space representations. In section 4 of this paper we will consider a few specific ones. See ([20]) for more details.

3 Balanced Representations

Let $\Sigma = (\mathcal{R}, \mathcal{R}^q, \mathcal{B})$ be a dynamical system and assume that $\mathcal{B} \in \mathbf{B}$. We will distinguish between the past and future behaviors $\mathcal{B}^-$ and $\mathcal{B}^+$ in that we examine the relative effect of past trajectories $w^- \in \mathcal{B}^-$ on their associated set of continuations $\mathcal{B}^+(w^-)$. For this purpose, we equip the past and future behavior of $\mathcal{B}$ with the structure of a Hilbert space and we introduce two operators which, in a sense, reflect the minimal dynamical effect which a past (future) trajectory exhibits on its set of compatible continuations (antecedents).

Consider the subsets $\mathcal{B}^-$ and $\mathcal{B}^+$. Introduce inner products

$$\begin{aligned} \langle \cdot, \cdot \rangle_- \ &: \ \mathcal{B}^- \times \mathcal{B}^- \to \mathcal{R} \\ \langle \cdot, \cdot \rangle_+ \ &: \ \mathcal{B}^+ \times \mathcal{B}^+ \to \mathcal{R} \end{aligned}$$

on $\mathcal{B}^-$ and $\mathcal{B}^+$, and assume that both $(\mathcal{B}^-, \langle \cdot, \cdot \rangle_-)$ and $(\mathcal{B}^+, \langle \cdot, \cdot \rangle_+)$ are Hilbert spaces. Hence, we assume that both $(\mathcal{B}^-, \langle \cdot, \cdot \rangle_-)$ and $(\mathcal{B}^+, \langle \cdot, \cdot \rangle_+)$ are complete positive definite inner product spaces. The induced norms on $\mathcal{B}^-$ and $\mathcal{B}^+$ will be denoted by $\| \cdot \|_-$ and $\| \cdot \|_+$, respectively.

Let $\Gamma_- : \mathcal{B}^- \to \mathcal{B}^+$ and $\Gamma_+ : \mathcal{B}^+ \to \mathcal{B}^-$ be defined as

$$\Gamma_-(w) \ := \ \arg\min \{ \ \| \tilde{w} \|_+ \mid \tilde{w} \in \mathcal{B}^+(w) \ \} \tag{3.1}$$

$$\Gamma_+(w) \ := \ \arg\min \{ \ \| \tilde{w} \|_- \mid \tilde{w} \in \mathcal{B}^-(w) \ \}. \tag{3.2}$$

Hence, Γ_- assigns to a trajectory $w \in \mathcal{B}^-$ the 'optimal response' or 'optimal continuation' $\tilde{w} \in \mathcal{B}^+$ which is compatible with w^-. A similar interpretation applies for Γ_+. The operators Γ_- and Γ_+ are unambiguously defined as is claimed by the following result.

Theorem 3.1 *Γ_- and Γ_+ are well defined, linear, bounded and continuous.*

For a proof we refer to [17] or the proof of Theorem 3.4. Since $\mathcal{B} \in \mathbf{B}$, the system Σ admits a linear time-invariant state space representation $\Sigma_s = (\mathcal{R}, \mathcal{R}^q, X, \mathcal{B}_s)$ with finite dimensional state space $X = \mathcal{R}^n$. See, e.g., [19]. Let Σ_s be such a representation and consider its state behavior $\mathcal{B}_s$. Let $x_0 \in X$ and denote by $\mathcal{B}(x_0)$ the set of all trajectories in $\mathcal{B}$ whose corresponding state trajectory passes through x_0 at time $t = 0$. Formally, define

$$\mathcal{B}(x_0) := \{w \in \mathcal{B} \mid \exists x : \ (w, x) \in \mathcal{B}_s \text{ and } x(0) = x_0\}$$

Clearly, $\mathcal{B}(x_0)$ may be empty in case no state trajectory passes through $x_0 \in X$. Note that

$$\mathcal{B} = \cup_{x \in X} \mathcal{B}(x)$$

and it should be observed that $\mathcal{B}(x_1)$ and $\mathcal{B}(x_2)$ may have a non-empty intersection whenever $x_1 \neq x_2$.

We first claim that for *minimal* state space representations of Σ, each $w \in \mathcal{B}$ *uniquely* determines an element $x_0 \in X$ such that $w \in \mathcal{B}(x_0)$. This means that we can retrieve the state vector $x(0)$ from observations on the external trajectories only. In fact, the state $x(0)$ can be retrieved from both past and future observations on the external trajetories, as is shown in the following theorem.

Theorem 3.2 *Let $\Sigma_s = (\mathcal{R}, \mathcal{R}^q, \mathcal{R}^n, \mathcal{B}_s)$ be a linear time-invariant state space representation of $(\mathcal{R}, \mathcal{R}^q, \mathcal{B})$, where $\mathcal{B} \in \mathbf{B}$. If the dimension n of the state space X of Σ_s is minimal among all state space representations of $\mathcal{B}$, then there exist linear surjective mappings $f_- : \mathcal{B}^- \to X$ and $f_+ : \mathcal{B}^+ \to X$ such that for all $x_0 \in X$*

$$\{ \ w \in \mathcal{B}(x_0) \ \} \iff \{ \ f_-(w^-) = x_0 = f_+(w^+) \ \}. \tag{3.3}$$

Proof. First observe that state minimality of Σ_s implies that

$$\{\mathcal{B}(x_1) = \mathcal{B}(x_2)\} \Rightarrow \{x_1 = x_2\}.$$

Infer from (2.2) that $\forall x_0 \in X, \ \ \mathcal{B}(x_0) = (\mathcal{B}(x_0))^- \wedge_0 (\mathcal{B}(x_0))^+$. Consequently,

$$\begin{aligned} \{(\mathcal{B}(x_1))^- = (\mathcal{B}(x_2))^-\} \quad &\Rightarrow \quad \{x_1 = x_2\} \\ \{(\mathcal{B}(x_1))^+ = (\mathcal{B}(x_2))^+\} \quad &\Rightarrow \quad \{x_1 = x_2\}. \end{aligned}$$

Equivalently, there exist mappings $f_- : \mathcal{B}^- \to X$ and $f_+ : \mathcal{B}^+ \to X$ such that (3.3) holds. Obviously, f_- and f_+ are linear by linearity of Σ_s and surjective as

$$\{x_0 \in X\} \Rightarrow \{\mathcal{B}(x_0) \neq \emptyset\}$$

by state minimality of Σ_s. □

The mappings f_- and f_+ have the interpretation to access the state of the system from past and future trajectories in the manifest (or external) behavior. Note that, since $\mathcal{B} = \cup_{x \in X} \mathcal{B}(x)$, the equivalence (3.3) implies that

$$\{\ w \in \mathcal{B}\ \} \Longleftrightarrow \{\ f_-(w^-) = f_+(w^+)\ \} \tag{3.4}$$

which shows that the common features of both past and future trajectories of $\mathcal{B}$ are reflected by means of the mappings f_- and f_+.

Let Σ_s be a state minimal representation of Σ and suppose that the mappings $f_- : \mathcal{B}^- \to X$ and $f_+ : \mathcal{B}^+ \to X$ are given as in Theorem 3.2. Let X^* be the algebraic dual of X, i.e., X^* consists of all bounded linear functionals defined on X. Then, clearly, the dual mappings f_-^* and f_+^* are well defined on X^*. We will be interested in the composite maps

$$\begin{aligned} f_- f_-^* &: X^* \to X \\ f_+ f_+^* &: X^* \to X. \end{aligned}$$

The following result will be used to define a balanced state space.

