The application of continuum damage mechanics to fatigue failure mechanisms

THE APPLICATION OF CONTINUUM DAMAGE MECHANICS TO FATIGUE FAILURE MECHANISMS

M.H.J.W. PAAS, P.J.G. SCHREURS and J.D. JANSSEN

Department of Mechanical Engineering,
Eindhoven University of Technology, The Netherlands

Abstract

Continuum Damage Mechanics is applied to the modeling of fatigue failure mechanisms. The resulting equations are solved by discretizing the time domain and the space domain. A tremendous reduction of computing times is achieved by uncoupling the constitutive equations and by employing an adaptive stepsize selection procedure. The failure behaviour of complex structures is studied.

1. Introduction

This paper is concerned with the modeling of fatigue failure mechanisms. For structures under fatigue loadings it is important to have accurate predictions of the expected lifetimes. In fatigue failure one is usually interested in the mechanisms prior to macrocrack initiation, since these events span most of the lifetime of a structure. In this paper Continuum Damage Mechanics is applied to model these mechanisms. First some basic CDM concepts are discussed. Then the constitutive equations are elaborated for isotropic damage states. The damage evolution law is developed for fatigue damage evolution, leading to a form that incorporates a number of well known literature models. The numerical solution process is discussed with special attention for the reduction of computational effort. The damage model and the numerical procedures are demonstrated with two examples.

2. Theory

In the following we restrict ourselves to isothermal conditions. It is assumed that the independent variables are given by the Green–Lagrange strain tensor $\mathbf{E}$, which is defined as $\mathbf{E} = 1/2(\mathbf{F}^c \cdot \mathbf{F} - \mathbf{I})$ with $\mathbf{F}$ the deformation tensor, its material time derivative $\dot{\mathbf{E}}$ and by a set of internal state variables $\underset{\sim}{z}$, representing dissipative mechanisms. Such variables can be scalar, vectorial and tensorial quantities. At every instant during the deformation of a continuum the so–called

J. F. Dijksman and F. T. M. Nieuwstadt (eds.), Integration of Theory and Applications in Applied Mechanics, 49–63.

balance laws have to be satisfied. The local forms of the laws of conservation of momentum, omitting body forces and inertial effects, and moment of momentum are given by

$$\vec{\nabla}.\sigma = \vec{0} \ ; \ \sigma = \sigma^{c} \tag{1}$$

where $\vec{\nabla}$ and σ denote, respectively, the gradient operator with respect to the deformed configuration and the Cauchy stress tensor. In addition to these equations the second law of thermodynamics requires

$$-\rho\dot{\psi} + \sigma : \mathbf{D} \geq 0 \tag{2}$$

where ψ is the specific Helmholtz free energy and D is the deformation rate tensor.

The material behaviour is characterized by constitutive equations specifying the dependence of the Cauchy stress tensor and the rate of internal variables on the independent variables. The constitutive variables have to satisfy a number of fundamental principles [1], which restrict the possible forms of the constitutive equations. Considerations of these principles and employment of the second law of thermodynamics yield

$$\sigma = \mathbf{J}^{-1}\mathbf{F}\cdot\mathbf{P}\cdot\mathbf{F}^{c} \ ; \ \mathbf{P} = \rho_{o}\frac{\partial\psi(\mathbf{E},\underset{\sim}{z})}{\partial\mathbf{E}} \tag{3}$$

$$-\frac{\partial\psi(\mathbf{E},\underset{\sim}{z})}{\partial z_{i}}\dot{z}_{i} = \underset{\sim}{X}^{T}\otimes\dot{\underset{\sim}{z}} \geq 0 \ ; \ \dot{\underset{\sim}{z}} = \underset{\sim}{f}(\mathbf{E}, \dot{\mathbf{E}}, \underset{\sim}{z}) \tag{4}$$

where **P** is the second Piola–Kirchhoff stress tensor. The quantities $\underset{\sim}{X}$ and $\dot{\underset{\sim}{z}}$ entering the Clausius–Duhem inequality are often designated as generalized irreversible forces and generalized fluxes. In irreversible thermodynamics it is postulated that the thermodynamic fluxes and forces are related to each other by functions, which can be derived from a potential ϕ and which contain as parameters the independent variables [2]. Thus, we accept that

$$\dot{\underset{\sim}{z}} = \nabla_{\underset{\sim}{X}}\,\phi(\underset{\sim}{X}, \underset{\sim}{\omega}) \ ; \ \underset{\sim}{\omega}^{T} = \{\mathbf{E}, \dot{\mathbf{E}}, \underset{\sim}{z}\} \tag{5}$$

Degradation of material properties is the result of initiation, growth and coalescence of microdefects. Within the context of Continuum Damage Mechanics (CDM) one may model this process by introducing an internal field variable defining the state of damage locally and recording its accumulation [3].

