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Plücker embedding of Grassmannian manifolds and the invariant theory of binary forms

by

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Abstract

In this note we show how some of the basic results of the classical invariant theory of binary forms - "the symbolical notation", "the Clebsch-Gordan formula", "the construction of covariants by transvectants, starting from the groundform(s)", "the first and second fundamental theorem" - can be derived from the description of functions on "Grassmannian manifolds" by Plücker coordinates.

1. On the Plücker embedding

Let C be a linear space over $\mathbb{C}$. Let V be a two dimensional linear space over $\mathbb{C}$ which is equipped with a non-trivial alternating bilinear form $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ - short: a 2-plane with form. Let S be the group of automorphisms of V respecting this form. Define the affine variety $Y = Y(C)$ to be the space Hom (V,C) of linear maps - unless stated otherwise we always consider only linear maps with finite dimensional support - from V to C.

(1.1) **Remark.** In terms of coordinates: $V = \mathbb{C}^2$, $\langle \binom{a_1}{a_2}, \binom{b_1}{b_2} \rangle = a_1 b_2 - a_2 b_1$ and $S = S\ell_2\mathbb{C}$. If $\dim C = d < \infty$ then $Y(C) \cong M_{d,2}\mathbb{C}$.

For each affine variety X we let A(X) denote its affine coordinate ring. If X is a linear space considered as an affine space then A(X) is of course the same thing as $S(X^*)$, the symmetric algebra of the dual of X; furthermore, we then let $(A(X))_n$ denote the n-part of A(X), that is, the space of homogeneous polynomial functions on X of degree n. We consider $C \wedge C$, the alternating product of C with itself, as an affine space. The quadratic polynomial functions

$$a \wedge b \,.\, c \wedge d + b \wedge c \,.\, a \wedge d + c \wedge a \,.\, b \wedge d$$

with a, b, c, d $\in C^*$ in the coordinate ring $A(C \wedge C) = S((C \wedge C)^*)$ generate a prime ideal. We define the affine variety $Z = Z(C) \subset (C \wedge C)$ to be the vanishing set of these functions. We define a morphism $\phi : Y \to Z$ by

$$\langle v, w \rangle \phi(f) = f(v) \wedge f(w)$$

for all $f \in Y$, $v, w \in V$. We let the group S act on the variety Y by

$$(sf)(v) = f(s^{-1} v)$$

for all $s \in S$, $f \in Y$, $v \in V$. This induces an action of S on A(Y); let $A(Y)^S$ denote the S-fixed part.

(1.2) **Theorem.** The morphism $\phi : Y \to Z$ induces an ismorphism $A(Z) \simeq A(Y)^S$.
Proof. This can easily be proved, using for example the techniques from [G & A] p 94-99 and it is probably a well-known result. ⊠.

(1.3) **Remark.** For a geometrical interpretation of the constructions of ϕ above we observe that it is - a variant of - the Plücker embedding of Grassmannians. To be more precise, it gives a bijection between the set of "2-planes in C with form" and the set $Z - \{0\}$, which is an affine variety with one point deleted. For each injective f in Y we get a 2-plane in C with form, by transporting structure via f. This induces a bijection between the set of S-orbits of injective f in Y and the set of 2-planes in C with form. Moreover ϕ induces a bijection between the set of S-orbits of injective f in Y and the set $Z - \{0\}$. All together, we thus get the promised bijection.

2. The symbolical notation

We define, for each finite set D, the $\mathbb{C}$-algebra $\mathcal{S}(D)$, "the symbolical algebra on the alphabet D", by

generators $(a,b) \quad a,b \in D \quad a \neq b$

and

relations (i) $(ab) = -(ba) \quad \forall\, a, b \in D$

(ii) $(ab)(cd) + (bc)(ad) + (ca)(bd) = 0$

$\forall\, a, b, c, d \in D$

Each product of elements of type (ab) is called a symbolical product. For each symbolical product σ in $\mathcal{S}(D)$ its valency is defined to be the vector $\underline{n} = (n_a)_{a \in D}$ where n_a is the number of occurrences of a in σ. Both of the defining relations are between symbolical products of the same valency. Therefore if we write $\mathcal{S}_{\underline{n}}(D)$ = the subspace of $\mathcal{S}(D)$ spanned by the symbolical products of valency $\underline{n}$, then

$$\mathcal{S}(D) = \bigoplus_{\underline{n}} \mathcal{S}_{\underline{n}}(D)$$

For each finite dimensional vectorspace C with chosen basis $\underset{\sim}{C}$ we let $\underset{\sim}{C}^* = \{c^* | c \in \underset{\sim}{C}\}$ denote the dual basis of C^*, the dual space of C. The coordinate ring A(Z(C)) of the variety Z(C) is then - by definition - canonically isomorphic to - the algebra $\mathcal{S}(\underset{\sim}{C}^*)$

We write $V_n = (A(V))_n$.