Theorem 3.3 *Let f_- and f_+ be as in Theorem 3.2 and let $\mathcal{B}^-$ and $\mathcal{B}^+$ be Hilbert spaces. Then the composite maps $f_- f_-^* : X^* \to X$ and $f_+ f_+^* : X^* \to X$ are nonsingular.*

Proof. Consider $f_- f_-^* : X^* \to X$ and let $x^* \in X^*$ be such that $f_- f_-^* x^* = 0$. It suffices to show that $x^* = 0$. To see this, observe that

$$\{f_- f_-^* x^* = 0\} \Rightarrow \{\langle f_-^* x^*, f_-^* x^* \rangle_- = 0\} \Rightarrow \{f_-^* x^* = 0\} \Rightarrow \{\text{im } f_- \subseteq \ker x^*\} \Rightarrow \{x^* = 0\}$$

where the last implication follows by surjectivity of f_-. Hence, $f_- f_-^*$ is nonsingular. The non-singularity of $f_+ f_+^*$ follows in a similar way. □

Hence, by Theorem 3.3, we have that

$$P := (f_- f_-^*)^{-1} : X \to X^* \tag{3.5}$$
$$Q := (f_+ f_+^*)^{-1} : X \to X^* \tag{3.6}$$

are well defined, real symmetric and positive definite operators. Hence, the mappings $f_- f_-^*$ and $f_+ f_+^*$ together with their inverses induce a natural identification between the state space X and its dual X^*. This leads in a natural way to inner products on the state space X by defining the quadratic forms

$$\begin{aligned} \langle x_1, x_2 \rangle_P &:= x_2^T P x_1 \\ \langle x_1, x_2 \rangle_Q &:= x_2^T Q x_1 \end{aligned}$$

We will refer to P and Q as the *past* and *future gramian* of Σ_s. We claim that both Γ_- and Γ_+ have discrete spectra whose nonzero elements can be expressed in terms of eigenvalues of the gramians (3.5) and (3.6).

Theorem 3.4 *Let f_- and f_+ be as in Theorem 3.2 and let the gramians P and Q be given by (3.5) and (3.6). Then,*

1. $\Gamma_- = f_+^*(f_+f_+^*)^{-1}f_-$.
2. *The spectrum $\sigma(\Gamma_-)$ of Γ_- is a pure point spectrum and the non-zero spectral values of Γ_- are given by $\lambda^{1/2}(P^{-1}Q)$.*
3. $\Gamma_+ = f_-^*(f_-f_-^*)^{-1}f_+$.
4. *The spectrum $\sigma(\Gamma_+)$ of Γ_+ is a pure point spectrum and the non-zero spectral values of Γ_+ are given by $\lambda^{1/2}(PQ^{-1})$.*

Proof. The proof is based on various results in least squares optimization theory. To prove statement 1, let $w \in \mathcal{B}^-$ and define $x := f_-(w)$. Then $\mathcal{B}^+(w) = (\mathcal{B}(x))^+$ are the continuations of w and, by Theorem 3.2, $\{\tilde{w}^+ \in \mathcal{B}^+(w)\} \Leftrightarrow \{f_+(\tilde{w}^+) = x\}$. Define

$$w^* := f_+^*(f_+f_+^*)^{-1}x = f_+^*(f_+f_+^*)^{-1}f_-w$$

which is well defined by Theorem 3.3, and observe that for any $\tilde{w} \in \mathcal{B}^+(w)$, $\tilde{w} \neq w^*$,

$$\begin{aligned} \langle \tilde{w},\, \tilde{w}\rangle_+ - \langle w^*,\, w^*\rangle_+ &= \langle \tilde{w},\, \tilde{w}\rangle_+ - 2\,\mathrm{Re}\langle \tilde{w},\, w^*\rangle_+ + \langle w^*,\, w^*\rangle_+ - 2\,\mathrm{Re}\langle w^* - \tilde{w},\, w^*\rangle_+ \\ &= \langle \tilde{w} - w^*,\, \tilde{w} - w^*\rangle_+ \;>\; 0, \end{aligned}$$

where we used that $\langle \tilde{w} - w^*,\, w^*\rangle_+ = 0$. Hence, w^* is the *unique* element in $\mathcal{B}^+(w)$ with the property that

$$w^* = \arg\min\,\{\|\,\tilde{w}\,\|_+ \mid \tilde{w} \in \mathcal{B}^+(w)\}.$$

(In fact, this shows that Γ_- is well defined as claimed by Theorem 3.1). As $w \in \mathcal{B}^-$ is arbitrary, this yields that $\Gamma_- = f_+^*(f_+f_+^*)^{-1}f_-$ as claimed.

2. Infer from statement 1 that $\Gamma_- = f_+^*Qf_-$ is a finite rank operator. Consequently, the spectrum of Γ_- is a countable set and every spectral value $0 \neq \sigma_i \in \sigma(\Gamma_-)$ is an eigenvalue of Γ_-. (See [6]). Suppose that σ_i is a singular value and w_i is a corresponding

singular vector of $\Gamma_-^*\Gamma_-$, i.e., $\Gamma_-^*\Gamma_- w_i = \sigma_i w_i$. Let $x_i := f_-(w_i)$ and note that $PQ^{-1}x_i = f_-\Gamma_-^*\Gamma_- w_i = \sigma_i x_i$. Hence, $\sigma(\Gamma_-^*\Gamma_-) \subseteq \sigma(P^{-1}Q)$. The converse inclusion is shown by observing that $f_-^* P x_i$ is an eigenfunction of $\Gamma_-^*\Gamma_-$ corresponding to an eigenvalue σ_i of $P^{-1}Q$. This yields the result.

Statements 3 and 4 are proven analogously. □

As the spectra of Γ_- and Γ_+ are defined by the inner product spaces $(\mathcal{B}_-, \langle\cdot,\cdot\rangle_-)$ and $(\mathcal{B}_+, \langle\cdot,\cdot\rangle_+)$, it follows that the eigenvalues $\lambda(P^{-1}Q)$ constitute a set of ***invariants*** associated with Σ. Note that $\sigma_i := \lambda_i^{1/2}(P^{-1}Q)$, $i = 1, \ldots, n$, are the singular values of Γ_-, as they appear as the eigenvalues of $\Gamma_-^*\Gamma_-$. Similarly, σ_i^{-1}, $i = 1, \ldots, n$ are the singular values of Γ_+.

A balanced state space is defined as follows.

Definition 3.1 *Let $\mathcal{B} \in \mathbf{B}$ and suppose that Σ_s is a minimal state space representation of $\Sigma = (\mathcal{R}, \mathcal{R}^q, \mathcal{B})$. The state space X of Σ_s is balanced with respect to the inner products $\langle\cdot,\cdot\rangle_-$ and $\langle\cdot,\cdot\rangle_+$ if the past and future gramians (3.5) and (3.6) satisfy $Q = P^{-1} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.*

Thus, in a balanced state space the contribution of a state $x \in X$ to the future behavior, as expressed by the quantity $x^T Q x$, is relatively large if and only if its contribution to the past, as expressed by $x^T P x$ is relatively small.

The following algorithm is well known (see e.g. [2]) and provides a straightforward way to obtain a balanced state space.

- Given the past and future gramians P and Q as defined by (3.5) and (3.6).
- Factorize Q as $Q = S_1^T S_1$.
- Define $P_1 := S_1 P^{-1} S_1^T$ and let $P_1 = S_2 \Lambda S_2^T$ be a singular value decomposition of P_1 with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n > 0$.
- Define $S := S_1^{-1} S_2 \Lambda^{\frac{1}{4}}$.

Then S is non-singular and it is easily seen that the basis transformation $x \to S^{-1}x$ results in the simultaneous congruence transformation

$$(P, Q) \to (S^T P S, S^T Q S) = (\Lambda^{-\frac{1}{2}}, \Lambda^{\frac{1}{2}}).$$

In particular, this proves the following

Theorem 3.5 *Let $\mathcal{B} \in \mathbf{B}$. Then for every pair of Hilbert spaces*

$$(\mathcal{B}^-, \langle\cdot,\cdot\rangle_-) \quad \text{and} \quad (\mathcal{B}^+, \langle\cdot,\cdot\rangle_+)$$

there exists a state space representation $\mathcal{B}_s$ of $\mathcal{B}$ which is balanced with respect to $\langle\cdot,\cdot\rangle_-$ and $\langle\cdot,\cdot\rangle_+$.

