

Robust design in structural acoustics, substructuring techniques and perturbation methods

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**Robust design in structural acoustics,
substructuring techniques and
perturbation methods**

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Report No. 2003.061

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Eindhoven, June 2003

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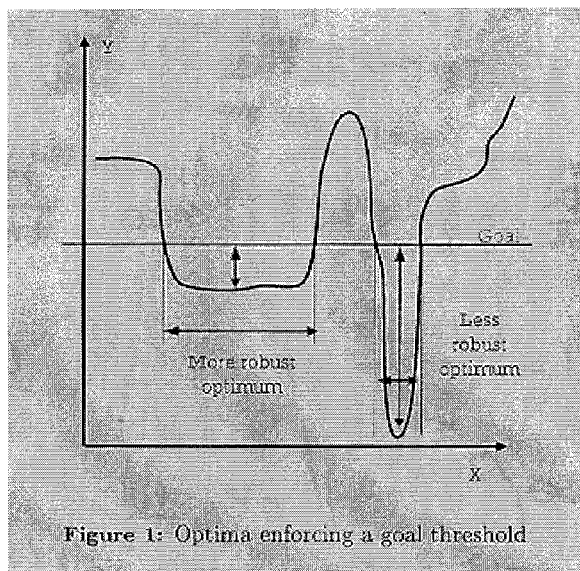
Wilbert Dijkhof*

June 19, 2003

1 Introduction

This is a part of the state of art report of the Ph.D.-project "Robust optimization in structural acoustics". This Ph.D.-project is part of a larger research program issued from a cooperation between TU/e and TNO TPD which has the general title "Robust design of low noise products". There are currently two Ph.D. projects involved. The companion project "Fast optimization in structural acoustics" is carried out by Francois DeBiesme, who is looking for efficient methods to compute and optimize the acoustic behavior of vibrating products.

When optimizing a vibrating structure in order to reduce the noise level, one has to deal with the fact that sometimes these optima are very sensitive to minor changes in its physical or geometrical properties. This is illustrated in figure 1:



"x" is some physical or geometrical parameter (like length, height, density, modulus or joint stiffness) which is modelled as being uncertain, "y" is the quantity which has to be minimized. In the example in figure 1, two local minima and an imposed threshold are plotted. In this case one is interested in those minima which are lower than the threshold. On the right hand side is the global minimum, but it is sensitive to minor changes in the physical/geometrical parameter "x". On the left hand side is the minima of interest, because that one is not very sensitive to minor

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changes in the physical/geometrical parameter "x". Generally, one wants to select those minima which are not very sensitive to changes in the physical or geometrical properties (i.e. have a low standard deviation).

Our interest is in minimizing quantities (in a robust way) like

1. the radiated acoustic power
2. vibration energy functions: like the average energy level in an element; (Anthony *et al.*, 2000), (Peeters, 2001)

The main topic in the thesis will be to provide instruments for robust optimization which are:

- *Computationally efficient.*
- *Generally applicable.* This implies that the instruments can be easily combined with the use of a standard FEM package, like Ansys for example. It also implies, that the estimation of the relevant statistics is not limited to that of the displacement field, but can also be used for determining the scatter in the radiated acoustic power or in the vibration energy levels.

In order to accomplish this an efficient structural analysis procedure combined with certain perturbation methods will be used. When using this, it doesn't matter which quantities must be optimized. Once the perturbations in the displacement field are obtained, the corresponding perturbations in the above mentioned quantities, like radiated acoustic power and vibration energy functions, can be "easily" obtained. Certain substructuring techniques, called *Component Mode Synthesis (CMS) methods*, will be used for the structural analysis. In (Sarkar and Ghanem, 2003) for example, the authors combine certain Component Mode Synthesis methods with the polynomial chaos decomposition. In general, the CMS methods have the following advantages:

- They are reduction methods (the number of degrees of freedom is reduced) resulting in decreased calculation times. This is advantageous, because when dealing with geometrically complex structures, application of the finite element method (FEM) may result in system matrices with an extremely high dimensions. On the other hand computers are getting faster and data storage capacity is continuously increasing, which implies that the analysis can be performed to a higher frequency range.
- Substructures can be analyzed and assessed independently. In aerospace applications for example, the substructures are often defined by the main contractor and produced by different subcontractors.
- There is a natural way of modelling the uncertainty using this technique. The uncertainty is in many cases only present in one or some of the substructures.
- The static solutions of the reduced equations are exact solutions of the original FEM equations (if static displacements fields are included).

The main idea of the *substructuring techniques* is that the structure is divided into different substructures (or components). The number of degrees of freedom (dofs) of each substructure is reduced and a dynamical analysis is performed on each of them. Finally the substructures are assembled again by coupling their interfaces forming the reduced coupled structure. Different substructuring methods have been developed. The most popular ones are Craig-Bampton reduction and Rubin reduction; (Craig Jr., 1995), (Rubin, 1975), (Mace and Shorter, 2001), (de Kraker, 2000), (Fey, 1992). In our study the Rubin method will be used, where the displacement field of the substructure, as will be explained in the following section, is approximated by a linear combination of kept free-interface modes, rigid body modes and residual flexibility modes. The main reason for using this method is that the modes can be handled experimentally in a better way than in the other CMS methods. An advantage of the Craig-Bampton method is that the coupling of the substructures is straightforward.

In section 2, the method of Rubin will be discussed. In particular it will be shown how the various modes are computed and how the substructures are connected. This method will be combined with two perturbation methods, which will be discussed in section 3. The first perturbation method makes use of the second order Taylor series (around the mean of some physical properties). As will be shown in subsection 3.2, this implies that the derivatives of the system matrices with respect to the uncertain parameters must be known. The necessary matrices in order to determine those derivatives can be obtained from the FEM package. The second perturbation method makes use of the fact that the perturbations in the global eigenvalues/eigenmodes are determined by the perturbations of the local eigenvalues. Originally this method has been derived for the Craig-Bampton method, see also (Craig Jr., 1995). In our study it is applied to the method of Rubin.

In this report the focus will be on the scatter in the displacement field of a vibrating structure, and not of the scatter in the acoustic power or in the energy levels. But since the considered uncertainty methods are perturbation methods, one could easily compute the scatter in the acoustic power or in the energy levels in a later stage.

2 CMS method of Rubin

2.1 Substructure analysis

In this subsection the dynamic equations of a linear substructure using the Finite Element Method (FEM) will be setup. In general, the dynamic behavior of a substructure is defined by the equation of motion (a linear partial differential equation with boundary conditions):

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u^{(s)}(\underline{x}, t) + \mathcal{F}^{(s)}(u^{(s)})(\underline{x}, t) &= f_V^{(s)}(\underline{x}, t), \quad \underline{x} \in \mathcal{D}^{(s)} \subset \mathbb{R}^3, \quad t \in \Theta^{(s)} \subset \mathbb{R} \\ \mathcal{B}^{(s)}(u^{(s)})(\underline{x}) &= f_B^{(s)}(\underline{x}), \quad \underline{x} \in \partial\mathcal{D}^{(s)} \quad \text{with} \quad \mathcal{B}^{(s)}(u^{(s)})(\underline{x}) = a^{(s)}(\underline{x})u^{(s)}(\underline{x}) + b^{(s)}(\underline{x})\frac{\partial u^{(s)}}{\partial n}(\underline{x}) \end{aligned} \quad (1)$$

The substructure is denoted by $s \in \{1, \dots, n\}$, the spatial coordinate by $\underline{x} = (x, y, z)$ and the time coordinate by t . The displacement is denoted by $u^{(s)}$, the volume load by $f_V^{(s)}$. $\mathcal{F}^{(s)}$ denotes a linear differential operator with respect to the place coordinate, $\mathcal{B}^{(s)}$ a linear boundary operator, $\partial\mathcal{D}^{(s)}$ the boundary of $\mathcal{D}^{(s)}$ and $f_B^{(s)}$ a load applied to the boundary. Performing a finite element discretisation on equation (1) results in the following equation:

$$M^{(s)} \frac{\partial^2}{\partial t^2} \underline{u}^{(s)}(t) + C^{(s)} \frac{\partial}{\partial t} \underline{u}^{(s)}(t) + K^{(s)} \underline{u}^{(s)}(t) = \underline{f}_V^{(s)}(t) \quad (2)$$

where the boundary condition is incorporated, $x_j \in \mathcal{D}^{(s)} \cup \partial\mathcal{D}^{(s)}$ and

$$\underline{u}^{(s)}(t) = \begin{bmatrix} u^{(s)}(x_1, t) \\ u^{(s)}(x_2, t) \\ \vdots \\ u^{(s)}(x_{n^{(s)}}, t) \end{bmatrix}, \quad \underline{f}_V^{(s)}(t) = \begin{bmatrix} f^{(s)}(x_1, t) \\ f^{(s)}(x_2, t) \\ \vdots \\ f^{(s)}(x_{n^{(s)}}, t) \end{bmatrix} \quad (3)$$

$n^{(s)}$ is the number of dofs of the discretised substructure, $K^{(s)} \in \mathbb{R}^{n^{(s)} \times n^{(s)}}$ is called the stiffness matrix, $C^{(s)} \in \mathbb{R}^{n^{(s)} \times n^{(s)}}$ the damping matrix and $M^{(s)} \in \mathbb{R}^{n^{(s)} \times n^{(s)}}$ the mass matrix. It is assumed that the excitation force is harmonic, that is

$$\underline{f}_V^{(s)}(t) = \underline{F}^{(s)} \exp(i\Omega t) \quad (4)$$

for some $\underline{F}^{(s)} \in \mathbb{R}^{n^{(s)}}$ and excitation frequency $\Omega \in \mathbb{R}^+$. Since the system is linear and there is damping present, it follows that the steady state response $u^{(s)}(t)$ is also harmonic

$$\underline{u}^{(s)}(t) = \underline{U}^{(s)} \exp(i\Omega t) \quad (5)$$

for some $\underline{U}^{(s)} \in \mathbb{C}^{n^{(s)}}$. Substitute the relations (4) and (5) in (2) results in:

$$(K^{(s)} + i\Omega C^{(s)} - \Omega^2 M^{(s)})\underline{U}^{(s)} = \underline{F}^{(s)} \quad (6)$$

implying that equation (2) is transformed into the frequency domain. It is assumed that the damping can be modelled as proportional damping, or modal damping. In those cases the eigenmodes of the damped and undamped substructure are equal. In the undamped case equation (6) simplifies to:

$$(K^{(s)} - \Omega^2 M^{(s)})\underline{U}^{(s)} = \underline{F}^{(s)} \quad (7)$$

The displacement vector $\underline{U}^{(s)}$ of substructure s is partitioned into boundary dofs $\underline{U}_b^{(s)}$ (where the dofs are subjected to external loads, or internal loads caused by connections with other substructures) and internal dofs $\underline{U}_i^{(s)}$ (where the dofs are not subjected to external/internal loads). $n_b^{(s)}$ is the number of boundary dofs, $n_i^{(s)}$ the number of internal dofs and $n^{(s)} = n_i^{(s)} + n_b^{(s)}$. The substructure stiffness matrix, mass matrix and load vector are partitioned in the same way. They are given by:

$$K^{(s)} = \begin{bmatrix} K_{ii}^{(s)} & K_{ib}^{(s)} \\ K_{bi}^{(s)} & K_{bb}^{(s)} \end{bmatrix}, \quad M^{(s)} = \begin{bmatrix} M_{ii}^{(s)} & M_{ib}^{(s)} \\ M_{bi}^{(s)} & M_{bb}^{(s)} \end{bmatrix}, \quad \underline{F}^{(s)} = \begin{bmatrix} \underline{0} \\ \underline{f}_b^{(s)} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{g}_b^{(s)} \end{bmatrix}, \quad \underline{U}^{(s)} = \begin{bmatrix} \underline{U}_i^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} \quad (8)$$

where $\underline{F}^{(s)}$ consists of an externally applied load $\underline{f}_b^{(s)}$ and a load $\underline{g}_b^{(s)}$ which is caused by the connection with adjacent substructures. To simplify the notation the superscript s will be omitted till it is needed again.