(2.1) **Theorem** ("Symbolical notation"). Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be disjoint sets. Then one can construct a canonical isomorphism of linear spaces

$$\mathrm{Hom}_S\left(\bigotimes_{a\in \underset{\sim}{A}} V_{n_a} , \bigotimes_{b\in \underset{\sim}{B}} V_{n_b} \right) = \mathcal{S}_{\underline{n}}(\underset{\sim}{A} \cup \underset{\sim}{B}^*) \quad \text{for all } \underline{n} = (n_a)_{a \in \underset{\sim}{A} \cup \underset{\sim}{B}^*}$$

where we have written $\bar{n}_{b^*} = n_b$ for all $b \in \underset{\sim}{B}$.

Proof. The promised isomorphism will be constructed by taking the composition of a number of canonical isomorphisms which are standard

$$\mathrm{Hom}\left(\bigotimes_{a\in \underset{\sim}{A}} A(V) , \bigotimes_{b\in \underset{\sim}{B}} A(V) \right) =$$

$$\mathrm{Hom}\left(A(\bigoplus_{a\in \underset{\sim}{A}} V), A(\bigoplus_{b\in \underset{\sim}{B}} V) \right)$$

$$\mathrm{Hom}\left(A\,(V \otimes A),\ A(V \otimes B)\right)$$

$$A\,(V \otimes A\,)^{*} \otimes A\,(V \otimes B)$$

$$A\,(V^{*} \otimes A^{*}) \otimes A\,(V \otimes B)$$

$$A\,(V^{*} \otimes A^{*} \oplus V \otimes B)$$

(using the canonical isomorphism $V^{*} \simeq V$)

$$A\left(V^{*} \otimes (A^{*} \oplus B)\right)$$

$$A\left(\mathrm{Hom}\,(V,\ A^{*} \oplus B)\right)$$

Taking S-fixed parts and using theorem (1.2) we get

$$\mathrm{Hom}_S\left(\bigotimes_{a \varepsilon \underset{\sim}{A}} A(V),\ \bigotimes_{b \varepsilon \underset{\sim}{B}} A(V)\right) \simeq A\left(Z(A^{*} \oplus B)\right) \simeq S\,(\underset{\sim}{A} \cup \underset{\sim}{B}^{*})$$

One readily verifies that this isomorphism restricts to an isomorpism

$$\mathrm{Hom}_S\left(\bigotimes_{a \varepsilon A} V_{n_a}\ ,\ \bigotimes_{b \varepsilon B} V_{n_b}\right) \simeq S_{\underline{n}}\,(\underset{\sim}{A} \cup \underset{\sim}{B}^{*})$$

for all $\underline{n} = (n_a)_{a\ \varepsilon\ \underset{\sim}{A} \cup \underset{\sim}{B}^{*}}$ ☒

Sometimes we will use the classical convention to write

$\underset{\sim}{A} = \{a,b,c,..\}$ and $\underset{\sim}{B}^{*} = \{x,y,z,..\}$

(2.2) **Examples**

(i) $(ax)^n \in \mathcal{S}(\{a,x\})$ corresponds to the identical map on V_n

(ii) $(ab)^i\,(ax)^{m-i}\,(bx)^{n-i} \in \mathcal{S}(\{a,b,x\})$ corresponds to the unique linear map $\tau_i : V_m \otimes V_n \to V_{m+n-2i}$ which sends

$$(\alpha^*)^m \otimes (\beta^*)^n \text{ to } (\alpha,\beta)^i\,(\alpha^*)^{m-i}\,(\beta^*)^{n-i}$$

for all $\alpha, \beta \in V$. Here for each $\alpha \in V$ the element $\alpha^* \in V^*$ is defined by $\alpha^*(\beta) = \langle \alpha, \beta \rangle$ for all $\beta \in V$. (We observe here that the image of the map $V \to V_p : \alpha \to (\alpha^*)^p$ spans the linear space V_p).

(2.3) **Remark.** Once one has seen example (ii) above, it is not hard to guess what the explicit definition of the inverse of the isomorphism of the corollary looks like in general. It is moreover easy to verify this by writing down explicitly all the isomorphisms used in the proof of the corollary.

