Dynamical analysis of linear passive networks with ideal diodes

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Eindhoven University of Technology
Department of Electrical Engineering

Dynamical analysis of linear passive networks with ideal diodes.
Part I: Well-posedness

W.P.M.H. Heemels, M.K. Çamlıbel,
J.M. Schumacher

March 2000

Measurement and Control Systems
Internal Report, 00 I/02

Paper submitted for publication elsewhere

Eindhoven, March 2000

Dynamical Analysis of Linear Passive Networks with Ideal Diodes. Part I: Well-posedness

W.P.M.H. Heemels[*]   M.K. Çamlıbel[†]   J.M. Schumacher[‡]

Abstract

The use of time-stepping methods for the simulation of the transient behaviour of circuit models with ideal diodes is by now well established. Little study has been made so far of the question in what sense the approximated time functions converge to the true solution of the network model. To answer this, one of course needs to establish first what should be understood by the transient "true solution" of such a network of a mixed discrete and continuous nature. In this paper we develop a solution concept and we prove that under mild conditions solutions exist, are unique, and are continuous except for a (possible) jump at the initial time. In a companion paper we discuss the convergence of a time-stepping algorithm.

1   Introduction

Nowadays switches like thyristors and diodes are used in electrical networks for a great variety of applications in both power engineering and signal processing. For the simulation of the transient behavior of such networks the switches are often modeled ideally [3, 27, 33, 45, 48]. It is well-known that ideal modeling causes the network model to be of a mixed discrete and continuous nature. In particular, the circuit evolves through multiple topologies (modes) depending on the (discrete) states of the diodes. The mode transitions are triggered by inequalities and may result in discontinuities and Dirac impulses in the network's variables, see e.g. [11, 27, 29, 33, 41, 45, 48]. Several numerical methods have been proposed to deal with these phenomena and simulation of circuits with nonsmooth characteristics is well established by now [3, 4, 14, 24, 25, 27, 38].

However, little attention has been paid to the question if and in what sense the computed time functions converge to the true solution of the network model. The simulation methods can be distinguished in two categories depending on whether or not the software attempts to find the

---

[*]Dept. of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, e-mail: w.p.m.h.heemels@tue.nl, tel: +31 40 2473587, fax: +31 40 2434582

[†]Dept. of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, and Dept. of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. On leave from Dept. of Electrical Eng., Istanbul Technical University, 80626 Maslak Istanbul, Turkey. e-mail: k.camlibel@tue.nl

[‡]Dept. of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail: jms@kub.nl

exact times at which events take place. The convergence of "event-tracking" methods might be inferred from a combination of standard results on the convergence of numerical algorithms for root finding and for simulation of smooth dynamical systems. For "time-stepping" methods, which are popular alternatives to event-tracking methods in a number of applications [4, 24, 38], the issue of convergence is less clear. It is the objective of this paper and the companion paper [6] to provide a rigorous basis for the use of time-stepping methods in the simulation of internally switched electrical circuits.

Before we are able to prove consistency – convergence of the approximated time functions to the exact solution of the network model – of a numerical routine, we have to establish what is meant by a transient "true solution" of a dynamical network with ideal diodes. In this paper we provide a mathematical framework that allows the precise formulation of a solution concept for these continuous/discrete networks. The framework will be borrowed from the theory of linear complementarity systems [20, 22, 26, 42, 43]. These systems can be seen as dynamical extensions of the *linear complementarity problem* [9], which (together with a number of variants) has been used extensively in the study of piecewise-linear electrical networks [4, 10, 14, 23–25, 44, 46].

The definition of true solutions is coupled to the question of well-posedness, i.e., existence and uniqueness of solutions of the network model. Much effort has been invested in considering existence and uniqueness of solutions to *static* (DC) models of electrical networks [7, 8, 15–17, 30, 31, 35, 36, 39, 40]. For the dynamic equivalent, the classical theory of ordinary differential equations guarantees existence and uniqueness of solutions under a Lipschitz continuity condition (see e.g. [47]). Here however we will be considering networks containing ideal diodes, for which such conditions are not fulfilled. The only papers known to the authors dealing with well-posedness for dynamic circuits containing non-Lipschitz elements are [12, 32]. However, the obtained results in [12, 32] do not cover the networks considered here, since an ideal diode cannot be reformulated as a current or voltage-controlled resistor.

The main purposes of the paper are the following.
(i) Define a mathematically precise solution concept for linear passive networks with diodes.
(ii) Prove (global) existence and uniqueness of solutions.
(iii) Establish regularity properties of the solutions. In particular, it will be rigorously proven that derivatives of Dirac impulses do not occur (even for inconsistent initial states) and Dirac impulses occur only at the initial time. Moreover, it will turn out that the set of switching times is a right-isolated set, meaning that following all time instants there exists a positive length time interval in which the diodes do not change their state. The companion paper [6] will use the obtained results to prove consistency of a transient simulation technique based on time-stepping.

The outline of the paper is as follows. In Section 2 linear passive networks with diodes will be reformulated as linear complementarity systems. In Section 3, we describe the evolution of the network model within a given mode (i.e., with a fixed state of the diodes). Next, an extension of the linear complementarity problem will be introduced, which will play an important role in the proof of well-posedness. In Section 5 the solution concept is introduced and the proof of global well-posedness is presented. Finally, we state the conclusions.

The following notations will be in force. $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \dots \}$, $\mathbb{R}$ the real numbers, $\mathbb{R}_+$ the nonnegative real numbers (including zero) and $\mathbb{C}$ the complex numbers. For

a positive integer l, $\bar{l}$ denotes the set $\{1, 2, \ldots, l\}$. If a is a (column) vector with k components, we denote its i-th component by a_i. $M^\top$ is the transpose of the matrix $M \in \mathbb{C}^{m \times n}$ and M^* denotes the complex conjugate transpose. A (not necessarily symmetric) matrix $M \in \mathbb{C}^{m \times m}$ is called nonnegative definite and we write $M \geqslant 0$ if Re $x^* M x = \frac{1}{2} x^* (M + M^*) x \geqslant 0$ for all $x \in \mathbb{C}^m$. In case strict inequality holds for all nonzero vectors x, we call the matrix positive definite and write $M > 0$. By $\mathcal{I}$ we denote the identity matrix of any dimension. Given $M \in \mathbb{R}^{k \times l}$ and two subsets $I \subseteq \bar{k}$ and $J \subseteq \bar{l}$, the (I, J)-submatrix of M is defined as $M_{IJ} := (M_{ij})_{i \in I, j \in J}$. In case $J = \bar{l}$, we also write $M_{I\bullet}$. If $I = \bar{k}$, the notation $M_{\bullet J}$ is sometimes used.

By $\mathbb{R}(s)$ we denote the field of real rational functions in one variable. $M(s) \in \mathbb{R}^{k \times l}(s)$ means that $M(s)$ is a $k \times l$ matrix with entries in $\mathbb{R}(s)$. A rational vector or matrix is called (strictly) proper, if for all entries the degree of the numerator is smaller than or equal to (strictly smaller than) the degree of the denominator.

A vector $u \in \mathbb{R}^k$ is called nonnegative (positive), and we write $u \geqslant 0$ ($u > 0$), if $u_i \geqslant 0$ ($u_i > 0$) for all $i \in \bar{k}$. If two vectors $u, y \in \mathbb{R}^k$ are orthogonal, i.e., $u^\top y = 0$, we write $u \perp y$. We write $u(s) \perp y(s)$ for two rational vectors $u(s), y(s) \in \mathbb{R}^k(s)$, if $u^\top(s) y(s) = 0$ for all $s \in \mathbb{C}$.

The set of arbitrarily often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^m$ is denoted by $C^\infty(\mathbb{R}; \mathbb{R}^m)$. $\mathcal{L}_2^k(t_0, t_1)$ denotes the set of all measurable functions v from (t_0, t_1) to $\mathbb{R}^k$ for which the integral $\int_{t_0}^{t_1} \|v(\tau)\|^2 d\tau$ is finite.

2 Linear passive networks with ideal diodes

As mentioned in e.g. [4, 25] linear electrical networks consisting of (linear) resistors, inductors, capacitors, gyrators, transformers (RLCGT) and ideal diodes can be described in a complementarity formulation. Indeed, the RLCGT-network can be described by the state space model

$$
\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) \tag{1a} \\
y(t) &= Cx(t) + Du(t) \tag{1b}
\end{align}
$$

under suitable conditions (the network does not contain all-capacitor loops or nodes with the only elements incident being inductors). See chapter 4 in [1] for more details. In (1) $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ denote real matrices of appropriate dimensions. Moreover, the pair (u_i, y_i) denotes the voltage-current variables at the connections to the diodes, i.e.,

$$u_i = -V_i, \ y_i = I_i \text{ or } u_i = I_i, \ y_i = -V_i,$$

where V_i and I_i are the voltage across and current through the i-th diode, respectively. The ideal diode characteristics are described by the relations

$$V_i \leqslant 0, \ I_i \geqslant 0, \ \{V_i = 0 \text{ or } I_i = 0\} \tag{2}$$
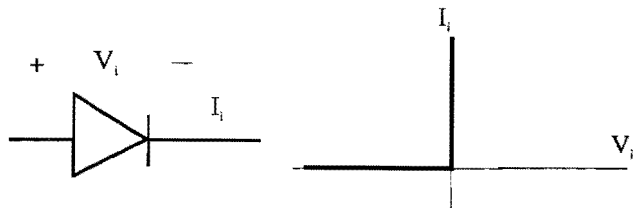
as shown in Figure 1.