If the damage state is isotropic, the damage variable can be characterized by a scalar quantity $D = D(\vec{x},t)$, for which $D = D(\vec{x},t_{o}) = D_{o} \geq 0$ corresponds to the intrinsically damaged state and $D = D(\vec{x},t_{c}) = D_{c} < 1$ corresponds to complete local rupture. Then the magnified or effective stress tensor is related to the Cauchy stress tensor by

$$\hat{\sigma} = \frac{\sigma}{1-D} \tag{6}$$

This equation is often referred to as the effective stress concept [4].

In addition the hypothesis of strain equivalence [4] is introduced, which states that the effective stress tensor for a damaged material can be derived from the Helmholtz free energy potential of a virgin material, i.e. $\psi^o(\mathbf{E}) = \psi(\mathbf{E}, D=0)$. The resulting constitutive equations are

$$\mathbf{P} = (1-D)\hat{\mathbf{P}} \; ; \; \hat{\mathbf{P}} = \rho_o \frac{\partial \psi^o}{\partial \mathbf{E}} \tag{7}$$

$$\dot{D} = \nabla_x \phi(X, \underset{\sim}{\omega}) \; ; \; X = -\frac{\partial \psi}{\partial D} = \psi^o(\mathbf{E}) \; ; \; \underset{\sim}{\omega}^T = \{\mathbf{E}, \dot{\mathbf{E}}, D\} \tag{8}$$

3. Fatigue damage modeling

Under fluctuating loads materials will fail at stress levels much lower than they can withstand under monotonic loading conditions. This phenomenon is called high cycle fatigue, where the term 'high cycle' refers to the large number of cycles that are usually required for failure to occur.

In this section the mechanisms involved in the fatigue failure of materials are described by CDM. Experimental observations [5] indicate that a proper evolution law for fatigue damage should display the following characteristics:

1. Damage is considered as the only relevant dissipative mechanism.
2. Irreversible nature of damage $\dot{D} \geq 0$, which follows from eqs. (4) and (8).
3. Nonlinear dependence of damage growth on loading and current damage state.
4. If D approaches a critical value, the damage growth becomes infinite, i.e. $\dot{D} \to \infty$.
5. Existence of a threshold below which no damage growth occurs.
6. For elastomers, glassy polymers and metals the damage should be independent of the loading frequency.

In fatigue experiments predominantly uniaxial loading conditions are considered. In order to generalize the experimental results to three dimensions, (scalar) equivalent quantities should be introduced. Here damage growth for a particular material is characterized by an equivalent strain $\bar{\epsilon} = \bar{\epsilon}(\mathbf{E})$. The following particularization of the damage evolution equation is proposed, which satisfies the requirements listed above with a minimal number of parameters

$$\dot{D} = \alpha \, H(\bar{\epsilon} - \epsilon_f) \, D^{\beta} \, \bar{\epsilon}^{\gamma} \, \dot{\bar{\epsilon}} \tag{9}$$

where α, $\gamma \geq 0$ are constants and β may depend on the range of loading $\beta(\Delta\bar{\epsilon}) \geq 0$ with $\Delta\bar{\epsilon} = \bar{\epsilon}_m - \bar{\epsilon}_l$. $H(\cdot)$ is the Heaviside step function, denoting that no damage growth occurs for $\bar{\epsilon} < \epsilon_f$. The evolution law will be elaborated for periodical loadings between the extremes $\bar{\epsilon}_l$ and $\bar{\epsilon}_m$. Integration over one period $[t_{i-1}, t_i]$ with $D(t_i) = D_i$ and $\Delta D_i = D_i - D_{i-1}$ yields

$$\int_{D_{i-1}}^{D_i} D^{-\beta} dD = \int_{\bar{\epsilon}_s}^{\bar{\epsilon}_m} \alpha \bar{\epsilon}^{\gamma} d\bar{\epsilon} \Rightarrow$$

$$\frac{1}{1-\beta}\left[\left[1 + \frac{\Delta D_i}{D_{i-1}}\right]^{1-\beta} - 1\right] = \frac{\alpha}{\gamma+1} D_{i-1}^{\beta-1}(\bar{\epsilon}_m^{\gamma+1} - \bar{\epsilon}_s^{\gamma+1}) \tag{10}$$

where $\bar{\epsilon}_s = \bar{\epsilon}_1$ if $\bar{\epsilon}_1 > \epsilon_f$ otherwise $\bar{\epsilon}_s = \epsilon_f$. Bearing in mind that $\frac{\Delta D_i}{D_{i-1}} << 1$ with $\Delta D_i = D_i - D_{i-1}$, the left–hand side of eq. (10) can be linearized, yielding

$$\Delta D_i = \delta D_{i-1}^{\beta} ; \delta = \frac{\alpha}{\omega}(\bar{\epsilon}_m^{\omega} - \bar{\epsilon}_s^{\omega}) ; \omega = \gamma+1. \tag{11}$$

