Finite subgroups of Lie groups : notes of a lecture and practice session at the European Summer School in Leiden, July 1, 1998

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1 Introduction

The primary goal of this lecture is to show how computer algebra software can be used to study concrete problems concerning Lie groups. A major, recently roughly completed, achievement is the determination of (the isomorphism type of) finite subgroups of the given exceptional Lie groups. Our secondary goal is to give a glimpse as to how this was done.

In §2, we start by an outline of the classification of finite subgroups of Lie groups of exceptional type. For, this is the motivation for the other topics to be dealt with.

In §3, we briefly review Weyl's character formula. Then, in §4, we exploit it to study elements of finite order in a given complex semisimple linear Lie group G. The outcome is a set of severe character restrictions for finite subgroups of G. They are important in the classification of finite groups having an embedding in G.

The rest of the lecture illustrates, by means of the example of the simple group $F = PSL(2,13)$ of order 1092 and the exceptional Lie group $G = G_2(\mathbf{C})$, how a finite group F can be shown to embed in G.

To this end, we give explicit descriptions of $PSL(2,13)$, together with some relevant representation theory, in §5, and next $G_2(\mathbf{C})$ in §6.

The following two sections are devoted to two construction methods for embeddings $E: F \to G$; in §7, we find a Lie group of type G_2 around F (the easy way), and in §8, we show how to find a subgroup of $G_2(\mathbf{C})$ isomorphic to $PSL(2,13)$ (the hard way).

In the practice session, we use the program LiE for the first part (up to the determination of possible subgroups for $G_2(\mathbf{C})$) and the systems GAP and Maple for the second part, e.g., for constructing an embedding of the simple group $PSL(2,13)$ into $G_2(\mathbf{C})$. The material for this session is interspersed with the text. We shall deal with the exercises written out in this text, and you are encouraged to acquire a copy of the LaTeX source of this text so as to be able to copy the displayed code as well as some undisplayed code at the end of the file.

1.1 Acknowledgments

Parts of the section on elements of finite order are taken from Stefan Grimm [16], other parts from joint work with Marc van Leeuwen.

Parts of the $PSL(2,13)$ work is done jointly with David Wales, for other purposes. I am grateful to David for discussions regarding the content of the lecture as well as his suggestions for improvement of early drafts.

GAP work and other programming assistance was generously supplied by Ronald de Man and Willem de Graaf. Thanks!

2 The classification

Let F be a finite group. We are interested in embeddings of F into a complex simple Lie group G. We shall often study G from an algebraic point of view, and regard it as an algebraic group (which is allowed, see [28]).

2.1 Classical groups

Usual representation theory helps us to decide whether there exists an embedding of F into $\mathrm{GL}(n, \mathbf{C})$. For such an embedding it is readily decided whether it leads to an embedding of F into $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{PSL}(n, \mathbf{C})$. Suppose that F has an irreducible representation $E : F \to \mathrm{GL}(n, \mathbf{C})$ with character χ. Then E is

- equivalent to an embedding in the closed Lie subgroup $\mathrm{O}(n, \mathbf{C})$ of $\mathrm{GL}(n, \mathbf{C})$ if and only if $\nu(\chi) = 1$, and
- equivalent to an embedding in the closed Lie subgroup $\mathrm{Sp}(n, \mathbf{C})$ of $\mathrm{GL}(n, \mathbf{C})$ if and only if $\nu(\chi) = -1$ (in which case n is even).

Here, ν is the so-called Frobenius-Schur index, which only takes values $0, -1, 1$ on irreducible characters:

$$\nu(\chi) = \frac{1}{|F|} \sum_{g \in F} \chi(g^2) = (\chi^{2+} - \chi^{2-}, 1).$$

(The notation χ^{2+} stands for the character on the symmetric tensor (of degree 2) of the representation E, and χ^{2-} for the character of the skew symmetric (alternating) part, also called the wedge product.) This roughly deals with the embedding of F in classical Lie groups. Incidentally, the case $\nu(\chi) = 0$ occurs if and only if the character is not real valued.
Exercise 1. Prove:

1. $\{\pm 1\}$ is a finite subgroup of $\mathrm{GL}(1, \mathbf{C})$ with Frobenius-Schur index 1.
2. $\{\pm 1, \pm i\}$ is a finite subgroup of $\mathrm{GL}(1, \mathbf{C})$ with Frobenius-Schur index 0.
3. The Frobenius-Schur index of the quaternion group inside $\mathrm{SL}(2, \mathbf{C})$ is -1. Here the *quaternion group* is understood to be the following subgroup of order 8 of $\mathrm{SL}(2, \mathbf{C})$:

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}.$$

2.2 Lie primitivity

The occurrence of orthogonal and symplectic groups between F and $\mathrm{GL}(n, \mathbf{C})$ already shows the importance of a maximality condition for $E(F)$.

Let $E : F \to G$ be a morphism of groups into a complex algebraic group G of positive dimension. We say that E is *Lie primitive* if there is no proper closed algebraic subgroup of G of positive dimension containing $E(F)$. We often just speak of the image of F under E and call this subgroup Lie primitive.

We have seen above that if an "irreducible" embedding of F into $\mathrm{SL}(n, \mathbf{C})$ is Lie primitive, its Frobenius Schur index is 0.

2.3 A general structure theorem

Like many theorems involving finite groups, the embedding problem can be reduced to one for simple groups. We recall from finite group theory that, for a prime number p, an *elementary abelian p-group* is a direct product of cyclic groups $\mathbf{Z}_p$ of order p.

Theorem. *Let F be a finite Lie primitive subgroup of a simple algebraic group G. Then one of the following three assertions holds.*

1. *F is the normalizer of an elementary abelian p-subgroup of G, where J is as in Table 1.*
2. *F has socle (product of all products of nonabelian simple subgroups of F which are normal)* $\mathrm{Alt}_5 \times \mathrm{Alt}_6$, *and G has type E_8,*
3. *F has a simple socle.*

In the first case, the elementary abelian groups J that occur are known, thanks to Alekseevskii [1]. He called them *Jordan subgroups*.

Table 1.
Jordan subgroups of simple complex Lie groups

G	J	$C_G(J)/J$	$N_G(J)/C_G(J)$
A_{p^n-1}, p a prime	$\mathbf{Z}_p^{2n}$	1	$Sp_{2n}(p)$
B_n, $n \geq 3$	$\mathbf{Z}_2^{2n}$	1	Sym_{2n+1}
$C_{2^{n-1}}$, $n \geq 2$	$\mathbf{Z}_2^{2n}$	1	$O_{2n}^-(2)$
$D_{2^{n-1}}$, $n \geq 3$	$\mathbf{Z}_2^{2n}$	1	$O_{2n}^+(2)$
D_{n+1}, $n \geq 4$	$\mathbf{Z}_2^{2n}$	1	Sym_{2n+2}
G_2	$\mathbf{Z}_2^3$	1	$SL_3(2)$
F_4	$\mathbf{Z}_3^3$	1	$SL_3(3)$
E_6	$\mathbf{Z}_3^3$	$\mathbf{Z}_3^3$	$SL_3(3)$
E_8	$\mathbf{Z}_5^3$	1	$SL_3(5)$
E_8	$\mathbf{Z}_2^5$	$\mathbf{Z}_2^{10}$	$SL_5(2)$
$D_4.\mathbf{Z}_3$ (G is not simple)	$\mathbf{Z}_2^3$	$\mathbf{Z}_2^6 \cdot \mathbf{Z}_3$	$SL_3(2)$

The second, very remarkable case, is due to Borovik [4, 5].

For the third case, the Classification of Finite Simple Groups is invoked, at least for the exceptional Lie groups, to determine the full list of possibilities. We shall go further into that now.

2.4 The exceptional groups

Recall that the exceptional complex Lie groups all occur in $E_8(\mathbf{C})$. In fact, we have

$$G_2(\mathbf{C}) < F_4(\mathbf{C}) < 3 \cdot E_6(\mathbf{C}) < 2 \cdot E_7(\mathbf{C}) < E_8(\mathbf{C}).$$

Here, $3 \cdot E_6(\mathbf{C})$ denotes the universal covering group of type E_6, which has a center of order 3, and similarly for E_7. If F is a simple group having an embedding in $E_6(\mathbf{C})$, it may actullay come from an embedding of a 3-fold central covering group of F embedding in $3 \cdot E_6(\mathbf{C})$ (and hence in $E_8(\mathbf{C})$. Thus, it makes sense to study finite simple groups having a nonsplit central extension embedding in one of the Lie groups occurring in the chain. Therefore, we look at the (nonabelian) finite simple groups themselves, as well as at central nonsplit extensions. Synoptical information regarding this category is contained in Table 2 below.