4 Structure of balanced representations

In this section we examine specific inner products on the past and future behaviors of a system Σ. It is shown how the past and future gramians can be explicitly evaluated by means of solutions of Riccati equations. Given these gramians, a balanced state space is obtained by applying the balancing algorithm of section 3.

Consider a state space system in *driving variable form* which is described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bv; \\ w &= Cx + Dv \end{aligned} \tag{4.1}$$

Here, $x : \mathcal{R} \to \mathcal{R}^n$ is the state, $v : \mathcal{R} \to \mathcal{R}^m$ denotes the *driving variable* and $w : \mathcal{R} \to \mathcal{R}^q$ is the external variable. (A, B, C, D) are real matrices of appropriate dimensions. This defines the behaviors

$$\begin{aligned} \mathcal{B}_s &:= \{ (w, x) : \mathcal{R} \to \mathcal{R}^q \times \mathcal{R}^n \mid x \text{ is abs. cont. and } \exists v \in \mathcal{L}^{loc} \text{ such that (4.1) holds} \} \\ \mathcal{B} &:= \{ w : \mathcal{R} \to \mathcal{R}^q \mid \exists x : \mathcal{R} \to \mathcal{R}^n \text{ such that } (w, x) \in \mathcal{B}_s \} \end{aligned}$$

Clearly, $\mathcal{B} \in \mathbf{B}$ and it is, by definition, represented by $\mathcal{B}_s$. Conversely, every $\mathcal{B} \in \mathbf{B}$ admits such a state space representation [19]. We will assume that $\mathcal{B}$ is controllable and that (4.1) is a *minimal* state space representation of $\mathcal{B}$ (in the sense that n and m are simultaneously minimal). As is shown in [17], this assumption is equivalent to the algebraic conditions that

1. D is injective,
2. (A, B) is controllable and
3. the pair $(C + DF, A + BF)$ is observable for all F.

Let $\mathcal{L}_2$ denote the Hilbert space of square integrable vector valued functions defined on $\mathcal{R}$. Define the $\mathcal{L}_2$ behaviors associated with (4.1) as

$$\begin{aligned} \mathcal{B}_s^2 &:= \mathcal{B}_s \cap \mathcal{L}_2^{q+n} \\ \mathcal{B}_2 &:= \mathcal{B} \cap \mathcal{L}_2^q \end{aligned}$$

Due to minimality of (4.1), it is possible [17] to prove that

$$\mathcal{B}_s^2 = \{(w, x) \in \mathcal{B}_s \mid w \in \mathcal{B}_2\}.$$

In other words, the state $x \in \mathcal{L}_2^n$ whenever $w \in \mathcal{B} \cap \mathcal{L}_2^q$. Moreover, the quadruple (A, B, C, D) defines a minimal representation $\mathcal{B}_s$ of $\mathcal{B}$ if and only if it defines a minimal representation $\mathcal{B}_s^2$ of $\mathcal{B}_2$. In the equivalence class of all (A, B, C, D) that represent $\mathcal{B}_2$ one can choose A such that $\sigma(A) \cap i\mathcal{R} = \emptyset$. In that case, for all $v \in \mathcal{L}_2^m$ there exists a *unique* $(w, x) \in \mathcal{L}_2^{q+n}$ such that (4.1) holds. In particular, this means that $\mathcal{B}_s^2$ (and hence $\mathcal{B}_2$) can be represented as the image of a map. See [17] for more details.

4.1 Riccati balancing

Consider the past and future $\mathcal{L}_2$ behaviors $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$ together with the usual inner products on $\mathcal{L}_2(\mathcal{R}^-, \mathcal{R}^q)$ and $\mathcal{L}_2(\mathcal{R}^+, \mathcal{R}^q)$, respectively. Associate with the quadruple (A, B, C, D) the algebraic Riccati equation

$$A^T K + KA - (B^T K + D^T C)^T (D^T D)^{-1} (B^T K + D^T C) + C^T C = 0 \tag{4.2}$$

The gramians of the state space system $\mathcal{B}_s^2$ are then characterized as follows.

Theorem 4.1 *If (4.1) defines a minimal state space system $\Sigma_s^2 = (\mathcal{R}, \mathcal{R}^q, \mathcal{R}^n, \mathcal{B}_s^2)$, then its past and future gramians are given by*

$$P = -K_- \quad \textit{and} \quad Q = K_+$$

respectively, where $K_+ = K_+^T > 0$ is the unique positive definite solution of (4.2) and $K_- = K_-^T < 0$ is the unique negative definite solution of (4.2).

Proof. By Theorem 3.3 and minimality of Σ_s^2, the map $f_+ : \mathcal{B}^+ \to X$ is surjective. Therefore, $Q := (f_+ f_+^*)^{-1} > 0$. Let $(w, x) \in \mathcal{B}_s^2$, $x(0) = x_0$ and observe that

$$x_0^T Q x_0 = \| \Gamma_-(w^-) \|^2 = \min_{\tilde{w} \in \mathcal{B}_2(x_0)} \| \tilde{w}^+ \|^2 .$$

Note that the right hand side of this expression defines a standard LQ problem. It is well known [18] that

$$\min_{\tilde{w} \in \mathcal{B}_2(x_0)} \| \tilde{w}^+ \|^2 = \| \tilde{w}^{opt} \|^2 = x_0^T K_+ x_0$$

where $K_+ > 0$ satisfies (4.2) and $\tilde{w}^{opt}$ is generated by the state feedback

$$v = -(DD^T)^{-1}(B^T K_+ + D^T C)x.$$

Since K_+ is the unique supremal solution of (4.2) [18], it follows that $Q = K_+$. A similar reasoning yields that $P = -K_-$. $\square$

The positive numbers $\sigma_i := \lambda_i^{1/2}(P^{-1}Q)$, $i = 1, \ldots, n$, with P and Q defined in Theorem 3.5 are the *LQG singular values* of the system. We emphasize that these numbers are *system invariants* that only depend on the choice of the inner products defined on the past and future behavior. In particular, the LQG singular values are independent of the particular state space representation (4.1) and will therefore be the same quantities for state space systems in descriptor form, input-state-output form, driving variable form, etc.

LQG singular values have been first introduced by Opdenacker and Jonckheere in [15]. They considered input-state-output representations, and showed that the positive square roots of the eigenvalues of $K_+K_-^{-1}$ are system invariants. In [15] K_+ and K_- occur as solutions to a linear quadratic control and filter problem. The relation of K_+ and K_- to LQ optimal control theory is easily seen by observing that for all $x_0 \in X$,

$$\begin{aligned} x_0^T K_+ x_0 &= \min_{w \in \mathcal{B}_2(x_0)^+} \| w \|^2_{\mathcal{L}_2^+} \\ -x_0^T K_- x_0 &= \min_{w \in \mathcal{B}_2(x_0)^-} \| w \|^2_{\mathcal{L}_2^-} \end{aligned}$$

We emphasize that from our analysis it follows that not the state space representation, but the inner products associated with $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$ define the LQG singular values. Riccati balanced representations associated with these singular values have found various applications in e.g. controller reduction. See [3] or [12] for more details.