The damage after N cycles must be computed by summation over all previous cycles. This can be circumvented by rewriting the the incremental damage growth per cycle ($\Delta N = 1$) as

$$\Delta D_i = \frac{\Delta D_i}{\Delta N} \Delta N = \frac{1}{f}\frac{\Delta D_i}{\Delta t} \simeq \frac{1}{f}\frac{dD}{dt}\Big|_{t_{i-1}} \equiv \frac{dD}{dN}\Big|_{t_{i-1}} \tag{12}$$

where f is the frequency of the periodical loading. Then the recurrence relation (11) is replaced by a differential equation. For a body that is loaded in n blocks with strain levels ranging between $\bar{\epsilon}_{1_k}$ and $\bar{\epsilon}_{m_k}$ with $\bar{\epsilon}_{m_k} > \epsilon_f$ for k = 1,...,n, the damage evolution for loading at block k is expressed as

$$\frac{dD}{dN} = \delta_k D^{\beta_k} ; \delta_k = \frac{\alpha}{\omega}(\bar{\epsilon}_{m_k}^{\omega} - \bar{\epsilon}_{s_k}^{\omega}) \tag{13}$$

In order to solve the differential equation (13), the initial damage $D(N=0) = D_o$ must be specified. This value depends on the local microstructure of the body under consideration. Due to the localization of damage in fatigue, the characterization of D_o becomes a difficult task. This problem is tackled in [6] and [7] by considering the initial damage D_o as a stochastic variable. In order to predict the damage state, a continuum is divided into a number of cells, of which the initial damage distribution can be determined, utilizing statistics of extremes. Fitting this distribution to experimental data the unknown model parameters are obtained.

Uncoupled equations

Under the assumption that the stress tensor is not influenced by the damage state until a critical damage state is reached at which local rupture occurs, we may write

$$P = [1 - D_c H(D - D_c)] \hat{P} \tag{14}$$

Then the damage mechanisms and elastic mechanisms become uncoupled and the

evolution law can be solved analytically. The damage at the end of block k after $N_k = N_{k-1} + \Delta N_k$ cycles is given by a recursive equation

$$D_k = \left[\Delta N_k / N_{c_k} + D_{k-1}^{1-\beta_k} \right]^{\frac{1}{1-\beta_k}} ; \; N_{c_k} = \frac{1}{(1-\beta_k)\delta_k} \tag{15}$$

If $\beta \neq 1$ does not depend on the strain amplitude, eq. (15) becomes

$$D(N) = \left[\sum_{k=1}^{n} \Delta N_k / N_{c_k} + D_o^{1-\beta} \right]^{\frac{1}{1-\beta}} \tag{16}$$

Let the initial damage be given by $D_o = 0$ and the critical damage by $D_c = 1$. Use of eq. (18) then gives

$$D(N) = \left[\sum_{k=1}^{n} \Delta N_k / N_{c_k} \right]^{\frac{1}{1-\beta}} \tag{17}$$

where N_{c_k} now corresponds to the number of cycles to failure for loading between the strain amplitudes $\bar{\epsilon}_{l_k}$ and $\bar{\epsilon}_{m_k}$. This form is called the modified Palmgren–Miner rule [8]. The choice of $\beta = 0$ results in the linear Palmgren–Miner damage rule

$$D(N) = \sum_{k=1}^{n} \Delta N_k / N_{c_k} \; ; \; N_{c_k}^{-1} = \delta_k \tag{18}$$

Because this rule does not require extensive information about the particular material being considered, it has found a widespread application in modern engineering science.

4. Numerical procedures

In the sequel the solution process for problems, involving inhomogeneous deformations will be discussed. Throughout the complete history of load application, the equilibrium relation must be satisfied. The equilibrium state is influenced by the current values of the damage variable. The damage evolution equation in turn depends on the deformation field. In order to establish a suitable form for the equilibrium equation the principle of weighted residuals is used. According to the principle of weighted residuals the equilibrium equation is equivalent to the requirement that at every instant and for all admissible weighting functions $\vec{w}$, the following integral equation must hold [9]

$$\int_V (\vec{\nabla} \cdot \boldsymbol{\sigma}) \cdot \vec{w} \, dV = 0 \quad \forall \, \vec{w} \tag{19}$$

where V is the deformed volume of the body. Using integration by parts and Gauss' theorem, the so–called weak form of the principle of weighted residuals is obtained. If this equation is transformed to the reference configuration, we have

$$\int_{V_o} (\vec{\nabla}_o \vec{w})^c : \mathbf{P} \cdot \mathbf{F}^c \, dV_o = \int_{S_o} \vec{p}_o \cdot \vec{w} \, dS_o \quad \forall \vec{w} \tag{20}$$

where the subscript $_o$ denotes that the quantities are defined in the reference configuration.