If the substructure contains a number of rigid body modes (say n_r), n_r internal dofs are selected which are able to suppress the rigid body modes when these n_r dofs are suppressed. One gets

$$\underline{U} = \begin{bmatrix} \underline{U}_i \\ \underline{U}_b \end{bmatrix} = \begin{bmatrix} \underline{U}_r \\ \underline{U}_w \\ \underline{U}_b \end{bmatrix} = \begin{bmatrix} \underline{U}_r \\ \underline{U}_E \end{bmatrix}, \quad n = n_i + n_b = n_r + n_w + n_b = n_r + n_E$$

Partitioning the stiffness matrix in (8) in the same way gives

$$K^{(s)} = \begin{bmatrix} K_{ii}^{(s)} & K_{ib}^{(s)} \\ K_{bi}^{(s)} & K_{bb}^{(s)} \end{bmatrix} = \begin{bmatrix} K_{rr}^{(s)} & K_{rw}^{(s)} & K_{rb}^{(s)} \\ K_{wr}^{(s)} & K_{ww}^{(s)} & K_{wb}^{(s)} \\ K_{br}^{(s)} & K_{bw}^{(s)} & K_{bb}^{(s)} \end{bmatrix} = \begin{bmatrix} K_{rr}^{(s)} & K_{rE}^{(s)} \\ K_{Er}^{(s)} & K_{EE}^{(s)} \end{bmatrix} \quad (9)$$

The mass matrix and the vector with loads are partitioned in the same way.

In the Component Mode Synthesis (CMS) method of Rubin (Rubin, 1975) the physical dofs are expanded with respect to a new basis (consisting of *free-interface modes*) and this basis is reduced by keeping a number n_K of free-interface modes (called *kept free-interface modes*) and approximating the remaining modes (called *deleted free-interface modes*) by so called *residual flexibility modes*. The last approximation is valid under the assumption that we are interested in the frequency range given by

$$0 \leq \Omega \leq \Omega_{n_K}$$

with Ω_{n_K} the eigenfrequency corresponding to the n_K -th eigenmode. The *physical dofs* $\underline{\Phi}$ can be approximated in terms a linear combination of *generalized dofs* \underline{q} using the **mode shape matrix** (or **reduction matrix**) $S \in \mathbb{R}^{n \times (n_K + n_b)}$ (composed of rigid body modes Φ_r , kept elastic free-interface modes Φ_k and residual flexibility modes Φ_b), which is given by

$$S = [\Phi_r \quad \Phi_k \quad \Phi_b]$$

as follows

$$\underline{U} = S\underline{q} \quad (10)$$

with

$$\underline{q} = \begin{bmatrix} \underline{q}_r \\ \underline{q}_k \\ \underline{q}_b \end{bmatrix}, \quad n_k = n_K - n_r$$

Applying the reduction to the frequency response equation (6) results in

$$(K_S + i\Omega C_S - \Omega^2 M_S)\underline{q} = \underline{F}_S \quad (11)$$

with $K_S = S'KS$, $C_S = S'CS$, $M_S = S'MS$ and $F_S = S'\underline{F}$. Note that K_S , M_S , C_S are all diagonal (the latter because of the proportional/modal damping assumption). The definition of the free-interface modes is given below:

Definition 2.1.1. free-interface modes, elastic free-interface modes, rigid body modes:

The free-interface modes are obtained by setting $\underline{F} = \underline{0}$ and solving for \underline{U} in equation (7) (in other words they are the eigenmodes ϕ_j of the corresponding eigenvalue problem):

$$(K - \Omega_j^2 M)\phi_j = \underline{0}, \quad j = 1, \dots, n \quad (12)$$

Since K is semi-positive definite and M positive definite the eigenvalues Ω_j^2 are non-negative. The eigenmodes with eigenvalue larger than zero are called **elastic free-interface modes** and the eigenmodes with eigenvalue equal to zero are called **rigid body modes**. \square

The free-interface modes will be normalized on the mass matrix, this implies

$$\Phi_k' M \Phi_k = I_{n_k}, \quad \Phi_k' K \Phi_k = \text{diag}(\Omega_j^2)$$

with $\Phi_k = [\phi_1 \ \dots \ \phi_{n_k}]$.

The rigid body modes can also be determined as follows. A set of n_r independent dofs which successively gets a unit displacement whereas the remaining $n_r - 1$ dofs of this set are suppressed is chosen. In other words the rigid body modes are the solution of

$$\begin{bmatrix} K_{rr} & K_{rE} \\ K_{Er} & K_{EE} \end{bmatrix} \begin{bmatrix} I_{n_r, n_r} \\ X \end{bmatrix} = \begin{bmatrix} O_{n_r, n_r} \\ O_{n_E, n_r} \end{bmatrix} \quad \text{with } K = \begin{bmatrix} K_{rr} & K_{rE} \\ K_{Er} & K_{EE} \end{bmatrix} \quad (13)$$

with n_r arbitrary. Although in practice for substructures without mechanisms

$$0 \leq n_r \leq 6 \quad \text{in three dimensions,} \quad 0 \leq n_r \leq 3 \quad \text{in two dimensions}$$

If for some n_r a solution exists, it is given by (as can be seen by solving the equation in formula (13))

$$\Phi_r^* = \begin{bmatrix} I_{n_r, n_r} \\ -K_{EE}^{-1} K_{Er} \end{bmatrix}$$

To orthogonalize the rigid body modes Φ_r^* one can proceed as follows. Since the dofs are chosen independently, they span the set \mathbb{R}^{n_r} , and it is possible to determine a basis for this set. Such a basis can be obtained as follows. Determine the mass-normalized eigenmodes \underline{x}_j of the matrix $M^* = (\Phi_r^*)' M \Phi_r^*$. That is

$$M^* \underline{x}_j = \gamma_j \underline{x}_j, \quad \text{where } \underline{x}_i' M^* \underline{x}_j = \delta_{i,j}, \quad X \equiv [\underline{x}_1 \ \dots \ \underline{x}_{n_r}]$$

The columns of $\Phi_r^* X$ form a basis of the set \mathbb{R}^{n_r} , since

$$(\Phi_r^* X)' M (\Phi_r^* X) = X' ((\Phi_r^*)' M \Phi_r^*) X = X' M^* X = I$$

Implying that the mass-orthogonalized rigid body modes becomes $\Phi_r = \Phi_r^* X$. As said earlier the main idea is to keep several elastic free-interface modes, say n_k (which are called the *kept elastic free-interface modes*) and to approximate the remaining ones by a couple of so called *residual flexibility modes*. A residual flexibility mode is the quasi static response (the meaning of this will be explained in definition (2.1.2)) on a static unit load acting on a boundary dof, from which the contribution of the kept free-interface modes has been removed. A precise definition is given below:

Definition 2.1.2. residual flexibility modes:

Define load columns for boundary load \underline{U}_b by the following $n \times n_b$ matrix

$$B = \begin{bmatrix} O_{n_i, n_b} \\ I_{n_b} \end{bmatrix} \quad (14)$$

The frequency response function of the undamped substructure which can be derived from equation (7) is given by

$$H(\Omega) = \sum_{j=1}^n \frac{\phi_j \phi_j'}{\Omega_j^2 - \Omega^2} = \sum_{j=0}^{n_k} \frac{\phi_j \phi_j'}{\Omega_j^2 - \Omega^2} + \sum_{j=n_k+1}^n \frac{\phi_j \phi_j'}{\Omega_j^2 - \Omega^2} \quad (15)$$

The frequency response function in (15) can be approximated by

$$H(\Omega) \approx \sum_{j=1}^{n_k} \frac{\phi_j \phi_j'}{\Omega_j^2 - \Omega^2} + \sum_{j=n_k+1}^n \frac{\phi_j \phi_j'}{\Omega_j^2} \quad (16)$$

assuming that the frequency range of interest is given by $0 \leq \Omega \leq \Omega_{n_k}$. The quasi-static response is given by

$$\sum_{j=n_r+1}^{n_k} \frac{\phi_j \phi_j'}{\Omega_j^2} + \sum_{j=n_k+1}^n \frac{\phi_j \phi_j'}{\Omega_j^2} = \Phi_k \Lambda_{kk}^{-1} \Phi_k' + \Phi_d \Lambda_{dd}^{-1} \Phi_d' = \Phi_E \Lambda_{EE}^{-1} \Phi_E' \equiv G_E$$

with $\Lambda_{kk} = \text{diag}(\Omega_{n_r+1}^2, \dots, \Omega_{n_k}^2)$, $\Lambda_{dd} = \text{diag}(\Omega_{n_k+1}^2, \dots, \Omega_n^2)$ and $\Lambda_{EE} = \text{diag}(\Lambda_{kk}, \Lambda_{dd})$

$$\Phi_k = \begin{bmatrix} \phi_{n_r+1} & \dots & \phi_{n_k} \end{bmatrix}, \quad \Phi_d = \begin{bmatrix} \phi_{n_k+1} & \dots & \phi_n \end{bmatrix} \quad \text{and} \quad \Phi_E = \begin{bmatrix} \Phi_k & \Phi_d \end{bmatrix}$$

The second sum in equation (16) is just the quasi-static response minus the contribution of the kept free-interface modes. The residual flexibility modes (as columns in a matrix) are defined by

$$\Phi_b = \sum_{j=n_k+1}^n \frac{\phi_j \phi_j'}{\Omega_j^2} B = \Phi_d \Lambda_{dd}^{-1} \Phi_d' B \quad (17)$$

□

It is not necessary to calculate all eigenvalues and eigenmodes in order to determine the residual flexibility modes. This would be computationally very inefficient. Instead the following theorem can be used

Theorem 2.1.1. Calculation residual flexibility modes:

It is very efficient to compute the residual flexibility modes

$$\Phi = (G_E - \Phi_k \Lambda_{kk}^{-1} \Phi_k') B \quad (18)$$

using $G_E = P'GP$ where

$$G \equiv \begin{bmatrix} O_{rr} & O_{rE} \\ O_{Er} & K_{EE}^{-1} \end{bmatrix}, \quad P \equiv I - M \Phi_r \Phi_r'$$

Note that if there are no rigid body modes, we just have

$$G = K_{EE}^{-1}, \quad P = I, \quad \text{and thus} \quad G_E = G = K_{EE}^{-1} = K^{-1}$$

Proof. Since the free-interface modes are orthonormalized with respect to the mass matrix we have $\Phi' M \Phi = I$ (and $\Phi' K \Phi = \Lambda$). It follows that $\Phi' M \Phi \Phi' = \Phi'$ and since Φ (and thus Φ') is invertible we obtain $M \Phi \Phi' = I$, or

$$M(\Phi_r \Phi_r' + \Phi_E \Phi_E') = I \quad (19)$$

Define $P \equiv I - M \Phi_r \Phi_r'$. It follows from equation (19) that

$$P = I - M \Phi_r \Phi_r' = M \Phi_E \Phi_E' = K \Phi_E \Lambda_{EE}^{-1} \Phi_E' \quad (20)$$

since $M \Phi_E \Lambda_{EE} = K \Phi_E$.

