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Deformations of operators

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Deformations of Operators

by

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Abstract

In this paper we consider versal families of operators. The theory of versal families of various kinds of objects and applications to the corresponding fields have been studied extensively by V.I. Arnold in [ARN II].

A more specialized paper of V.I. Arnold, which inspired us to study families of operators, is [ARN I]. This paper deals with families of matrices. In chapter I we shall give a short description of the theory in [ARN I].

Chapter III deals with deformations of Hilbert-Schmidt operators and in this chapter we shall prove a generalization of one of the theorems in [ARN I]. As a preparation we investigate some properties of operators on Hilbert space in chapter II.

AMS Subject Classification: 47A55, 47B10.

B. Basic Notions and Notations

In this paper some fundamental theorems on Functional Analysis and Differential Geometry are used. Most of the basic concepts and theorems on Functional Analysis, used in this paper, can be found in every textbook on this subject. For example in [DUN I] and [DUN II] or in [HIL]. Some of them, which are more specific and deal with operators on Hilbert space, can be found as exercises in [HAL I].

The basic concepts of Differential Geometry such as Differential Calculus and the theory of Manifolds can be found essentially in [LAN] and [ABR]. The theory of finite dimensional Manifolds is essentially in [GOL].

To cause no ambiguity we want to give here some definitions and notations of rather fundamental concepts, which are defined and notated in many (slightly) different ways.

Functional Analysis

Throughout this paper $\mathcal{H}$ will denote a separable infinite dimensional Hilbert space. The letters E and F will denote Banach spaces. A subset V of E is a subspace of E iff V is a linear space and V is closed in E. Let V be a subspace of E, V splits in E iff V has a closed complement $V' \subset E$, i.e. a subspace V' such that $E = V \oplus V'$.

$\mathcal{L}(E \to F)$ denotes the Banach space of bounded linear operators from E into F. The norm on $\mathcal{L}(E \to F)$ is given by

$$\|A\| := \sup_{x \in E, \|x\|=1} \|Ax\| .$$

Furthermore, if $A \in \mathcal{L}(E \to F)$, Ker(A) is the subspace $\overset{\leftarrow}{A}(0) \subset E$. The linear manifold $\mathrm{Ran}(A) \subset F$ is the set $\{Ax \mid x \in E\}$.

$\mathcal{L}(E)$ denotes the Banach space of bounded linear operators of E into itself. If $A \in \mathcal{L}(E)$, $\sigma(A)$ denotes the spectrum of A. If $\lambda \in \mathbb{C}\backslash\sigma(A)$ then $R(\lambda,A) :=$ $:= (\lambda I - A)^{-1}$ is the resolvent of A. (I stands for the identity operator.) The theory of spectral sets and operator functions, such as projections defined with contour integrals, can be found in [DUN I], Ch. VII.

If $A \in \mathcal{L}(\mathcal{H})$ then A^* will denote the adjoint of A.

In this paper we use two different topologies on $\mathcal{L}(\mathcal{H})$, the uniform operator topology induced by the norm and sometimes the strong operator topology (see [HAL I], Ch. 11 and Ch. 12).

Differential Geometry

In this paper the derivative of a map should be thought of as a linear operator.

B1. Definition (differentiable mapping). If f is a continuous map from an open subset $U \subset E$ into F and $x \in U$ then f is differentiable at x iff there is a bounded linear map $D_x f \in \mathcal{L}(E \to F)$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - (D_x f)h\|}{\|h\|} = 0 .$$

The linear map $D_x f$ is necessarily unique. f is of class C^1 in U (notation $f \in C^1(U \to F)$) iff f is differentiable at each point of U and the map $x \to D_x f$ is continuous from U into $\mathcal{L}(E \to F)$ (norm topology) (see also [LAN], Ch. I, § 3 and [ABR], Ch. I, § 1).