Figure 1: The ideal diode characteristic.

By suitable substitutions the following system description is obtained:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3a}$$

$$y(t) = Cx(t) + Du(t) \tag{3b}$$

$$0 \leqslant y(t) \quad \perp \quad u(t) \geqslant 0. \tag{3c}$$

In this formulation $t \in \mathbb{R}_+$ denotes the time variable, $x(t)$ the state, and $u(t)$ and $y(t)$ the complementarity variables at time t. The system (3) is called a *linear complementarity system*. System descriptions of this form were introduced in [42] and were further studied in [20,22,26,43]. We use the notation LCS(A, B, C, D) to indicate the system given by (3). Note that (3c) is equivalent to

$$y_i(t) \geqslant 0, \quad u_i(t) \geqslant 0, \quad \{y_i(t) = 0 \text{ or } u_i(t) = 0\}$$

for all $i \in \bar{k}$.

Since (3a)-(3b) is a model for the RLCGT-multiport network consisting of resistors, capacitors, inductors, gyrators and transformers, the quadruple (A, B, C, D) is passive (or in the terms of [49], *dissipative* with respect to the supply rate $u^\mathsf{T} y$) in the following sense.

Definition 2.1 [49] A system (A, B, C, D) given by (1) is called *passive*, or *dissipative* with respect to the supply rate $u^\mathsf{T} y$, if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$, (a *storage function*), such that for all $t_0 \leqslant t_1$ and all time functions $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$ satisfying (1) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\mathsf{T}(t)y(t)dt \geqslant V(x(t_1)).$$

The above inequality is called the *dissipation inequality*. The storage function represents a notion of "stored energy" in the network.

A technical assumption that we will often use is the following.

Assumption 2.2 B has full column rank and (A, B, C) is a minimal representation, i.e., the matrices $[B \ AB \ \ldots \ A^{n-1}B]$ and $[C^\mathsf{T} \ A^\mathsf{T}C^\mathsf{T} \ldots (A^\mathsf{T})^{n-1}C^\mathsf{T}]$ have full rank.

The minimality of the system description (A, B, C) is standard in the literature on dissipative dynamical systems, see e.g. [49]. The following proposition gives several equivalent characterizations of passivity.

4

Proposition 2.3 *[49] Consider a system (A, B, C, D) in which (A, B, C) is a minimal representation. The following statements are equivalent.*

- *(A, B, C, D) is passive.*

- *The transfer matrix $G(s) := C(s\mathcal{I} - A)^{-1}B + D$ is positive real, i.e., $x^*[G(\lambda) + G^*(\lambda)]x \geqslant 0$ for all complex vectors x and all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\ \lambda > 0$ and λ is not an eigenvalue of A.*

- *The matrix inequalities*

$$
\begin{pmatrix}
-A^\top K - KA & -KB + C^\top \\
-B^\top K + C & D + D^\top
\end{pmatrix} \geqslant 0 \tag{4a}
$$

and

$$
K = K^\top \geqslant 0 \tag{4b}
$$

have a solution K.

Moreover, in case (A, B, C, D) is passive, all solutions to the linear matrix inequalities (4) are positive definite and K is a solution to (4a) if and only if $V(x) = \frac{1}{2}x^\top Kx$ defines a storage function of the system (A, B, C, D).

We note the following consequence of the definition of passivity, which will be used below.

Lemma 2.4 *Consider a system (A, B, C, D) in which (A, B, C) is a minimal representation and (A, B, C, D) is passive. If $v \in \mathbb{R}^k$ satisfies $(D + D^\top)v = 0$ (or equivalently, $v^\top Dv = 0$), then $C^\top v = KBv$ for any K satisfying (4a).*

Proof. According to Proposition 2.3, passivity of the system implies the existence of a symmetric $K > 0$ such that

$$
\begin{bmatrix}
A^\top K + KA & KB - C^\top \\
B^\top K - C & -(D + D^\top)
\end{bmatrix} \leqslant 0. \tag{5}
$$

Premultiplication of (5) by $(\gamma z^\top \ v^\top)$ and postmultiplication by $(\gamma z^\top \ v^\top)^\top$ for arbitrary $z \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ yields

$$
\gamma^2 z^\top (A^\top K + KA)z + 2\gamma z^\top (KB - C^\top)v \leqslant 0
$$

due to $(D + D^\top)v = 0$. Considering this expression as an inequality for a quadratic form in γ, we find that $z^\top (KB - C^\top)v = 0$. Since z is arbitrary, we obtain

$$
(KB - C^\top)v = 0. \tag{6}
$$

$\square$

3 Dynamics in a given mode

Equation (3c) implies that, for all t, and for every $i = 1, \ldots, k$ $u_i(t) = 0$ or $y_i(t) = 0$ must be satisfied (diode is conducting or blocking). This results in a multimodal system with 2^k modes, where each mode is characterized by a subset I of $\bar{k}$, indicating that $y_i(t) = 0$ if $i \in I$ and $u_i(t) = 0$ if $i \in I^c$ with $I^c := \{i \in \bar{k} \mid i \notin I\}$. For each such mode (also called "topology," "configuration," or "discrete state") the laws of motion are given by differential and algebraic equations (DAEs). Specifically, in mode I they are given by (we omit the time arguments for brevity)

$$
\begin{aligned}
\dot{x} &= Ax + Bu & \text{(7a)} \\
y &= Cx + Du & \text{(7b)} \\
y_i &= 0, \ i \in I & \text{(7c)} \\
u_i &= 0, \ i \in I^c. & \text{(7d)}
\end{aligned}
$$

The mode will vary during the time evolution of the system (diodes go from conducting to blocking or vice versa). The system evolves in a certain mode as long as the inequality conditions in (3c) are satisfied. At the event of a mode transition, the system may in principle display jumps of the state variable x. Jumping phenomena are well-known in the theory of unilaterally constrained mechanical systems [5], where at impacts the change of velocity of the colliding bodies is often modelled as being instantaneous. These discontinuous and impulsive motions are also observed in electrical networks (see e.g. [11, 27, 29, 33, 41, 45, 48]) and consequently, a distributional framework will be needed to obtain a mathematically precise solution concept. We restrict ourselves to the Dirac distribution denoted by δ and its derivatives, where $\delta^{(i)}$ denotes the i-th (distributional) derivative of δ.

Definition 3.1 [18] An *impulsive-smooth distribution* is a distribution u of the form u $=$ u$_{imp}$ + u$_{reg}$, where

- u$_{imp}$ is a linear combination of δ and its derivatives, i.e.,

$$
\mathrm{u}_{imp} = \sum_{i=0}^{l} u^{-i} \delta^{(i)}
$$

for vectors $u^{-i} \in \mathbb{R}^k$, $i = 0, \ldots, l$ and

- u$_{reg}$ is an arbitrarily often differentiable function from $(0, \infty)$ to $\mathbb{R}^k$ such that $\mathrm{u}_{reg}^{(m)}(0+) :=$ $\lim_{t \downarrow 0} \frac{d^m \mathrm{u}_{reg}}{dt^m}(t)$ exists and is finite for all $m = 0, 1, 2, \ldots$.

The class of impulsive-smooth distributions is denoted by C_{imp}^k. For a distribution u $\in C_{imp}^k$, u$_{imp}$ is called the impulsive part and u$_{reg}$ is called the smooth part. In case u$_{imp} = 0$ we call u a *regular* or *smooth* distribution. If the Laplace transform of an impulsive-smooth distribution is rational, we call the distribution of *Bohl type* or a *Bohl distribution*. For a smooth Bohl distribution, we will use the term *Bohl function*.

6

We also would like to introduce the notion of the derivative of an impulsive-smooth distribution.

Definition 3.2 Let u be an impulsive-smooth distribution that can be written as $u = u_{imp} + u_{reg}$, where

$$u_{imp} = \sum_{i=0}^{l} u^{-i} \delta^{(i)}$$

for vectors $u^{-i} \in \mathbb{R}^k$, $i = 0, \dots, l$ and u_{reg} is the smooth part. The derivative of u is denoted by $\dot{u}$ and defined by

$$\dot{u} = \sum_{i=0}^{l} u^{-i} \delta^{(i+1)} + u_{reg}(0+)\delta + \dot{u}_{reg}, \tag{8}$$

where $\dot{u}_{reg}$ denotes the usual derivative of a function on $(0, \infty)$.