Define

$$G \equiv \begin{bmatrix} O_{rr} & O_{rE} \\ O_{Er} & K_{EE}^{-1} \end{bmatrix}$$

Then it follows

$$K' G K = \begin{bmatrix} O_{rr} & K_{rE} K_{EE}^{-1} \\ O_{Er} & I_{EE}^{-1} \end{bmatrix} K = \begin{bmatrix} K_{rr} & K_{rE} \\ K_{Er} & K_{EE} \end{bmatrix} = K \quad (21)$$

Note that $\Phi' K \Phi = \Lambda$ implies

$$\Phi_E' K \Phi_E = \Lambda_{EE} \quad (22)$$

Finally, it follows that

$$P' G P \stackrel{(20)}{=} \Phi_E \Lambda_{EE}^{-1} \Phi_E' K' G K \Phi_E \Lambda_{EE}^{-1} \Phi_E' \stackrel{(21)}{=} \Phi_E \Lambda_{EE}^{-1} \Phi_E' K \Phi_E \Lambda_{EE}^{-1} \Phi_E' \stackrel{(22)}{=} \Phi_E \Lambda_{EE}^{-1} \Phi_E' = G_E$$

The equations (16) and (17) conclude the proof. \square

Note that P is easy to compute, which implies that G_E and thus Φ_b will be easy to compute. In the following subsection we will explain a method to connect the substructures. Note further that the residual flexibility modes will introduce inaccurate eigenfrequencies and eigenmodes above the cut-off frequency Ω_K in the reduced substructure equations.

2.2 The coupling procedure of Martinez

In this subsection it will be explained how the substructures can be coupled using the coupling procedure of Martinez (Martinez and Miller, 1985), (de Kraker, 2000). The superscript s is used again to denote the substructure. As can be seen by looking at the reduction matrix the coupling of two substructures is not straightforward, because the generalized dofs $\underline{q}^{(s)}$ don't explicitly contain the physical boundary dofs $\underline{U}_b^{(s)}$. In the coupling procedure of Martinez these boundary dofs are recovered again. Recalling the internal dofs $\underline{U}_i^{(s)}$ and the boundary dofs $\underline{U}_b^{(s)}$ one can rewrite equation (10) as

$$\begin{bmatrix} \underline{U}_i^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} = \begin{bmatrix} \Phi_{i,r} & \Phi_{i,k} & \Phi_{i,b} \\ \Phi_{b,r} & \Phi_{b,k} & \Phi_{b,b} \end{bmatrix} \begin{bmatrix} \underline{q}_r^{(s)} \\ \underline{q}_k^{(s)} \\ \underline{q}_b^{(s)} \end{bmatrix} \quad (23)$$

Solving for $\underline{q}_b^{(s)}$ in the second equation gives

$$\underline{q}_b^{(s)} = \Phi_{b,b}^{-1} \underline{U}_b^{(s)} - \Phi_{b,b}^{-1} \Phi_{b,k} \underline{q}_k^{(s)} - \Phi_{b,b}^{-1} \Phi_{b,r} \underline{q}_r^{(s)}$$

It follows that

$$\begin{bmatrix} \underline{q}_r^{(s)} \\ \underline{q}_k^{(s)} \\ \underline{q}_b^{(s)} \end{bmatrix} = \begin{bmatrix} I_{r,r} & O_{r,k} & O_{r,b} \\ O_{k,r} & I_{k,k} & O_{k,b} \\ -\Phi_{b,b}^{-1} \Phi_{b,r} & -\Phi_{b,b}^{-1} \Phi_{b,k} & \Phi_{b,b}^{-1} \end{bmatrix} \begin{bmatrix} \underline{q}_r^{(s)} \\ \underline{q}_k^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} \quad (24)$$

Combining equation (23) and (24) gives:

$$\begin{aligned} \begin{bmatrix} \underline{U}_i^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} &= \begin{bmatrix} \Phi_{i,r} & \Phi_{i,k} & \Phi_{i,b} \\ \Phi_{b,r} & \Phi_{b,k} & \Phi_{b,b} \end{bmatrix} \begin{bmatrix} I_{r,r} & O_{r,k} & O_{r,b} \\ O_{k,r} & I_{k,k} & O_{k,b} \\ -\Phi_{b,b}^{-1}\Phi_{b,r} & -\Phi_{b,b}^{-1}\Phi_{b,k} & \Phi_{b,b}^{-1} \end{bmatrix} \begin{bmatrix} \underline{q}_r^{(s)} \\ \underline{q}_k^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} \\ &\equiv S^{(s)} S_2^{(s)} \begin{bmatrix} \underline{q}_i^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} \equiv [Z_i^{(s)} Z_b^{(s)}] \begin{bmatrix} \underline{q}_i^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} \equiv Z^{(s)} \begin{bmatrix} \underline{q}_i^{(s)} \\ \underline{U}_b^{(s)} \end{bmatrix} \equiv Z^{(s)} \underline{q}_Z^{(s)} \text{ with } \underline{q}_i^{(s)} = \begin{bmatrix} \underline{q}_r^{(s)} \\ \underline{q}_k^{(s)} \end{bmatrix} \end{aligned} \quad (25)$$

Substituting equation (25) in equation (6) and left multiplying by $Z'^{(s)}$ gives:

$$(K_Z^{(s)} + i\Omega C_Z^{(s)} - \Omega^2 M_Z^{(s)}) \underline{q}_Z^{(s)}(\Omega) = \underline{F}_Z^{(s)} \quad (26)$$

with

$$K_Z^{(s)} \equiv Z'^{(s)} K^{(s)} Z^{(s)} \equiv \begin{bmatrix} k_{ii}^{(s)} & k_{ib}^{(s)} \\ k_{bi}^{(s)} & k_{bb}^{(s)} \end{bmatrix} = (S_2)^{(s)} K_S^{(s)} S_2^{(s)} \quad (27)$$

$$C_Z^{(s)} \equiv Z'^{(s)} C^{(s)} Z^{(s)} \equiv \begin{bmatrix} c_{ii}^{(s)} & c_{ib}^{(s)} \\ c_{bi}^{(s)} & c_{bb}^{(s)} \end{bmatrix} = (S_2)^{(s)} C_S^{(s)} S_2^{(s)} \quad (28)$$

$$M_Z^{(s)} \equiv Z'^{(s)} M^{(s)} Z^{(s)} \equiv \begin{bmatrix} m_{ii}^{(s)} & m_{ib}^{(s)} \\ m_{bi}^{(s)} & m_{bb}^{(s)} \end{bmatrix} = (S_2)^{(s)} M_S^{(s)} S_2^{(s)} \quad (29)$$

$$\underline{F}_Z^{(s)} \equiv Z'^{(s)} \underline{F}^{(s)} \equiv \begin{bmatrix} \underline{f}_i^{(s)} \\ \underline{f}_b^{(s)} \end{bmatrix} = (S_2)^{(s)} \underline{F}_S^{(s)} \quad (30)$$

Note that $k_{ii}^{(s)}$, $c_{ii}^{(s)}$ and $m_{ii}^{(s)}$ are not diagonal because the basis, S , is multiplied with a new transformation (resulting in modes which are not orthogonal anymore). By connecting the substructures (including their boundary conditions), the coupled system equations are obtained

$$(K_Z + i\Omega C_Z - \Omega^2 M_Z) \underline{q}_Z(\Omega) = \underline{F}_Z + C' \underline{\lambda} \quad \text{with the constraint} \quad C \underline{q}_Z(\Omega) = \underline{0} \quad (31)$$

The constraint equation ensures compatibility between substructure interfaces. The system matrices are given by

$$K_Z = \text{diag}(K_Z^{(1)}, \dots, K_Z^{(N)}), \quad C_Z = \text{diag}(C_Z^{(1)}, \dots, C_Z^{(N)}), \quad M_Z = \text{diag}(M_Z^{(1)}, \dots, M_Z^{(N)}) \quad (32)$$

The local modal degrees of freedom (or modal participation factors), the modal forces (or generalized forces) and the Lagrange multipliers are given by:

$$\underline{q}_Z = \begin{bmatrix} \underline{q}_i^{(1)} \\ \underline{U}_b^{(1)} \\ \underline{q}_i^{(2)} \\ \underline{U}_b^{(2)} \\ \vdots \\ \underline{q}_i^{(N)} \\ \underline{U}_b^{(N)} \end{bmatrix}, \quad \underline{F}_Z = \begin{bmatrix} \underline{f}_i^{(1)} \\ \underline{f}_b^{(1)} \\ \underline{f}_i^{(2)} \\ \underline{f}_b^{(2)} \\ \vdots \\ \underline{f}_i^{(N)} \\ \underline{f}_b^{(N)} \end{bmatrix}, \quad \underline{\lambda} = \text{a vector consisting of Lagrange multipliers}$$

Note that $\underline{\lambda}$ can be identified with the internal forces which ensure equilibrium of interface loads. Note further that the matrix C contains only elements of the set $\{-1, 0, 1\}$, because all interface dofs are explicitly present in \underline{q}_Z . To determine the system of equations without Lagrange multipliers, a basis of the null-space of C must be determined. That is all vectors \underline{x}_m such that $C \underline{x}_m = \underline{0}$. Those vectors define a matrix L :

$$L = [\underline{x}_1 \quad \underline{x}_2 \quad \dots \quad \underline{x}_M]$$

with the property $CL = 0$. This gives us new coordinates \underline{w} such that $\underline{q}_Z = L\underline{w}$. Substituting $\underline{q}_Z = L\underline{w}$ into equation (31) and left multiplying with L' gives

$$L'(K_Z + i\Omega C_Z - \Omega^2 M_Z)L\underline{w}(\Omega) = L'\underline{F}_Z + L'C'\underline{\lambda} \quad (33)$$

Using that $CL = 0$ and $L'C' = (CL)'$, the coupled system without constraints is obtained

$$L'(K_Z + i\Omega C_Z - \Omega^2 M_Z)L\underline{w}(\Omega) = L'\underline{F}_Z \quad \text{or} \quad (K_L + i\Omega C_L - \Omega^2 M_L)\underline{w}(\Omega) = \underline{F}_L \quad (34)$$

with

$$K_L = L'K_ZL \equiv \begin{bmatrix} K_{ii} & K_{ib} \\ K'_{ib} & K_{bb} \end{bmatrix}, \quad M_L = L'M_ZL \equiv \begin{bmatrix} M_{ii} & M_{ib} \\ M'_{ib} & M_{bb} \end{bmatrix}, \quad \underline{F}_L = L'\underline{F}_Z \equiv \begin{bmatrix} \underline{F}_i \\ \underline{F}_b \end{bmatrix} \quad (35)$$

$$C_L = L'C_ZL \equiv \begin{bmatrix} C_{ii} & C_{ib} \\ C'_{ib} & C_{bb} \end{bmatrix} \quad (36)$$