B2. Definition (submanifold). Suppose M is a C^r-manifold. A subset $N \subset M$ is a C^r-submanifold iff at every point $x \in N$ there is an admissible chart (i.e. compatible with the atlas of M) (U_x, φ) such that $\varphi(U_x) = V_1 \times V_2$, where V_1 and V_2 are open neighbourhoods of the origins in the Banach spaces F_1 respectively F_2, such that $\varphi(x) = (0,0)$ and $\varphi(U_x \cap N) = V_1 \times \{0\}$ (see also [LAN] Ch. II, § 2 and [ABR], Ch. IV, § 17).

B3. Remark. Note that for $x \in N$ the tangent space $T_x N$ to N at x is a splitting subspace of the tangent space $T_x M$ to M at x (see [ABR], Ch. II, § 17, p. 45).

B4. Definition (double splitting map). Suppose f is a map from the (C^p, $p \geq 1$) manifold M_1, into the (C^q, $q \geq 1$) manifold M_2, differentiable at $x \in M_1$. Then f is called double splitting at $x \in M_1$ iff

B4.1. $\mathrm{Ker}(D_x f)$ splits in $T_x M_1$.

B4.2. $\mathrm{Ran}(D_x f)$ is closed and splits in $T_{f(x)} M_2$.

B5. Definition (transversality of a map and a submanifold). Let N be a (C^p, $p \geq 1$) submanifold of the manifold M. Suppose f is a map from the (C^q, $q \geq 1$) manifold Λ into M, differentiable at $\lambda \in \Lambda$. f is called transversal to N at λ iff

B5.1. $f(\lambda) \in N$.

B5.2. f is double splitting at λ.

B5.3. $\mathrm{Ran}(D_\lambda f)$ contains a closed complement of $T_{f(\lambda)}N$ in $T_{f(\lambda)}M$.

f is minimal transversal to N at λ iff f is transversal and

$$T_{f(\lambda)}M = T_{f(\lambda)}N \oplus (D_\lambda f)T_\lambda \Lambda \ .$$

B6. Definition (transversality of two submanifolds). Suppose N_1 and N_2 are both submanifolds of the C^p-manifold M. N_1 is transversal to N_2 at x iff

B6.1. $x \in N_1 \cap N_2$.

B6.2. $T_xM = T_xN_1 + T_xN_2$.

N_1 is minimal transversal to N_2 at x if the sum in B6.2 is a direct sum ($T_xN_1 \cap T_xN_2 = \{0\}$).

B7. Remark. In many books transversality is defined as follows: f is transversal to the submanifold N at $\lambda \in \Lambda$ iff $f(\lambda) \notin N$ or B5.1, B5.2, B5.3.

I. The theory of Arnold. Deformations of Matrices

§ 0. Introduction

In this chapter we shall give a short description of the theory of versal deformations of matrices given by Arnold in [ARN I]. This description covers the sections § 2, § 3 and § 4. We give an additional result in § 5.

§ 1. Holomorphic mappings in $\mathbb{C}^{n\times n}$. Instability of the Jordan normal form

In this section we consider holomorphic mappings from an open subset of $\mathbb{C}^p$ into the matrixalgebra $\mathbb{C}^{n\times n}$ ($\mathbb{C}^{n\times n}$ is the algebra consisting of all $n \times n$ complex matrices, provided with the usual operations).

Holomorphy is defined in the usual way, that is

1.1. Definition. A map $A: U \to \mathbb{C}^{n\times n}$, where U is open in $\mathbb{C}^p$, is holomorphic in U iff each $\lambda_0 \in U$ has a neighbourhood where $A(\lambda)$ can be developed in a power series

$$A(\lambda) = \sum_{|\alpha|=0}^{\infty} A_\alpha(\lambda - \lambda_0)^\alpha$$

convergent in some matrix-norm; here α is the multi-index $(\alpha_1,\alpha_2,\ldots,\alpha_p)$; $|\alpha| := \alpha_1 +\ldots+ \alpha_p$; $A_\alpha \in \mathbb{C}^{n\times n}$ and $(\lambda - \lambda_0)^\alpha := (\lambda_1 - \lambda_{01})^{\alpha_1}\ldots(\lambda_p - \lambda_{0p})^{\alpha_p}$.