Table 2. Nonabelian simple groups L a central extension of which embeds in a complex Lie group of exceptional type X_n.

X_n	L
G_2	Alt_5, Alt_6, $L(2,7)$, $L(2,8)$, $L(2,13)$, $U(3,3)$
F_4	Alt_7, Alt_8, Alt_9, $L(2,17)$, $L(2,25)$, $L(2,27)$, $L(3,3)$, ${}^3D_4(2)$, $U(4,2)$, $O(7,2)$, $O^+(8,2)$
E_6	Alt_{10}, Alt_{11}, $L(2,11)$, $L(2,19)$, $L(3,4)$, $U(4,3)$, ${}^2F_4(2)'$, M_{11}, J_2
E_7	Alt_{12}, Alt_{13}, $L(2,29)$, $L(2,37)$, $U(3,8)$, M_{12}
E_8	Alt_{14}, Alt_{15}, Alt_{16}, Alt_{17}, $L(2,16)$, $L(2,31)$, $L(2,41)$, $L(2,32)$, $L(2,49)$, $L(2,61)$, $L(3,5)$, $Sp(4,5)$, $G_2(3)$, $Sz(8)$

The following result gives two kinds of information which can be read from this table.

Theorem. *Let L be a finite simple group and let G be a simple algebraic group of exceptional type X_n.*

1. *If L occurs on a line corresponding to X_n in Table 2, then a central extension of it embeds in $G(\mathbf{C})$.*
2. *If X_n is as in some line of Table 2 and L appears neither in the line corresponding to X_n nor in a line above it, then no central extension of L embeds in $G(\mathbf{C})$.*

Some warnings are in order.

- The theorem does not say anything about Lie primitivity. For instance, the group Alt_6 embeds in a central extension of $A_2(\mathbf{C})$ and hence in $G_2(\mathbf{C})$. But there is no Lie primitive subgroup of $G_2(\mathbf{C})$ isomorphic to a central extension of Alt_6.
- Also, no claims about the exact number of inequivalent embeddings are being made. (In general, a group homomorphism $E : F \to G$ is said to be *equivalent* to $E' : F \to G$ if there exists $g \in G$ such that E' maps f to $gE(f)g^{-1}$ $(f \in F)$.) For instance, we shall discuss below an embedding of PSL$(2,13)$ in $G_2(\mathbf{C})$, but there are embeddings of PSL$(2,13)$ in $F_4(\mathbf{C})$ that do not factor through an embedding in $G_2(\mathbf{C})$.

The proof of the theorem (or rather the more detailed version, where more information is given on the possible embeddings) consists of two parts: one establishing that the groups listed are the only ones and the other that each of the groups listed occurs. We shall illustrate both parts.

The result summarizes work by several people, cf. [15, 18] for surveys, and, for instance, [17, 19, 20, 14, 12, 13, 27, 7] for details.

3 Weyl's character formula

To begin, we note that images of elements of F in G, being of finite order, are semisimple, and hence embeddable in a torus. Thus, we can determine their character values in representations of G by restriction. This leads us to the well-known formula for characters of a maximal torus T of G on irreducible modules.

Let W be the Weyl group of G. Denote by P the weight lattice, and by P^+ the dominant weights with respect to T. Denote by sn the sign character of W, by ρ the half sum of the positive roots. Furthermore, for $\mu \in P$, write ε_μ to denote the "formal character" μ, so that we can view the group algebra over P as the linear span of all ε_μ. In this group algebra, we have the formal character ch_λ of T on V_λ, the irreducible representation of G with highest weight $\lambda \in P^+$; it is the sum of all ε_μ for μ a linear character of T, with multiplicities given by the following formula.

Theorem (Weyl's character formula). *If $\lambda \in P^+$ then*

$$ch_\lambda = \frac{\sum_{\sigma \in W} sn(\sigma)\varepsilon_{\sigma(\lambda+\rho)}}{\sum_{\sigma \in W} sn(\sigma)\varepsilon_{\sigma\rho}}.$$

Other formulas exist for these characters. In LiE, the character of T on V_λ can be obtained via Demazure's routine. For instance, `Demazure([1,0],G2)` shows that on the representation for $G_2(\mathbf{C})$ with highest weight [1,0], the character $\chi_{[1,0]}$ is

```
1X[-2, 1] +1X[-1, 0] +1X[-1, 1] +1X[ 0, 0] +
1X[ 1,-1] +1X[ 1, 0] +1X[ 2,-1]
```

which, upon replacement of `X[1,0]` by λ and `X[0,1]` by μ, reads:

$$\chi_{[1,0]} = \lambda^{-2}\mu + \lambda^{-1} + \lambda^{-1}\mu + 1 + \lambda\mu^{-1} + \lambda + \lambda^2\mu^{-1}.$$

Corollary (Weyl's dimension formula). *If $\lambda \in P^+$ then*

$$\dim V_\lambda = \frac{\prod_{\alpha \succ 0}(\lambda + \rho, \alpha)}{\prod_{\alpha \succ 0}(\rho, \alpha)}.$$

In LiE, `dim([1,0],G2)` gives 7, in accordance with the value obtained from the above G_2 character $\chi_{[1,0]}$ by substitution of 1 for both λ and μ.

Exercise 2. Show (by hand) that $\dim V_\rho = 2^n$, where n is the number of positive roots of W. Check it for some cases using LiE, e.g., via

```
dim([1,1,1,1],D4) = 2^(n_rows(pos_roots(D4)))
```

or, if you prefer to loop over some groups:

```
conjecture = 1
for i=1 to 4 do
   for j=4 to 8 do g = Lie_group(i,j);
       setdefault g;
       if dim(all_one(j)) == 2^(n_pos_roots)
       then print("conjecture verified for"); print(g)
       else conjecture = 0; print("counterexample found for"); print(g)
       fi
   od
od
```

Exercise 3. It may be difficult to prove the following formula (cf. [23]):

$$2^r V_\rho \otimes V_\rho = \Lambda^* g,$$

where r is the Lie rank of G, and g is the Lie algebra of a complex simple Lie group G, that is, the adjoint G-module.

But you can verify it for very low-dimensional Lie groups using LiE:

```
verify_kostant (grp g) = {
  loc verify = 1;
  loc  a = adjoint(g);
  loc d = dim(g);
  loc r = Lie_rank(g);
  rhs = 2 * X(null(r));
  if d%2 == 0 then
    for i=1 to d/2-1 do rhs = rhs + 2*plethysm(all_one(i),a,g) od;
    rhs = rhs +  plethysm(all_one(d/2),a,g)
  else
    for i=1 to (d-1)/2 do rhs = rhs + 2*plethysm(all_one(i),a,g) od
  fi;
  lhs = 2^r * tensor(all_one(r),all_one(r),g);
  print("LHS="); print(lhs); print("RHS="); print(rhs);
  if lhs == rhs then print("they are equal, as required")
  else print("error: not equal as required"); verify = 0
  fi; verify
}

verify_kostant(G2); verify_kostant(A2)

#create more object hangers for bigger computations:
maxobjects 100000
maxnodes 100000

verify_kostant(A3)
```

It is hard to verify the formula in this way for higher dimensional Lie algebras because `plethysm(all_one(i),...)` requires steeply increasing time and space as `i` grows.

3.1 Branching

If H is a closed Lie subgroup of G, the LiE function `branch` tells you how the character of G on V_λ decomposes into a sum of high weight representations of H. However, you will have to tell the system how the fundamental characters decompose; this is done in the so-called *restriction matrix*.

For instance, in branching from G_2 to its fundamental Lie subgroup of type A_2 (whose root system is the closed subsystem of G_2 of the long roots), the restriction matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(the columns might be interchanged depending on the chosen ordering, because of the symmetry of the Dynkin diagram of type A_2). If `m` is this matrix, then `branch(v,A_2,m,G_2)` gives the highest weights of the $A_2(\mathbf{C})$-representation into which the $G_2(\mathbf{C})$ representation with highest weight `v` decomposes.

3.2 Subtori

In LiE, the fundamental weights ω_i are used to describe elements of the maximal torus T. Recall that weights (elements of P) are group morphisms $T \to \mathbf{C}^*$ (also called linear characters). In particular, a weight λ can be evaluated at an element $t \in T$; we write t^λ for the resulting value. The set of fundamental weights form a complete set of coordinates in the sense that any element $t \in T$ is uniquely determined by the values t^{ω_i} for $i = 1, \ldots, r$.