Lemma 3.3 *Consider the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. Then the following holds.*

1. *For all $I \subseteq \bar{k}$ and for all initial states x_0, there exists a unique solution $(u, x, y) \in C_{imp}^{k+n+k}$ satisfying the dynamics for mode I given by*

$$\dot{x} = Ax + Bu + x_0 \delta \tag{9a}$$
$$y = Cx + Du \tag{9b}$$
$$y_i = 0, \ i \in I \tag{9c}$$
$$u_i = 0, \ i \in I^c \tag{9d}$$

as equalities of distributions. We denote this solution by $(u^{x_0,I}, x^{x_0,I}, y^{x_0,I})$.

2. *For all modes I there exist matrices F^I and K^I such that for all initial states x_0 the smooth parts $(u, x, y) := (u_{reg}^{x_0,I}, x_{reg}^{x_0,I}, y_{reg}^{x_0,I})$ of $(u^{x_0,I}, x^{x_0,I}, y^{x_0,I})$ are Bohl functions and satisfy*

$$\dot{x} = F^I x \tag{10}$$
$$u = K^I x \tag{11}$$
$$y = Cx + Du. \tag{12}$$

The matrices F^I and K^I only depend on the mode I and not on the particular x_0 at hand.

Proof.

1. The existence and uniqueness of a solution for (9) for all initial states x_0 is equivalent to the transfer matrix $G_{II} := C_{I\bullet}(s\mathcal{I} - A)^{-1}B_{\bullet I} + D_{II}$ being invertible as a rational matrix [18, Prop. 3.23, Thm. 3.24, Thm. 3.26]. This can also be seen from (17)-(18) below. Suppose

on the contrary that $\det G_{II}(s) \equiv 0$. Then there exists a rational vector $v(s) \not\equiv 0$ such that $G_{II}(s)v(s) \equiv 0$. Take $\sigma > 0$ such that $v(\sigma) \neq 0$ and $\sigma \mathcal{I} - A$ is invertible. Define $\bar{u}$ as

$$\bar{u}_i := \begin{cases} 0 & \text{if } i \notin I \\ v_i(\sigma) & \text{if } i \in I \end{cases}$$

The triple

$$u(t) = \bar{u}e^{\sigma t} \tag{13}$$

$$x(t) = (\sigma \mathcal{I} - A)^{-1} B \bar{u} e^{\sigma t} \tag{14}$$

$$y(t) = G(\sigma) \bar{u} e^{\sigma t} \tag{15}$$

satisfies the system equations (1), where $G(s) = C(s\mathcal{I} - A)^{-1}B + D$. Since (A, B, C, D) is passive, there exists a $K > 0$ such that the dissipation inequality

$$x^\top(t_0)Kx(t_0) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geqslant x^\top(t_1)Kx(t_1) \tag{16}$$

holds for all t_0 and t_1 with $t_1 \geqslant t_0$. It can be verified that $u^\top(t)y(t) = e^{2\sigma t}\bar{u}^\top G(\sigma)\bar{u} = e^{2\sigma t}v(\sigma)^\top G_{II}(\sigma)v(\sigma) = 0$ for all t. By letting t_0 tend to $-\infty$, (16) results in

$$0 \geqslant x^\top(t_1)Kx(t_1)$$

for all t_1. Because $K > 0$, this implies that $x(t_1) = 0$ for all t_1. From (14) it follows that $B\bar{u} = 0$. Since B is of full column rank, $\bar{u} = 0$ and hence also $v(\sigma) = 0$. We reached a contradiction and consequently proved the first statement.

2. This statement follows from [18, Thm. 3.10]. □

Remark 3.4 In terms of Definition 3.2 in [22] the first property of Theorem 3.3 states that all modes are *autonomous*. To be specific, mode I is called autonomous (see also [22, Lemma 3.3]) if for all initial states x_0 there exists a unique impulsive-smooth solution to (9).

Remark 3.5 The positive realness of $G(s)$ implies that $G(\sigma)$ is nonnegative definite for all $\sigma > 0$. Since a nonnegative definite matrix has only nonnegative principal minors [9, p. 153] and $\det G_{II}(s) \not\equiv 0$ (as shown in the proof of Lemma 3.3), it follows that there exists a $\sigma_0 \in \mathbb{R}$ such that for all $\sigma \geqslant \sigma_0$ the principal minors of $G(\sigma)$ are positive, i.e., $\det G_{II}(\sigma) > 0$ for all $I \subseteq \bar{k}$. In terms of [9, Def. 3.3.1] this means that $G(\sigma)$ is a P-matrix for all sufficiently large σ.

The solutions $(\mathrm{u}^{x_0,I}, \mathrm{x}^{x_0,I}, \mathrm{y}^{x_0,I})$ have *rational* Laplace transforms, denoted by $(\hat{\mathrm{u}}^{x_0,I}(s), \hat{\mathrm{x}}^{x_0,I}(s), \hat{\mathrm{y}}^{x_0,I}($ which satisfy

$$s\hat{\mathrm{x}}^{x_0,I}(s) = A\hat{\mathrm{x}}^{x_0,I}(s) + B\hat{\mathrm{u}}^{x_0,I}(s) + x_0 \tag{17a}$$

$$\hat{\mathrm{y}}^{x_0,I}(s) = C\hat{\mathrm{x}}^{x_0,I}(s) + D\hat{\mathrm{u}}^{x_0,I}(s) \tag{17b}$$

$$\hat{\mathrm{y}}_I^{x_0,I}(s) = 0 \tag{17c}$$

$$\hat{\mathrm{u}}_{I^c}^{x_0,I}(s) = 0. \tag{17d}$$

We introduce $G(s) = C(s\mathcal{I} - A)^{-1}B + D$ and $R(s) = C(s\mathcal{I} - A)^{-1}$. Since $G_{II}(s)$ is invertible as a rational matrix (see the proof of Lemma 3.3), the equations (17) can be solved explicitly. This yields that the Laplace transforms $(\hat{u}^{x_0,I}(s), \hat{x}^{x_0,I}(s), \hat{y}^{x_0,I}(s))$ are given by

$$\hat{u}_I^{x_0,I}(s) = -G_{II}^{-1}(s)R_{I\bullet}(s)x_0 \tag{18a}$$

$$\hat{u}_{I^c}^{x_0,I}(s) = 0 \tag{18b}$$

$$\hat{x}^{x_0,I}(s) = (s\mathcal{I} - A)^{-1}Bx_0 + (s\mathcal{I} - A)^{-1}B\hat{u}^{x_0,I}(s) \tag{18c}$$

$$\hat{y}_{I^c}^{x_0,I}(s) = [R_{I^c\bullet}(s) - G_{I^cI}(s)G_{II}^{-1}(s)R_{I\bullet}(s)]x_0 \tag{18d}$$

$$\hat{y}_I^{x_0,I}(s) = 0. \tag{18e}$$

Hence, the solutions of the mode dynamics (9) are one-to-one related (by the Laplace transform and its inverse) to solutions satisfying (17). On the basis of this relation, we can prove that only Dirac impulses (and not its derivatives) show up in passive electrical networks with diodes. Note that this statement is implied by the fact that the Laplace transforms $(\hat{u}^{x_0,I}(s), \hat{x}^{x_0,I}(s), \hat{y}^{x_0,I}(s))$ are proper for any $x_0 \in \mathbb{R}^n$ and $I \subseteq \bar{k}$.

Theorem 3.6 *Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. Then for each $x_0 \in \mathbb{R}^n$ and $I \subseteq \bar{k}$ the Laplace transform $\hat{u}^{x_0,I}(s)$ is proper.*

Proof. Denote $\hat{u}^{x_0,I}(s)$ by $u(s)$ for brevity. The triple

$$\bar{u}(t) = u(\sigma)e^{\sigma t} \tag{19}$$

$$\bar{x}(t) = (\sigma\mathcal{I} - A)^{-1}Bu(\sigma)e^{\sigma t} \tag{20}$$

$$\bar{y}(t) = G(\sigma)u(\sigma)e^{\sigma t} \tag{21}$$

satisfies (1) for all $\sigma \in \mathbb{R}$ such that $\sigma\mathcal{I} - A$ is nonsingular. It follows from passivity that there exists a $K > 0$ such that for all t_1 and t_0 with $t_1 \geqslant t_0$

$$\bar{x}^\top(t_1)K\bar{x}(t_1) - \bar{x}^\top(t_0)K\bar{x}(t_0) \leqslant \int_{t_0}^{t_1} \bar{u}^\top(t)\bar{y}(t)dt. \tag{22}$$

By substituting (19)-(21) into the dissipation inequality (22), one obtains

$$u^\top(\sigma)B^\top(\sigma\mathcal{I} - A)^{-\top}K(\sigma\mathcal{I} - A)^{-1}Bu(\sigma) \leqslant \frac{1}{2\sigma}u^\top(\sigma)G(\sigma)u(\sigma). \tag{23}$$