The requirement that the principle of weighted residuals must be satisfied at every instant will be relaxed and replaced by the requirement that this must be true for a discrete number of times $t = t_0, t_1, \ldots, t_n$. The time discretization results in an incremental solution process. It is assumed that the solutions up to time t_n are known and that the solution at $t_{n+1} = t_n + \Delta t_n$ is to be determined.

At time t_{n+1} the integral equation is solved numerically. An iterative procedure, for determining the position vector field and its related quantities, is derived by writing all unknown quantities as the sum of an approximation of and a deviation from the exact solution. In the sequel we denote the actual values of a quantity q at time t_{n+1} by $q(t_{n+1})$; an approximation for $q(t_{n+1})$ obtained in the a th iteration is denoted by q^a_{n+1} and its deviation by δq. Then the weighted residuals formulation takes the form

$$\begin{aligned} &\int_{V_o} (\vec{\nabla}_o \cdot \vec{w})^c : (\mathbf{P}^a_{n+1} \cdot {}^4\mathbf{I}^c + {}^4\mathbf{N}^a_{n+1}) : (\vec{\nabla}_o \delta \vec{x}) \, dV_o = \\ &-\int_{V_o} (\vec{\nabla}_o \cdot \vec{w})^c : \mathbf{P}^a_{n+1} \cdot \mathbf{F}^{a\,c}_{n+1} \, dV_o + \int_{S_o} \vec{p}_{o_{n+1}} \cdot \vec{w} \, dS_o \quad \forall \vec{w} \end{aligned} \tag{21}$$

${}^4\mathbf{N}$ is defined by

$${}^4\mathbf{N} : \mathbf{A} = \tfrac{1}{2} [\mathbf{P}_{\mathbf{E}} : \{\mathbf{F} \cdot \mathbf{A}^c + \mathbf{A} \cdot \mathbf{F}\}] \cdot \mathbf{F} \quad \forall \mathbf{A} \tag{22}$$

where $\mathbf{P}_{\mathbf{E}}$ denotes the partial derivative of $\mathbf{P}$ with respect to $\mathbf{E}$.

With regard to the constitutive equations, it is assumed that the elastic response can be described by a linear stress–strain relationship. Furthermore the damage evolution equation (13) is cast into a more general form. Then the constitutive equations are written as

$$\mathbf{P} = (1-D)\,{}^4\mathbf{C}^o : \mathbf{E} \; ; \; \dot{D} = F(\mathbf{E}, D) \tag{23}$$

for the coupled set of equations, and

$$\mathbf{P} = (1-D_c H(D - D_c))\,{}^4\mathbf{C}^o : \mathbf{E} \; ; \; \dot{D} = F(D; \mathbf{E} = \mathbf{E}(t_o)) \tag{24}$$

for the uncoupled set of equations.

Integration of the evolution equation

An implicit integration method is chosen in order to account for changes in the variables due to the coupling with the weigthed residuals equation. A further advantage of implicit methods is their unconditional stability, implying that errors in intermediate results have little influence on the final result for any stepsize Δt, which therefore need to be selected only on considerations of accuracy. Under the assumption that the solution process has been completed up till time t_n and that the function F can be approximated by a linear polynomial between two successive times t_n and t_{n+1}, the so–called trapezium rule is obtained

$$D_{n+1}^{a+1} = D_n + \tfrac{1}{2}\Delta t_n[F_n + F_{n+1}^a] \tag{25}$$

For the initial approximation D_{n+1}^0 we have $F_{n+1}^0 = F_n$ and the trapezium rule reduces to Euler's method.

In order to deal effectively with computing times, some mechanism for automatically changing the stepsize as the integration proceeds, should be employed. The problem is to ascertain what the stepsize should be before the start of the next integration step. This is performed by estimating the truncation error for a step and depending on its value, adjust the current stepsize either upward or downward. The local truncation error for $D(t_{n+1})$ is defined by

$$L(t_{n+1}) = D(t_{n+1}) - D_{n+1} \tag{26}$$

The stepsize must be computed with an explicit integration method. A straightforward method is obtained by expansion of $D(t)$ into a Taylor series in the neighbourhood of D_n. Comparing the results of two expansions of successive orders, respectively k and (k+1), an approximation for the local truncation error L_{n+1}, that is made when the kth order integration method is used, is given by

$$L_{n+1} = {}^{k+1}D_{n+1} - {}^{k}D_{n+1} = \frac{1}{(k+1)!} F_n^{(k)} (\Delta t_n)^{k+1} \tag{27}$$

where $F_n^{(k)}$ is the kth derivative of F with respect to time.