In LiE, a 1-dimensional subtorus of T is given by a vector $[a_1, \ldots, a_r, 0]$; it represents the subtorus $\{t(\ell) \in T \mid \ell \in \mathbf{C}^*\}$ for which $t(\ell)^{\omega_i} = \ell^{a_i}$ for $i = 1, \ldots, r$. The restriction matrix needed for such a 1-dimensional torus in branching is essentially obtained by transposition of the vector after removal of the last component (which contains 0, for reasons as yet unexplained). For instance, the LiE session

```
t=[1,0,0]
branch([1,0],T1,*[t-3],G2)
```

gives us `1X[-2] +2X[-1] +1X[ 0] +2X[ 1] +1X[ 2]`, meaning that the character of the torus `t` on the 7-dimensional representation for G_2 is $\lambda^{-2} + 2\lambda^{-1} + 1 + 2\lambda + \lambda^2$.

3.3 Example: $SL(n, \mathbf{C})$

For the special linear group $SL(n, \mathbf{C})$ there is a much more familiar way to describe a toral element, namely by its diagonal entries in diagonalised form. If t is a diagonal matrix with entries $(t_1, \ldots, t_n)$ on the main diagonal in the standard representation, then the values of the fundamental weights ω_i on t are given by

$$t^{\omega_i} = \prod_{j=1}^{i} t_j.$$

Therefore let t be a toral element of $SL(n, \mathbf{C})$, with matrix eigenvalues $[\zeta^{b_1}, \ldots, \zeta^{b_n}]$, where ζ is a parameter in $\mathbf{C}^*$ (note that $\sum_{j=1}^{n} b_j \equiv 0$ since $t \in SL(n, \mathbf{C})$). Then t can be represented in LiE by applying the following function `mk_toral` (an abbreviation for 'make toral element'), to the vector $[b_1, \ldots, b_n]$:

```
mk_toral(vec b) = {loc n=size(b);
    if (b * *[all_one(n)])[1] != 0 then
      print("error: not in SLn")
    fi;
    for i=2 to n-1 do b[i] = (b[i-1]+b[i]) od ; b[n]=0;
    b
}
```

Exercise 4. Compute the restriction matrix for the embedding of $G_2(\mathbf{C})$ into $\mathrm{GL}(7, \mathbf{C})$ resulting from the 7-dimensional representation of G_2 with highest weight [1,0].

4 Elements of finite order

Elements of finite order are semisimple and embed in a torus. Since we are, for the moment, only interested in such elements up to conjugacy, and since all maximal tori are conjugate, we can view them as element of the maximal torus T.

There are two ways of describing elements of finite order, often called EFOs for short. One way is used in LiE, the other is called Kac coordinates. Since the use of each is advantageous, we discuss them both.

4.1 EFOs in LiE

Recall the description of toral elements given in §3.2. Although LiE cannot represent arbitrary complex toral elements, it can represent torus elements for which all t^{ω_i} are roots of unity. To this end, a vector $[a_1, \ldots, a_r, d]$ in LiE represents the element $t \in T$ for which $t^{\omega_i} = e^{2\pi \mathrm{i} a_i/d} = \zeta_d^{a_i}$ for $i = 1, \ldots, r$, where $\zeta_d = e^{2\pi \mathrm{i}/d}$ is a canonical d-th root of unity.

It follows from this description that any a_i may be taken modulo d, and that all the entries (including the final d) may be multiplied by a common non-zero factor, without changing the indicated toral element.

The use of this representation of a finite element is that we can perform two routines on EFOs in LiE:

- the `spectrum`, providing its eigenvalues with multiplicities on a highest weight module
- the `centr_type(t)`, centralizer type, which is the type of (reductive) group that the centralizer of `t` is.

For instance, the following computation for a toral element t of order 2 in $SL(5, \mathbf{C})$:

```
setdefault A_4; t=[1,0,0,0,2]; sr=[1,0,0,0]
spectrum(sr,t)
```

returns $3X[0]+2X[1]$, showing that t has 3 eigenvalues 1, and 2 eigenvalues -1 in the standard representation. It is therefore conjugate to the element `mk_toral([0,0,0,1,1],2)`, which equals $[0,0,0,1,2]$. The element t itself can be obtained as an image of `mk_toral` by an appropriate permutation of the eigenvalues: we have `t == mk_toral([1,1,0,0,0],2)`.

Continuing with this example, we find that `centr_type(t)`, returns $A_1A_2T_1$, and that the centraliser `centr_type(t-5+0)` of the one parameter subgroup containing it is A_2T_2.

Exercise 5. Interpret these results in terms of linear algebra.

Exercise 6. Verify that the function `spectrum` may be simulated as follows:

```
spec(pol p; vec t) = loc r=size(t); branch(p,T_1,*[t-r])%[t[r]]
```

4.2 The Kac method

The second explicit description of EFOs in T is more convenient for finding representatives of their conjugacy classes under the Weyl group W.

The lattice $Q = \oplus_{i=1}^{r} \mathbf{Z}\alpha_i$, spanned by the (fundamental) roots, is the so-called *root lattice* of G with respect to T. It is a sublattice of the *weight lattice* P, spanned by the (fundamental) weights, which can be canonically identified with the (algebraic) multiplicative character group $X(T)$ of T.

We write $\mathbf{t}^* = X(T) \otimes_{\mathbf{Z}} \mathbf{R}$, and $\langle\cdot,\cdot\rangle : \mathbf{t}^* \times \mathbf{t} \to \mathbf{R}$ for the natural pairing of $\mathbf{t}$, the real Cartan subalgebra of g corresponding to T, with its dual $\mathbf{t}^*$. By definition of the W action on $\mathbf{t}^*$ it is W-invariant. The lattices

$$\hat{Q} := \{\lambda \in \mathbf{t} \mid \forall \gamma \in P \mid \langle\gamma, \lambda\rangle \in \mathbf{Z}\} = \ker(\exp 2\pi i)$$

and

$$\hat{P} := \{\lambda \in \mathbf{t} \mid \forall \gamma \in Q \mid \langle\gamma, \lambda\rangle \in \mathbf{Z}\} = \ker(\mathrm{Ad} \circ \exp 2\pi i)$$

are called the *coroot lattice* and *coweight lattice*, respectively. Thus we obtain the W-equivariant exact sequence

$$0 \longrightarrow \hat{Q} \longrightarrow \mathbf{t} \longrightarrow T \longrightarrow 1, \tag{1}$$

where the third map is the morphism $\exp 2\pi i(\cdot)$ The kernel of the adjoint representation is the center $Z(G)$ of G which is a finite subgroup of T. Thus, $Z(G) \cong \hat{P}/\hat{Q}$ and W operates trivially on P/Q and $\hat{P}/\hat{Q}$. Furthermore, $[P:Q] = [\hat{P}:\hat{Q}] = |Z(G)|$.

We use the above exact sequence to identify the elements of finite order in T with elements in $\hat{Q} \otimes \mathbf{Q}$:

$$T_n := \{x \in T \mid x^n = 1\} = \frac{1}{n}\hat{Q}/\hat{Q}. \tag{2}$$

Exercise 7. For an EFO $x \in T$ the smallest n such that $\mathrm{Ad}\,(x)^n = 1$, is called the *adjoint order* of x. Let T_n^{adj} denote the elements in T of adjoint order dividing n. Establish $|T_n^{adj}| = |Z(G)|n^r$ and $|T_n| = n^r$.

This presentation can be exploited to handle conjugacy. As we have seen before, all conjugacy classes of EFO meet T_n. Two elements $x = e^{2\pi i X}$ and $y = e^{2\pi i Y}$, $X, Y \in \mathbf{t}$, are conjugate in G if and only if there is $w \in W$ such that $wX \equiv Y \pmod{\hat{Q}}$. The latter equivalence calls for the use of the *affine Weyl group*, $\widetilde{W}$, which is the semidirect product of W and $\hat{Q}$. It is the subgroup of the affine automorphisms of $\mathbf{t}$ generated by W and translations by elements of $\hat{Q}$. We can now rephrase the above condition for conjugacy:

$$x = e^{2\pi i X} \text{ and } y = e^{2\pi i Y} \text{ are conjugate in } G \Longleftrightarrow \exists\, \tilde{w} \in \widetilde{W} \text{ such that } \tilde{w}X = Y.$$

The group $\widetilde{W}$ is generated by W together with the reflection in the hyperplane $\{X \mid \langle -\alpha_0, X\rangle = 1\}$, where

$$-\alpha_0 = \sum_{i=1}^{r} n_i \alpha_i,$$

is the *highest root* of the root system Δ. Putting $n_0 = 1$, we can write this as $n_0\alpha_0 + \cdots + n_r\alpha_r = 0$. The numbers $n_0, \ldots, n_r$ are called the *marks* of G.