Since $K > 0$, B has full column rank, and $(\sigma\mathcal{I} - A)^{-1} = \frac{1}{\sigma}\mathcal{I} + \frac{1}{\sigma^2}A + \frac{1}{\sigma^3}A^2 + \ldots$ is strictly proper, there exists an $\alpha > 0$ such that

$$\frac{\alpha}{\sigma^2}\|u(\sigma)\|^2 \leqslant u^\top(\sigma)B^\top(\sigma\mathcal{I} - A)^{-\top}K(\sigma\mathcal{I} - A)^{-1}Bu(\sigma) \tag{24}$$

for all sufficiently large σ. We know from (17) that $u^\top(s)y(s) = 0$, where $y(s) := \hat{y}^{x_0,I} = C(s\mathcal{I} - A)^{-1}x_0 + G(s)u(s)$. Hence, the right-hand side of (23) satisfies

$$\frac{1}{2\sigma}u^\top(\sigma)G(\sigma)u(\sigma) = -\frac{1}{2\sigma}u^\top(\sigma)C(\sigma\mathcal{I} - A)^{-1}x_0$$

$$\leqslant \frac{1}{2\sigma}\|C(\sigma\mathcal{I} - A)^{-1}x_0\|\|u(\sigma)\|$$

$$\leqslant \frac{\beta}{2\sigma^2}\|u(\sigma)\|\|x_0\|. \tag{25}$$

9

The last inequality follows from the existence of a $\beta > 0$ such that $\|C(\sigma\mathcal{I} - A)^{-1}\| \leqslant \frac{\beta}{\sigma}$ for all sufficiently large σ. Thus, (23), (24) and (25) yield

$$\|u(\sigma)\| \leqslant \frac{\beta}{2\alpha}\|x_0\| \tag{26}$$

for all sufficiently large σ. Hence, $u(s)$ must be proper. $\qquad\square$

The fact that solutions of linear passive networks with ideal diodes do not contain derivatives of Dirac impulses is widely believed true on "intuitive" grounds, but the authors are not aware of any previous rigorous proof. The framework proposed here makes it possible to prove the intuition.

To summarize the discussion so far, it has been shown that instead of considering impulsive-smooth distributions as the solution space within a mode, we can restrict ourselves to Bohl distributions with impulsive part containing only Dirac impulses and not its derivatives (i.e., Bohl distributions with *proper* rational Laplace transforms).

Consider a solution to (9) for mode I and initial state x_0. An important observation is that a nontrivial impulsive part of $u^{x_0,I}$ will result in a re-initialization (jump) of the state. If $u_{imp} = u^0\delta$ (i.e., $u^0 = \lim_{s\to\infty} \hat{u}^{x_0,I}(s)$), then a jump will take place according to

$$x_{reg}(0+) := \lim_{t\downarrow 0} x_{reg}(t) = x_0 + Bu^0. \tag{27}$$

The proof can be found in [18].

The following properties can be proven for the impulsive part of an impulsive-smooth distribution satisfying the mode dynamics.

Lemma 3.7 *Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption 2.2 is satisfied and (A,B,C,D) represents a passive system. Consider the impulsive-smooth solution $(u^{x_0,I}, x^{x_0,I}, y^{x_0,I})$ to (9) for mode I and initial state x_0. The impulsive part $u^{x_0,I}_{imp}$ is given by $u^0\delta$ for some vector $u^0 \in \mathbb{R}^k$ that satisfies $u^{0\top}Du^0 = 0$ and $u^{0\top}C(x_0+Bu^0) = 0$.*

Proof. As stated before, the properness of $\hat{u}^{x_0,I}(s)$ implies that $u^{x_0,I}_{imp} = u^0\delta$ with $u^0 = \lim_{s\to\infty} \hat{u}^{x_0,I}(s)$. For brevity we will denote $\hat{u}^{x_0,I}(s)$ by $u(s)$ and $\hat{y}^{x_0,I}(s)$ by $y(s)$ in this proof. Take the power series expansion of $u(s)$ around infinity as

$$u(s) = u^0 + u^1 s^{-1} + u^2 s^{-2} + \cdots \tag{28}$$

Because for all i either $u_i(s) \equiv 0$ or $y_i(s) \equiv 0$, we have that

$$u^\top(s)y(s) = u^\top(s)[C(s\mathcal{I} - A)^{-1}x_0 + G(s)u(s)] = 0.$$

Substituting (28) into this equality and considering the coefficients corresponding to s^0 and s^{-1} yield

$$u^{0\top}Du^0 = 0 \tag{29}$$

$$u^{0\top}Cx_0 + u^{0\top}Du^1 + u^{1\top}Du^0 + u^{0\top}CBu^0 = 0. \tag{30}$$

10

The relation (29) implies that

$$(D + D^\top)u^0 = 0. \tag{31}$$

Now, (30) and (31) give

$$u^{0\top}Cx_0 + u^{0\top}CBu^0 = 0, \tag{32}$$

which establishes together with (29) the desired identities. □

4 The rational complementarity problem

In the previous section the dynamics within a mode (i.e., with a fixed state of the diodes) has been considered, while the inequality conditions have been neglected. However, a solution $(\mathbf{u}^{x_0,I}, \mathbf{x}^{x_0,I}, \mathbf{y}^{x_0,I})$ within a mode (9) will in general only be valid for a limited amount of time, since a change of mode (diode going from conducting to blocking or vice versa) may be triggered by the inequality constraints. Therefore, we would like to express some kind of "local nonnegativity." We call a (smooth) Bohl function v *initially nonnegative* if there exists an $\varepsilon > 0$ such that $v(t) \geqslant 0$ for all $t \in [0, \varepsilon)$. Note that a Bohl function v is initially nonnegative if and only if there exists a $\sigma_0 \in \mathbb{R}$ such that its Laplace transform $\hat{v}(\sigma) \geqslant 0$ for all $\sigma \geqslant \sigma_0$. Hence, there is a connection between small time values for time functions and large values for the indeterminate s in the Laplace transform. This fact is closely related to the well-known initial value theorem (see e.g. [13]). The definition of initial nonnegativity for Bohl distributions will be based on this observation (see also [20, 22]).

Definition 4.1 We call a Bohl distribution v *initially nonnegative*, if its Laplace transform $\hat{v}(s)$ satisfies $\hat{v}(\sigma) \geqslant 0$ for all sufficiently large real σ.

Remark 4.2 To relate the definition to the time domain, note that a scalar-valued Bohl distribution v without derivatives of the Dirac impulse (i.e., $\mathbf{v}_{imp} = v^0\delta$ for some $v^0 \in \mathbb{R}$) is initially nonnegative if and only if

1. $v^0 > 0$, or

2. $v^0 = 0$ and there exists an $\varepsilon > 0$ such that $v_{reg}(t) \geqslant 0$ for all $t \in [0, \varepsilon)$.

Definition 4.3 We call a Bohl distribution $(\mathbf{u}, \mathbf{x}, \mathbf{y}) \in C_{imp}^{k+n+k}$ an *initial solution* to (3) with initial state x_0, if there exists an $I \subseteq \bar{k}$ such that

1. $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ satisfies (9) for mode I and initial state x_0 in the distributional sense and

2. $\mathbf{u}, \mathbf{y}$ are initially nonnegative.

According to Lemma 3.3 condition 1 means that $(\mathbf{u}, \mathbf{x}, \mathbf{y}) = (\mathbf{u}^{x_0,I}, \mathbf{x}^{x_0,I}, \mathbf{y}^{x_0,I})$ for an LCS with (A, B, C, D) passive and satisfying Assumption 2.2.

11

Example 4.4 Consider the system $\dot{x}(t) = u(t)$, $y(t) = x(t)$ together with (3c). This represents a system consisting of a capacitor connected to a diode. The current in the network is equal to u and the voltage across the capacitor is equal to $y = x$. For initial state $x(0) = x_0 = 1$, (u, x, y) with $u = 0$ (no current) and $y(t) = x(t) = 1$ for all $t \in \mathbb{R}$ is an initial solution. This corresponds to the case that the diode is always blocking and there is no (nonzero) current in the network. To demonstrate that the distributional framework is needed, consider the initial state $x_0 = -1$, for which (u, x, y) with $u = \delta$, $x(t) = y(t) = 0$, $t > 0$ is the unique initial solution. This corresponds to an instantaneous discharge of the capacitor at time instant 0. Note that a state jump occurs at time 0 from -1 to 0.