4.2 H_∞ balancing

Consider the ubiquitous input-state-output system described by the equations

$$\begin{aligned} \dot{x} &= A'x + B'u \\ y &= C'x + D'u \; ; \end{aligned} \tag{4.3}$$

and assume that $\sigma(A') \subset \mathcal{C}^- := \{s \in \mathcal{C} \mid \operatorname{Re}(s) < 0\}$. With $w := \operatorname{col}(u, y)$ viewed as the external variables, this defines the behavior

$$\mathcal{B} := \{ \, (u, y) \in \mathcal{L}^{loc} \mid \exists x \text{ abs. continuous such that (4.3) holds} \}$$

Suppose that $\mathcal{B}$ is controllable and assume that (4.3) defines a minimal state space representation of $\mathcal{B}$. This is equivalent to assuming that the pair (A, B) is controllable and the pair (C, A) is observable. Let $H : \mathcal{R} \to \mathcal{R}^{m \times p}$ be the convolution kernel

$$H(t) := \begin{cases} C' \exp(A't)B' + D'\delta(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} .$$

and let $G(s) := C'(Is - A')^{-1}B' + D'$ be the transfer function associated with (4.3). Fix $\gamma > 0$ such that the H_∞ norm

$$\| G \|_{H_\infty} = \sup_{u \in \mathcal{L}_2} \frac{\| H * u \|_{\mathcal{L}_2}}{\| u \|_{\mathcal{L}_2}} \; < \; \gamma.$$

Here, '$*$' denotes convolution and $y := H * u$ is a well defined element of $\mathcal{L}_2^p$ for all $u \in \mathcal{L}_2^m$. Define the $\mathcal{L}_2$ behavior of (4.3) as $\mathcal{B}_2 := \mathcal{B} \cap \mathcal{L}_2^q$ and consider a trajectory $w = \operatorname{col}(u, y) \in \mathcal{B}_2$. Minimality of the state space representation (4.3) implies that the corresponding state trajectory x belongs to $\mathcal{L}_2$ and is uniquely determined by

$$x(t) = \int_{-\infty}^{t} \exp(A'(t - t'))B'u(t')dt', \; t \in \mathcal{R}.$$

Consequently, we can write

$$y(t) = C' \int_{-\infty}^{t} \exp(A'(t-t'))B'u(t')dt' + D'u(t), \quad t < 0 \tag{4.4}$$
$$y(t) = C' \exp(A't)x(0) + C' \int_{0}^{t} \exp(A'(t-t'))B'u(t')dt' + D'u(t), \quad t \geq 0$$

which shows that the past y^- of the output is a function of u^-, whereas the future y^+ is a function of u^+ and the state x at time $t = 0$.

Consider the past and future behaviors $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$ and let $w^- = \mathrm{col}(u^-, y^-) \in \mathcal{B}_2^-$ and $w^+ = \mathrm{col}(u^+, y^+) \in \mathcal{B}_2^+$. Then, by (4.4), w^+ can be uniquely decomposed as

$$w^+ = \begin{pmatrix} u^+ \\ y^+ \end{pmatrix} = \begin{pmatrix} 0 \\ y_0^+ \end{pmatrix} + \begin{pmatrix} u^+ \\ y_u^+ \end{pmatrix}, \tag{4.5}$$

where, for $t \geq 0$,

$$y_0^+(t) := C' \exp(A't)x(0) \text{ and}$$
$$y_u^+(t) := C' \int_0^t \exp(A'(t-t'))B'u(t')dt' + D'u(t)$$

Note that both $\mathrm{col}(0, y_0^+)$ and $\mathrm{col}(u^+, y_u^+)$ are elements of $\mathcal{B}_2^+$. Using this decomposition, we make $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$ normed spaces by introducing

$$\| w^- \|_-^2 := \gamma^2 \| u^- \|_{\mathcal{L}_2^-}^2 - \| y^- \|_{\mathcal{L}_2^-}^2$$
$$\| w^+ \|_+^2 := \| y_0^+ \|_{\mathcal{L}_2^+}^2 + \gamma^2 \| u^+ \|_{\mathcal{L}_2^+}^2 - \| y_u^+ \|_{\mathcal{L}_2^+}^2 .$$

It is easy to see that, by definition of γ, these indeed define norms on $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$, respectively. We obtain the Hilbert spaces $(\mathcal{B}^-, \langle \cdot, \cdot \rangle_-)$ and $(\mathcal{B}^+, \langle \cdot, \cdot \rangle_+)$ by putting

$$\langle w_1, w_2 \rangle_- := (\| w_1 + w_2 \|_-^2 - \| w_1 - w_2 \|_-^2)/4$$
$$\langle w_1, w_2 \rangle_+ := (\| w_1 + w_2 \|_+^2 - \| w_1 - w_2 \|_+^2)/4.$$

In order to characterize the past and future gramians we introduce the algebraic Riccati equation

$$A'^T K + KA' + (B'^T K + D'^T C')^T (\gamma^2 I - D'^T D')^{-1} (B'^T K + D'^T C') + C'^T C' = 0 \tag{4.6}$$

A solution K of (4.6) will be called *stabilizing* (*anti-stabilizing*) if

$$\sigma(A' - (\gamma^2 I - D'^T D')^{-1}(B'^T K + D'^T C')) \subset \mathcal{C}^- \quad (\text{resp.} \quad \subset \mathcal{C}^+)$$

Furthermore, let the *observability gramian* M associated with (4.3) be defined as

$$M := \int_0^\infty \exp(A'^T t) C'^T C' \exp(A't)dt$$

The past and future gramians of this system are then characterized as follows.

Theorem 4.2 *Suppose that the equations (4.3) define a minimal state space representation of* $\Sigma = (\mathcal{R}, \mathcal{R}^q, \mathcal{B}_2)$. *Its past and future gramians are given by*

$$P = -K_-, \quad \text{and} \quad Q = K_+ + M$$

respectively, where $K_+ = K_+^T$ *is the unique stabilizing solution of (4.6) and* $K_- = K_-^T$ *is the unique anti-stabilizing solution of (4.6).*

Proof. Let K be a solution of (4.6). Differentiating $x^T K x$ along solutions of (4.3) yields that

$$\begin{aligned} \frac{d}{dt} x^T K x &= - \| y \|^2 + \gamma^2 \| u \|^2 - \\ &- \| (\gamma^2 I - D'^T D')^{-1/2} u - (\gamma^2 I - D'^T D')^{-3/2} (B'^T K + D'^T C') x \|^2 . \end{aligned}$$

Let $x_0 \in X$ and consider the decomposition (4.5) of $w^+ = \operatorname{col}(u^+, y^+) \in \mathcal{B}_2^+(x_0)$. Then, $\| y_0^+ \|_{\mathcal{L}_2^+} = x_0^T M x_0$ and we find that

$$\begin{aligned} \| w^+ \|_+^2 &= x_0^T (K + M) x_0 \\ &+ \| (\gamma^2 I - D'^T D')^{-1/2} u - (\gamma^2 I - D'^T D')^{-3/2} (B'^T K + D'^T C') x \|_{\mathcal{L}_2^+}^2 \end{aligned}$$

Consequently, the future of $w \in \mathcal{B}_2(x_0)$ has minimal $\| \cdot \|_+$ norm if and only if

$$u = (\gamma^2 I - D'^T D')^{-1} (B'^T K + D'^T C') x,$$

belongs to $\mathcal{L}_2^+$. Equivalently, if $K = K_+ = K_+^T$ is the stabilizing solution of (4.6). Since $x_0 \in X$ is arbitrary and $\| \Gamma_-(w^-) \|_+^2 = x_0^T Q x_0 = x_0^T (K_+ + M) x_0$, it follows that $Q = K_+ + M$. A similar argument yields that $P = -K_-$. □

As for Riccati balanced representations, we remark that $\gamma_i := \lambda_i^{1/2}(P^{-1}Q), \;\; i = 1, \ldots, n$, with P and Q defined as in Theorem 4.1 are system invariants that coincide with the non-zero spectral values of Γ_-. The positive numbers $\{\gamma_i\}_{i=1,\ldots,n}$ are called the H_∞*−singular values* of the system.