A suitable criterion for the stepsize selection is obtained by requiring that the relative errors have a constant value ϵ. Hence, we write

$$\epsilon = \frac{|{}^{k+1}D_{n+1} - {}^{k}D_{n+1}|}{{}^{k+1}D_n} = \frac{|L_{n+1}|}{{}^{k+1}D_n} \tag{28}$$

Using eq. (27) and eq. (28) the stepsize is given by

$$\Delta t_n = \left[\epsilon\,(k+1)!\;{}^{k+1}D_n\;|F_n^{(k)}|^{-1}\right]^{\frac{1}{k+1}} \tag{29}$$

For k = 1, i.e. an initial approximation for the damage at t_{n+1} is computed with Euler's method, the desired stepsize is

$$\Delta t_n = \left[2\epsilon\, {}^2D_n \left| F_n^{(1)} \right|^{-1}\right]^{\frac{1}{2}} \tag{30}$$

$$F_n^{(1)} = [F_D^T(E_n, D_n)\, F_n + F_E(E_n, D_n) : \dot{E}]$$

Finite element equations

The integral form (21) permits the solution of the unknown position vector field to be approximated with the finite element method (FEM). The weighting functions are chosen according to the Galerkin method [9], which implies that the weighting functions and the position vectors are interpolated identically. As a result of this discretization eq. (21) can be written as a summation over all elements

$$\sum_e \overset{\sim}{\delta w}^{eT} \cdot \underline{K}^e \cdot \overset{\sim}{\delta x}^e = -\sum_e \overset{\sim}{\delta w}^{eT} \cdot \overset{\sim}{\vec{n}}^e + \sum_e \overset{\sim}{\delta w}^{eT} \cdot \overset{\sim}{\vec{h}}^e \tag{31}$$

$$\underline{K}^e = \int_{V_0^e} (\vec{\nabla}_0 \overset{\sim}{\phi}) \cdot (P_{n+1}^a \cdot {}^4I^c + {}^4N_{n+1}^a) \cdot (\vec{\nabla}_0 \overset{\sim}{\phi}^T)\, dV_0$$

$$\overset{\sim}{\vec{n}}^e = \int_{V_0^e} (\vec{\nabla}_0 \overset{\sim}{\phi}) \cdot P_{n+1}^a \cdot F_{n+1}^{ac}\, dV_0$$

$$\overset{\sim}{\vec{h}}^e = \int_{S_0^e} \overset{\sim}{\phi}\, \vec{p}_{0_{n+1}}\, dS_0$$

where $\overset{\sim}{\delta x}^e$ and $\overset{\sim}{\delta w}^e$ are columns containing the iterative changes in the nodal position vectors and weighting functions of element e, and $\overset{\sim}{\phi}$ is the corresponding column of interpolation functions; $\underline{K}^e$ denotes the element stiffness matrix and the columns $\overset{\sim}{\vec{n}}^e$ and $\overset{\sim}{\vec{h}}^e$ respectively store the internal and external nodal forces. Assemblage of all element stiffness matrices and internal and external element forces finally leads to

$$\underline{K} \cdot \overset{\sim}{\delta x} = -\overset{\sim}{\vec{n}} + \overset{\sim}{\vec{h}} = \overset{\sim}{\vec{r}} \tag{32}$$

In each iteration we calculate an out of balance load vector $\overset{\sim}{\vec{r}}$ and the stiffness matrix $\underline{K}$, which yields an iterative change in the nodal point position vectors. Then new approximations of the quantities that depend upon the nodal point position vectors and the internal variables can be determined. The iterative process is continued until the out of balance load vector or the iterative changes $\delta \tilde{x}$ are sufficiently small.

The solution strategies are summarized below for times larger than t_o.

1 Δt_n
$t_{n+1} \longleftarrow t_n + \Delta t_n$
D^1_{n+1}
if$(\exists\, i \in \{1,\ldots,4.n_e\} \mid {}_iD^a_{n+1} > D_c)$ *go to 2*
if(*equations are uncoupled*) *then*
$n \longleftarrow n+1$
go to 1
else
2 $\underset{\sim}{K}(\underset{\sim}{x}^a_{n+1}, D^a_{n+1})\ \delta\underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{x}^a_{n+1}, D^a_{n+1})$
$\underset{\sim}{x}^a_{n+1} \longleftarrow \underset{\sim}{x}^a_{n+1} + \delta\underset{\sim}{x}$
$\underset{\sim}{F}^a_{n+1}, \underset{\sim}{E}^a_{n+1}, \underset{\sim}{\sigma}^a_{n+1}$
if$(|\underset{\sim}{r}| < \alpha)$ *then*
$n \longleftarrow n+1$
go to 1
else
$a \longleftarrow a+1$
D^a_{n+1}
go to 2
endif
endif

A four–node isoparametric element was chosen for the evaluation of the element stiffness matrix and the element nodal forces. Plane stress conditions were assumed and the deformation in the thickness direction was considered homogeneous. The interpolation functions are bilinear functions, which depend on the isoparametric coordinates ξ_1 and ξ_2.