Note that \underline{F}_L contains only elements of $\underline{f}_b^{(s)}$ and that the interface loads $\underline{g}_b^{(s)}$ cancel out. In general, if the substructures are weakly or proportionally damped, it does not imply that the global system is also weakly or proportionally damped. But if this is the case, then the solution of equation (34) is given by

$$\underline{w}(\Omega) = H(\Omega)\underline{F}_L \quad \text{with} \quad H(\Omega) = \sum_{j=1}^p \frac{\underline{\phi}_{global,j} \underline{\phi}'_{global,j}}{m_j(\Omega_{global,j}^2 + 2\xi_j \Omega_{global,j} \Omega - \Omega^2)} \quad (37)$$

with $\{\underline{\phi}_{global,j}\}_j$ and $\{\Omega_{global,j}\}_j$ the solutions of the eigenvalue problem

$$(K_L - \Omega_{global,j}^2 M_L)\underline{\phi}_{global,j} = 0$$

with

$$\Omega_{global,j}^2 = \frac{k_j}{m_j} = \frac{T'K_L T[j,j]}{T'M_L T[j,j]}, \quad c_j = T'C_L T[j,j], \quad T = [\underline{\phi}_{global,1} \quad \dots \quad \underline{\phi}_{global,p}]$$

The modal damping factor is given by

$$\xi_j = \frac{c_j}{2\Omega_{global,j} m_j}$$

Finally the reduced system dofs \underline{w} can be expressed in terms of the physical degrees of freedom $\underline{U}^{(s)}$ of the system, using the transformations $\underline{q}_Z = L\underline{w}$ and $\underline{U}^{(s)} = Z^{(s)}\underline{q}_Z^{(s)}$

$$\underline{U} \equiv \begin{bmatrix} \underline{U}^{(1)} \\ \underline{U}^{(2)} \\ \vdots \\ \underline{U}^{(N)} \end{bmatrix} = \begin{bmatrix} Z^{(1)}L^{(1)} \\ Z^{(2)}L^{(2)} \\ \vdots \\ Z^{(N)}L^{(N)} \end{bmatrix} \underline{w} \equiv Z\underline{w} \quad (38)$$

To illustrate the method of Rubin we will consider two simple examples in the next subsection.

2.3 Two examples

The first example focusses only on the reduction.

Example 2.3.1. single beam:

Consider an aluminum beam system consisting of only one substructure (as in figure 1). This beam is clamped on one side and free on the other side. The structure is modelled in Ansys with ten beam3-elements of equal length. This 2d-element has three dofs at each node (translation in

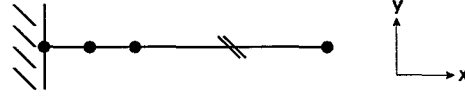


Figure 1: beam; ten elements

the x and y direction, and rotation about the z-axis). The (symmetric) element matrices in local coordinates are given by:

$$K^{(e)} = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ * & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ * & * & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ * & * & * & AE/L & 0 & 0 \\ * & * & * & * & 12EI/L^3 & -6EI/L^2 \\ * & * & * & * & * & 4EI/L \end{bmatrix} \quad (39)$$

and

$$M^{(e)} = \frac{\rho AL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ * & 156 & 22L & 0 & 54 & -13L \\ * & * & 4L^2 & 0 & 13L & -3L^2 \\ * & * & * & 140 & 0 & 0 \\ * & * & * & * & 156 & -22L \\ * & * & * & * & * & 4L^2 \end{bmatrix} \quad (40)$$

The physical properties of the beam are listed in the following table:

height (h)	0.05 m
width (b)	0.025 m
Young's modulus (Y)	$7 * 10^{10} \text{ N/m}^2$
density (ρ)	2700 kg/m^3
length ($l = 10L$)	1 m
area of cross section (A)	$h * b \text{ m}^2$
second moment of area (I)	$h^3 * b/12 \text{ m}^4$

Assembling the element matrices results in matrices K and M with size 30×30 . Note that this structure contains 30 free-interfaces modes and no rigid body modes. It is assumed that the beam will be loaded at the three dofs of its free end. In this case the beam has three boundary dofs ($U_{11}, V_{11}, \theta_{11}$). Further it is assumed that the frequency content of the loading is below 1300 Hz.

The system is reduced by calculating four (elastic) free-interface modes. This implies that only the first four eigenvalues will be correct. This results in reduced system matrices K_Z and M_Z with size 7×7 (four kept free-interface modes plus three boundary dofs). The eigenvalues are calculated respectively analytically, with Ansys and from the reduced system. They are given in the following table (in Hz):

analytic	Ansys	reduced system	mode type
41.126	41.106	41.107	bending
257.73	256.88	256.88	bending
721.66	716.10	716.10	bending
1272.9	1274.2	1274.2	axial
1414.2	1394.9	1594.0	bending

Note that the fourth eigenfrequency corresponds to a longitudinal mode while others correspond to bending modes.

Example 2.3.2. two beams example:

The second example focusses on both the reduction and the substructuring. Two aluminum beams assembled under an angle of 45 degrees are considered (one end is clamped, the other is free), as in figure 2. Both beams (ten elements each) are modelled using the same properties as in the

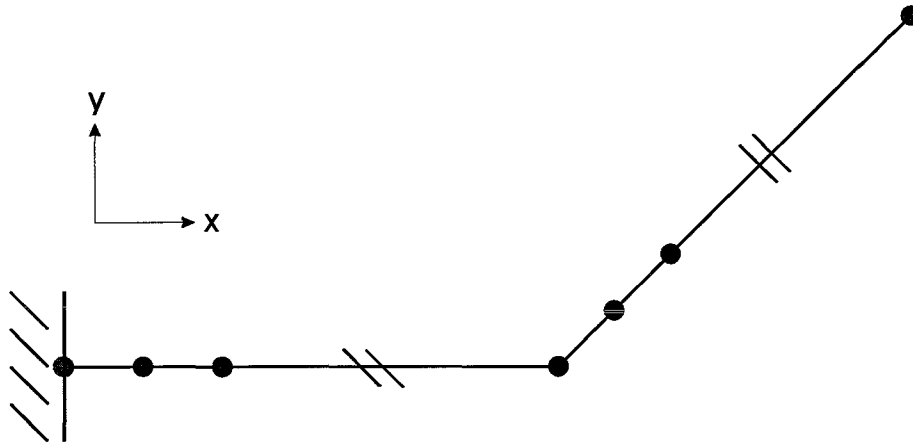


Figure 2: two beams; ten elements each

previous example. It is assumed that the second beam will be loaded at the three dofs of its free end. It is further assumed that the frequency content of the loading of the global structure is below 1300 Hz.

The substructures are reduced by calculating four elastic free-interface modes for the first substructure, and three for the second one. This results in reduced global system matrices K_Z and M_Z with size 19×19 . The type of modes are given in the following table:

modes	substructure 1	substructure 2
rigid body modes	0	3
residual flexibility modes	3	6
kept elastic free-interface modes	4	3

The local eigenvalues are given in the following table (in Hz):

substructure 1	substructure 2
41.107	0.005 \approx 0
256.88	0.005 \approx 0
716.10	0.005 \approx 0
1274.2	260.37
1594.0	713.51
5057.0	1388.7
7928.1	2454.2

Note that every set of local eigenvalues must be accurate up to 1300 Hz. The global eigenvalues are given in the following table

Ansys	reduced system	relative error (%)
11.009	11.009	\approx 0
51.321	51.321	\approx 0
179.99	179.99	\approx 0
294.25	294.26	\approx 0
579.72	579.75	\approx 0
698.01	698.03	\approx 0
1011.6	1013.0	0.14
1208.8	1238.8	2.4
1528.7	1612.0	5.2

As one can see, the global eigenvalues are indeed accurately described up to 1300 Hz (although the relative error in the eight global eigenvalue is a bit large).

3 Perturbation methods

In this section it is assumed that a certain physical or geometrical property of one (or more) substructure(s) is uncertain and the corresponding scatter in the displacement is computed. The mean and standard deviation will be estimated using perturbation techniques.

In subsection 3.2, a perturbation technique will be discussed, where the displacement is developed in a Taylor series about the mean of the uncertain physical/geometrical properties. Those properties are modelled as random variables. The concept of random variable will be discussed in subsection 3.1.

In subsection 3.3, another perturbation technique will be discussed, where the local eigenvalues will be perturbed. In order to generate those perturbations, it will be assumed that the local eigenvalues are modelled as being a random field (this concept will be discussed in subsection 3.1), prescribed by a certain joint probability density. The Karhunen-Loeve (KL) expansion will be used to generate those perturbations. This is possible because it converts random fields (or correlated random variables) into a set of uncorrelated random variables. The Karhunen-Loeve expansion will also be briefly discussed in the following subsection.

3.1 Generation of random variables

In this subsection, the concepts of random variable, random field, Gaussian field and Karhunen-Loeve expansion will be briefly discussed, see also (Balakrishnan, 1995).

Let's consider a physical example, and discuss the necessary concepts:

$$\begin{cases} \frac{d^2}{dx^2}U(x, \Omega) + \frac{\rho}{Y}\Omega^2U(x, \Omega) = 0 & x \in]0, 1[, \Omega \in \mathbb{R}^+ \\ U(0, \Omega) = 0 & \Omega \in \mathbb{R}^+ \\ \frac{d}{dx}U(1, \Omega) = \frac{1}{AY}F & \Omega \in \mathbb{R}^+ \end{cases} \quad (41)$$

This differential equation describes the longitudinal waves in a bar clamped on one side in a rigid wall, and excited by a harmonic force F . Young's modulus Y is an uncertain parameter, that is a random variable $Y : \Theta \rightarrow \mathbb{R}$ defined on a probability space $(\Theta, \Sigma, \mathcal{P})$. Let's assume that Y is Gaussian distributed, which implies that the probability function $\mathcal{P} : \Sigma \rightarrow [0, 1]$ is given by

$$\mathcal{P}(Y \leq y) = \int_{-\infty}^y \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

with $\mu = \mathbb{E}[Y]$ the mean and $\sigma^2 = \mathbb{E}[(Y - \mathbb{E}[Y])^2]$ the variance of Y .

Let's consider an other example: Suppose the local eigenvalues $\{\Omega_{local,j}^2\}_j$ of a substructure are determined, and those local eigenvalues are uncertain. As they are correlated, they can be turned into a random field

$$\Omega_{local}^2 : \{1, 2, \dots, n_K\} \times \Sigma \rightarrow \mathbb{R}$$

defined on a probability space $(\Theta, \Sigma, \mathcal{P})$. Let's assume that the random field Ω_{local}^2 is Gaussian distributed, which implies that the probability function $\mathcal{P} : \Sigma \rightarrow [0, 1]$ is in this case given by

$$\mathcal{P}(Y_1 \leq y_1, \dots, Y_{n_K} \leq y_{n_K}) = \int_{-\infty}^{y_1} \dots \int_{-\infty}^{y_{n_K}} \frac{1}{(\det \Lambda)^{\frac{1}{2}} (\sqrt{2\pi})^{n_K}} \exp\left(-\frac{1}{2}[\Lambda^{-1}(\underline{x}-\underline{m}), (\underline{x}-\underline{m})]\right) dx_1 dx_{n_K}$$

with $\underline{x}[k] = x_k$, \underline{m} is the mean, and Λ the covariance matrix of Y .