Remark. The H_∞ singular values, defined in this way, denote *open-loop* quantities in the sense that a compensator for Σ is not taken into consideration. In [12] and [11] closed-loop configurations are considered and H_∞ singular values are defined as the positive square roots of the matrix product XY, where X and Y are the unique positive definite solutions of

$$\begin{aligned} A'^T X + X A' - (1 - \gamma^{-2}) X B' B'^T X + C'^T C' &= 0 \\ A' Y + Y A'^T - (1 - \gamma^{-2}) Y C'^T C' Y + B' B'^T &= 0 \end{aligned}$$

where it is assumed that $D' = 0$ and the maximal eigenvalue $\lambda_1(XY) < \gamma^2$.

Remark. A similar analysis can be carried out for anti-stable systems, i.e., for systems with $\sigma(A') \subset \mathcal{C}^+$. More generally, one can define a set of $\mathcal{L}_\infty$ singular values related to the state space system (4.3) if $\sigma(A') \cap i\mathcal{R} = \emptyset$. In that case, the analysis of this section will be symmetric with respect to the sets $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$. However, we will not pursue the details here.

4.3 Lyapunov balancing

Consider again the state space system described by (4.3). Suppose that the system is minimal and asymptotically stable, i.e., $\sigma(A') \subset \mathcal{C}^-$. Define the controllability and observability gramians W and M as the unique positive definite solutions of the Lyapunov equations

$$\begin{aligned} A'W + WA'^T + B'B'^T &= 0 \\ A'^T M + MA' + C'^T C' &= 0 \end{aligned} \tag{4.7}$$

It is well known that the eigenvalues of the product WM are similarity invariants and that their square roots, $\{\mu_i\}_{i=1,\ldots,n}$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, are the Hankel singular values [2] of the Hankel operator induced by the input-output map (4.3). We will show that the Hankel singular values μ_i, $i = 1, \ldots, n$, can be obtained as a special case of our setting. For this purpose, let $\varepsilon > 0$ and define the following norms on the past and future behavior $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$.

$$\begin{aligned} \| w \|_-^2 &:= \| u \|_{\mathcal{L}_2^-}^2 + \varepsilon^2 \| y \|_{\mathcal{L}_2^-}^2 \\ \| w \|_+^2 &:= \frac{1}{\varepsilon^2} \| u \|_{\mathcal{L}_2^+}^2 + \| y \|_{\mathcal{L}_2^+}^2 . \end{aligned}$$

Like in section (4.2), $\mathcal{B}_2^-$ and $\mathcal{B}_2^+$ can be given a Hilbert space structure using these norms. Note that for $\varepsilon = 1$ we obtain that the singular values of Γ_- coincide with the LQG singular values defined in section 4.1.

A similar analysis as before shows that for $\varepsilon > 0$ the past and future gramians associated with (4.3) are given by

$$P_\varepsilon = -\varepsilon^2 K_\varepsilon^- \ , \quad Q_\varepsilon = K_\varepsilon^+ \tag{4.8}$$

where K_ε^- and K_ε^+ are, respectively, the minimum and the maximum solution (in the sense of real symmetric matrices) of the Riccati equation

$$A'^T K + KA' - (B'^T K + D'^T C')^T (D'^T D' + \frac{1}{\varepsilon^2} I)^{-1} (B'^T K + D'TC') + C'^T C' = 0$$

The following result claims that for $\varepsilon \to 0$ the past and future gramians of a stable input-output system converge to the classical controllability and observability gramian W and M.

Theorem 4.3 *Let the equations (4.3) define a minimal state space representation of $(\mathcal{R}, \mathcal{R}^q, \mathcal{B}_2)$ and let $\sigma_i(\varepsilon)$, $i = 1, \ldots, n$, denote the singular values of Γ_-. Suppose that the singular values are ordered according to $\sigma_1(\varepsilon) \geq \sigma_2(\varepsilon) \geq \ldots \geq \sigma_n(\varepsilon) > 0$. If $\sigma(A') \subset C^-$ then*

$$\begin{aligned}\lim_{\varepsilon \to 0} P_\varepsilon &= W^{-1}\\ \lim_{\varepsilon \to 0} Q_\varepsilon &= M.\end{aligned}$$

Moreover, in that case $\lim_{\varepsilon \to 0} \sigma_i(\varepsilon) = \mu_i$ for all $i = 1, \ldots, n$.

Proof. Let $\varepsilon > 0$, $x \in X$ and note that

$$x^T P_\varepsilon x = \min_{w \in \mathcal{B}_2(x)^-} \| w^- \|_-^2 = \min_{w \in \mathcal{B}_2(x)^-} \| u \|_{\mathcal{L}_2^-}^2 + \varepsilon^2 \| y \|_{\mathcal{L}_2^-}^2 \tag{4.9}$$

$$x^T Q_\varepsilon x = \min_{w \in \mathcal{B}_2(x)^+} \| w^+ \|_+^2 = \min_{w \in \mathcal{B}_2(x)^+} \frac{1}{\varepsilon^2} \| u \|_{\mathcal{L}_2^+}^2 + \| y \|_{\mathcal{L}_2^+}^2 . \tag{4.10}$$

First observe that, for any $x \in X$, $x^T P_\varepsilon x$ and $x^T Q_\varepsilon x$ are, respectively, nonincreasing and nondecreasing if $\varepsilon \to 0$. Second, note that by taking $\varepsilon = 0$ in (4.9), and $u = 0$ in (4.10), we obtain that $x^T P_\varepsilon x \geq x^T W^{-1} x$ and $x^T Q_\varepsilon x \leq x^T M x$. Conclude from this that both $\lim_{\varepsilon \to 0} x^T P_\varepsilon x$ and $\lim_{\varepsilon \to 0} x^T Q_\varepsilon x$ exist. Interchanging the order of 'lim' and 'min' then yields that $\forall x \in X$

$$\lim_{\varepsilon \to 0} x^T P_\varepsilon x = x^T W^{-1} x \quad \text{and} \quad \lim_{\varepsilon \to 0} x^T Q_\varepsilon x = x^T M x.$$

Symmetry of $P_\varepsilon, W, Q_\varepsilon$ and M then yields the result. □

From Theorem 4.3 we conclude that the classical controllability and observability gramians $W > 0$ and $M > 0$ associated with the minimal state space system (4.3) can be obtained as a limiting case of Riccati balanced systems. Note that, for any $x_0 \in X$,

$$x_0^T W^{-1} x_0 = \min_{(u,y) \in \mathcal{B}_2(x_0)} \| u \|_{\mathcal{L}_2^-}^2 \tag{4.11}$$

$$x_0^T M x_0 = \min_{(0,y) \in \mathcal{B}_2(x_0)} \| y \|_{\mathcal{L}_2^+}^2 . \tag{4.12}$$

In order to define Lyapunov balanced state space systems in a behavioral context, equations (4.11) suggest a more direct approach by taking the norms $\| u \|_{\mathcal{L}_2^-}$ on the past behavior $\mathcal{B}_2^-$ and $\| y \|_{\mathcal{L}_2^+}$ on the future trajectories in

$$\mathcal{B}_2 \cap \{ w = (y^T, u^T)^T \mid u = 0 \}.$$

For asymptotically stable systems $\| u \|_{\mathcal{L}_2^-}$ induces a Hilbert space structure on $\mathcal{B}_2^-$. However, $(\mathcal{B}_2^+, \| y \|_{\mathcal{L}_2^+})$ is not a normed space so that no Hilbert space structure can be induced on $\mathcal{B}_2^+$ in this way.

5 Model approximation

In this section we discuss the model approximation problem for systems in $\mathbf{B}$ and develop approximation procedures which are based on balanced representations.

Consider the model class $\mathbf{B}$. For $\mathcal{B} \in \mathbf{B}$, let $c(\mathcal{B})$, the *complexity* of $\mathcal{B}$, denote the minimal dimension of the state space among the set of all state space representations of $\mathcal{B}$. The model approximation problem in $\mathbf{B}$ then amounts to reducing the dimension $n = c(\mathcal{B})$ of the state space of a system $\mathcal{B} \in \mathbf{B}$, so as to obtain an approximate system $\mathcal{B}_{red} \in \mathbf{B}$ of complexity $k = c(\mathcal{B}_{red}) < n$ that is, in some sense, close to $\mathcal{B}$.