5. Examples

Example 1. Consider a plate, which is loaded in x–direction at x=L with a distributed periodical force, which varies between zero and a positive value q. The plate has an initial damage D_o. The external loading will cause the plate to deteriorate. The stress state in the plate is uni–axial. For an isotropic linear elastic material we find

$$E_{22} = E_{33} = -\nu E_{11}\ ;\ \hat{P}_{11} = \frac{P_{11}}{1-D} = E\, E_{11} \tag{33}$$

The Cauchy stress σ_{11} is obtained as

$$\hat{\sigma}_{11} = \frac{\sigma_{11}}{1-D} = J^{-1}\lambda_1^2\, \hat{P}_{11} \tag{34}$$

where λ_i is the elongation factor with respect to direction i. Using eq. (33), (34) and $\sigma_{11} = q/t = q/\lambda_3 t_o$ with t the thickness of the plate, the elongation factor λ_1 is described by

$$\alpha^2\lambda_1^2(\lambda_1^2-1)^2 = \lambda_3^2 = 1 - \nu(\lambda_1^2-1) \tag{35}$$

where $\alpha = (E/2q)t_o(1-D) = \alpha(D)$. The displacement field can be solved from the preceding equation.

The damage evolution is described by eq. (13), where the magnitude of the equivalent strain varies between zero and an extreme value. The equivalent effective strain is chosen as

$$\bar{\epsilon} = \sqrt{\Sigma\ [\langle\epsilon_i\rangle^2 + \tilde{h}\langle-\epsilon_i\rangle^2]} \tag{36}$$

where ϵ_i are principal strains and $\tilde{h} \in [0,1]$ is a parameter, which accounts for the fact that compressive strains are less harmfull than tensile strains.

Through the parameter α in eq. (35) the displacement field is dependent upon the current damage state. Thus the damage evolution equation is a nonlinear initial value problem, for which no actual solution is available. When the elastic and dissipative mechanisms are uncoupled according to eq. (14), an analytical solution can be established as (see eq. (16))

$$D(N) = [\frac{\alpha}{\omega}(1-\beta)\ \bar{\epsilon}_m^{\omega}\ N + D_o^{1-\beta}]^{\frac{1}{1-\beta}} \tag{37}$$

The material parameters used in the test example are summarized in table 1. It is noted that these values correspond to Polystyrene.

Table 1 Data used in examples

B = 25 mm	$E = 3\cdot10^3$ Nmm^{-2}	$\alpha = 1.35\cdot10^{-8}$
L = 25 mm	$\nu = 0.25$	$\beta = 1.4$
t = 0.05 mm	h = 0.2	$\omega = 3.6$
q = 17.2 Nmm^{-1}	$D_o = 2.4\cdot10^{-5}$	

In Fig. 1 the damage is depicted as a function of the number of cycles for a relative error $\epsilon = 0.025$. The numerical solutions for the coupled and uncoupled equations are shown together with the analytical solution for the uncoupled equations (37). It appears that the numerical integration works well, since the numerical and analytical solutions for the uncoupled set of equations almost coincide. The small deviations between the coupled and uncoupled solutions indicate that the application of the uncoupled equations should be favoured with regard to computational effort.

In Fig. 2 the equivalent strain is shown as a function of the number of cycles. For the uncoupled equations the equivalent strain has a constant value untill a critical damage level has been reached. For the coupled equations the equivalent strain is a continuously increasing function following the damage evolution.

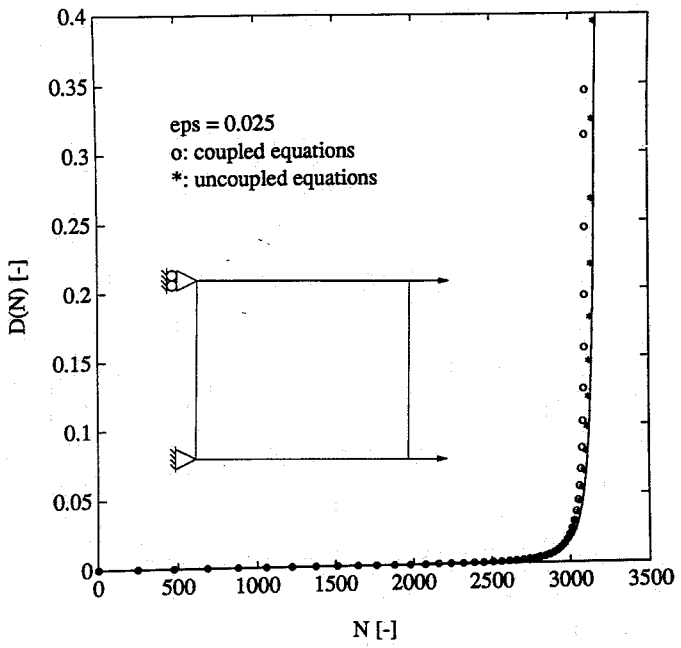

Fig. 1 Damage as a function of the number of cycles ($\epsilon = 0.025$)