Let's have a closer look at the used concepts. A triple $(\Theta, \Sigma, \mathcal{P})$ is called a probability space with Θ the space of outcomes of a stochastic experiment, Σ the space of events on Θ and $\mathcal{P} : \Sigma \rightarrow [0, 1]$ the probability function which assigns a probability to an event. In other words, the basic idea is that one can assign probabilities to every event. The formal definition of 'probability space' can be found in the appendix A.1.1. The definitions of a random variable and a random field are as follows

Definition 3.1.1. random variable, random field:

Let $(\Theta, \Sigma, \mathcal{P})$ be a probability space. A random variable is a mapping $X : \Theta \rightarrow \mathbb{R}$ which takes (real) values with a certain probability, that is, it describes random fluctuations. A random variable is a special case of a random field. Random fields are used to describe random (both in space and time) fluctuations. Basically, a random field $f : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ with $\mathcal{D} \subset \mathbb{R}^n$ is just a collection of random variables. It is called discrete if its domain \mathcal{D} is countable, and continuous if its domain \mathcal{D} is an interval in \mathbb{R} . The formal definitions can be found in the appendix A.1.2. \square

Let's have a closer look at an example of a random variable which is Gaussian distributed

Definition 3.1.2. Gaussian variable:

A random variable $X : \Theta \rightarrow \mathbb{R}$ is Gaussian (or normal) with parameters μ and σ^2 (denoted as $X \sim N(\mu, \sigma^2)$) if the probability density of X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \Theta = \mathbb{R}$$

The corresponding distribution $F_X : \mathcal{B} \rightarrow [0, 1]$ is given by

$$F_X(B) = \mathcal{P}(X \in B) = \int_{B \in \mathcal{B}} f_X(x) dx$$

Recall that the mean of X is given by $\mathbb{E}[X] = \mu$ and the variance of X is given by $\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sigma^2$, with $\mathbb{E}[g(X)] = \int_{\Theta} g(x) f_X(x) dx$ for any descent function $g : \Theta \rightarrow \mathbb{R}$. \square

Let's have a closer look at an example of a random field which is Gaussian distributed

Definition 3.1.3. Gaussian field:

Let $\underline{r} \equiv (r_1, r_2, \dots, r_M) \in \mathcal{D}^M$ with $\mathcal{D} \subseteq \mathbb{R}$. A random field $g : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ is called Gaussian if the joint distribution of $\{g(r_j, \cdot)\}_{j=1}^M$ is Gaussian for all $\underline{r} \in \mathcal{D}^M$ and all $M \in \mathbb{N}^+$. The joint distribution of $\{X(r_j, \cdot)\}_{j=1}^M$ is given by $F_{X_{\underline{r}}}(\underline{x}) = \mathcal{P}(X_{r_1} \leq x_1, \dots, X_{r_M} \leq x_M)$ with $\underline{x} \equiv (x_1, \dots, x_M) \in \mathbb{R}^M$ and $\underline{r} \in \mathcal{D}^M$. The corresponding joint probability density is given by

$$f_X(\underline{x}) = \frac{1}{(\det \Lambda)^{\frac{1}{2}} (\sqrt{2\pi})^M} \exp\left(-\frac{1}{2} [\Lambda^{-1}(\underline{x} - \underline{m}), (\underline{x} - \underline{m})]\right)$$

where $\underline{x} \in \mathbb{R}^M$, $\underline{m} \in \mathbb{R}^M$ is the mean, and $\Lambda \in \mathbb{R}^{M \times M}$ the covariance matrix given by

$$\underline{m} = \mathbb{E}[\underline{Y}], \quad \Lambda = \mathbb{E}[(\underline{Y} - \underline{m})(\underline{Y} - \underline{m})']$$

with $\underline{Y}[j] \equiv X(r_j, \cdot)$ for $j = 1, \dots, M$. The mean of a vector of random variables is defined as the vector of the means of the random variables, $\underline{m}[j] = \mathbb{E}[X(r_j, \cdot)]$. \square

The Karhunen-Loeve expansion can only be applied to zero-mean random fields. It is not always the case that a random field and 'the same random field minus its mean' has the same covariance matrix. But in case of Gaussian fields this is true, and this fact will be used at a later stage:

Lemma 3.1.1. Gaussian fields:

Let g be a Gaussian field with finite mean $m(\underline{r}) \equiv \mathbb{E}[g(\underline{r}, \cdot)]$ and let $h(\underline{r}, \cdot) = g(\underline{r}, \cdot) - \mathbb{E}[g(\underline{r}, \cdot)]$, it follows that h is also a Gaussian field with joint probability density with zero mean and the same covariance as g . \square

Having seen the definitions of random variable and random field, it will be discussed how to generate realizations of a Gaussian field (given its mean and covariance). The Karhunen-Loeve (KL) expansion is a tool that can be used for this purpose. It is used to separate the random and deterministic part (thus place dependent part) of a random field.

Definition 3.1.4. Karhunen-Loeve expansion:

The KL-expansion is a decomposition of a zero mean random field in a denumerable (countable) set of random variables, it's given by

$$h(\underline{r}, \theta) = \sum_{j=0}^{\infty} \sqrt{\lambda_j} \xi_j(\theta) f_j(\underline{r}), \quad \underline{r} \in \mathcal{D} \subset \mathbb{R}^n, \theta \in \Theta \quad (42)$$

with $\{\xi_j\}_j$ zero mean orthonormal random variables (i.e. $(\xi_i, \xi_j) = \mathbb{E}[\xi_i \xi_j] = \delta_{i,j}$), $\{\lambda_j\}_j$ the eigenvalues and $\{f_j(\underline{r})\}_j$ the orthonormalized eigenfunctions of an integral operator corresponding to the covariance function of h . This decomposition exists if the following criteria are met: the covariance of h is positive definite, symmetric and continuous, and \mathcal{D} is compact (closed, bounded subset of \mathbb{R}^n). In particular if h is Gaussian then $\{\xi_j\}_j$ is jointly Gaussian and thus ξ_j is Gaussian for every j . The eigenvalues λ_j and eigenfunctions f_j in the KL-expansion are the solutions of the following eigenvalue problem

$$\int_{\mathcal{D}} \text{Cov}[h(x), h(y)] f_j(x) dx = \lambda_j f_j(y) \quad (43)$$

□

Recall that a covariance functions is always symmetric and positive semi-definite. If it is just positive definite, though also continuous, the corresponding random field has a KL-expansion.

The following example will be used in subsection 3.3, see also (Ghanem and Spanos, 1990). Suppose that the covariance function of a Gaussian field $g : \{1, 2, \dots, n_K\} \rightarrow \mathbb{R}$ is given by

$$\text{Cov}[g(x), g(y)] = \exp(-c|x - y|), \quad (x, y) \in \{1, 2, \dots, n_K\} \times \{1, 2, \dots, n_K\} \quad (44)$$

where c is called the correlation length (since it reflects the rate at which the correlation decays between two points of the field).

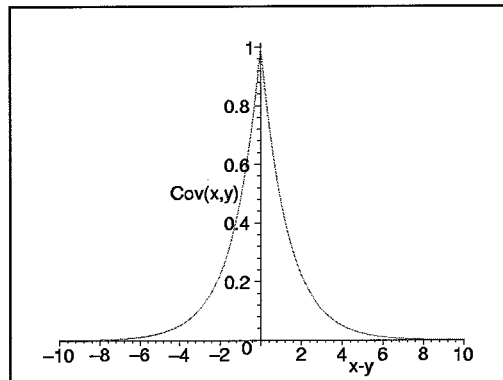


Figure 3: covariance function; $c=0.75$

In lemma 3.1.1 it is stated that $h = g - \mathbb{E}[g]$ is also Gaussian with the same covariance function as g . This covariance function satisfies the criteria mentioned in definition 3.1.4, which are: positive definite, symmetric and continuous, and the distribution of g is fixed by its first two moments (which is the case for Gaussian fields). This implies that h has a KL-expansion, as in equation (42), with convergence in distribution. Since h is a discrete random field, the integral eigenvalue problem in equation (43) can be turned into a "normal" eigenvalue problem

$$\begin{aligned} \int_{\{1,2,\dots,n_K\}} \text{Cov}[h(x), h(y)] f_j(x) dx &= \sum_{s=1}^{n_K} \text{Cov}[h(s), h(y)] f_j(s) \\ &= \sum_{s=1}^{n_K} C[s, y] f_j(s) = \sum_{s=1}^{n_K} C[y, s] f_j(s) = C[y, *] \underline{f}_j \quad (45) \end{aligned}$$

with the covariance matrix C given by $C[s, y] = \text{Cov}[h(s), h(y)]$ and $\underline{f}_j[s] = f_j(s)$. Hence, for every $y \in \{1, 2, \dots, n_K\}$, the following holds

$$\lambda_j \underline{f}_j(y) = C[y, *] \underline{f}_j$$

and thus

$$C \underline{f}_j = \lambda_j \underline{f}_j \quad (46)$$

3.2 Computing statistics using second order Taylor series

In this subsection the mean and standard deviation will be estimated using a second order Taylor series (about the mean of the random variable), see for example (Matthies *et al.*, 1997). Recall that the Taylor series of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ (about $x = h$) is given by

$$f(x) = f(h) + (x - h)Df(h) + \frac{1}{2}(x - h)^2 D^2 f(h) + O(|x - h|^3) \text{ with } D^k f(h) = \left. \frac{\partial^k f(x)}{\partial x^k} \right|_{x=h} \quad (47)$$

Let's consider equation (34), thus

$$B(\Omega)\underline{U}(\Omega) = \underline{F} \text{ with } B(\Omega) \equiv K + i\Omega C - \Omega^2 M \quad (48)$$

where the subscript L is omitted for simplicity. Let α be some uncertain physical or geometrical property of the substructure, which is modelled as a random variable $\alpha : \Theta \rightarrow \mathbb{R}$ on some probability space $(\Theta, \Sigma, \mathcal{P})$. In general the stiffness matrix, mass matrix, generalized force and thus the solution $\underline{U}(\Omega)$ depend on α (this will be denoted as $B(\alpha, \Omega)$, $\underline{F}(\alpha)$ and $\underline{U}(\alpha, \Omega)$, etc.). Regarding the damping matrix C , the easiest is to consider the scatter in C to be independent of the scatter in K and M (thus C is independent of α), although this is not necessary. For simplicity, let's assume for a moment that C is deterministic. Equation (48) becomes

$$B(\alpha, \Omega)\underline{U}(\alpha, \Omega) = \underline{F}(\alpha) \text{ with } B(\alpha, \Omega) \equiv K(\alpha) + i\Omega C - \Omega^2 M(\alpha) \quad (49)$$

The mappings which depend on α will be developed in a second order Taylor series (as in (47)) with $x = \alpha$ and $h = \mathbb{E}[\alpha]$. The left hand side and right hand side of (49) will become polynomials in $\alpha - \mathbb{E}[\alpha]$, and that implies that the coefficients of $(\alpha - \mathbb{E}[\alpha])^k$ have to be equal to zero. For the constant terms (with $k = 0$) we have

$$B(\mathbb{E}[\alpha], \Omega)\underline{U}(\mathbb{E}[\alpha], \Omega) = \underline{F}(\mathbb{E}[\alpha]) \quad (50)$$

For the linear terms (with $k = 1$) we have

$$B(\mathbb{E}[\alpha], \Omega)D\underline{U}(\mathbb{E}[\alpha], \Omega) + DB(\mathbb{E}[\alpha], \Omega)\underline{U}(\mathbb{E}[\alpha], \Omega) = D\underline{F}(\mathbb{E}[\alpha]) \quad (51)$$