Therefore, the simplex

$$F := \{X \in \mathbf{t} \mid \langle \alpha_i, X\rangle \geq 0,\ i = 1, \ldots, r \text{ and } \langle -\alpha_0, X\rangle \leq 1\} \tag{3}$$

is a fundamental region for $\widetilde{W}$ in $\mathbf{t}$, and we have a bijective correspondence

$$\begin{array}{ccc} \{\text{points of } \frac{1}{n}\hat{Q} \cap F\} & \stackrel{1-1}{\longleftrightarrow} & \{\text{conjugacy classes of EFO in } T_n\} \\ X & \mapsto & e^{2\pi i X} \end{array} \tag{4}$$

Using the fundamental coweights $\widehat{\omega_1}, \ldots, \widehat{\omega_r}$, which are dual to the fundamental (also called simple) roots $\alpha_1, \ldots, \alpha_r$, we can express any element $X \in \frac{1}{m}\hat{Q}$, for $m \in \mathbf{N}$, as

$$X = \frac{1}{m}\sum_{i=1}^{r} s_i \hat{\omega}_i, \text{ where } s_i \in \mathbf{Z}. \tag{5}$$

Clearly the $r+1$ tuple $(m, s_1, \ldots, s_r)$ is uniquely determined by X if we assume in addition that they have no common factor, i.e., $\gcd(m, s_1, \ldots, s_r) = 1$. The element X belongs to the fundamental region F if $\langle \alpha_i, X\rangle = \frac{1}{m}s_i \geq 0$ for $i = 1, 2, \ldots, r$ and $\langle -\alpha_0, X\rangle = \frac{1}{m}\sum_{i=1}^{r} n_i s_i \leq 1$. This leads to the following definition:

Let X be as in (5) and define $s_0 := m - \sum_{i=1}^{r} n_i s_i$. Then

$$X \in F \quad \Longleftrightarrow \quad s_i \geq 0 \text{ for } i = 0, 1, \ldots, r$$

If X has coordinates $[s_0, \ldots, s_r]$ then the adjoint order of X is m/d where $m := \sum_{i=0}^{r} n_i s_i$ and $d := \gcd(s_0, \ldots, s_r)$. (This follows from the fact that the fundamental coweights span the coweight lattice $\hat{P}$.)

We shall call $\mathrm{s} = [s_0, \ldots, s_r]$ the *Kac coordinates* of X, although *barycentric* would also be an appropriate name. Then

$$X := X(\mathrm{s}) := \frac{1}{m} \sum_{i=1}^{r} s_i \hat{\omega}_i \quad \text{where} \quad m := \sum_{i=0}^{r} n_i s_i.$$

We have found a bijection between the conjugacy classes in T_m^{adj}, the subgroup of T of all elements whose adjoint order is a divisor of m, and the set of $r+1$-tuples of nonnegative integers $\mathrm{s} = [s_0, \ldots, s_r]$ with $\sum_{i=0}^{r} n_i s_i = m$:

$$\{\mathrm{s} \mid \sum_{i=0}^{r} n_i s_i = m,\, s_i \geq 0\} \stackrel{1-1}{\longleftrightarrow} T_m^{adj}/W.$$

$$\mathrm{s} \mapsto X(\mathrm{s})$$

Exercise 8. Prove that the full order of an element $X \in \hat{Q} \otimes \mathbf{Q}$ with coordinates s such that $\gcd(s_0, \ldots, s_r) = 1$ is the least integer n such that $nX \in \hat{Q}$. Since X has adjoint order $m = \sum n_i s_i$, we see that n is a multiple of m and that $d := n/m$ is the order of $mX \pmod{\hat{Q}}$. Establish that

$$d = \frac{\det C}{\gcd((s_1, \ldots, s_r)C^{-\top} \det C, \det C)}.$$

[Hint: In order to determine d we have to express $mX = \sum s_i \hat{\omega}_i$ in the basis $\hat{\alpha}_1, \ldots, \hat{\alpha}_r$. The transpose of the Cartan matrix C describes the change of basis in $\mathfrak{t}$ from the $\hat{\alpha}$-basis to the $\hat{\omega}$-basis. Now d is the smallest integer such that $d(s_1, \ldots, s_r)C^{-\top}$ has integer coefficients.] Verify that the matrix $(\det C)C^{-1} \pmod{\det C}$ describes the center $Z(G)$ by generators.

4.3 From Kac to LiE

For EFOs, we describe transformations back and forth from Kac's coordinates to LiE's for semisimple elements. In a first reading, you can use these sections as black boxes, and just use the transformations `kac_to_ss` and `ss_to_kac`, together with `efos` as indicated in Exercise 9.

First we give `kac_to_ss`, a routine for transforming Kac's coordinates to LiE's notation. The special (non-linear) Kac coordinate is placed at the end of the vector.

```
kac_to_ss(vec kac; grp g) =
{ if n_comp(g)!=1 || Lie_rank(g[0])!=0 then
error("Simple group required.")
  fi
; loc marks=high_root(g)^[1]; loc denom=(kac*marks)*det_Cartan(g)
; loc coroot_coords=(kac-(size(kac)))* *i_Cartan(g)
#; print("Intermediate result:"); print(coroot_coords^[denom])
; loc common=gcd(coroot_coords^[denom]); denom=denom/common
; ((coroot_coords/common)%denom) ^ [denom]
}

kac_to_ss(mat m; grp g) =
{ for i=1 to n_rows(m) do m[i]=kac_to_ss(m[i],g) od; m }
```

4.4 From LiE to Kac

The transformation back, ss_to_kac, involves bringing an element of finite order into the fundamental domain for W, which is essentially the algorithm of dominant for the affine Weyl group associated to the dual of g, performed on vectors with rational coefficients. We cannot use the built-in routine for dominant because it does not deal with affine groups. So we implement the algorithm completely by hand. Eventually we must transform from coroot coordinates to coweight coordinates.

```
ss_to_kac(vec ss; grp g) =
{ loc denom=ss[size(ss)]; loc v=ss-size(ss)
; loc r=Lie_rank(g); loc c=Cartan(g)
; loc h=high_root(g); loc e=id(r)
; loc a=h*c # high_root expressed in weight coordinates
; loc a0=h; for i=1 to r do a0[i]=a0[i]*norm(e[i],g) od
; a0=a0/norm(h,g) # high_root^ expressed in coroot coordinates
; loc i=1
; while 1
  do loc level_i=v*c[i] # v evaluated on simple root alpha_i
  ; if level_i<0 then v[i]+= -level_i; if i>3 then i+= -2 else i=1 fi
    else if i<r then i+=1
    else # now v is dominant
      loc level_0=v*a # v evaluated on high_root
    ; if level_0<=denom then break fi # in fundamental region
    ; v+= -(level_0/(2*denom))*denom*a0
# translate integer multiple of a0
    ; level_0=v*a # or level_0=level_0%(2*denom)
    ; if level_0>denom then v+= (denom-level_0)*a0 fi
    ; i=1 # restart dominant making loop
    fi fi
  od
#; print("Representative in fundamental domain:"); print(v^[denom])
; loc s=v* *c # transform to coweight coordinates
; loc kac=s^[denom-s*h]
# add final (non-linear) Kac coordinate
; kac/gcd(kac) # eliminate any common factor
}

ss_to_kac(mat m; grp g) =
{ for i=1 to n_rows(m) do m[i]=ss_to_kac(m[i],g) od; m }
```

4.5 Finding EFOs in LiE

The following LiE routines are used to generate all elements of a given finite order m in the fundamental domain. They enumerate all vectors h of non-negative numbers such that h*high_root <= m; the final Kac coordinate can then be set to m-h*high_root. This generates all elements of adjoint order a divisor of m; we can filter the result to obtain all elements of adjoint or full order exactly m.

```
efos(int m; grp g) = do_efo(0)
ad_efos(int m; grp g) = do_efo(1)

do_efo(int for_ad) =
{ loc r=Lie_rank(g); loc mark=high_root(g)
```

```
; loc h=null(r+1); loc result=null(0,r+1)
; gen_efo(1,m); result
}

gen_efo(int i,rem)=
if i>r
then h[r+1]=rem # fill the last Kac coordinate with remainder
; if if for_ad then gcd(h)==1 else kac_to_ss(h,g)[r+1]==m fi # filter
  then result+=h
  fi
else # try all values for h[i] that don't overflow rem, and recurse
  for j=0 to rem/mark[i] do h[i]=j; gen_efo(i+1,rem-j*mark[i]) od
fi
```

Exercise 9. Use LiE to compile a list of all elements of order 5 in E_8 with corresponding centralizer type and eigenvalue decomposition in the adjoint representation. [Hint: Use `efos(5,E8)` to determine the conjugacy classes of elements of order 13 in $G_2(\mathbf{C})$, and `kac_to_ss` to bring them into shape for application of `centr_type` and `spectrum`.]