We emphasize that an initial solution only satisfies the equations (3) in the following local sense. In case an initial solution has a nontrivial impulsive part, only the re-initialization as given in (27) forms a piece of the global solution. If the initial solution (u, x, y) is smooth, the largest interval on which (u, x, y) satisfies the equations (3) is equal to $[0, \varepsilon)$, where ε is given by

$$\varepsilon := \inf\{t > 0 \mid u_{reg,i}(t) < 0 \text{ or } y_{reg,i}(t) < 0 \text{ for some } i \in \bar{k}\}. \tag{33}$$

Example 4.5 Consider the network shown in Figure 2 with $R_1 = 2\,\Omega$, $R_2 = 1\,\Omega$, $L = 1\,H$ and $C = 1\,F$. We introduce the variables x_1 as the voltage across the capacitor, x_2 the current through the inductor, $-u$ the voltage across the diode and y the current through the diode. The system is governed by the equations

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 2x_2 + u \\
y &= x_2 + u
\end{aligned}$$

together with the complementarity conditions (3c). For initial condition $x_1(0) = -1$, $x_2(0) = 2$,
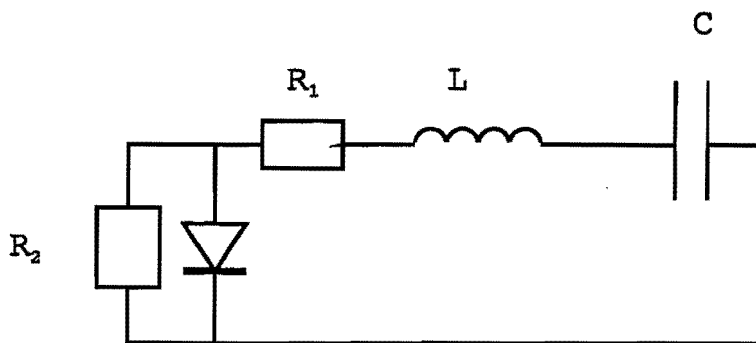
Figure 2: A simple network.

it can be verified that the unique initial solution (in the conducting mode) is given by $u = 0$, $x_1(t) = (t - 1)e^{-t}$, $y(t) = x_2(t) = (2 - t)e^{-t}$, $t > 0$. This initial solution forms a part of the (global) solution on the interval $[0, \varepsilon) = [0, 2)$. The time $t = 2$ is determined by the violation of the inequality constraint $y(t) \geqslant 0$ corresponding to the current through the diode becoming negative. This causes the diode to go from conducting to blocking. To determine the next part of the global solution, we have to find a continuation from initial state $x(2) = (e^{-2}, 0)^\top$, i.e., determining an initial solution with initial state $(e^{-2}, 0)^\top$.

12

Even when a solution within some mode exists and is unique given an initial state, it still might be possible that different modes give rise to different initial solutions. It is also possible that there are no initial solutions at all, i.e., no solution within a mode satisfies the initial nonnegativity conditions. We will start our investigation of well-posedness for linear passive complementarity systems by studying existence and uniqueness of initial solutions. An important tool in existence and uniqueness of initial solutions is the *rational complementarity problem* (RCP) [20, 43].

Definition 4.6 (The rational complementarity problem) Let the vector $x_0 \in \mathbb{R}^n$ and matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ be given. The *rational complementarity problem* $\mathrm{RCP}(x_0, A, B, C, D)$ is the problem of finding rational k-vectors $u(s) \in \mathbb{R}^k(s)$ and $y(s) \in \mathbb{R}^k(s)$ such that

1. for all $s \in \mathbb{C}$

$$y(s) = C(s\mathcal{I} - A)^{-1}x_0 + [C(s\mathcal{I} - A)^{-1}B + D]u(s) \text{ and } u(s) \perp y(s), \qquad (34)$$

and

2. there exists a $\sigma_0 \in \mathbb{R}$ satisfying for all $\sigma > \sigma_0$

$$y(\sigma) \geqslant 0 \quad \text{and} \quad u(\sigma) \geqslant 0. \qquad (35)$$

Any pair of rational vectors $(u(s), y(s))$ satisfying the above conditions is said to be a *solution* to $\mathrm{RCP}(x_0, A, B, C, D)$. If A, B, C and D are clear from the context, we also write $\mathrm{RCP}(x_0)$ for brevity.

From the definition of initial nonnegativity and (17), the following important relation is clear from [22].

Theorem 4.7 *Consider the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ and assume that all modes of $LCS(A, B, C, D)$ are autonomous (see Remark 3.4). Then the following statements hold.*

- *All initial solutions are of Bohl type.*

- *There is a one-to-one correspondence between initial solutions to (3) and solutions to $RCP(x_0)$. More specifically, $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ is an initial solution to (3) if and only if its Laplace transform $(\hat{\mathrm{u}}(s), \hat{\mathrm{x}}(s), \hat{\mathrm{y}}(s))$ is such that $(\hat{\mathrm{u}}(s), \hat{\mathrm{y}}(s))$ is a solution to $RCP(x_0)$ and*

$$\hat{\mathrm{x}}(s) = (s\mathcal{I} - A)^{-1}x_0 + (s\mathcal{I} - A)^{-1}B\hat{\mathrm{u}}(s). \qquad (36)$$

- *The following statements are equivalent.*

  1. *There exists a unique initial solution to $LCS(A, B, C, D)$ for initial state x_0.*

  2. *$RCP(x_0)$ has a unique solution.*

13

- *The initial solution is smooth if and only if the corresponding solution to RCP(x_0) is strictly proper. Similarly, the initial solution has an impulsive part containing only Dirac distributions (and not its derivatives) if and only if the corresponding solution to RCP(x_0) is proper.*

As a consequence, studying existence and uniqueness of initial solutions is equivalent to studying existence and uniqueness of solutions to RCPs. In [20] necessary and sufficient conditions for existence and uniqueness of solutions to RCPs have been presented in terms of families of *linear complementarity problems* (cf. Definition 4.10 below). Based on this relation and the literature on linear complementarity problems the following result has been proven in [20].

Theorem 4.8 *Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. Then RCP(x_0) has a unique solution for all x_0.*

Theorem 4.7 now yields the following.

Theorem 4.9 *Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. From each initial state x_0 there exists exactly one initial solution to LCS(A, B, C, D).*

According to Theorem 4.7 there exists a one-to-one relation between initial solutions and solutions to RCP. Since strictly proper Laplace transforms correspond to smooth Bohl distributions (without Dirac impulses and jumps of the state variable), it is interesting to characterize the set of initial states for which the corresponding solution to the RCP is strictly proper. In the following theorem such an explicit characterization will be given. To formulate the theorem, we need the following concepts.

Definition 4.10 Let a real vector $q \in \mathbb{R}^k$ and a real matrix $M \in \mathbb{R}^{k \times k}$ be given. The linear complementarity problem with data q and M (LCP(q, M)) is the problem of finding a real vector $z \in \mathbb{R}^k$ such that $0 \leqslant z \perp (q + Mz) \geqslant 0$. Any such vector z is called a solution to LCP(q, M).

For an extensive survey on LCPs, we refer to [9]. The set of all solutions z to LCP(q, M) will be denoted by SOL(q, M).

Remark 4.11 If $(u(s), y(s))$ is a solution to RCP(x_0, A, B, C, D), then $u(\sigma)$ is a solution to LCP($C(\sigma \mathcal{I} - A)^{-1} x_0, G(\sigma)$) for all sufficiently large (real) σ, where $G(s) = C(s\mathcal{I} - A)^{-1}B + D$.

Remark 4.12 Several times we shall employ the following standard observation on solutions of LCP. If $z_i \in$ SOL(q_i, M_i) with $i \in \{1, 2\}$ then

$$(z_1 - z_2)^\top ((q_1 + M_1 z_1) - (q_2 + M_2 z_2))$$
$$= -z_1^\top (q_2 + M_2 z_2) - z_2^\top (q_1 + M_1 z_1) \leqslant 0.$$

Finally, a *dual cone* is defined as follows [9].

Definition 4.13 Let Q be a nonempty set in $\mathbb{R}^k$. The *dual cone* of Q, denoted by Q^*, is defined as the set

$$Q^* = \{w \in \mathbb{R}^k \mid w^\top v \geqslant 0 \text{ for all } v \in Q\}.$$

Theorem 4.14 *Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. Denote the solution set of LCP$(0, D)$ by $Q := SOL(0, D)$. Furthermore, let $(u_{x_0}(s), y_{x_0}(s))$ be the (unique) solution to RCP(x_0). The following assertions hold:*

1. *For all $x_0 \in \mathbb{R}^n$, $C(x_0 + Bu^0) \in Q^*$ where $u^0 = \lim_{s \to \infty} u_{x_0}(s)$.*

2. *$u_{x_0}(s)$ is strictly proper if and only if $Cx_0 \in Q^*$.*

3. *$\lim_{s \to \infty} u_{x_0}(s) \in Q$.*

Proof.

1: In view of Remark 4.11 and Remark 4.12, we have for each $v \in Q := SOL(0, D)$ that

$$(u_{x_0}(\sigma) - v)^\top (C(\sigma\mathcal{I} - A)^{-1}x_0 + G(\sigma)u_{x_0}(\sigma) - Dv) \leqslant 0$$

for all sufficiently large σ. Since $D \geqslant 0$ ((4a) yields $D + D^\top \geqslant 0$) and $G(\sigma) = C(\sigma\mathcal{I} - A)^{-1}B + D$, we obtain

$$(u_{x_0}(\sigma) - v)^\top (C(\sigma\mathcal{I} - A)^{-1}x_0 + C(\sigma\mathcal{I} - A)^{-1}Bu_{x_0}(\sigma)) \leqslant 0 \tag{37}$$

for all sufficiently large σ. Multiplying this relation by σ and letting σ tend to infinity yields, since $u_{x_0}(s)$ is proper,

$$(u^0 - v)^\top (Cx_0 + CBu^0) \leqslant 0$$

It follows from Lemma 3.7 that $v^\top(Cx_0 + CBu^0) \geqslant 0$ for all $v \in Q$ and thus $C(x_0 + Bu^0) \in Q^*$.