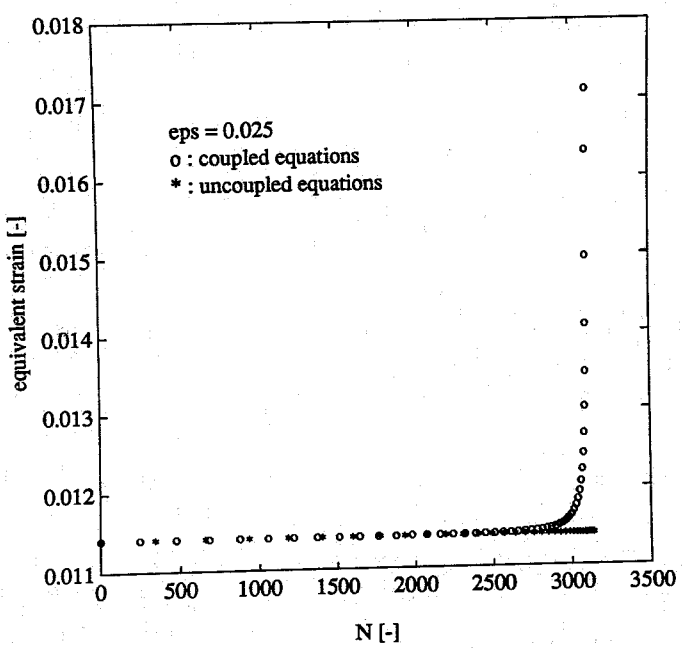

Fig. 2 Equivalent strain as function of the number of cycles ($\epsilon = 0.025$)

Next the stepsize selection procedure is illustrated for $\epsilon = 0.025$. The discrete times, at which the damage and deformation state were calculated, are marked. In Fig. 3 the stepsize is shown as a function of the number of steps. The stepsize selection procedure performs well, computing relatively large steps when the damage is changing slowly and continuously decreasing steps as the damage increases. The deviations between the stepsize selection for the coupled and uncoupled set of equations results from a different first derivative in eq. (30), implying that a more conservative estimate for the coupled set of equations must be made due to the dependence on the strain rate. The additional reduction in computational effort by the stepsize selection procedure can easily be imagined. Since without any stepsize selection procedure a correct description of the damage curve would require a very small stepsize, which is completely dictated by the explosive growth towards the end of the lifetime. For the example at hand, a 25–fold reduction in computing times is obtained using the uncoupled equations, whereas a tenfold reduction is obtained with the stepsize selection procedure.

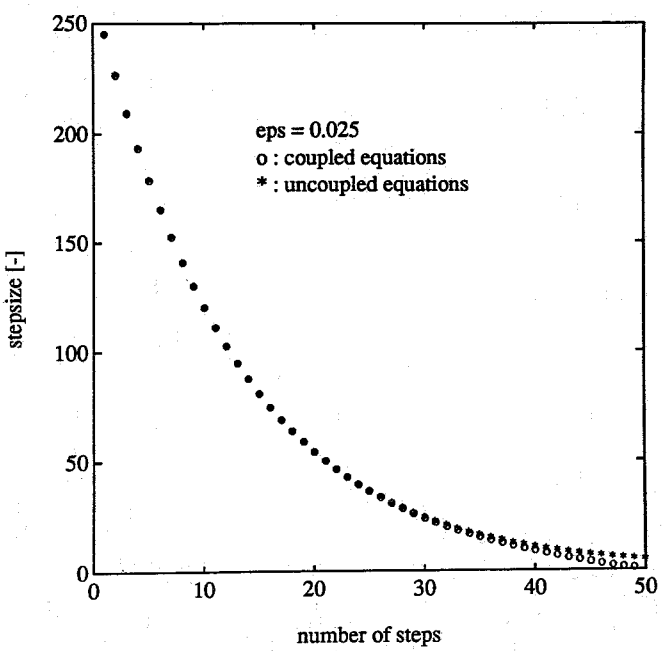

Fig. 3 Stepsize as a function of the number of steps

Example 2. Consider the structure in Fig. 4, which is loaded periodically at the edge y = h with a maximum distributed force q = 1.2 Nmm^{-1}. As a result of the dynamic loading a progressive detoriation and corresponding stiffness reduction of the structure occurs. The behaviour of the plate is studied with a local approach of fracture [10]. Here a macrocrack is represented as a zone in a finite element mesh in which the rigidity has been decreased severely. Macrocrack growth is identified with the growth of the damaged zone. It is remarked that in case of a local

approach of fracture the uncoupled equations are of major importance, provided their solutions do not differ too much from the actual solutions.

In Figs. 5 a,b the Von Mises stresses are shown in the neighbourhood of the hole at two discrete times; the initial state (N = 0) and the state after N = 1100 cycles. The number of cycles until the damage state in Fig. 5b has been reached, strongly depends on the intrinsic damage state in the plate. For a detailed discussion about initial damage distributions in solids, the reader is referred to [6] and [7].