We will derive results for model approximation based on the method of balanced truncations. A major criticism for this heuristic technique is that it is not clear whether the resulting reduced order models are optimal in some metric defined on $\mathbf{B}$. However, we will provide bounds on specific distance measures between the given and the reduced order system.

Consider a minimal driving variable state space representation (4.1) of a system $\mathcal{B} \in \mathbf{B}$ and let $K_+ = K_+^T > 0$ be the maximal solution of the algebraic Riccati equation (4.2). Define

$$F := -(D^TD)^{-1}(B^TK_+ + D^TC)$$

And let $R \in \mathcal{R}^{m\times m}$ be any non-singular matrix such that $RR^T = (D^TD)^{-1}$. It is then easy to verify that the driving variable system

$$\begin{aligned} \dot{x} &= A'x + B'v \\ w &= C'x + D'v \end{aligned} \tag{5.1}$$

with $A' = A + BF$, $B' = BR$, $C' = C + DF$ and $D' = DR$ also represents $\mathcal{B}$. Moreover, (5.1) is a minimal representation of $\mathcal{B}$ and it has the property that for all $v \in \mathcal{L}_2^m$ there exist a unique pair $(w, x) \in \mathcal{L}_2^{q+n}$ such that (5.1) is satisfied and $\| v \|_{\mathcal{L}_2} = \| w \|_{\mathcal{L}_2}$. That is, the mapping $\Psi : v \in \mathcal{L}_2^m \to w \in \mathcal{L}_2^q$ defined by the equations (5.1) is isometric. Stated otherwise, the *transfer function* $G(s) := C'(Is - A')^{-1}B' + D'$ is inner, i.e., it is stable and $G^*(i\omega)G(i\omega) = I$ for all $\omega \in \mathcal{R}$. Let this state space system be given and assume that its state space is balanced with respect to the standard inner products of $\mathcal{L}_2(\mathcal{R}^-, \mathcal{R}^q)$ and $\mathcal{L}_2(\mathcal{R}^+, \mathcal{R}^q)$ defined on the past and future behavior of $\mathcal{B}$, respectively. (See section 4.1).

Hence, the singular values of the operator Γ_- are given by the LQG singular values $\{\sigma_i\}_{i=1,\dots,n}$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ as defined and characterized in section 4.1. Let $n := c(\mathcal{B})$ and let $k < n$. Partition the state vector x of (5.1) as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $x_1 \in \mathcal{R}^k$. Partition (A', B', C') conformally as

$$A' = \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix}, \; B' = \begin{pmatrix} B'_1 \\ B'_2 \end{pmatrix}, \; C' = (C'_1 \; C'_2), \tag{5.2}$$

Write $X = X_1 \oplus X_2$ with $X_1 = \mathcal{R}^k$ and observe that the subspace $X_1 \subset X$ of the state space of (5.1) contains those states x_1 for which both $x_1^T P x_1$ is relatively small and $x_1^T Q x_1$ is relatively large. This suggests that the subspace X_2 is of less relevance in assessing the relative contribution of state space components to the interaction between the system and its environment. The driving variable system

$$\begin{aligned} \dot{x}_1 &= A'_{11} x_1 + B'_1 v \\ w &= C'_1 x + D' v \end{aligned} \tag{5.3}$$

will be called a *k-th order balanced approximant* of $\mathcal{B}_s$. Let $\mathcal{B}_s(A'_{11}, B'_1, C'_1, D')$ denote its state space behavior. Its induced external behavior $\mathcal{B}_{red}$ is regarded as a feasible approximant of $\mathcal{B}$. Clearly, $c(\mathcal{B}_{red}) \leq k$.

Remark. This method is obviously asymmetric with respect to time. Indeed, a similar reasoning may be applied when considering the operator Γ_+, in which case the subspace X_2 is regarded as to determine the dominant subspace of the state space.

In the next theorem we show that for the k-th order balanced approximant $\mathcal{B}_{red}$ an explicit upperbound can be given on the L_2 norm of the error system.

Theorem 5.1 *Suppose that the state space system $\mathcal{B}_s := \mathcal{B}_s(A', B', C', D')$, defined by (5.1), is balanced with singular values*

$$\sigma_1 \geq \ldots \geq \sigma_k > \sigma_{k+1} \geq \ldots \geq \sigma_n$$

and suppose that the associated transfer function $G(s) = C'(Is - A')^{-1}B' + D'$ is inner. Let $\mathcal{B}_s(A'_{11}, B'_1, C'_1, D')$ be a $k - th$ order balanced approximant of $\mathcal{B}_s$. Then,

1. *$\mathcal{B}_s(A'_{11}, B'_1, C'_1, D')$ is balanced with respect to the standard inner products on $\mathcal{L}_2^-$ and $\mathcal{L}_2^+$.*
2. *The transfer function $G_{red}(s) := C'_1(Is - A'_{11})^{-1}B'_1 + D'$ is inner, i.e., it is stable and $G^*_{red}(i\omega)G_{red}(i\omega) = I$ for all $\omega \in \mathcal{R}$.*
3. *With $G(s) := C'(Is - A')^{-1}B' + D'$, there holds that*

$$\| G - G_{red} \|_{H_\infty} \leq 2 \sum_{i=k+1}^{n} \sigma_i (1 + \sigma_i^2)^{-\frac{1}{2}}.$$

Proof. 1. It is straightforward to verify that for any k-th order balanced approximant $\mathcal{B}_s(A'_{11}, B'_1, C'_1, D'_1)$ the Riccati equation

$$A'^T_{11}K_1 + K_1A'_{11} + (B'^T_1K_1 + D'^TC'_1)^T(D'^TD')^{-1}(B'^T_1K_1 + D'^TC'_1) + C'^T_1C'_1 = 0,$$

admits real symmetric solutions

$$K_+ = \mathrm{diag}(\sigma_1, \ldots, \sigma_k) \text{ and } K_- = -\mathrm{diag}(\sigma_1^{-1}, \ldots, \sigma_k^{-1}).$$

The driving variable state space representation $\mathcal{B}_s(A'_{11}, B'_1, C'_1, D'_1)$ is then balanced by Theorem 4.1.

2. It is well known (see, e.g., Theorem 5.1 in [2]) that G_{red} is inner if and only if $\sigma(A'_{11}) \subset \mathbb{C}^-$ and for some $K = K^T$ there holds

$$A'^T_{11}K + KA'_{11} + C'^T_1C'_1 = 0, B'^T_1K + D'^TC'_1 = 0,\ D'^TD' = I. \tag{5.4}$$

Clearly, $D'^TD' = I$. Next, consider the future gramian $Q = \mathrm{diag}(\sigma_1, \cdots, \sigma_n)$ and observe that, by construction of (A', B', C', D'),

$$A'^TQ + QA' + C'^TC' = 0, \quad B'^TQ + D'^TC' = 0.$$

From this expression it follows that (5.4) holds for $K = \mathrm{diag}(\sigma_1, \cdots, \sigma_k)$. It therefore remains to show that $\sigma(A'_{11}) \subset \mathbb{C}^-$. To see this, we apply a result of [16] on standard (Lyapunov) balanced truncations. Let $W = W^T > 0$ and $M = M^T > 0$ be the controllability and observability gramians associated with the triple (A', B', C'). We have seen that $Q = M = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. Moreover, it is straightforward to verify that the past gramian $P = W^{-1} - Q$. Infer from this that

$$P^{-1}Q = \mathrm{diag}(\sigma_1^2, \ldots \sigma_n^2) = (I - WM)^{-1}WM = (M^{-1}W^{-1} - I)^{-1}. \tag{5.5}$$

In particular this observation implies that $WM = \mathrm{diag}(\gamma_1^2, \cdots, \gamma_n^2)$, where $1 \geq \gamma_1 \geq \cdots \geq \gamma_n > 0$ where

$$\gamma_i = \sigma_i(1 + \sigma_i^2)^{-\frac{1}{2}}, \quad i = 1, \cdots, n.$$

As $P^{-1}Q$ is diagonal and positive definite, we first conclude that $1 > \gamma_1$. Now, consider the balanced approximant $(A'_{11}, B'_1, C'_1, D')$. Since, by assumption, $\sigma_k > \sigma_{k+1}$, it follows that also $\gamma_k > \gamma_{k+1}$. From [16] we infer that $\sigma(A'_{11}) \subset \mathbb{C}^-$, as desired.