2: "only if": Suppose $u_{x_0}(s)$ is strictly proper. Statement *1* and $u^0 = 0$ yield $Cx_0 \in Q^*$.

"if": Suppose that $Cx_0 \in Q^*$. From Lemma 2.4 and Lemma 3.7 we obtain that

$$u^{0\top}Du^0 = 0 \tag{38}$$
$$u^{0\top}Cx_0 + u^{0\top}CBu^0 = 0 \tag{39}$$
$$(KB - C^\top)u^0 = 0 \tag{40}$$

Since $(u_{x_0}(s), y_{x_0}(s))$ is the solution to RCP(x_0), $u^0 \geqslant 0$ and $Du^0 \geqslant 0$. Together with (38), this gives $u^0 = \lim_{s \to \infty} u_{x_0}(s) \in Q$ (this proves statement *3*).

From (40), we obtain $u^{0\top}CBu^0 = u^{0\top}B^\top KBu^0$. Since $u^0 \in Q$ and $Cx_0 \in Q^*$, (39) gives

$$0 \geqslant -u^{0\top}Cx_0 = u^{0\top}CBu^0 = u^{0\top}B^\top KBu^0 \geqslant 0.$$

15

Finally, positive definiteness of K and the full column rank of B imply $u^0 = 0$, i.e., $u_{x_0}(s)$ is strictly proper.

3: This has already been shown in the proof of statement *2*. □

Theorem 4.14 has several immediate consequences. First, we introduce a definition.

Definition 4.15 A state x_0 is called *regular* for LCS(A, B, C, D), if the corresponding initial solution is smooth. The collection of regular states for a given quadruple (A, B, C, D) is denoted by $\mathcal{R}$.

Since strictly proper Laplace transforms correspond to smooth Bohl distributions, statement *2* in Theorem 4.14 gives a characterization of the regular states: $x_0 \in \mathcal{R}$ if and only if $Cx_0 \in \mathcal{Q}^*$ with $\mathcal{Q} = \mathrm{SOL}(0, D)$. As we shall see, this characterization plays a key role in the proof of global existence of solutions as the set of such initial states will be proven to be invariant under the dynamics.

According to [9, Cor. 3.8.10 and Thm 3.1.7 (c)] and because $D \geqslant 0$ one has $Cx_0 \in \mathcal{Q}^*$ if and only if LCP(Cx_0, D) is solvable. Hence, a test for deciding the regularity of an initial state consist of determining whether or not a certain LCP has a solution. In [3] it is stated that a well-designed circuit does not contain Dirac impulses. As a consequence, the characterization of $\mathcal{R}$ forms a verification of the synthesis of the network containing diodes.

To give an idea about the structure of the cones $\mathcal{Q}^*$ and $\mathcal{R}$, a few examples are in order.

Example 4.16 Consider the following situations. In each case we assume that the quadruple (A, B, C, D) is passive and satisfies Assumption 2.2.

(a) If $D = 0$, then $\mathcal{Q} = \mathbb{R}_+^k$ and $\mathcal{Q}^* = \mathbb{R}_+^k$. Hence, $\mathcal{R} = \{x_0 \in \mathbb{R}^n \mid Cx_0 \geqslant 0\}$.

(b) If $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\mathcal{Q} = \{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_1 \geqslant 0 \text{ and } u_2 = 0\}$. Consequently, $\mathcal{Q}^* = \{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid y_1 \geqslant 0\}$ and thus $\mathcal{R} = \{x_0 \in \mathbb{R}^n \mid C_{1\bullet}x_0 \geqslant 0\}$.

(c) If D is positive definite, it follows that $\mathcal{Q} = \{0\}$, which implies that $\mathcal{Q}^* = \mathbb{R}^k$ and thus $\mathcal{R} = \mathbb{R}^n$.

A direct implication of the statements *1* and *2* in Theorem 4.14 is that, if smooth continuation is not possible for x_0, it is possible after one re-initialization. Indeed, by (27) the state after re-initialization is equal to $x_0 + Bu^0$, if the impulsive part of the (unique) initial solution is equal to $u^0\delta$. According to the fact that the Laplace transform of an initial solution is a solution to the corresponding RCP (which is automatically proper), it follows that $\lim_{s\to\infty} u_{x_0}(s) = u^0$ is indeed the coefficient determining the impulsive part. Since $C(x_0 + Bu^0) \in \mathcal{Q}^*$, it follows from statement *2* that $x_0 + Bu^0$ is a regular state. Hence, from $x_0 + Bu^0$ there exists a smooth initial solution. To summarize this discussion, we formulate a local existence result.

Theorem 4.17 *Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. For all initial states x_0, there exists a unique Bohl distribution $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ defined on $[0, \varepsilon)$ for some $\varepsilon > 0$ satisfying that*

1. *there exists an initial solution $(\bar{\mathrm{u}}, \bar{\mathrm{x}}, \bar{\mathrm{y}})$ such that*

$$(\mathrm{u}_{imp}, \mathrm{x}_{imp}, \mathrm{y}_{imp}) = (\bar{\mathrm{u}}_{imp}, \bar{\mathrm{x}}_{imp}, \bar{\mathrm{y}}_{imp}).$$

   with $\bar{\mathrm{u}}_{imp} = u^0 \delta$ for some $u^0 \in \mathbb{R}^k$,

2. *$\mathrm{x}_{reg}(0+) = x_0 + Bu^0$, and*

3. *for all $t \in (0, \varepsilon)$*

$$
\begin{aligned}
\mathrm{x}_{reg}(t) &= \mathrm{x}_{reg}(0+) + \int_0^t [A\mathrm{x}_{reg}(\tau) + B\mathrm{u}_{reg}(\tau)]d\tau \\
\mathrm{y}_{reg}(t) &= C\mathrm{x}_{reg}(t) + D\mathrm{u}_{reg}(t) \\
0 &\leqslant \mathrm{u}_{reg}(t) \perp \mathrm{y}_{reg}(t) \geqslant 0.
\end{aligned}
$$

5 Solution concept and global well-posedness

In [20, 22] a (global) solution concept has been introduced that is based on concatenation of initial solutions. In principle, this allows impulses at any mode transition time (necessary for e.g. unilaterally constrained mechanical systems). In the context of linear passive electrical networks with diodes, such a general notion of solution will not be needed. In fact, the solution concept as formulated in Theorem 4.17 will be extended such that mode changes are possible. This will be achieved by dropping the Bohl requirement and allowing $\mathcal{L}_2$ functions as regular parts. The function space $\mathcal{L}_\delta(0, T)$ consists of the distributions of the form $\mathrm{u} = \mathrm{u}_{imp} + \mathrm{u}_{reg}$, where $\mathrm{u}_{imp} = u^0 \delta$ with $u_0 \in \mathbb{R}$ and $\mathrm{u}_{reg} \in \mathcal{L}_2(0, T)$.

Definition 5.1 Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. Let a time horizon $T > 0$ and initial state x_0 be given. $(\mathrm{u}, \mathrm{x}, \mathrm{y}) \in \mathcal{L}_\delta^{k+n+k}(0, T)$ is called a solution to $\mathrm{LCS}(A, B, C, D)$ on $[0, T]$, if

1. there exists an initial solution $(\bar{\mathrm{u}}, \bar{\mathrm{x}}, \bar{\mathrm{y}})$ such that

$$(\mathrm{u}_{imp}, \mathrm{x}_{imp}, \mathrm{y}_{imp}) = (\bar{\mathrm{u}}_{imp}, \bar{\mathrm{x}}_{imp}, \bar{\mathrm{y}}_{imp}),$$

2. $\mathrm{x}_{reg}(0+) = x_0 + Bu^0$ with $u^0 \in \mathbb{R}^k$ given by $\bar{\mathrm{u}}_{imp} = u^0 \delta$, and

3. for almost all $t \in (0, T)$

$$
\begin{aligned}
\mathrm{x}_{reg}(t) &= \mathrm{x}_{reg}(0+) + \int_0^t [A\mathrm{x}_{reg}(\tau) + B\mathrm{u}_{reg}(\tau)]d\tau \\
\mathrm{y}_{reg}(t) &= C\mathrm{x}_{reg}(t) + D\mathrm{u}_{reg}(t) \\
0 &\leqslant \mathrm{u}_{reg}(t) \perp \mathrm{y}_{reg}(t) \geqslant 0.
\end{aligned}
$$

We have already proven local well-posedness (Theorem 4.17). The question arises whether global well-posedness is also guaranteed.