Exercise 10. Give semisimple elements u, t, w of G_2 with the following spectrum (=eigenvalues with multiplicities) in the 8-dimensional representation $V_{[0,0]} \oplus V_{[1,0]}$ of dimension $1+7=8$. Here ζ is a primitive 13-th root of 1, and ω a primitive cube root of 1.

$$\begin{array}{rcl} u & : & 1,1,\zeta^2,\zeta^5,\zeta^6,\zeta^7,\zeta^8,\zeta^{11} \\ t & : & 1,1,-1,-1,\omega,\omega^2,-\omega,-\omega^2\,. \\ w & : & 1,1,1,1,-1,-1,-1,-1 \end{array}$$

4.6 Implications for finite subgroups

Two strong restrictions regarding finite subgroups of G are:

1. every finite cyclic subgroup of G is embeddable in T and so the EFO theory can be applied to find its spectrum in any of the highest weight representations of G.

2. any nilpotent subgroup is embeddable in $N_G(T)$, the normalizer in G of T. (A result of Borel and Serre [3].) The group $N_G(T)$ is an extension of T by the Weyl group W, and so is well controlled. It proves for instance that $\mathbf{Z}_2^4$ is not embeddable in $G_2(\mathbf{C})$, nor $\mathbf{Z}_p^3$ for $p > 2$. (The fact that $\mathbf{Z}_2^3$ does embed is very remarkable, see Table 1!)

Using observations of this kind, and the Classification of Finite Simple Groups, the list of subgroups of G with a simple socle can be easily brought down to a finite number of simple groups.

Exercise 11. Suppose $\mathrm{PSL}(2,p)$, with p an odd prime number, embeds in the simple Lie group G. Prove that $(p-1)/2$ divides $|W|$. [Hint: use the existence in $\mathrm{PSL}(2,p)$ of a subgroup of order $p(p-1)/2$.]

Exercise 12. Suppose Alt_n, with $n \in \mathbf{N}$, embeds in the simple Lie group $G_2(\mathbf{C})$. Prove that $n < 6$. [Hint, you only have to show Alt_6 does not embed. Study characters of a possible embedding, using information about EFOs.]

5 The abstract group PSL(2,13)

Our sample project will be to embed $F = \mathrm{PSL}(2,13)$ in $G = G_2(\mathbf{C})$. Thus we have to focus on definitions of both groups involved. We start with F.

The group PSL(2, 13) is the so-called fractional linear group over the field of order 13. It can be defined as the quotient of the group of all 2×2 matrices of determinant 1 over $\mathbf{Z}_{13}$ by its centre, which is $\{\pm I_2\}$. When representing elements of this group, we will work with 2×2 matrices, identifying each matrix m, with its negative $-m$.

We distinguish the following three generating matrices:

$$u := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t := \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}, \quad w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These matrices satisfy the relations

$$u^{13} = t^6 = w^2 = 1,$$
$$(uw)^3 = tut^{-1}u^9 = (wt)^2 = 1.$$

Moreover, this is a presentation of F by generators and relations: F is isomorphic to the quotient of the free group on the letters u, t, w by the normal subgroup generated by the left hand sides of the above equations.

Exercise 13. Verify this fact using a Todd-Coxeter enumeration in GAP.

5.1 Representations of PSL(2, 13)

Before studying how $F = \mathrm{PSL}(2,13)$ could possibly embed in $G_2(\mathbf{C})$, we recall representation theory for F. Each representation is a finite sum of irreducibles. A representation is uniquely determined by its character (that is, the composition of the representation with the trace, whence a function on F, constant on each conjugacy class of F).

Since the character of a sum representation is the sum of the characters of the constituents, it suffices to know the irreducible characters (that is, the characters of the irreducibles). There are finitely many such characters, and usually they are listed in the so-called character table. The table for F is given below. We have written η for a primitive 7-th root of 1, and ζ for a primitive 13-th root of 1. Also, we have abbreviated

$$a = -\zeta^2 - \zeta^5 - \zeta^6 - \zeta^7 - \zeta^8 - \zeta^{11}, \quad b = -\zeta - \zeta^3 - \zeta^4 - \zeta^9 - \zeta^{10} - \zeta^{12}.$$

Table 3. Character table for PSL(2,13)

orders	1	13	13	6	3	2	7	7	7
centralizers	1092	13	13	6	6	12	7	7	7
classes	1	84	84	182	182	91	156	156	156
	1	1	1	1	1	1	1	1	1
	12	-1	-1	0	0	0	$-\eta-\eta^6$	$-\eta^2-\eta^5$	$-\eta^3-\eta^4$
	12	-1	-1	0	0	0	$-\eta^2-\eta^5$	$-\eta^3-\eta^4$	$-\eta-\eta^6$
	12	-1	-1	0	0	0	$-\eta^3-\eta^4$	$-\eta-\eta^6$	$-\eta^2-\eta^5$
	13	0	0	1	1	1	-1	-1	-1
	14	1	1	1	-1	-2	0	0	0
	14	1	1	-1	-1	2	0	0	0
	7	a	b	-1	1	-1	0	0	0
	7	b	a	-1	1	-1	0	0	0

Thus, there are irreducible characters of degrees 1, 12 (3 times), 13 , 14 (twice), 7 (twice). A familiar check is that the sum of squares of the degrees is the group order:

$$1^2 + 3 \cdot 12^2 + 13^2 + 2 \cdot 14^2 + 2 \cdot 7^2 = 1092 = |\mathrm{PSL}(2,13)|.$$

This character table can be found within GAP, e.g., as follows.

```
ct := CharTable(''PSL'',2,13);
DisplayCharTable(ct);
```

Exercise 14. Prove that $a + b = 1$ and that $r = a - b$ satisfies $r^2 = 13$. Conclude that a and b are real valued and that the Frobenius-Schur indices of the irreducible 7-dimensional representations are 1.

5.2 The 7-dimensional representations for PSL(2,13)

We describe one of the irreducible 7-dimensional representations explicitly. They are defined over $\mathbf{Q}(\sqrt{13})$. (Observe that the numbers a and b of Table 3 belong to this field.) First we give a representation defined over $\mathbf{Q}(\zeta)$, where ζ is a primitive 13-th root of 1. (Observe that $\sqrt{13}$ belongs to $\mathbf{Q}(\zeta)$.) It is known how to construct such representations, see e.g., [26]. We give three 8×8 matrices for u, t, and w, respectively.