For the quadratic terms (with $k = 2$) we have

$$B(\mathbb{E}[\alpha], \Omega)D^2\underline{U}(\mathbb{E}[\alpha], \Omega) + 2DB(\mathbb{E}[\alpha], \Omega)D\underline{U}(\mathbb{E}[\alpha], \Omega) + D^2B(\mathbb{E}[\alpha], \Omega)\underline{U}(\mathbb{E}[\alpha], \Omega) = D^2\underline{F}(\mathbb{E}[\alpha]) \quad (52)$$

Note that the solution of (50) is obtained by an analysis of the corresponding deterministic system. This is done by computing the frequency response function as in formula (37). Thus

$$\underline{U}(\mathbb{E}[\alpha], \Omega) = H(\mathbb{E}[\alpha], \Omega)\underline{F}(\mathbb{E}[\alpha]) \text{ with } H(\mathbb{E}[\alpha], \Omega) \equiv B^{-1}(\mathbb{E}[\alpha], \Omega) \quad (53)$$

After computing this frequency response function we can solve for $D\underline{U}(\mathbb{E}[\alpha], \Omega)$ and $D^2\underline{U}(\mathbb{E}[\alpha], \Omega)$ using the equations (51) and (52), they are given by:

$$D\underline{U}(\mathbb{E}[\alpha], \Omega) = H(\mathbb{E}[\alpha], \Omega)(D\underline{F}(\mathbb{E}[\alpha]) - DB(\mathbb{E}[\alpha], \Omega)\underline{U}(\mathbb{E}[\alpha], \Omega)) \quad (54)$$

and

$$D^2\underline{U}(\mathbb{E}[\alpha], \Omega) = H(\mathbb{E}[\alpha], \Omega)(D^2\underline{F}(\mathbb{E}[\alpha]) - 2DB(\mathbb{E}[\alpha], \Omega)D\underline{U}(\mathbb{E}[\alpha], \Omega) - D^2B(\mathbb{E}[\alpha], \Omega)\underline{U}(\mathbb{E}[\alpha], \Omega)) \quad (55)$$

The last three equations are substituted in the Taylor approximation of $\underline{U}(\mathbb{E}[\alpha], \Omega)$ (see equation (47)) to obtain the approximation:

$$\underline{U}(\alpha, \Omega) \approx \underline{U}(\mathbb{E}[\alpha], \Omega) + (\alpha - \mathbb{E}[\alpha])D\underline{U}(\mathbb{E}[\alpha], \Omega) + \frac{1}{2}(\alpha - \mathbb{E}[\alpha])^2 D^2\underline{U}(\mathbb{E}[\alpha], \Omega) \quad (56)$$

with

$$D^k \underline{U}(\mathbb{E}[\alpha], \Omega) = \left. \frac{\partial^k \underline{U}(\alpha, \Omega)}{\partial \alpha^k} \right|_{\alpha=\mathbb{E}[\alpha]}$$

Collecting all terms of second order and taking the mean we obtain:

$$\mathbb{E}[\underline{U}(\alpha)] = \underline{U}(\mathbb{E}[\alpha]) + \frac{1}{2} \text{Var}[\alpha] D^2 \underline{U}(\mathbb{E}[\alpha]) \quad (57)$$

and similar for the second moment:

$$\begin{aligned} \mathbb{E}[\underline{U}(\alpha)\underline{U}'(\alpha)] &\approx \underline{U}(\mathbb{E}[\alpha])\underline{U}'(\mathbb{E}[\alpha]) + \frac{1}{2} \text{Var}[\alpha] (\underline{U}(\mathbb{E}[\alpha])\cdot D^2 \underline{U}'(\mathbb{E}[\alpha]) + D^2 \underline{U}(\mathbb{E}[\alpha])\cdot \underline{U}'(\mathbb{E}[\alpha])) \\ &\quad + \text{Var}[\alpha] D\underline{U}(\mathbb{E}[\alpha])\cdot D\underline{U}'(\mathbb{E}[\alpha]) \end{aligned} \quad (58)$$

The covariance function is given by

$$\text{Cov}[\underline{U}(\alpha), \underline{U}'(\alpha)] = \mathbb{E}[\underline{U}(\alpha)\underline{U}'(\alpha)] - \mathbb{E}[\underline{U}(\alpha)]\mathbb{E}[\underline{U}'(\alpha)] \quad (59)$$

As can be seen in the derivation above, the following quantities needs to be computed: $H(\mathbb{E}[\alpha], \Omega)$, $B(\mathbb{E}[\alpha], \Omega)$, $DB(\mathbb{E}[\alpha], \Omega)$ and $D^2B(\mathbb{E}[\alpha], \Omega)$. $H(\mathbb{E}[\alpha], \Omega)$ follows from the deterministic analysis, which must be performed first. The other quantities can be determined directly (from equation (32) and (35)):

$$B(\alpha, \Omega) = K_L(\alpha) + i\Omega C_L - \Omega^2 M_L(\alpha)$$

$$DB(\alpha, \Omega) = L' \text{diag}(O, \dots, O, DB_Z^{(v)}(\alpha, \Omega), O, \dots, O)L$$

where the v -th substructure depends on α , that is $B_Z^{(v)}(\Omega) = B_Z^{(v)}(\alpha, \Omega)$ and similar

$$D^2B(\alpha, \Omega) = L' \text{diag}(O, \dots, O, D^2B_Z^{(v)}(\alpha, \Omega), O, \dots, O)L$$

The derivatives of $B_Z^{(v)}(\alpha, \Omega)$ can be derived in Ansys, using that

$$DB_Z^{(v)}(\mathbb{E}[\alpha], \Omega) \approx \frac{B_Z^{(v)}(\mathbb{E}[\alpha] + h, \Omega) - B_Z^{(v)}(\mathbb{E}[\alpha], \Omega)}{h}, \text{ with } h \text{ small} \quad (60)$$

This implies that two analyses must be performed in Ansys, one with the value $\mathbb{E}[\alpha] + h$ and one with $\mathbb{E}[\alpha]$. For the second derivative $D^2B_Z^{(v)}(\mathbb{E}[\alpha], \Omega)$ one needs three analyses, namely

$$B_Z^{(v)}(\mathbb{E}[\alpha], \Omega), \quad B_Z^{(v)}(\mathbb{E}[\alpha] + h, \Omega) \quad \text{and} \quad B_Z^{(v)}(\mathbb{E}[\alpha] + 2h, \Omega)$$

If one wants to model more uncertain properties (for example damping), similar formulas can be obtained (see also equation (78) of the appendix). A flowchart of the used equations is given in figure 4.

3.3 Local modal perturbation method: applied to Rubin

Mace and Shorter (Mace and Shorter, 2001) derived the local modal perturbation method using the CMS method of Craig and Bampton. In this subsection the local modal perturbation method using the method of Rubin will be derived, and it will be shown that under some weak approximation a similar result can be obtained as Mace and Shorter did.

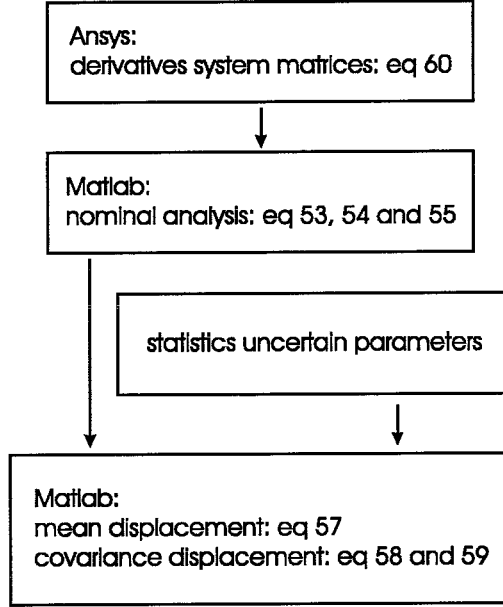


Figure 4: flowchart: Taylor series

Recall equation (34). The corresponding eigenvalue problem is given by

$$(K_L - \Omega_{global,j}^2 M_L) \underline{\phi}_{global,j} = \underline{0} \quad (61)$$

$\{\Omega_{global,j}^2\}_j$ are the global eigenvalues and $\{\underline{\phi}_{global,j}\}_j$ the global eigenmodes of the structure. Assuming that the system matrices depend on an uncertain parameter (modelled as a random variable), say $\alpha : \Theta \rightarrow \mathbb{R}$, equation (61) becomes

$$(K_L(\alpha) - \Omega_{global,j}^2(\alpha) M_L(\alpha)) \underline{\phi}_{global,j}(\alpha) = \underline{0} \quad (62)$$

Multiplying both sides of equation (62) with $\underline{\phi}'_{global,j}$ and differentiating with respect to α gives, see also (Plaut and Huseyin, 1972) or (Balmes, 1998):

$$D_\alpha \Omega_{global,j}^2(\alpha) = \underline{\phi}'_{global,j}(\alpha) (D_\alpha K_L(\alpha) - \Omega_{global,j}^2(\alpha) D_\alpha M_L(\alpha)) \underline{\phi}_{global,j}(\alpha) \quad (63)$$

with as in the previous subsection $D_\alpha = \frac{\partial}{\partial \alpha}$. Recall that $K_L(\alpha)$ is given by equation (35)

$$K_L(\alpha) = L' \text{diag}(K_Z^{(1)}, \dots, K_Z^{(v)}(\alpha), \dots, K_Z^{(n)}) L \quad (64)$$

where for all s it holds that $K_Z^{(s)} = (S_2')^{(s)} K_S^{(s)} S_2^{(s)}$ and $M_Z^{(s)} = (S_2')^{(s)} M_S^{(s)} S_2^{(s)}$ (see equation (27)). Note that $K_S^{(s)}$ is a diagonal matrix containing the local eigenvalues $\{(\Omega_{local,j}^{(s)})^2\}_j$ and $M_S^{(s)}$ is the identity matrix, corresponding to the eigenvalue problem of equation (11)

$$(K_S^{(s)} - (\Omega_{local,j}^{(s)})^2 M_S^{(s)}) \underline{\phi}_{local,j}^{(s)} = \underline{0} \quad (65)$$

It is assumed that the v -th substructure matrices depend on α , and the others don't. Similar to what Mace and Shorter did, it is assumed that $S_2^{(v)}$ is independent of α (of course this is just an approximation, and it remains to be seen how accurate it is). Differentiating K_L with respect to α and using that $S_2^{(v)}$ is independent of α , gives

$$\begin{aligned} D_\alpha K_L(\alpha) &= L' \text{diag}(O, \dots, O, (S_2')^{(s)} D_\alpha K_S^{(v)}(\alpha) S_2^{(v)}, O, \dots, O) L \\ &= L'[* , v] (S_2')^{(v)} D_\alpha K_S^{(v)}(\alpha) S_2^{(v)} L[j , *] = (S_2^{(v)} L[* , v])' D_\alpha K_S^{(v)}(\alpha) (S_2^{(v)} L[* , v]) \end{aligned} \quad (66)$$

with $L'[* , v]$ the v -th block column of L' . Since $K_L(\alpha)$ is block-diagonal, it follows that

$$\begin{aligned}
& \underline{\phi}'_{global,j}(\alpha) D_\alpha K_L(\alpha) \underline{\phi}_{global,j}(\alpha) = \underline{\phi}'_{global,j}(\alpha) (S_2^{(v)} L[* , v])' D_\alpha K_S^{(v)}(\alpha) (S_2^{(v)} L[* , v]) \underline{\phi}_{global,j}(\alpha) \\
& = (S_2^{(v)} L[* , v] \underline{\phi}_{global,j}(\alpha))' D_\alpha K_S^{(v)}(\alpha) (S_2^{(v)} L[* , v] \underline{\phi}_{global,j}(\alpha)) \\
& = (S_2^{(v)} L[* , v] \underline{\phi}_{global,j}(\alpha))' \begin{bmatrix} D_\alpha(\Omega_{local,1}^2)^{(v)}(\alpha) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_\alpha(\Omega_{local,n^{(v)}}^2)^{(v)}(\alpha) \end{bmatrix} (S_2^{(v)} L[* , v] \underline{\phi}_{global,j}(\alpha)) \\
& \equiv \underline{w}'_j(\alpha) \begin{bmatrix} D_\alpha(\Omega_{local,1}^2)^{(v)}(\alpha) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_\alpha(\Omega_{local,n^{(v)}}^2)^{(v)}(\alpha) \end{bmatrix} \underline{w}_j(\alpha) = \sum_{t=1}^{n^{(v)}} \underline{w}_j(\alpha)[t]^2 (\Omega_{local,t}^2)^{(v)}(\alpha)
\end{aligned} \tag{67}$$

with

$$\underline{w}_j(\alpha) \equiv S_2^{(v)} L[* , v] \underline{\phi}_{global,j}(\alpha)$$