5.1 Global existence

We now come to the main existence result of this paper.

Theorem 5.2 *Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. Then, for all initial states x_0 and all $T > 0$ the system $LCS(A, B, C, D)$ has a solution on $[0, T]$ in the sense of Definition 5.1.*

Proof. The construction of a solution will be based on concatenation of initial solutions. Theorem 4.17 implies that a solution (u, x, y) exists on $[0, \tau_1]$ (take τ_1 as large as possible, i.e., equal to ε as in (33)) from initial state x_0. Note that $x(0+) \in \mathcal{R}$ and that $(u_{reg}, x_{reg}, y_{reg})$ is part of a smooth initial solution with initial state $x_{reg}(0+)$. Since $t \mapsto (u_{reg}, x_{reg}, y_{reg})(t + \rho)$ forms a smooth initial solution for any $\rho \in (0, \tau_1)$, we have that $x_{reg}(\rho) \in \mathcal{R}$ for all $\rho \in (0, \tau_1)$. Since $(u_{reg}, x_{reg}, y_{reg})$ is a Bohl function, the limit $\lim_{t \uparrow \tau_1} x_{reg}(t) = x_{reg}(\tau_1)$ exists. The closedness of $\mathcal{R}$ (follows from statement 2 in Theorem 4.14) implies that $x(\tau_1) \in \mathcal{R}$. Due to local existence of solutions and $x(\tau_1) \in \mathcal{R}$, there exists a smooth continuation (a smooth initial solution) from $x(\tau_1)$ that defines a solution on $[0, \tau_2]$ with $\tau_2 > \tau_1$. This construction can be repeated as long as the limit $\lim_{t \uparrow \tau} x(t)$ exists, where $[0, \tau]$ is the time-interval on which a solution has been generated so far. An obstruction to the existence of a global solution (on $[0, T]$) can only be that the intervals of continuation $[\tau_i, \tau_{i+1})$ are getting smaller and smaller such that $\lim_{i \to \infty} \tau_i = \tau^* < T$ and $\lim_{t \uparrow \tau^*} x(t)$ does not exist. To complete the proof we will show the existence of the latter limit under any circumstances.

Suppose the maximal interval on which a solution (u, x, y) can be defined is $[0, \tau^*)$, $\tau^* < T$. According to Lemma 3.3 there is at most exponential growth ($\dot{x} = F^I x$) between mode changes. For shortness we drop the subscript *reg* in the remainder of the proof. Since x is continuous on $(0, \tau^*)$, this implies that x is bounded (say $\|x(t)\| \leqslant M$ for all $t \in [0, \tau^*)$). On an interval $(s, t) \subseteq [0, \tau^*)$ where (u, x, y) is governed by the dynamics $\dot{x} = F^I x$ of mode I, the following estimate holds

$$\|x(t) - x(s)\| = \|e^{F^I(t-s)} x(s) - x(s)\| \leqslant c_I \mid t - s \mid \|x(s)\| \leqslant c_I M \mid t - s \mid . \tag{41}$$

Indeed, note that the matrix function $t \mapsto \frac{e^{tF^I} - \mathcal{I}}{t}$ is bounded (by c_I) on $[0, \tau^*)$. Hence, for $(s, t) \subseteq [0, \tau^*)$ with x possibly evolving through several modes we get from (41) that

$$\|x(t) - x(s)\| \leqslant M \max_{I \subseteq \bar{k}} c_I \mid t - s \mid .$$

This implies that x is Lipschitz continuous on $[0, \tau^*)$ and thus also uniformly continuous. It follows from a standard result in mathematical analysis [37, ex. 4.13] that $x^* := \lim_{t \uparrow \tau^*} x(t)$ exists. From the construction above it can be derived that $x(t) \in \mathcal{R}$ for all $t \in [0, \tau^*)$ and hence,

18

$x^* \in \mathcal{R}$, which implies that smooth continuation is possible (local existence) from x^* beyond τ^*. This contradicts the definition of τ^*. Hence, existence of a solution on $[0, T]$ is guaranteed. $\square$

5.2 Uniqueness

It can easily be seen that the solutions obtained by the construction in Theorem 5.2 must be unique, because the initial solutions are unique (see e.g. [20]). However, it might be possible that a different construction yields other solutions. The following theorem states that this is not the case.

Theorem 5.3 *Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption 2.2 is satisfied and (A, B, C, D) represents a passive system. Then for all initial states x_0 and all final times $T > 0$ there exists at most one solution $(\mathrm{u}, \mathrm{x}, \mathrm{y}) \in \mathcal{L}_{\delta}^{k+n+k}[0, T]$ to $LCS(A, B, C, D)$ in the sense of Definition 5.1.*

Proof. Suppose that two solutions $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ and $(\mathrm{u}', \mathrm{x}', \mathrm{y}')$ exist in the sense of Definition 5.1. According to Corollary 4.9 there exists exactly one initial solution from the initial state x_0. This implies that the impulsive parts of $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ and $(\mathrm{u}', \mathrm{x}', \mathrm{y}')$ must be the same and moreover, that the re-initialization from x_0 must be unique so that $\mathrm{x}(0+) = \mathrm{x}'(0+)$. Clearly, $(\mathrm{u} - \mathrm{u}', \mathrm{x} - \mathrm{x}', \mathrm{y} - \mathrm{y}')$ satisfies (1) from initial state 0 and is smooth. The dissipation inequality yields

$$\int_0^t [\mathrm{u}(\tau) - \mathrm{u}'(\tau)]^\top [\mathrm{y}(\tau) - \mathrm{y}'(\tau)] d\tau \geqslant$$
$$[\mathrm{x}(t) - \mathrm{x}'(t)]^\top K [\mathrm{x}(t) - \mathrm{x}'(t)]$$

for all $t \in (0, \infty)$. From the fact that $\mathrm{u}, \mathrm{u}', \mathrm{y}$ and y' are nonnegative almost everywhere and the complementarity of (u, y) and $(\mathrm{u}', \mathrm{y}')$, we obtain

$$\int_0^t [\mathrm{u}(\tau) - \mathrm{u}'(\tau)]^\top [\mathrm{y}(\tau) - \mathrm{y}'(\tau)] d\tau \leqslant 0.$$

Hence,

$$[\mathrm{x}(t) - \mathrm{x}'(t)]^\top K [\mathrm{x}(t) - \mathrm{x}'(t)] \leqslant 0$$

for all $t \in (0, \infty)$. Since $K > 0$, we obtain $\mathrm{x}(t) = \mathrm{x}'(t)$ for all t. Since B is of full column rank, it follows that $\mathrm{u} = \mathrm{u}'$ and $\mathrm{y} = \mathrm{y}'$ almost everywhere. $\square$

Since the global solution is unique, the solution must be equal to the one constructed in the proof of Theorem 5.2. This characterizes the nature of solutions to linear passive complementarity systems. Between mode changes the trajectories are of Bohl type and thus real-analytic. Moreover, the set $\mathcal{E}$ of mode transition times is right-isolated, i.e., for all $\tau \in \mathcal{E}$ there exists an $\alpha > 0$ such that $(\tau, \tau + \alpha) \cap \mathcal{E}$ is empty.

19

Remark 5.4 The fact that the set of mode transition times $\mathcal{E}$ is right-isolated can also be formulated as follows: there do no exist left-accumulation points[1] of mode transition times in the solutions defined by Definition 5.1. However, we cannot exclude the existence of right-accumulation points in general on the basis of this paper. Using a result in [21] it can be proven that for a linear passive network with one diode satisfying Assumption 2.2 and $D = 0$ also right-accumulations do not occur.

6 Conclusions

Linear passive electrical circuits with ideal diodes have been studied in the context of linear complementarity systems, with the aim of establishing a rigorous basis for the analysis of numerical methods for the transient simulation of switched electrical networks. The companion paper [6] deals with the question whether the approximated time functions obtained by a time-stepping method [4, 24, 38] converge to the true transient solution of the network model. To answer such a question, one needs of course a definition of what should be understood by the transient true solution. This question has been dealt with in this paper and formal proofs were given for the existence and uniqueness of solutions. Moreover, several regularity properties of the solutions have been proven which will be used in the companion paper [6]. In particular, it has been shown that derivatives of Dirac impulses do not occur and that Dirac impulses happen only at the initial time instant; also the set of regular states has been exactly characterized.

Networks with internally triggered switches have discrete as well as continuous characteristics. From this point of view, the paper proposes a systematic modeling framework and a precise notion of solution for a class of networks of such a mixed nature. Systems consisting of continuous dynamics (differential equations) and switching logic are sometimes called "hybrid systems" and receive currently much attention from both control theorists [2, 28] and computer scientists [34]. Hybrid systems are encountered in various research programs ranging from switching controllers, unilaterally constrained mechanical systems, piecewise linear systems, and switched electrical networks to hydraulic systems with valves. Since the underlying problems for these systems are essentially the same, all these research programs may benefit from a general theory as is currently being developed for complementarity systems.