The same definition is used if $\mathbb{C}^{n\times n}$ is replaced by any Banach space (then the sum must be convergent in the Banach space norm).

1.2. Remark. It is well-known that A is holomorphic iff for every bounded linear functional L on the Banach space, the mapping $\lambda \to L(A(\lambda))$ is holomorphic from $\mathbb{C}^p$ into $\mathbb{C}$ (see [HIL], Ch. III, § 2). In our case ($\mathbb{C}^{n\times n}$) this implies that all entries of $A(\lambda)$, $a_{ij}(\lambda)$, are holomorphic functions of λ. The converse is also true of course.

1.3. Instability of the Jordan normal from

Suppose $A(\lambda)$ is a holomorphic map (which will also be called a family) from $\mathbb{C}^p$ into $\mathbb{C}^{n\times n}$. If $A(\lambda)$ is reduced to its Jordan normal form $J(\lambda)$ (see [GAN], Ch. VII, § 7), then in general, $J(\lambda)$ is not a holomorphic function of λ; $J(\lambda)$ sometimes depends even discontinuously on λ.

1.3.1. Example. Define

$$A(\lambda) = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}; \ \lambda \in \mathbb{C} .$$

The Jordan normal form of $A(\lambda)$ is given by

$$J(\lambda) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ if } \lambda \neq 0$$

and

$$J(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } \lambda = 0 .$$

In this example the smoothness of the family is lost by reducing the family to its Jordan normal form. So, if a matrix is only known approximately it is unwise to reduce it to the Jordan normal form. In studying smooth families, we are therefore interested in normal forms to which a family can be reduced without losing the smoothness.

§ 2. Deformations of matrices

2.1. Definition. A deformation of a matrix A_0 is a map $A: \Lambda \to \mathbb{C}^{n\times n}$ with $A(0) = A_0$ and holomorphic in an open set containing the origin of Λ. The space Λ ($= \mathbb{C}^p$ for some $p \in \mathbb{N}$) is called the base of the deformation (or the base of the family).

2.2. Definition. Two deformations of A_0, $A(\lambda)$ and $B(\lambda)$, are said to be similar, if there is a deformation $C(\lambda)$ of the identity matrix such that

$$A(\lambda) = C(\lambda)B(\lambda)C^{-1}(\lambda)$$

for λ in some open set in Λ containing 0 ($C^{-1}(\lambda)$ means $(C(\lambda))^{-1}$).
In other words: the germ of $A(\lambda)$ at $\lambda = 0$ is the germ of $C(\lambda)B(\lambda)C^{-1}(\lambda)$ at $\lambda = 0$.

2.3. Definition. If $A(\lambda)$ is a deformation of A_0, depending on k complex parameters, and $\varphi: \mathbb{C}^\ell \to \mathbb{C}^k$ is a map which is holomorphic in a neighbourhood of $0 \in \mathbb{C}^\ell$ and satisfies $\varphi(0) = 0$, then we call $A(\varphi(\mu))$ the deformation of A_0 induced by A under φ; clearly $A(\varphi(\mu))$ depends on ℓ parameters.

2.4. Definition. A deformation $A(\lambda)$ of A_0 is called versal iff every deformation $B(\mu)$ of A_0 is similar to a deformation induced by A under a suitable change of the parameters i.e. if there exist $\ell \in \mathbb{N}$, a function $\varphi: \mathbb{C}^\ell \to \mathbb{C}^k$ with $\varphi(0) = 0$ and a deformation $C(\mu)$ of the identity matrix such that

$$B(\mu) = C(\mu)A(\varphi(\mu))C^{-1}(\mu).$$

2.5. Example.

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \text{ is a versal deformation of } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which depends on 4 complex parameters.