3. The Clebsch - Gordan formula

(3.1.) **Theorem** ("The Clebsch - Gordan formula"). V_n is an irreducible $\mathbb{C}[S]$-module for all $n \in \mathbb{N}_0$ and one can construct a canonical isomorphism of $\mathbb{C}[S]$- modules

$$V_m \otimes V_n \simeq \bigoplus_{i=0}^{\min(m,n)} V_{m+n-2i} \quad \text{for all } m, n \in \mathbb{N}_0$$

Proof. To prove that V_n is irreducible, we have to show

$$\mathrm{Hom}_S(V_n, V_n) = \mathbb{C}\, \mathrm{id}_{V_n} .$$

By theorem (2.1) and example (2.2)(i) this is equivalent to

$$\mathcal{S}_{n,n}(\{a,x\}) = \mathbb{C}\,(ax)^n$$

the truth of which is evident.

Moreover, again by theorem (2.1) we have

$$\mathrm{Hom}_{\mathcal{S}}(V_m \otimes V_n , V_{m+n-2i}) = \mathcal{S}_{m,n,m+n-2i}(\{a,b,x\});$$

to determine this space we have to find the symbolical products $(ab)^p\,(ax)^q\,(bx)^r$ with $p + q = m$, $p + r = n$ and $q + r = m + n - 2i$; there is one and only one: $(ab)^i\,(ax)^{m-i}\,(bx)^{n-i}$, so the space above is

$$\mathbb{C}\,(ab)^i\,(ax)^{m-i}\,(bx)^{n-i}$$

Let τ_i denote the element of $\mathrm{Hom}_S(V_m \otimes V_n , V_{m+n-2i})$ which corresponds to $(ab)^i\,(ax)^{m-i}\,(bx)^{n-i}$; $\tau_i = 0$ - see example (2.2) (ii) above - so as V_{m+n-2i} is an irreducible module, τ_i is surjective. In fact we can even conclude that the morphism of $\mathbb{C}[S]$ - modules $\bigoplus_i \tau_i$ from $V_m \otimes V_n$ to

$\bigoplus_{i=0}^{\min(m,n)} V_{m+n-2i}$ is surjective and so, by dimension counting, also injective ∎.

We mention without proof that the $\mathbb{C}[S]$ modules W which are isomorphic to direct sums of modules V_n with n variable can be characterized by the property that the defining

map $S \times W \to W$ is rational. Therefore such modules are called rational. Now we give a description of the ring R_S of virtual characters of "rational" representations.
We write

$$\left(1 - XY + Y^2\right)^{-1} = \sum_{n=0}^{\infty} f_n(X)\ Y^n$$

(3.2) **Corollary.** There is a unique isomorphism of rings $R_S = \mathbb{Z}[X]$ for which the charater ρ_n of V_n corresponds to f_n for all $n \in \mathbb{N}_o$.

Proof. On the one hand one has $f_o = 1$, $f_1 = X$ and $f_n = X f_{n-1} - f_{n-2}$ for all $n \geq 2$, on the other hand, by theorem (3.1) one has
$\rho_o = 1$ and $\rho_1\ \rho_{n-1} = \rho_n + \rho_{n-2}$ so $\rho_n = \rho_1\ \rho_{n-1} - \rho_{n-2}$ for all $n \geq 2$. So the morphism of rings $i: \mathbb{Z}[X] \to R_S$ which is defined by $i\big(\phi(X)\big) = \phi(\rho_1)$ for all $\phi(X) \in \mathbb{Z}[X]$ has the property that $i(f_n) = \phi_n$ for all $n \in \mathbb{N}_o$.
Moreover, by induction, f_n is a polynomial of degree n with leading coefficient 1. Therefore the f_n form a $\mathbb{Z}$-basis of $\mathbb{Z}[X]$; as furthermore the ρ_n form a $\mathbb{Z}$-basis of R_S and $i(f_n) = \rho_n$ for all $n \in \mathbb{N}_o$, we conclude that i is an isomorphism ⊠

4. Construction of the covariants by transvectants, starting from the groundform(s).

Let $C_{n,p,d}$ denote the linear space of S-equivariant polynomial maps from V_n to V_p which are homogeneous of degree d, that is, $C_{n,p,d} = \big((A(V_n))_d \otimes V_p\big)^S$. Its elements are called-homogeneous-covariants of level n, order p and degree d. Covariants of order p=0 are called invariants. The direct sum $C_n = \bigoplus_{p,d} C_{n,p,d}$ is a subalgebra of the algebra of maps from V_n to $A(V)$; it is called the algebra of covariants of level n.