$$u \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$t \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$w \mapsto \frac{1}{13} \left[\begin{array}{c}
\left[\tfrac{1}{2}(13 - \zeta + \zeta^2 - \zeta^3 - \zeta^4 + \zeta^5 + \zeta^6 + \zeta^7 + \zeta^8 - \zeta^9 - \zeta^{10} + \zeta^{11} - \zeta^{12}),\right. \\
\%1, \%1, \%2, \%2, \%2, \%1, \\
\left.\tfrac{1}{2}(13 + \zeta - \zeta^2 + \zeta^3 + \zeta^4 - \zeta^5 - \zeta^6 - \zeta^7 - \zeta^8 + \zeta^9 + \zeta^{10} - \zeta^{11} + \zeta^{12})\right] \\
\left[\%1, 2\zeta + 2\zeta^2 + 3\zeta^3 + 2\zeta^4 + 4\zeta^5 + 4\zeta^8 + 2\zeta^9 + 3\zeta^{10} + 2\zeta^{11} + 2\zeta^{12},\right. \\
\%3, \%4, 4\zeta + 3\zeta^2 + 2\zeta^3 + 2\zeta^5 + 2\zeta^6 + 2\zeta^7 + 2\zeta^8 + 2\zeta^{10} + 3\zeta^{11} + 4\zeta^{12}, \\
\left.\%5, \%6, \%2\right] \\
\left[\%1, \%3, 2\zeta + 4\zeta^2 + 2\zeta^3 + 3\zeta^4 + 2\zeta^6 + 2\zeta^7 + 3\zeta^9 + 2\zeta^{10} + 4\zeta^{11} + 2\zeta^{12}, \%7,\right. \\
\left.\%5, 2\zeta^2 + 4\zeta^3 + 2\zeta^4 + 2\zeta^5 + 3\zeta^6 + 3\zeta^7 + 2\zeta^8 + 2\zeta^9 + 4\zeta^{10} + 2\zeta^{11}, \%8, \%2\right] \\
[\%2, \%4, \%7, \%3, \%6, \%8, \%5, \%1] \\
\left[\%2, 4\zeta + 3\zeta^2 + 2\zeta^3 + 2\zeta^5 + 2\zeta^6 + 2\zeta^7 + 2\zeta^8 + 2\zeta^{10} + 3\zeta^{11} + 4\zeta^{12}, \%5, \%6,\right. \\
2\zeta + 2\zeta^2 + 3\zeta^3 + 2\zeta^4 + 4\zeta^5 + 4\zeta^8 + 2\zeta^9 + 3\zeta^{10} + 2\zeta^{11} + 2\zeta^{12}, \%3, \\
\left.\%4, \%1\right] \\
\left[\%2, \%5, 2\zeta^2 + 4\zeta^3 + 2\zeta^4 + 2\zeta^5 + 3\zeta^6 + 3\zeta^7 + 2\zeta^8 + 2\zeta^9 + 4\zeta^{10} + 2\zeta^{11}, \%8,\right. \\
\left.\%3, 2\zeta + 4\zeta^2 + 2\zeta^3 + 3\zeta^4 + 2\zeta^6 + 2\zeta^7 + 3\zeta^9 + 2\zeta^{10} + 4\zeta^{11} + 2\zeta^{12}, \%7, \%1\right] \\
[\%1, \%6, \%8, \%5, \%4, \%7, \%3, \%2] \\
[\tfrac{1}{2}(13 + \zeta - \zeta^2 + \zeta^3 + \zeta^4 - \zeta^5 - \zeta^6 - \zeta^7 - \zeta^8 + \zeta^9 + \zeta^{10} - \zeta^{11} + \zeta^{12}), \\
\%2, \%2, \%1, \%1, \%1, \\
\%2, \tfrac{1}{2}(13 - \zeta + \zeta^2 - \zeta^3 - \zeta^4 + \zeta^5 + \zeta^6 + \zeta^7 + \zeta^8 - \zeta^9 - \zeta^{10} + \zeta^{11} - \zeta^{12})]
\end{array} \right]$$

$$\begin{aligned}
\text{where}\quad \%1 &:= -\zeta+\zeta^2-\zeta^3-\zeta^4+\zeta^5+\zeta^6+\zeta^7+\zeta^8-\zeta^9-\zeta^{10}+\zeta^{11}-\zeta^{12}\\
\%2 &:= \zeta-\zeta^2+\zeta^3+\zeta^4-\zeta^5-\zeta^6-\zeta^7-\zeta^8+\zeta^9+\zeta^{10}-\zeta^{11}+\zeta^{12}\\
\%3 &:= 3\zeta+2\zeta^3+2\zeta^4+2\zeta^5+4\zeta^6+4\zeta^7+2\zeta^8+2\zeta^9+2\zeta^{10}+3\zeta^{12}\\
\%4 &:= -2\zeta-4\zeta^2-2\zeta^3-3\zeta^4-2\zeta^6-2\zeta^7-3\zeta^9-2\zeta^{10}-4\zeta^{11}-2\zeta^{12}\\
\%5 &:= 2\zeta+2\zeta^2+4\zeta^4+3\zeta^5+2\zeta^6+2\zeta^7+3\zeta^8+4\zeta^9+2\zeta^{11}+2\zeta^{12}\\
\%6 &:= -2\zeta^2-4\zeta^3-2\zeta^4-2\zeta^5-3\zeta^6-3\zeta^7-2\zeta^8-2\zeta^9-4\zeta^{10}-2\zeta^{11}\\
\%7 &:= -2\zeta-2\zeta^2-3\zeta^3-2\zeta^4-4\zeta^5-4\zeta^8-2\zeta^9-3\zeta^{10}-2\zeta^{11}-2\zeta^{12}\\
\%8 &:= -4\zeta-3\zeta^2-2\zeta^3-2\zeta^5-2\zeta^6-2\zeta^7-2\zeta^8-2\zeta^{10}-3\zeta^{11}-4\zeta^{12}
\end{aligned}$$

These have the following properties:

- they satisfy the relations u, t, and w specified above, and so they indeed generate a group isomorphic to PSL$(2,13)$;
- they fix the vector e_1+e_8 (sum of two standard basis vectors);
- they leave invariant the linear span of $e_2,\ldots,e_7$ and e_1-e_8, and so they give rise to a 7-dimensional representation.

Exercise 15. Check that the eigenvalues of the 8×8 matrices for u, t, and w are as indicated in the exercise at the end of §4. Use Table 3 to compute the Frobenius-Schur index (cf. 2.1) of the 7 dimensional subrepresentation, and conclude that u,t,w generate a subgroup of $O(8,\mathbf{C})$.

6 The Cayley algebra

There are at least three ways of defining $G_2(\mathbf{C})$ explicitly:

1. Write out the multiplication of the Lie algebra of type G_2 on a 14 dimensional vector space, for instance using the Chevalley basis. Then $G_2(\mathbf{C})$ is the automorphism group of this Lie algebra.
2. Write out a trilinear alternating form on a 7-dimensional vector space $\mathbf{C}^7$, whose stabilizer is irreducible on the underlying space $\mathbf{C}^7$; then this stabilizer is $G_2(\mathbf{C})$.
3. Write out the multiplication of the octonions on an 8 dimensional vector space. Its automorphism group will then be $G_2(\mathbf{C})$.

Below we take the latter approach. Later, we shall also consider the second approach. The first approach has been studied in [25]. An explicit embedding can also be found in [12].

6.1 Multiplication

Recall the following classical concepts regarding the vector space $\mathbf{C}^3$:

- the standard inner product of $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ is $x\cdot y = x_1y_1+x_2y_2+x_3y_3$. It is bilinear and symmetric.
- the standard outer product of x and y is the vector $x\times y=(x_2y_3-x_3y_2, x_3y_1-x_1y_3, x_1y_2-x_2y_1)$. It is bilinear and anti-symmetric.

We define $\mathbf{O}(\mathbf{C})$ as the set of matrices

$$x = \begin{pmatrix} x_1 & x_{234} \\ x_{567} & x_8 \end{pmatrix}$$

where $x_1, x_8 \in \mathbf{C}$ and $x_{234} = (x_2, x_3, x_4), x_{567} = (x_5, x_6, x_7) \in \mathbf{C}^3$. We provide $\mathbf{O}(\mathbf{C})$ with the usual (entrywise) vector space structure over $\mathbf{C}$ and with the following *nonassociative* multiplication:

$$x\,y = \begin{pmatrix} x_1 & x_{234} \\ x_{567} & x_8 \end{pmatrix} \begin{pmatrix} y_1 & y_{234} \\ y_{567} & y_8 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 y_1 - x_{234} \cdot y_{567} & x_1 y_{234} + y_8 x_{234} + x_{567} \times y_{567} \\ y_1 x_{567} + x_8 y_{567} + x_{234} \times y_{234} & x_8 y_8 - x_{567} \cdot y_{234} \end{pmatrix}.$$

It is clear that $\mathbf{O}$ is an algebra over $\mathbf{C}$ and that

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $x\,1 = 1\,x = x$ for all $x \in \mathbf{O}$; so we may identify $\mathbf{C}$ with the subalgebra $\mathbf{C}1$ of $\mathbf{O}$.

The quadratic form $Q : \mathbf{O} \to \mathbf{C}$ given by

$$Q(x) = x_1 x_8 + x_{234} \cdot x_{567} = x_1 x_8 + x_2 x_5 + x_3 x_6 + x_4 x_7$$

is obviously non-degenerate. It has a remarkable relation with the algebra structure on $\mathbf{O}$:

Lemma. *The quadratic form Q on $\mathbf{O}$ satisfies $Q(x\,y) = Q(x)Q(y)$ for all $x, y \in \mathbf{O}$.*

Straightforward calculation will provide a proof. It is also useful to test an implementation of the Cayley product by this rule.

Remark. An algebra of dimension 8 over $\mathbf{C}$ provided with a 1 and a non-degenerate quadratic form as in the lemma, can be shown to be unique up to isomorphism. This means that every property of $\mathbf{O}$ can be obtained by merely using the identity of the lemma for explicit computations. See, e.g., [30].