3. The last statement of the theorem follows from the observation that, by (5.5), the state space system defined by (A', B', C') satisfies $WM = \mathrm{diag}(\gamma_1^2, \ldots, \gamma_n^2)$, while $M = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. Hence, also W is a diagonal matrix and a state space transformation $x \to S^{-1}x$ with $S = (I + M^2)^{1/4}$ achieves that $M = W = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$. Since S is a diagonal matrix, the state space X_k of the k-th order balanced approximant $\mathcal{B}_s(A'_{11}, B'_1, C'_1, D')$ coincides with the state space obtained from a k-th order balanced

truncation of a Lyapunov balanced representation of $\mathcal{B}$. This implies that a $k-th$ order truncation of a Lyapunov balanced representation of $\mathcal{B}$ also coincides with tha quadruple $(A'_{11}, B'_1, C'_1, D')$. Consequently, the controllability and observability gramians W_k and M_k, associated with the triple (A'_{11}, B'_1, C'_1) satisfy $W_k M_k = \text{diag}(\gamma_1^2, \ldots, \gamma_k^2)$. and $(A'_{11}, B'_1, C'_1, D')$ defines a standard input-state-output system with corresponding (Hankel-) singular values $\gamma_1, \ldots, \gamma_k$. The result then follows from [2], Theorem 9.6. □

It follows from the proof of Theorem 5.1 that whenever the state space system

$$\mathcal{B}_s(A', B', C', D')$$

is balanced with respect to the standard inner products on $\mathcal{L}_2^-$ and $\mathcal{L}_2^+$, then the k−th order balanced truncation $(A'_{11}, B'_1, C'_1, D')$ coincides with the k−th order system which is obtained by truncating the standard (Lyapunov) balanced triple (A', B', C') for which the associated controllability and observability gramians W and M are equal and diagonal. In the notation of the above theorem, the positive real numbers

$$\gamma_i := \sigma_i(1 + \sigma_i^2)^{-\frac{1}{2}}, \quad i = 1, \ldots, n$$

are, in fact, the square roots of the eigenvalues of the matrix WM. This interesting connection between balanced truncations of LQG balanced state space systems and truncations obtained from Lyapunov balanced representations has been pursued by several authors. See, e.g. [14] [8] for more details.

We finally remark that the transfer function G_{red} in statements 2 and 3 of Theorem 5.1 has the interpretation of a generator of a graph of a reduced order system. That is, the trajectories of the external behavior associated with G_{red} are given by the image of G_{red} when viewed as an operator $G_{red} : L_2 \to L_2$ defined as $w = G_{red}v$. where w and v are to be interpreted in the frequency domain.

6 Conclusions

In this paper we developed the concept of a balanced state space using the behavioral framework of systems theory. The intrinsic property of state is to split the past and future behavior of a linear time-invariant system. This property is formalized in a set theoretic context and is used to introduce a concept of balancing without reference to specific equations that describe the dynamic behavior of the system. Apart from the generality of this set-up, this has the advantage that the property of a balanced state space is well defined for a wide variety of state space representations, including input-state-output representations, descriptor systems, state space systems in driving variable form, etc. The past and future behavior of a system have been viewed as Hilbert spaces in which the corresponding norms quantize the effect that past trajectories exhibit on their continuations and the effect future trajectories exhibit on their antecedents. We showed that this quantification naturally leads to an identification between the state space of the system and its algebraic dual by means of two gramians: the past and the future gramian of the system. In a balanced state space these gramians are required to be diagonal and each others inverses. It has been proved that for the class of linear time-invariant and finite dimensional systems, balanced representations always exist. In fact, the so called Riccati balanced state space representations appear in a very natural and convincing way using this framework. H_∞ balanced representations have been introduced and we discussed how the prevailing notion of (Lyapunov) balanced representations can be obtained as a special case of our setting.

The concept of H_∞ balancing can be generalized so as to incorporate non-stable systems. In the line of section 4.2 one can easily formalize an extension to define an $\mathcal{L}_\infty$ balanced state space representation. Other generalizations can be made to infinite dimensional systems, dissipative systems or non-linear systems. However, these generalizations have not been pursued here and are the topic of future research.

References

[1] Aoki, M.
"State Space Modeling of Time Series,"
Springer, Berlin, 1987.

[2] Glover, K.
"All Optimal Hankel Norm Approximations of Linear Multivariable Systems and their L_∞ Error Bounds,"
International Journal of Control, Vol. 39, (1984). p. 1115–1193.

[3] Jonckheere, E.A. and L.M. Silverman
"A New Set of System Invariants for Linear Systems –Application to Reduced Order Compensator Design,"
IEEE Transactions on Automatic Control, Vol. 28, (1982). p. 953–964.

[4] Fuhrmann, P.A. and R. Ober,
"A Functional Approach to LQG Balancing,"
International Journal of Control, Vol. 57, (1993). No. 3, p. 627–741.

[5] Fuhrmann, P.A.
"A Polynomial Approach to Hankel Norm and Balanced Approximations,"
Linear Algebra and its Applications, Vol. 146, (1991). p. 133–220.

[6] Kato, T.
"Perturbation Theory for Linear Operators,"
Springer, New York, 1966.

[7] Kenney, C. and G. Hewer,
"Necessary Conditions for Balancing Unstable Systems,"
IEEE Transactions on Automatic Control, Vol. 32, (1987). p. 157–160.

[8] McFarlane, D.C. and K. Glover,
"Robust Controller Design Using Normalized Coprime Factor Plant Descriptions,"
Lecture Notes in Control and Information Sciences, Vol. 138. Springer, Berlin, 1989.

[9] Moore, B.C.
"Singular Value Analysis of Linear Systems, Part I and II,"
Department of Electrical Engineering, University of Toronto, Toronto, Ont., Systems and Control Report 7801 and 7802, July 1978; also in Proceedings IEEE Conference on Decision and Control, San Diego, January 10–12, 1979. p. 66–73.

[10] Moore, B.C.
"Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction,"
IEEE Transactions on Automatic Control, Vol. 26, (1981). p. 17–32.

[11] Mustafa, D.
"H_∞ Characteristic Values,"
Proceedings 28th IEEE Conference on Decision and Control, Florida, December 13–15, 1989. p. 1483–1487.

[12] Mustafa, D. and K. Glover,
"Controller Reduction by H_∞ Balanced Truncation,"
IEEE Transactions on Automatic Control, Vol. 36, (1991). p. 668–682.

[13] Ober, R.
"Balanced Realizations: Canonical Form, Parametrization, Model Reduction,"
International Journal of Control, Vol. 46, No. 2, (1987). p. 643–670.

[14] Ober, R. and D. McFarlane,
"Balanced Canonical Forms for Minimal Systems: a Normalized Coprime Factor Approach,"
Linear Algebra and its Applications, Vol. 124, (1989). p. 23–64.

[15] Opdenacker, Ph. and E.A. Jonckheere,
"LQG Balancing and Reduced LQG Compensation of Symmetric Passive Systems,"
International Journal of Control, Vol. 41, No. 1, (1985). p. 73–109.

[16] Pernebo, L. and L.M. Silverman,
"Model Reduction via Balanced State Space Representations,"
IEEE Transactions on Automatic Control, Vol. 27, (1982). p. 382–387.