Since $M_S^{(v)}$ is the identity matrix (and thus independent of α), it follows that

$$D_\alpha M_L(\alpha) = O$$

which implies

$$\underline{\phi}'_{global,j}(\alpha) D_\alpha M_L(\alpha) \underline{\phi}_{global,j}(\alpha) = O \tag{68}$$

Substituting equations in (67) and (68) in equation (63) results in

$$D_\alpha \Omega_{global,j}^2(\alpha) = \sum_{t=1}^{n^{(v)}} \underline{w}_j(\alpha)[t]^2 D_\alpha(\Omega_{local,t}^2)^{(v)}(\alpha) \tag{69}$$

Since $D_\alpha f(\alpha) \approx \frac{f(\alpha+h) - f(\alpha)}{\alpha+h-\alpha} \equiv \frac{\delta f(\alpha;h)}{\delta h}$ for small h , it follows that (using the notation of Mace and Shorter, and after multiplying equation (69) with δh)

$$\delta \Omega_{global,j}^2(\alpha; h) = \sum_{t=1}^{n^{(v)}} \underline{w}_j(\alpha)[t]^2 \delta(\Omega_{local,t}^2)^{(v)}(\alpha; h) \tag{70}$$

This equation (70) gives the perturbation in the global eigenvalues, as a result of a perturbation in the local eigenvalues. In order to determine the perturbation in the global eigenmodes, one starts with the following formula, as derived in (Plaut and Huseyin, 1972) or (Balmes, 1998)

$$\begin{aligned}
D_\alpha \underline{\phi}'_{global,j}(\alpha) &= -\frac{1}{2} (\underline{\phi}'_{global,j}(\alpha) D_\alpha M_L(\alpha) \underline{\phi}_{global,j}(\alpha)) \underline{\phi}_{global,k}(\alpha) \\
&+ \sum_{k=1, k \neq j}^n \frac{\underline{\phi}'_{global,k}(\alpha) (D_\alpha K_L(\alpha) - \Omega_{global,j}^2(\alpha) D_\alpha M_L(\alpha)) \underline{\phi}_{global,j}(\alpha)}{\Omega_{global,j}^2(\alpha) - \Omega_{global,k}^2(\alpha)} \underline{\phi}_{global,k}(\alpha)
\end{aligned} \tag{71}$$

Substituting equations (67) and (68) in (71) we obtain

$$D_\alpha \underline{\phi}'_{global,j}(\alpha) = \sum_{k=1, k \neq j}^n \sum_{t=1}^{n^{(v)}} \frac{\underline{w}_k(\alpha)[t] \underline{w}_j(\alpha)[t]}{\Omega_{global,j}^2(\alpha) - \Omega_{global,k}^2(\alpha)} \underline{\phi}_{global,k}(\alpha) D_\alpha(\Omega_{local,t}^2)^{(v)}(\alpha) \tag{72}$$

Using the notation of Mace and Shorter

$$\delta \underline{\phi}'_{global,j}(\alpha; h) = \sum_{k=1, k \neq j}^n \sum_{t=1}^{n^{(v)}} \frac{\underline{w}_k(\alpha)[t] \underline{w}_j(\alpha)[t]}{\Omega_{global,j}^2(\alpha) - \Omega_{global,k}^2(\alpha)} \underline{\phi}_{global,k}(\alpha) \delta(\Omega_{local,t}^2)^{(v)}(\alpha; h) \tag{73}$$

The general idea of Mace and Shorter is to determine the nominal system first, that is $\Omega_{global,j}^2(\mathbb{E}[\alpha])$, $\phi_{global,j}(\mathbb{E}[\alpha])$, and then to perturb the global eigenvalues/eigenmodes which are generated by the perturbations of the local eigenvalues. These perturbations are given in the equations (70) and (73). The perturbations in the local eigenvalues can be obtained in the following way:

Since $x \mapsto \Omega_{local,t}^2(x)$ is continuous (on some bounded interval $\subset \mathbb{R}^+$), it follows that $\Omega_{local,t}^2(\alpha)$ is also a random variable (defined on the same probability space as α), see appendix (A.1.3). From this follows that $\Omega_{local}^2(\alpha) : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ is a discrete random field with $\mathcal{D} = \{1, \dots, n_K\}$ and

$$\Omega_{local}^2(\alpha)(t, \theta) := \Omega_{local,t}^2(\alpha(\theta)), \quad t \in \mathcal{D}, \quad \theta \in \Theta$$

The dependency of the local eigenvalues on α will be dropped, and it will be assumed that the local eigenvalues form a Gaussian field with a certain mean and covariance matrix. The mean is determined in the nominal analysis and the covariance matrix given in equation (44) is used (where c can be chosen freely):

$$C[s, y] = \exp(-c|s - y|), \quad (s, y) \in \{1, \dots, n_K\} \times \{1, \dots, n_K\} \quad (74)$$

Finally note that, assuming that $h \equiv \alpha - \mathbb{E}[\alpha]$ is small,

$$\delta(\Omega_{local,t}^2)^{(v)}(\alpha; h) = \delta h \cdot D(\Omega_{local,t}^2)^{(v)}(\alpha) = (\alpha - \mathbb{E}[\alpha]) \frac{(\Omega_{local,t}^2)^{(v)}(\alpha) - (\Omega_{local,t}^2)^{(v)}(\mathbb{E}[\alpha])}{\mathbb{E}[\alpha]} \quad (75)$$

where $(\Omega_{local,t}^2)^{(v)}(\alpha)$ is obtained by its truncated KL-expansion:

$$\Omega_{local}^2(t, \theta) \approx \mathbb{E}[\Omega_{local}^2(t)] + \sum_{j=0}^N \sqrt{\lambda_j} \xi_j(\theta) f_j(t), \quad t \in \{1, \dots, n_K\}, \quad \theta \in \Theta = \mathbb{R} \quad (76)$$

Using the eigenfunctions f_j and eigenvalues λ_j from the covariance matrix given in equation (46). The independent Gaussian variables ξ_j can be generated in Matlab.

In appendix (A.1.2) is described how to obtain the correct probability space on which the sum in the righthand side of equation (76) is used to generate the perturbations in the local eigenvalues. Once these are obtained, equations (70) and (73) the perturbations in the global eigenvalues and eigenmodes (note that the corresponding perturbations in the local eigenmodes are not needed here). Those perturbations will be used to generate perturbations of the frequency response function, from which an estimation of its mean and standard deviation can be obtained. A flowchart of the used equations is given in figure 5.

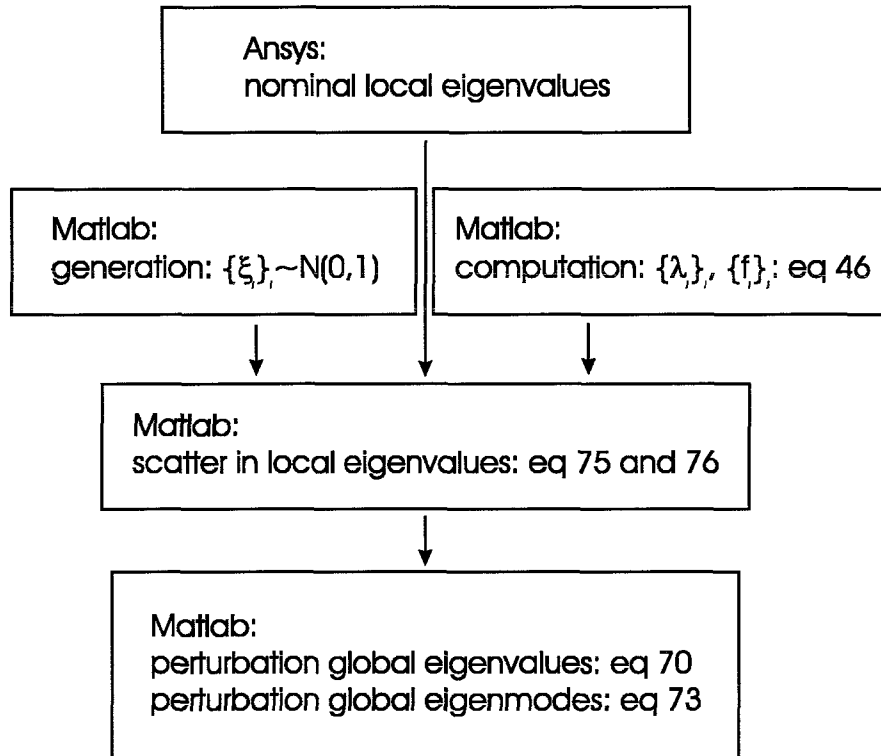


Figure 5: flowchart: local modal perturbation method

4 Conclusions

In this report, instruments are provided for robust optimization, which are

- Computationally efficient.
- Generally applicable. This implies that the instruments can be easily combined with the use of a standard FEM package like Ansys. It also implies, that the estimation of the relevant statistics is not limited to that of the displacement field, but can also be used for determining the scatter in the acoustic power or in the energy levels, or other quantities.

In order to accomplish this, the Component Mode Synthesis method of Rubin was used and discussed in this report. This CMS was combined with two perturbation methods: second order Taylor statistics and Local Modal Perturbation method. The last combination is new. When using these methods, it doesn't matter which quantities must be optimized. Once one obtained the perturbations in the displacement field, one can "easily" obtain the corresponding perturbations in the above mentioned quantities, like acoustic power and energy functions. Several conclusions are expected about the two perturbation methods, but some of them have to be tested after the implementation of the methods:

1. Second order Taylor statistics:

advantages:

- General applicable.
- The needed derivatives (of the stiffness or mass matrix) can be obtained from Ansys.
- Computationally efficient.

disadvantages:

- Requires small standard deviation (say 3 percent of the mean) of the physical/geometrical parameters.

2. Local Modal Perturbation method:
advantages:

- General applicable.
- Computationally efficient.

disadvantages:

- Requires small standard deviation (say 3 percent of the mean) of the local eigenvalues.
- It is assumed that the transformation matrix $S_2^{(v)}$ (see equation (24), and the remark above equation (66)) is deterministic. It remains to be checked how accurate this assumption is.
- The generation of the local eigenvalues is not trivial. The main difficulty is that they are correlated. The correlation is not exactly known in practice. This is solved by assuming a suitable correlation of the local eigenvalues, in which a correlation length is introduced. This length can be changed, if one wants to assume a stronger or weaker correlation.