Exercise 16. Verify that the following Maple code implements the Cayley algebra. [Hint: use the above remark.]

```
with(linalg);
cayprod := proc(x,y) local o2,o5, ans;
  o2 := outerprod(vector(3,[x[5],x[6],x[7]]),vector(3,[y[5],y[6],y[7]]));
  o5 := outerprod(vector(3,[x[2],x[3],x[4]]),vector(3,[y[2],y[3],y[4]]));
  ans:= vector(8,
    [x[1]*y[1] - innerprod([x[2],x[3],x[4]],[y[5],y[6],y[7]]),
    x[1]*y[2] + y[8]*x[2] + o2[1],
    x[1]*y[3] + y[8]*x[3] + o2[2],
    x[1]*y[4] + y[8]*x[4] + o2[3],
    y[1]*x[5] + x[8]*y[5] + o5[1],
    y[1]*x[6] + x[8]*y[6] + o5[2],
    y[1]*x[7] + x[8]*y[7] + o5[3],
    x[8]*y[8] - innerprod([x[5],x[6],x[7]],[y[2],y[3],y[4]])]
  );
  evalm(ans)
end;

innerprod := proc(xx,yy) local ans, i;
```

```
  ans := 0;
  for i to 3 do ans := ans + xx[i]*yy[i] od;
  ans
end;

outerprod := proc(xx,yy) local ans;
  ans := vector(3);
  ans[1] := xx[2]*yy[3]-xx[3]*yy[2];
  ans[2] := xx[3]*yy[1]-xx[1]*yy[3];
  ans[3] := xx[1]*yy[2]-xx[2]*yy[1];
  evalm(ans)
end;


quadform := proc(x) ;
  x[1]*x[8] + x[2]*x[5] + x[3]*x[6] + x[4]*x[7]
end;
```

6.2 Automorphisms of O

We first point out some elements of Aut $\mathbf{O}$.

Obviously, for $g \in SL(3, \mathbf{C})$ the map

$$s_g : \begin{pmatrix} a & v \\ w & b \end{pmatrix} \mapsto \begin{pmatrix} a & g\,v \\ g^{\#}w & b \end{pmatrix} \quad (a, b \in \mathbf{C}; v, w \in \mathbf{C}^3)$$

where $g^{\#} = (g^{-1})^{\top}$, defines an automorphism of $\mathbf{O}$. Here is another obvious automorphism $\tau \in \mathrm{Aut}\,\mathbf{O}$:

$$\tau : \begin{pmatrix} a & v \\ w & b \end{pmatrix} \mapsto \begin{pmatrix} b & w \\ v & a \end{pmatrix} \quad (a, b \in \mathbf{C}; v, w \in \mathbf{C}^3).$$

Theorem. *The group $G = \mathrm{Aut}\,(\mathbf{O})$ of automorphisms of the Cayley algebra $\mathbf{O}$ over $\mathbf{C}$ is a simple complex algebraic group of type G_2.*

A proof can be given by determining the Lie algebra of derivations which preserve the multiplicative structure. Determining the Lie algebra is easy in view of the linearity of the equations involved:

$$\text{the Lie algebra } = \{8{\times}8 \text{ matrices } g \mid \forall\, x, y \in \mathbf{O} : g(xy) = (gx)y + x(gy)\}.$$

Exercise 17. Determine a basis for this Lie algebra. Can you also prove that it is simple of type G_2?

We shall write some elements of G as 8×8-matrices with respect to the basis suggested above: e_i is the vector x with all $x_j = 0$ but for x_i, which is 1. In particular, $1 = e_1 + e_8$ is fixed by each element of G.

A maximal torus of G comes from a maximal torus for the subgroup $SL_3(\mathbf{C})$ we have already found:

$$T = \left\{\mathrm{diag}(1, \lambda, \mu/\lambda^2, \lambda/\mu, 1/\lambda, \lambda^2/\mu, \mu/\lambda, 1) \mid \lambda, \mu \in \mathbf{C} \setminus \{0\}\right\},$$

where diag means diagonal matrix with indicated entries. Compare this with the character found for G_2 on $V_{[1,0]}$ as an example following Weyl's character formula (at beginning of 3)!

Let us find its normalizer. Set

$$r_1 = s_{(12)}\tau \quad , \text{ and } \; r_2 = s_{(23)},$$

where $\pi \in \mathrm{Sym}_3$ (also) denotes the 3×3-matrix g determined by $ge_i = e_{\pi(i)}$ for $i = 1, 2, 3$. Then $R = \{r_1, r_2\}$ is a generating set of a subgroup W of G which is isomorphic to the dihedral group of order 12, and normalizes T. Since only the identity fixed T pointwise, W is isomorphic to and acts on T as the Weyl group of type G_2.

Exercise 18. Given the above Cayley multiplication in Maple, the following is a simple test for membership of $G_2(\mathbf{C})$ for an invertible 8×8 matrix g (the outcome should be the zero vector). Try it out, e.g, on parametrized torus elements, or on the matrices u and t in §5.2 (we do not recommend trying w; there are more efficient ways)!

```
grpeqs := proc(g) local x,y,z, gx,gy,gz;
  x := vector(8);
  y := vector(8);
  z := cayprod(x,y);
  gx := evalm(g &* x);
  gy := evalm(g &* y);
  gz := evalm(g &* z);
  evalm(cayprod(gx,gy)-gz)
end;
```

Exercise 19. Let $(\cdot|\cdot)$ denote the bilinear form corresponding to Q, that is,

$$(x|y) = Q(x+y) - Q(x) - Q(y).$$

The restriction to $1^\perp$ of the trilinear form

$$(x, y, z) \mapsto (x \cdot y|z)$$

is *alternating*, that is,

$$(x \cdot y|z) = -(y \cdot x|z) = (y \cdot z|x).$$

1. Prove this, and write out the form explicitly on the basis $e_1 - e_8$, $e_2, \ldots, e_7$.
2. Moreover, the stabilizer in $\mathrm{GL}(7, \mathbf{C})$ of this form is $\mathrm{Aut}(\mathbf{O})$. Prove this, e.g. by checking that the Lie algebra stabilising the form has dimension 14.

7 Building G_2 from $\mathrm{PSL}(2, 13)$

Up to equivalence ($\mathrm{GL}(7, \mathbf{C})$ orbits) there is a unique 1-dimensional space of 3-linear alternating forms whose stabilizer in $\mathrm{GL}(7, \mathbf{C})$ is irreducible. It is found in the previous exercise. The stabilizer of this form is $G_2(\mathbf{C})$. These facts are known from the classification of alternating trilinear forms on $\mathbf{C}^7$, see for instance [8]. Since $\mathrm{PSL}(2, 13)$ is irreducible and stabilizes a trilinear alternating form, it embeds in $G_2(\mathbf{C})$. In this section, we turn this existence proof into an explicit construction.

7.1 A character computation

The fact that $\mathrm{PSL}(2, 13)$ has an invariant trilinear alternating form can be checked by calculating the inner product $(\chi^{3-}, 1)$, where χ^{3-} is the character of $\mathrm{PSL}(2, 13)$ on the linear space of trilinear alternating forms. Explicitly, for $g \in \mathrm{PSL}(2, 13)$,

$$\chi^{3-}(g) = \frac{\chi(g)^3 - 3\chi(g^2)\chi(g) + 2\chi(g^3)}{6}.$$

Exercise 20. Compute the values $\chi^{3-}(g)$ for g in PSL(2,13) and χ a 7-dimensional irreducible character. Conclude that it decomposes into

$$1_a + 13_a + 14_b + 7_{\{a \text{ or } b\}},$$

where 14_b stands for the second 14-dimensional irreducible character from Table 3, and so on. The verification that the trivial representation occurs in the third exterior power shows that PSL(2,13) leaves fixed a trilinear alternating form. This observation suffices for a proof that PSL(2, 13) embeds in $G_2(\mathbf{C})$.

Hint: the following GAP commands will find the character and actually test if there is a group invariant trilinear alternating form on the 7-dimensional representation space.

```
ct := CharTable("PSL",2,13);
chars := ct.irreducibles;
chi := chars[8];
sym3 := Symmetrisations(ct,[chi],3);
#the first component corresponds to the wedge product:
sym3decomp := MatScalarProducts(ct,chars,[sym3[1]])[1];
Print("Dim of group invariant trilinear altern. forms is: ");
Print(sym3decomp)[1]);Print(".\n");
```

7.2 Constructing the form

Using GAP, we shall search for the PSL(2, 13) invariant form and thus find the group G_2 explicitly as an overgroup of PSL(2, 13). The steps are

1. Take the three 7-dimensional matrices for u, t, w obtained from the 8-dimensional ones above by chopping off the fixed vector $1 = e_1 + e_8$.
2. Write down the 3-linear alternating form invariant under F.
3. If you do not wish to use the classification of trilinear alternating forms mentioned at the beginning of this section (7), you can determine the Lie algebra stabilizer of the form found, and decide that it is of type G_2. It then follows from the computations that PSL(2, 13) acts as a group of automorphisms of this Lie algebra and so belongs to its automorphism group, $G_2(\mathbf{C})$.