(4.1) **Examples of covariants**

(i) $id_{V_n} \in C_{n,n,1}$ ("the groundform")

(ii) if $\phi \in C_{n,p,d}$ and $\psi \in C_{n,q,e}$ then, for

all $0 \leq i \leq \min(p,q)$, their i-th transvectant $(\phi,\psi)_i \in C_{n,p+q-2i,d+e}$ is defined to be the composition

$$V_n \xrightarrow{\Delta} V_n \otimes V_n \xrightarrow{\phi \otimes \psi} V_p \otimes V_q \xrightarrow{\tau_i} V_{p+q-2i}$$

where Δ denotes the diagonal map $x \to x \otimes x$.

It is convenient to define τ_i and so $(\ ,\ )_i$ to be the zero map if $i > \min(p,q)$. Thus one can extend the i-th transvectant to a bilinear map $(\ ,\ )_i : C_n \times C_n \to C_n$. If we write

$$C_{n,-,d} = \bigoplus_{p \in \mathbb{N}_0} C_{n,p,d} \qquad \text{so } C_{n,-,d} = \left(A(V_n)_d \otimes A(V)\right)^S,$$

then $(\ ,\ )_i$ restricts to a map

$$C_{n,-,d} \times C_{n,-,e} \to C_{n,-,d+e}.$$

(4.2) **Proposition.** $C_{n,-,1} = \mathbb{C}\, id_{V_n}$ and the linear map

$$\bigoplus_{i\in\mathbb{N}_o} C_{n,-,d} \otimes C_{n,-,e} \to C_{n,-,d+e} \quad \text{defined}$$

by $(\phi_i \otimes \psi_i)_i \to \sum_i \phi_i \otimes \psi_i$ is surjective for all $d, e \in \mathbb{N}_o$.

Proof. The first statement is obvious - see theorem (3.1) and example (i) above - and, using that the canonical maps $A(V_n)_d \otimes A(V_n)_e \to A(V_n)_{d+e}$ are surjective, it suffices for the proof of the second statement, to show that the map

$$F_{M,N} : \bigoplus_i L_S(M,A(V)) \otimes L_S(N,A(V)) \to L_S(M \otimes N,A(V))$$

given by $F_{M,N}\left(\bigoplus_i f_i \otimes g_i\right) = \sum_i \tau_i \circ (f_i \otimes g_i)$
is surjective for $M = (A(V_n)_d)^*$ and $N = (A(V_n)_e)^*$
Now, for irreducible modules $M = V_p$, $N = V_q$ the map $F_{M,N}$ is equal, by theorem (3.1), to the map

$$\bigoplus_{i\in\mathbb{N}_o} \mathbb{C} \otimes \mathbb{C} \to \bigoplus_{i=o}^{\min(p,q)} \mathbb{C} \quad \text{defined by } (x_i \otimes y_i)_{i\in\mathbb{N}_o} \to (x_i y_i)_{o\leq i\leq \min(p,q)},$$ which is

clearly surjective. It follows that $F_{M,N}$ is surjective for arbitrary rational $\mathbb{C}[S]$ - modules M, N ∎

(4.3) **Corollary**

(i) C_n has no proper subspace B with $id_{V_n} \in B$ and $(B,B)_i \subset B$ for all $i \in \mathbb{N}_o$

or equivalently

(ii) Define inductively the sets of covariants

$$A_1 = \{id_{V_n}\}$$
$$A_{k+1} = A_k \cup \{(\phi,\psi)_i \mid \phi,\psi \in A_k,\ i \leq \min(\text{order}\phi, \text{order}\psi)\} \quad \text{for all } k \in \mathbb{N}.$$

Then C_n is spanned by the union of all A_k.

5. The first and second fundamental theorem.

For each finite group G and each $\mathbb{C}[G]$ - module A we let A^G denote the largest submodule of A on which G acts trivially and we let A_G denote the largest quotient module of A on which G acts trivially. We recall that the map $A^G \to A_G$ which is the composite of the canonical injection $A^G \to A$ and the canonical surjection $A \twoheadrightarrow A_G$ is an isomorphism.