Several hypothesis can be formulated. They will be tested in a later stage using suitable numerical experiments.

- If the uncertainty in the physical/geometrical parameters is known (and you have few of them), then the first method should be used, since this will be more accurate.
- If many parameters are uncertain, then the second method should be used. The first method requires many analyses from Ansys, basically three analyses for every uncertain parameter *times* the number of uncertain parameters. In the case, an analysis means getting the system matrices for a certain parameter set.
- If the dependency of the local eigenvalues on the physical or geometrical parameters is not known, obviously the second method must be used.

Implementation:

The next step is to implement the perturbation methods in Matlab, and compare them with a Monte Carlo Simulation. Differences in accuracy, computational efficiency and limitations will be considered. The analysis in (Sarkar and Ghanem, 2003) might be used for a comparison, but also a list will be made with other suitable structure considered in the literature.

A Appendix: Some theory of random fields

A.1 random variable, random field

The definition of a probability space is as follows

Definition A.1.1. probability space and σ -algebra:

A triple $(\Theta, \Sigma, \mathcal{P})$ is called a probability space with Θ a sample space (also called the space of outcomes), Σ the σ -algebra of events on Θ (also called the event space) and $\mathcal{P} : \Sigma \rightarrow [0, 1]$ the probability measure. Recall that a σ -algebra is defined as follows:

A collection Σ of subsets of a set Θ is called a σ -algebra (in Θ) if:

1. $\Theta \in \Sigma$
2. $A \in \Sigma \Rightarrow A^c \in \Sigma$ with $A^c = \Theta \setminus A$
3. with a series $(A_j) \in \Sigma \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \Sigma$

□

The most trivial example of a σ -algebra is the powerset $P(\Theta)$ of some set Θ , that is, $P(\Theta)$ consists of all possible subsets of Θ . By verifying the axioms above, it can be seen that $P(\Theta)$ is indeed a σ -algebra on Θ . The definition of a random variable is as follows

Definition A.1.2. random variable:

Let $(\Theta, \Sigma, \mathcal{P})$ be a probability space. A function $X : \Theta \rightarrow \mathbb{R}$ on this probability space is called a real-valued random variable if X is a $\Sigma - \mathcal{B}$ -measurable function, which just means that $\{X \in B\} \in \Sigma$ for every $B \subseteq \mathbb{R}$. Note that $\{X \in B\}$ is an abbreviation of $\{X(s) | s \in \Theta\}$. □

The basic idea is that one can assign probabilities to every set $\{X \in B\} \in \Sigma$. A random field is just a collection of random variables, formally

Definition A.1.3. random field:

A function $g : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ with $\mathcal{D} \subseteq \mathbb{R}^n$ and $n \geq 1$ is called a real-valued random field if $g(\underline{r}, \cdot) : \Theta \rightarrow \mathbb{R}$ is a random variable for every $\underline{r} \in \mathcal{D}$. The field is called discrete if \mathcal{D} is countable, and continuous if it is an interval in \mathbb{R} . □

The following theorem described how one can find a σ -algebra such that a mapping $X : \Theta \rightarrow \mathbb{R}$ can be turned into a random variable.

Theorem A.1.1. random variable and σ -algebra:

Suppose we have a mapping $X : \Theta \rightarrow \mathbb{R}$ then we can always find a σ -algebra Σ such that X becomes a $\Sigma - \mathcal{B}$ -measurable mapping. The σ -algebra $\Sigma = X^{-1}(\mathcal{B})$ does the trick.

Proof. First we will show that $X^{-1}(\mathcal{B})$ is actually a σ -algebra:

1. $\Theta \in X^{-1}(\mathcal{B})$ since $X(\Theta) \subseteq \mathbb{R}$ and $\mathbb{R} \in \mathcal{B}$
2. Suppose $A \in X^{-1}(\mathcal{B})$ then it follows that $A^c \in X^{-1}(\mathcal{B})$ since $(X^{-1}(\mathcal{B}))^c = X^{-1}(\mathcal{B}^c)$ for all $B \in \mathcal{B}$ with $B^c = \mathbb{R} \setminus B$
3. with a series $(A_j) \in X^{-1}(\mathcal{B}) \Rightarrow \bigcup_{j=1}^{\infty} A_j \in X^{-1}(\mathcal{B})$ since $\bigcup_j X^{-1}(B_j) = X^{-1}(\bigcup_j B_j)$ for all $\{B_j\}_j \in \mathcal{B}$

It follows that X becomes a $X^{-1}(\mathcal{B}) - \mathcal{B}$ -measurable mapping since $X^{-1}(B) \in X^{-1}(\mathcal{B})$ for all $B \in \mathcal{B}$. □

In subsection 3.3 a random field is generated by using the Karhunen-Loeve expansion. As can be seen in equation (76), it is needed to generate $N + 1$ independent (Gaussian distributed) random variables $\{\xi_j\}_{j=1}^N$ on some probability space $(\Theta, \Sigma, \mathcal{P})$. The appropriate σ -algebra Σ is given in the following theorem

Theorem A.1.2. the σ -algebra $\Sigma(H)$

Let ξ_1, \dots, ξ_n be a basis of H (a Gaussian Hilbert space) with mappings $\xi_j : \Theta \rightarrow \mathbb{R}$. We are interested in the smallest σ -algebra \mathcal{U} such that all the mappings $\xi_j : \Theta \rightarrow \mathbb{R}$ are \mathcal{U} - \mathcal{B} -measurable mappings. Let $V \equiv \cup_{j=1}^n \xi_j^{-1}(\mathcal{B})$. The set V generates a σ -algebra denoted as $\mathcal{U}(V)$, it's by definition the smallest σ -algebra with $V \subseteq \mathcal{U}(V)$. This σ -algebra is denoted as $\Sigma(H)$ or as $\Sigma(\xi_1, \dots, \xi_n)$ and it's the smallest σ -algebra such that $\xi_j : \Theta \rightarrow \mathbb{R}$ are $\Sigma(H)$ - \mathcal{B} -measurable mappings.

Proof. A mapping $\xi_k : \Theta \rightarrow \mathbb{R}$ is $\Sigma(H)$ - \mathcal{B} -measurable if

$$\xi_k^{-1}(A) \in \Sigma(H) \text{ for all } A \in \mathcal{B}$$

Let $A \in \mathcal{B}$ then it follows that

$$\xi_k^{-1}(A) \in \xi_k^{-1}(\mathcal{B})$$

or

$$\xi_k^{-1}(A) \in V$$

Since $V \subseteq \Sigma(H)$ it follows that

$$\xi_k^{-1}(A) \in \Sigma(H)$$

This is true for all $k = 1, \dots, n$. □

Finally, a continuous functions of a random variable is a random variable

Theorem A.1.3. continuous functions of a random variable:

Let $X : \Theta \rightarrow \mathbb{R}$ be a random variable defined on the probability space $(\Theta, \Sigma, \mathcal{P})$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then it follows that $f(X) : \Theta \rightarrow \mathbb{R}$ given by

$$f(X)(\theta) := f(X(\theta))$$

is a random variable, defined on the same probability space as X .

Proof. This follows from the fact that a continuous functions is measurable, and the composition of two measurable functions is measurable. □

A.2 Multi-dimensional Taylor series

Recall that the (multi-dimensional) Taylor series of a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ (about $\underline{x} = \underline{h}$) is given by

$$f(\underline{x}) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\sum_{k=1}^m (x_k - h_k) D_k \right)^j f(\underline{h}) \text{ with } D_k^j f(\underline{h}) = \frac{\partial^j f(\underline{x})}{\partial x_k^j} \Big|_{\underline{x}=\underline{h}} \quad (77)$$

where $\underline{x} = (x_1, \dots, x_m)$ and $\underline{h} = (h_1, \dots, h_m)$. For example, with $m = 2$, $x_1 = \alpha_1$, $x_2 = \alpha_2$, $h_1 = \mathbb{E}[\alpha_1]$ and $h_2 = \mathbb{E}[\alpha_2]$, the equivalent of equation (56) becomes

$$\begin{aligned} \underline{U}(\underline{\alpha}) &\approx \underline{U}(\mathbb{E}[\underline{\alpha}]) + (\alpha_1 - \mathbb{E}[\alpha_1]) D_1 \underline{U}(\mathbb{E}[\underline{\alpha}]) + (\alpha_2 - \mathbb{E}[\alpha_2]) D_2 \underline{U}(\mathbb{E}[\underline{\alpha}]) \\ &+ \frac{1}{2} (\alpha_1 - \mathbb{E}[\alpha_1])^2 D_1^2 \underline{U}(\mathbb{E}[\underline{\alpha}]) + (\alpha_1 - \mathbb{E}[\alpha_1]) (\alpha_2 - \mathbb{E}[\alpha_2]) D_1 D_2 \underline{U}(\mathbb{E}[\underline{\alpha}]) + \frac{1}{2} (\alpha_2 - \mathbb{E}[\alpha_2])^2 D_2^2 \underline{U}(\mathbb{E}[\underline{\alpha}]) \end{aligned} \quad (78)$$

Taking the mean of $\underline{U}(\underline{\alpha})$ in equation (78) results in

$$\mathbb{E}[\underline{U}(\underline{\alpha})] \approx \underline{U}(\mathbb{E}[\underline{\alpha}]) + \frac{1}{2} \text{Var}[\alpha_1] D_1^2 \underline{U}(\mathbb{E}[\underline{\alpha}]) + \frac{1}{2} \text{Var}[\alpha_2] D_2^2 \underline{U}(\mathbb{E}[\underline{\alpha}]) \quad (79)$$

Taking the covariance of $\underline{U}(\underline{\alpha})$ in equation (78) with its transpose is given by

$$\begin{aligned} \mathbb{E}[\underline{U}(\underline{\alpha})\underline{U}'(\underline{\alpha})] &\approx \underline{U}(\mathbb{E}[\underline{\alpha}])\underline{U}'(\mathbb{E}[\underline{\alpha}]) + \frac{1}{2} \text{Var}[\alpha_1] \underline{U}(\mathbb{E}[\underline{\alpha}]) \cdot D_1^2 \underline{U}'(\mathbb{E}[\underline{\alpha}]) + \frac{1}{2} \text{Var}[\alpha_2] \underline{U}(\mathbb{E}[\underline{\alpha}]) \cdot D_2^2 \underline{U}'(\mathbb{E}[\underline{\alpha}]) \\ &+ \frac{1}{2} \text{Var}[\alpha_1] D_1^2 \underline{U}(\mathbb{E}[\underline{\alpha}]) \cdot \underline{U}'(\mathbb{E}[\underline{\alpha}]) + \frac{1}{2} \text{Var}[\alpha_2] D_2^2 \underline{U}(\mathbb{E}[\underline{\alpha}]) \cdot \underline{U}'(\mathbb{E}[\underline{\alpha}]) \\ &+ \text{Var}[\alpha_1] \text{Var}[\alpha_2] D_1 D_2 \underline{U}(\mathbb{E}[\underline{\alpha}]) D_1 D_2 \underline{U}'(\mathbb{E}[\underline{\alpha}]) \quad (80) \end{aligned}$$

Note that equation (79) and (80) are valid if α_1 and α_2 are independent.

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