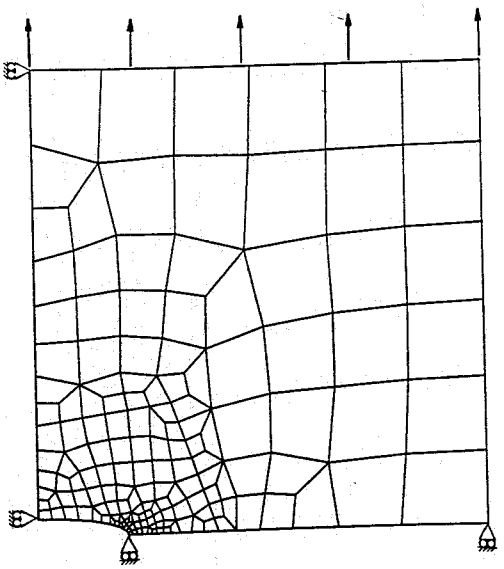

Fig. 4 Plate with an elliptical hole

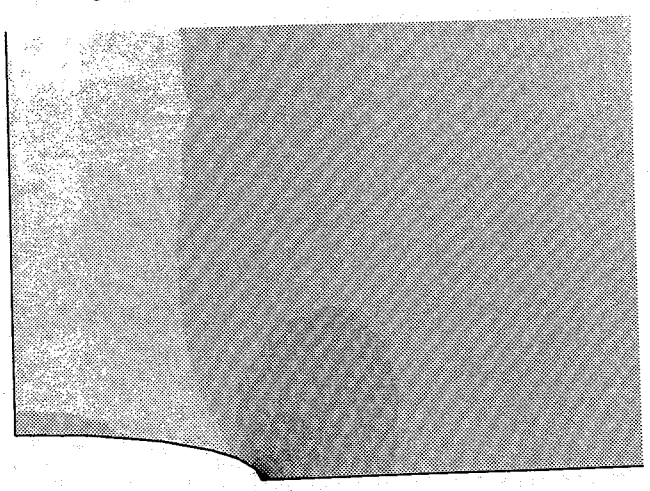

Fig. 5a Stress distribution after N = 0 cycles

Fig. 5b Stress distribution after N = 1100 cycles

6.Conclusions

Using thermodynamics with internal state variables general forms for the constitutive equations were derived. These equations were specified further using CDM. Isotropic damage states were considered. A damage evolution law was developed for the modeling of fatigue damage. Here damage was considered to be the only dissipative mechanism. The resulting equation could be reduced to some well known cumulative damage models in literature.

The equations were solved numerically by discretizing the time domain and the space domain using finite element procedures. These equations were coupled, resulting in excessive computing times. Special emphasis was placed on the reduction of computing times. This was established by uncoupling the constitutive equations and by employing an adaptive stepsize selection procedure. The performance of the integration procedures in combination with the adaptive stepsize selection procedure was tested. It appeared that for damage curves, which show a sudden explosive increase, the uncoupling of the constitutive equations yields accurate solutions. The failure behaviour of complex structures was studied with the Finite Element Method using a local approach of fracture.

References

[1] Malvern, L.E. (1969) *Introduction to the mechanics of a continuous medium.* Prentice–Hall, Englewood Cliffs, N.J..

[2] Kestin, J. and Bataille J. (1978) *Irreversible thermodynamics of continua and internal variables,* in Continuum models of discrete systems, Provan, J.W., ed., Univ. of Waterloo Press.

[3] Krajcinovic D. (1984) Continuum Damage Mechanics. *Appl. Mech. Rev.* **37**,

1–6.

[4] Chaboche J.L. (1988) Continuum Damage Mechanics: Part 1 – general concepts. *J. appl. Mech.* **55**, 59–64.

[5] Lemaitre J. and Dufailly J. (1987) Damage measurements. *Engng Fracture Mech.* 28, 643–661.

[6] Paas, M.H.J.W., Oomens, C.W.J., Schreurs, P.J.G., and Janssen, J.D. The mechanical behaviour of continuous media with stochastic damage. Accepted for publication in *Engng. Fract. Mech.*

[7] Paas, M.H.J.W., Schreurs, P.J.G., Oomens, C.W.J., and Janssen, J.D. (1989) A stochastic approach to fatigue damage modeling. *Mechanics and mechanisms of damage in composites and multimaterials*, D. Baptiste ed., 22–36.

[8] Hwang, W., and Han, K.S. (1986) Cumulative damage models and multi stress fatigue life prediction. *J. Comp. Mat.* **20**, 125–153.

[9] Zienkiewicz, O.C. (1977) The Finite Element Method, Mc Graw–Hill.

[10] Lemaitre, J. (1986) Local approach of fracture. *Engng Fract. Mech.* **25**, 523–537.