Let $\underset{\sim}{A}$ and $\underset{\sim}{X}$ be disjoint finite sets with $|\underset{\sim}{A}| = d$ and $\underset{\sim}{X} = \{x\}$. Let $\Sigma(\underset{\sim}{A})$ be the group of all permutations of $\underset{\sim}{A}$. Then $\Sigma(\underset{\sim}{A})$ acts on the algebra $\mathcal{S}(\underset{\sim}{A} \cup \underset{\sim}{X})$, which was defined in section 2, via its action on $\underset{\sim}{A}$ and the trivial action on $\underset{\sim}{X}$:

$$\sigma(ab) = (\sigma a\ \sigma b) \text{ for all } \sigma \in \Sigma(\underset{\sim}{A}), \ a, b \in \underset{\sim}{A} \cup \underset{\sim}{X}.$$

We will write $\tilde{\mathcal{S}}_{n,p,d}(\underset{\sim}{A};x) = (\mathcal{S}_{\underbrace{n,\dots,n}_{d \text{ times}},p}(\underset{\sim}{A} \cup \underset{\sim}{X}))_{\Sigma(\underset{\sim}{A})}$.

(5.1) **Theorem** ("the first and second fundamental theorem"). One can construct a canonical isomorphism

$$C_{n,p,d} \simeq \tilde{S}_{n,p,d}(\underset{\sim}{A};x).$$

Proof. $C_{n,p,d}$

$$\simeq \left(A(V_n)\right)_d \otimes V_p \,)^S \text{ (by definition)}$$

$$\simeq \left(\left(\bigotimes_{a\in \underset{\sim}{A}} V_n^*\right)_{\Sigma(\underset{\sim}{A})} \otimes V_p\right)^S$$

$$\simeq \left(\left(\bigotimes_{a\in \underset{\sim}{A}} V_n^*\right)^{\Sigma(\underset{\sim}{A})} \otimes V_p\right)^S$$

$$\simeq \left(\left(\bigotimes_{a\in \underset{\sim}{A}} V_n\right)^* \otimes V_p\right)^{S \times \Sigma(\underset{\sim}{A})} \text{ (as the actions of S and } \Sigma(\underset{\sim}{A}) \text{ commute)}$$

$$\simeq \mathrm{Hom}\left(\bigotimes_{a\in \underset{\sim}{A}} V_n, V_p\right)^{S \times \Sigma(\underset{\sim}{A})}$$

$$\simeq \left(\mathrm{Hom}_S\left(\bigotimes_{a\in \underset{\sim}{A}} V_n, V_p\right)\right)_{\Sigma(\underset{\sim}{A})}$$

$$\simeq S_{\underbrace{n,\dots,n}_{d \text{ times}},p}\left(\underset{\sim}{A} \cup \{x\}\right)_{\Sigma(\underset{\sim}{A})}$$

$$\simeq \tilde{S}_{n,p,d}(\underset{\sim}{A};x) \qquad \boxtimes$$

(5.2) **Remark.** As far as explicit formulas are concerned, it will suffice to give one example:

$$(ab)^i\ (ax)^{n-i}\ (bx)^{n-i} \in \tilde{\mathcal{S}}_{n,n,2n-2i}\left(\{a,b,x\}\right)$$

corresponds to the composition of the map

$$V_n \to V_n \otimes V_n \text{ which sends } u \text{ to } u \otimes u \text{ for all } u \in V_n$$

and the unique linear map

$$V_n \otimes V_n \to V_{2n-2i} \text{ which sends } (\alpha^*)^n \otimes (\beta^*)^n \text{ to } (\alpha,\beta)^i\ (\alpha^*)^{n-i}\ (\beta^*)^{n-i}$$

for all α, $\beta \in V$, with notations as in section 1.

(5.3) **Remark.** The surjectivity of the inverse of the isomorphism of theorem (5.1) is usually called "the first fundamental theorem of invariant theory" and its injectivity "the second fundamental theorem of invariant theory".

(5.4) **Remark.** Everything in this and the previous section generalizes without difficulty to "covariants of several ground forms". It will suffice to give here only some of the definitions. Let G be a finite set. Let $\underline{n} = (n_g)_{g\in G}$ and $\underline{d} = (d_g)_{g\in G}$ be vectors with all coefficients in $\mathbb{N}_o$ and let $p\in\mathbb{N}_o$. Let $C_{\underline{n},\ p,\ \underline{d}}$ denote the linear space of S-equivariant polynomial maps from $V_{\underline{n}} = \prod_{g\in G} V_{n_g}$ to V_p which are homogeneous of degree $\underline{d}$. Its elements are called – homogeneous –

covariants of level $\underline{n}$, order p and degree $\underline{d}$. For each $g \in G$, let π_g be the projection from $V_{\underline{n}}$ onto V_{n_g} and let $\underline{e}_g$ be the vector $(\delta_{g,h})_{h \in G}$ (δ = Kronecker delta). Then $\pi_g \in C_{\underline{n}, n_g, \underline{e}_g}$. The π_g , $g \in G$, are called the ground forms of level $\underline{n}$.

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