Acknowledgement

---

[1]A point τ is called a left-accumulation point of $\mathcal{E} \subseteq \mathbb{R}$, if there exists a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ with $\tau_i \in \mathcal{E}$ such that $\tau_i > \tau$ and $\lim_{i\to\infty} \tau_i = \tau$. A right-accumulation point is defined by changing ">" into "<".

20

References

[1] B.D.O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis. A Modern Systems Theory Approach.* Pentice-Hall, Englewood Cliffs, New Jersey, 1973.

[2] P.J. Antsaklis and A. Nerode (guest eds.). Special issue on hybrid control systems. *IEEE Transactions on Automatic Control,* 43(4), 1998.

[3] D. Bedrosian and J. Vlach. Time-domain analysis of networks with internally controlled switches. *IEEE Trans. Circuits and Systems-I,* 39(3):199–212, 1992.

[4] W.M.G. van Bokhoven. *Piecewise Linear Modelling and Analysis.* Kluwer, Deventer, The Netherlands, 1981.

[5] B. Brogliato. *Nonsmooth Impact Mechanics. Models, Dynamics and Control,* volume 220 of *Lecture Notes in Control and Information Sciences.* Springer, London, 1996.

[6] M.K. Çamlıbel, W.P.M.H. Heemels, and J.M. Schumacher. Dynamical analysis of linear passive networks with ideal diodes. Part II: Consistency of a time-stepping method. *Submitted to IEEE Transactions on Circuits and Systems-I,* 2000.

[7] M.J. Chien. Existence and computation of DC solutions of nonlinear networks in a bounded set. *IEEE Trans. Circuits and Systems,* 23(11):655–663, 1976.

[8] M. Ciampa, P. Terreni, and M. Poletti. Conditions for the existence and uniqueness of DC solutions of networks containing nonlinear opamps with ideal model. In *IEEE International Symposium on Circuits and Systems,* volume 4, pages 2498–2501, 1993.

[9] R.W. Cottle, J.-S. Pang, and R.E. Stone. *The Linear Complementarity Problem.* Academic Press, Inc., Boston, 1992.

[10] B. De Moor, L. Vandenberghe, and J. Vandewalle. The generalized linear complementarity problem and an algorithm to find all its solutions. *Mathematical Programming,* 57(3):415–426, 1992.

[11] A. Dervisoglu. State equations and initial values in active RLC networks. *IEEE Trans. Circuit Theory,* 18:544–547, 1971.

[12] C.A. Desoer and J. Katzenelson. Nonlinear RLC networks. *The Bell System Technical Journal,* 44:161–198, 1965.

[13] J.J. DiStefano, A.R. Stubberud, and I.J. Williams. *Theory and problems of feedback and control systems.* Schaum's outline series. McGraw-Hill, 1967.

[14] J.T.J. van Eijndhoven. Solving the linear complementarity problem in circuit simulation. *SIAM Journal on Control and Optimization,* 24(5):1050–1062, 1986.

[15] T. Fujisawa and E.S. Kuh. Some results on existence and uniqueness of solutions of nonlinear networks. *IEEE Trans. on Circuit Theory,* 18(5):501–506, 1971.

[16] T. Fujisawa and E.S. Kuh. Piecewise-linear theory of nonlinear networks. *SIAM Journal on Applied Mathematics,* 22:307–328, 1972.

[17] M. Green and A.N. Willson Jr. On the uniqueness of a circuit's DC operating point when its transistors have variable current gains. *IEEE Trans. Circuits and Systems*, 36(12):1521–1528, 1989.

[18] M.L.J. Hautus and L.M. Silverman. System structure and singular control. *Linear Algebra and its Applications*, 50:369–402, 1983.

[19] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Dissipative systems and complementarity conditions. In *Proceedings of the 37-th IEEE Conf. on Decision and Control'98*, Tampa (USA), pages 4127–4132, 1998.

[20] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. The rational complementarity problem. *Linear Algebra and its Applications*, 294:93–135, 1999.

[21] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Well-posedness of linear complementarity systems. In *38-th IEEE Conf. on Decision and Control*, Phoenix (USA), pages 3037–3042, 1999.

[22] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Linear complementarity systems. to appear in *SIAM Journal on Applied Mathematics*, 2000.

[23] T.A.M. Kevenaar and D.M.W. Leenaerts. Comparison of piecewise-linear model descriptions. *IEEE Transactions on Circuit and Systems-I*, 39(12):996–1004, 1992.

[24] D.M.W. Leenaerts. On linear dynamic complementarity systems. *IEEE Trans. Circuits and Systems I: Fundamental Theory and Applications*, 46(8):1022 –1026, Aug. 1999.

[25] D.M.W. Leenaerts and W.M.G. van Bokhoven. *Piecewise linear modelling and analysis*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[26] Y.J. Lootsma, A.J. van der Schaft, and M.K. Çamlıbel. Uniqueness of solutions of relay systems. *Automatica*, 35(3):467–478, 1999.

[27] A. Massarini, U. Reggiani, and K. Kazimierczuk. Analysis of networks with ideal switches by state equations. *IEEE Trans. Circuits and Systems-I*, 44(8):692–697, 1997.

[28] A.S. Morse, C.C. Pantelides, S.S. Sastry, and J.M. Schumacher (guest eds.). Special issue on hybrid systems. *Automatica*, 35(3), 1999.

[29] Y. Murakami. A method for the formulation and solution of circuits composed of switches and linear RLC networks. *IEEE Trans. Circuits and Systems*, 34:496–509, 1987.

[30] R.O. Nielsen and A.N. Willson Jr. Topological criteria for establishing the uniqueness of solutions. *IEEE Trans. Circuits and Systems*, 24(7):349–362, 1977.

[31] T. Ohtsuki, T. Fujisawa, and S. Kumagai. Existence theorems and a solution algorithm for piecewise-linear resistor networks. *SIAM Journal on Mathematical Analysis*, 8(1):69–99, 1977.

[32] T. Ohtsuki and H. Watanabe. State-variable analysis of RLC networks containing nonlinear coupling elements. *IEEE Trans. on Circuit Theory*, 18(1):26–38, 1969.

[33] A. Opal and J. Vlach. Consistent initial conditions of nonlinear networks with switches. *IEEE Trans. Circuits and Systems*, 38(7):698–710, 1991.

[34] A. Pnueli and J. Sifakis (guest eds.). Special issue on hybrid systems. *Theoretical Computer Science* 138, 1995.

[35] V.C. Prasad. Necessary and sufficient condition for uniqueness of solutions of certain piecewise linear resistive networks containing transistors and diodes. *Electronic Letters*, 28(13), 1992.

[36] V.C. Prasad and V.P. Prakash. Existence, uniqueness and determination of solutions of certain piecewise linear resistive networks. In *IEEE International Symposium on Circuits and Systems*, volume 2, pages 1470–1473, 1990.

[37] W. Rudin. *Principles of mathematical analysis*. McGraw-Hill, 1976.

[38] I.W. Sandberg. Theorems on the computation of the transient response of nonlinear networks containing transistors and diodes. *Bell System Technical Journal*, 49:1739–1776, 1970.

[39] I.W. Sandberg and A.N. Willson Jr. Some theorems on the properties of DC equations of nonlinear networks. *Bell System Technical Journal*, 48:1–35, 1969.

[40] I.W. Sandberg and A.N. Willson Jr. Existence and uniqueness of solutions for the equations of nonlinear DC networks. *SIAM Journal on Applied Mathematics*, 22:173–186, 1972.

[41] S.S. Sastry and C.A. Desoer. Jump behaviour of circuits and systems. *IEEE Trans. Circuits and Systems*, 28(12):1109–1124, 1981.

[42] A.J. van der Schaft and J.M. Schumacher. The complementary-slackness class of hybrid systems. *Mathematics of Control, Signals and Systems*, 9:266–301, 1996.

[43] A.J. van der Schaft and J.M. Schumacher. Complementarity modeling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, 1998.

[44] S.N. Stevens and P.-M. Lin. Analysis of piecewise-linear resistive networks using complementary pivot theory. *IEEE Transactions on Circuits and Systems*, 28(5):429–441, 1981.

[45] J. Tolsa and M. Salichs. Analysis of linear networks with inconsistent initial conditions. *IEEE Trans. Circuits and Systems-I*, 40(12):885–894, 1993.

[46] L. Vandenberghe, B.L. De Moor, and J. Vandewalle. The generalized linear complementarity problem and an algorithm to find all solutions. *IEEE Transactions on Circuits and Systems*, 36(11):1382–1391, 1989.

[47] M. Vidyasagar. *Nonlinear System Analysis*. Prentice Hall, London, 1992.

[48] J. Vlach, J.M. Wojciechowski, and A. Opal. Analysis of nonlinear networks with inconsistent initial conditions. *IEEE Transactions on Circuits and Systems-I*, 42(4):195–200, 1995.

[49] J.C. Willems. Dissipative dynamical systems. *Archive for Rational Mechanics and Analysis*, 45:321–393, 1972.