[17] Weiland, S.
"Theory of Approximation and Disturbance Attenuation for Linear Systems,"
Ph.D. Thesis, Groningen University, the Netherlands, 1991.

[18] Willems, J.C.
"Least Squares Stationary Optimal Control and the Algebraic Riccati Equation,"
IEEE Transactions on Automatic Control, Vol. 16, No. 6, (1971). p. 621–634.

[19] Willems, J.C.
"From Time Series to Linear System –Part I: Finite Dimensional Linear Time-Invariant Systems,"
Automatica, Vol. 22, No. 5, (1986). p. 561–580.

[20] Willems, J.C.
"Models for Dynamics,"
In: Dynamics Reported, Vol. 2, Ed. U. Kirchgraber, H.O. Walter; John Wiley & Sons Ltd and B.G. Teubner, (1989). p. 171–269.

[21] Willems, J.C.
"Paradigms and Puzzles in the Theory of Dynamical Systems,"
IEEE Transactions on Automatic Control, Vol. 36, (1991). p. 259–294.

 ISSN 0167-9708
Coden: TEUEDE

Faculty of Electrical Engineering

(252) Chen, J. and P.J.I. de Maagt, M.H.A.J. Herben
WIDE-ANGLE RADIATION PATTERN CALCULATION OF PARABOLOIDAL REFLECTOR ANTENNAS: A comparative study.
EUT Report 91-E-252. 1991. ISBN 90-6144-252-4

(253) Haan, S.W.H. de
A PWM CURRENT-SOURCE INVERTER FOR INTERCONNECTION BETWEEN A PHOTOVOLTAIC ARRAY AND THE UTILITY LINE.
EUT Report 91-E-253. 1991. ISBN 90-6144-253-2

(254) Velde, M. van de and P.J.M. Cluitmans
EEG ANALYSIS FOR MONITORING OF ANESTHETIC DEPTH.
EUT Report 91-E-254. 1991. ISBN 90-6144-254-0

(255) Smolders, A.B.
AN EFFICIENT METHOD FOR ANALYZING MICROSTRIP ANTENNAS WITH A DIELECTRIC COVER USING A SPECTRAL DOMAIN MOMENT METHOD.
EUT Report 91-E-255. 1991. ISBN 90-6144-255-9

(256) Backx, A.C.P.M. and A.A.H. Damen
IDENTIFICATION FOR THE CONTROL OF MIMO INDUSTRIAL PROCESSES.
EUT Report 91-E-256. 1991. ISBN 90-6144-256-7

(257) Maagt, P.J.I. de and H.G. ter Morsche, J.L.M. van den Broek
A SPATIAL RECONSTRUCTION TECHNIQUE APPLICABLE TO MICROWAVE RADIOMETRY.
EUT Report 92-E-257. 1992. ISBN 90-6144-257-5

(258) Vleeshouwers, J.M.
DERIVATION OF A MODEL OF THE EXCITER OF A BRUSHLESS SYNCHRONOUS MACHINE.
EUT Report 92-E-258. 1992. ISBN 90-6144-258-3

(259) Orlov, V.B.
DEFECT MOTION AS THE ORIGIN OF THE 1/F CONDUCTANCE NOISE IN SOLIDS.
EUT Report 92-E-259. 1992. ISBN 90-6144-259-1

(260) Rooijackers, J.E.
ALGORITHMS FOR SPEECH CODING SYSTEMS BASED ON LINEAR PREDICTION.
EUT Report 92-E-260. 1992. ISBN 90-6144-260-5

(261) Boom, T.J.J. van den and A.A.H. Damen, Martin Klompstra
IDENTIFICATION FOR ROBUST CONTROL USING AN H-infinity NORM.
EUT Report 92-E-261. 1992. ISBN 90-6144-261-3

(262) Groten, M. and W. van Etten
LASER LINEWIDTH MEASUREMENT IN THE PRESENCE OF RIN AND USING THE RECIRCULATING SELF HETERODYNE METHOD.
EUT Report 92-E-262. 1992. ISBN 90-6144-262-1

(263) Smolders, A.B.
RIGOROUS ANALYSIS OF THICK MICROSTRIP ANTENNAS AND WIRE ANTENNAS EMBEDDED IN A SUBSTRATE.
EUT Report 92-E-263. 1992. ISBN 90-6144-263-X

(264) Freriks, L.W. and P.J.M. Cluitmans, M.J. van Gils
THE ADAPTIVE RESONANCE THEORY NETWORK: (Clustering-) behaviour in relation with brainstem auditory evoked potential patterns.
EUT Report 92-E-264. 1992. ISBN 90-6144-264-8

(265) Wellen, J.S. and F. Karouta, M.F.C. Schemmann, E. Smalbrugge, L.M.F. Kaufmann
MANUFACTURING AND CHARACTERIZATION OF GAAS/ALGAAS MULTIPLE QUANTUMWELL RIDGE WAVEGUIDE LASERS.
EUT Report 92-E-265. 1992. ISBN 90-6144-265-6

(266) Cluitmans, L.J.M.
USING GENETIC ALGORITHMS FOR SCHEDULING DATA FLOW GRAPHS.
EUT Report 92-E-266. 1992. ISBN 90-6144-266-4

(267) Józwiak, L. and A.P.H. van Dijk
A METHOD FOR GENERAL SIMULTANEOUS FULL DECOMPOSITION OF SEQUENTIAL MACHINES: Algorithms and implementation.
EUT Report 92-E-267. 1992. ISBN 90-6144-267-2

(268) Boom, H. van den and W. van Etten, W.H.C. de Krom, P. van Bennekom, F. Huijskens, L. Niessen, F. de Leijer
AN OPTICAL ASK AND FSK PHASE DIVERSITY TRANSMISSION SYSTEM.
EUT Report 92-E-268. 1992. ISBN 90-6144-268-0

(269) Putten, P.H.A. van der
MULTIDISCIPLINAIR SPECIFICEREN EN ONTWERPEN VAN MICROELEKTRONICA IN PRODUKTEN (in Dutch).
EUT Report 93-E-269. 1993. ISBN 90-6144-269-9

(270) Bloks, R.H.J.
PROGRIL: A language for the definition of protocol grammars.
EUT Report 93-E-270. 1993. ISBN 90-6144-270-2

(271) Bloks, R.H.J.
CODE GENERATION FOR THE ATTRIBUTE EVALUATOR OF THE PROTOCOL ENGINE GRAMMAR PROCESSOR UNIT.
EUT Report 93-E-271. 1993. ISBN 90-6144-271-0

(272) Yan, Keping and E.M. van Veldhuizen
FLUE GAS CLEANING BY PULSE CORONA STREAMER.
EUT Report 93-E-272. 1993. ISBN 90-6144-272-9

(273) Smolders, A.B.
FINITE STACKED MICROSTRIP ARRAYS WITH THICK SUBSTRATES.
EUT Report 93-E-273. 1993. ISBN 90-6144-273-7

(274) Bollen, M.H.J. and M.A. van Houten
ON INSULAR POWER SYSTEMS: Drawing up an inventory of phenomena and research possibilities.
EUT Report 93-E-274. 1993. ISBN 90-6144-274-5

(275) Deursen, A.P.J. van
ELECTROMAGNETIC COMPATIBILITY: Part 5, installation and mitigation guidelines, section 3, cabling and wiring.
EUT Report 93-E-275. 1993. ISBN 90-6144-275-3

(276) Bollen, M.H.J.
LITERATURE SEARCH FOR RELIABILITY DATA OF COMPONENTS IN ELECTRIC DISTRIBUTION NETWORKS.
EUT Report 93-E-276. 1993. ISBN 90-6144-276-1

(277) Weiland, Siep
A BEHAVIORAL APPROACH TO BALANCED REPRESENTATIONS OF DYNAMICAL SYSTEMS.
EUT Report 93-E-277. 1993. ISBN 90-6144-277-X