The second step can be carried out with the following code for producing the action of u, t, w on the 35-dimensional space $\Lambda^3\mathbf{C}^7$, and computing the intersection of the kernels of $u-1$, $t-1$, and $w-1$ in this action. Here is the corresponding GAP code. The three matrices are to be found at the end of the source file.

```
bw3:=Combinations([1..7],3);

normalize:=function(v)  local s,tmp;
  s:=1;
  if v[1]=v[2] or v[2]=v[3] or v[1]=v[3] then
    return([]);
  fi;
  if v[1]>v[2] then
    tmp:=v[1];
    v[1]:=v[2];
    v[2]:=tmp;
    s:=-s;
  fi;
  if v[2]>v[3] then
```

```
    tmp:=v[2];
    v[2]:=v[3];
    v[3]:=tmp;
    s:=-s;
    if v[1]>v[2] then
      tmp:=v[1];
      v[1]:=v[2];
      v[2]:=tmp;
      s:=-s;
    fi;
  fi;
  return([s,Position(bw3,v)]);
end;

wedge3:=function(a)  local i,j,k,l,aw,v,x;
  aw:=List([1..35],i->List([1..35],i->0));
  for i in [1..35] do
    v:=bw3[i];
    for j in [1..7] do
      for k in [1..7] do
for l in [1..7] do
  x:=normalize([j,k,l]);
  if x<>[] then
    aw[x[2]][i]:=aw[x[2]][i]+x[1]*a[j][v[1]]*a[k][v[2]]*a[l][v[3]];
  fi;
od;
      od;
    od;
  od;
  return(aw);
end;

#Read the 7-dim u,t,w  matrices over Q(zeta)  from file
Read("L213.g");

I := IdentityMat(7);

#make wedge matrices of size 35
wu := wedge3(u); wt := wedge3(t); ww := wedge3(w);
wI := wedge3(I);

#determinde fixed spaces
ku := NullspaceMat(TransposedMat(wu - wI));
kt := NullspaceMat(TransposedMat(wt - wI));
kw := NullspaceMat(TransposedMat(ww - wI));

#transform to vector spaces:
vu := VectorSpace(ku,C);
vt := VectorSpace(kt,C);
vw := VectorSpace(kw,C);


#determine the intersection of fixed spaces:
formspace := Intersection(vu,vt,vw);
```

```
if Dimension(formspace) <> 1 then
   Print("surprise: expected dim to be 1\n");
fi;

form := formspace.basis.vectors[1];

Print("The invariant form found is:\n");
for i in [1..35] do if form[i] <> 0 then
  Print("+",form[i],"*",bw3[i]); fi;
od;
Print("\n");
```

8 Building PSL(2,13) inside $G_2(\mathrm{C})$

We shall now start from the Cayley algebra of §6 and construct PSL(2,13) as a group of automorphisms of it.

8.1 Spectrum analysis

Using observations of the kind displayed in §4.6, one can derive that the subgroup $B = \langle u, t\rangle$ of order $13 \cdot 6 = 78$ of $F = \mathrm{PSL}(2,13)$ must embed in $N_G(T)$, with u embedding in T.

Let us first look at the possible elements of order 6 in G, using LiE.

```
setdefault G2
read efo-file
m = efos(6)

for r row n do print(r); s = kac_to_ss(r);
       print(s); print(centr_type(s));
               print(spectrum([1,0],s)); print(" ")
od
```

```
     [0,1,4]
     [1,2,6]
     A1T1
     3X[0] +2X[1] +2X[5]

     [1,0,3]
     [2,3,6]
     A1T1
     1X[0] +2X[1] +1X[2] +1X[4] +2X[5]

     [1,1,1]
     [3,5,6]
     T2
     1X[0] +1X[1] +1X[2] +2X[3] +1X[4] +1X[5]
```

This spectrum analysis, compared with the eigenvalues of t on the 7-dimensional irreducible representation of $\mathrm{PSL}(2,13)$ (cf. 5.1), give that we must have the latter case: the centralizer of t in G must be a maximal torus. Since t cannot embed in the same torus T in which u is embedded, we must search for t in the W part of $N_G(T)$. There is, up to conjugacy, one element of order 6 (inducing order 6 action on T), viz. $t = r_1 r_2$, corresponding to the 8×8 matrix given in §5.2.

Exercise 21. Perform the same spectrum analysis for u and find its centralizer in G.

Exercise 22. Verify that the 8-dimensional diagonal matrix for u given in §5.2, is the only matrix of order 13, up to powers in T, which satisfies $tut^{-1} = u^4$. Conclude that the embedding of B into G is unique up to conjugacy in G and algebraic conjugacy (interchanging the two 6-dimensional irreducible representations of B).

So far, we have identified the subgroup B of PSL(2,13) as the subgroup of G generated by the two matrices for u and t.

8.2 Finding the third transformation generating PSL(2,13) in G

We are left with finding an element $w \in G$ satisfying the defining relations for PSL(2,13) together with the matrices for u, t that were already found. To this end, we proceed as follows: we first focus on $w \in G$ in its role of inverting t, that is $wtw^{-1} = t^{-1}$.

We can easily find an element $w_0 \in G$ with this behaviour:

$$w_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $w \in w_0C$, where $C = C_G(t)$ is the centralizer of t in G. The analysis above tells that C is isomorphic (conjugate in fact) to a maximal torus.

All sorts of equations can now be written down for $c \in C$ so that $w = w_0c$ is as required. The obvious ones come from the defining relations of PSL(2,13). But they may be hard, as they are of relatively high degree. One could set up a set of equations for c and find an element $w := w_0c$ such that $F = \langle u, t, w\rangle$ is isomorphic to PSL(2,13). For instance, the traces of elements of the form u^iw can be determined using Table 3. They lead to linear equations in the parameters for C.

However, there is a better method. We can use the matrix for w as given in §5.2. The centralizer in GL(7, **C**) (the subgroup of GL(**O**) fixing $1 = e_1 + e_8$ and $1^\perp$) of $B = \langle u, t\rangle$ is a 2-dimensional group D of diagonal matrices. Since B embeds in Aut(**O**), one may just look in the set of D conjugates of w for an element which is inside Aut(**O**). Thus, writing w_1 for the matrix w of §5.2, we look for $w \in$ GL(7, **C**) of the from dw_1d^{-1} for some $d \in D$, which satisfies $(we_i)(we_j) = w(e_ie_j)$ for $i, j = 1, \ldots, 8$. This will give manageable equations for d.

Exercise nextex. Carry out this suggestion and find $w \in$ Aut(**O**) such that $\langle B, w\rangle$ is isomorphic to PSL(2,13).

Exercise 23. Can you determine the normalizer of F in G? Recall that it must be finite for F to be Lie primitive.

9 Concluding Remarks

9.1 Open problems

The following two questions have not been completely solved:

1. Conjugacy: how many equivalence classes are there?

2. Minimal field: what is the minimal field over which an embedding in a form of G_2 can be realized?

Both problems are solved for PSL(2, 13) in $G_2(\mathbf{C})$, but not for all embeddings. As for Problem 1, the number is known to be finite, cf. [29]. As for Problem 2, The minimal defining field is $\mathbf{Q}(\sqrt{13})$, cf. [10]. But the definition of a minimal defining field is delicate! It comes with a particular form of the group G in which F embeds, which need not be the particular form over $\mathbf{Q}$ one may have started with.

9.2 Not dealt with..

For almost each of the groups appearing in Table 2, there is a paper (preprint) dealing with its embedding. Sometimes, the embedding need not be established explicitly: just like for our sample case, §7, one can derive from character theoretic arguments that the group must embed in a certain Lie group.

Sometimes the group embeds via a well controlled subgroup. For instance Alt_6 has a 3-dimensional projective representation, and so a nonsplit central extension of Alt_6 embeds in $3 \cdot A_2(\mathbf{C})$, from which we deduce that it embeds in $G_2(\mathbf{C})$.

But there are cases for which the analogous work of §8 has to be carried out completely in order to prove that it embeds. Now for bigger Lie groups than $G_2(\mathbf{C})$, this requires more tricks to bound time and space of computer use than we have been able to display.

There are topics such as integral representations of the finite groups involved, modular representations, that is, analogous questions over finite fields (which are also relevant to some of the construction methods for characteristic 0) that we have not touched upon and which have been extensively studied as well.

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