2.6. Example.

$$\begin{bmatrix} 1+\lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ is a 2-parameter versal deformation of } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

2.7. Definition. Let $N \subset \mathbb{C}^{n\times n}$ be a complex analytic submanifold of the complex analytic manifold $M := \mathbb{C}^{n\times n}$ (analytic manifold means that the charts are open subsets of $\mathbb{C}^{n\times n}$ and the chart-functions are holomorphic in the sense of definition 1.1). Let $A: \Lambda \to M$ be holomorphic in a neighbourhood of $\lambda \in \Lambda$. Then the map A is said to be transversal to N at $\lambda \in \Lambda$ iff

2.7.1. $\quad T_{A(\lambda)}M = (D_\lambda A)T_\lambda\Lambda + T_{A(\lambda)}N$.

$T_{A(\lambda)}M$ is the tangent space of M at $A(\lambda)$, $T_\lambda\Lambda$ is the tangent space of Λ at λ and $T_{A(\lambda)}N$ is the tangent space of N at $A(\lambda)$ which is a subspace of $T_{A(\lambda)}M$.

2.8. Remark. The reader should compare definition 2.7 with definition B5 and note that since Λ and M are both finite dimensional, $D_\lambda A$ is automatically double splitting at any point $\lambda \in \Lambda$. The sum in 2.7.1 is not necessarily direct. It is possible, although it is not very interesting, that $\dim T_{A(\lambda)}M = \dim T_{A(\lambda)}N = \dim(D_\lambda A)T_\lambda\Lambda = n^2$.

§ 3. The orbit and the centralizer

Consider the space of all n × n matrices $\mathbb{C}^{n\times n}$ and the (Lie) group G of all non-singular matrices. G is an open set in $\mathbb{C}^{n\times n}$ containing the identity matrix e. It is well known that G is a connected analytic submanifold of $\mathbb{C}^{n\times n}$.

3.1. Remark. Note that the group of non-singular matrices in $\mathbb{R}^{n\times n}$ is not connected.

3.2. Definition. If A_0 is fixed in $\mathbb{C}^{n\times n}$ we define the map $\alpha_{A_0} : G \to \mathbb{C}^{n\times n}$ by

$$\alpha_{A_0}(g) := gA_0g^{-1}; \; g \in G .$$

$\alpha_{A_0}(G)$ is an analytic submanifold of $\mathbb{C}^{n\times n}$ and it is called the orbit N of A_0 under the action of the group G (see [GIB]).
α_{A_0} is a holomorphic map, the derivative in e: $D_e\alpha_{A_0}$ is a map from the Lie-algebra T_eG (= $\mathbb{C}^{n\times n}$) into $T_{A_0}\mathbb{C}^{n\times n}$ (= $\mathbb{C}^{n\times n}$), and satisfies

$$(D_e\alpha_{A_0})C = [C,A_0] = CA_0 - A_0C .$$

The derivative $D_{g_0}\alpha_{A_0}$ at an arbitrary point $g_0 \in G$ is given by

$$D_{g_0}\alpha_{A_0}(h) = g_0[g_0^{-1}h,A_0]g_0^{-1} .$$

The proof of this statement is left to the reader. In Chapter II, § 5 we shall prove an analogous result.

3.3. Remark. If A and B are linear operators $[A,B] := AB - BA$ is called the commutator of A and B.
For the sake of brevity we shall write Ad_{A_0} for $D_e\alpha_{A_0}$.

3.4. Definition. The kernel of the linear map Ad_{A_0} is called the centralizer of A_0 and is denoted by $Z(A_0)$. It consists of all the matrices that commute with A_0.
The range of Ad_{A_0} is the tangent space to the orbit N of A_0 at A_0.

3.5. Lemma. The codimension of the orbit is the dimension of the centralizer.

Proof. Since $T_e G$ and $T_{A_0}\mathbb{C}^{n\times n}$ are both vectorspaces with the same dimension, n^2, and Ad_{A_0} is linear we have

$$\dim(\mathrm{Ker}\ Ad_{A_0}) + \dim(\mathrm{Ran}\ Ad_{A_0}) = n^2 .$$

Hence

$$\mathrm{codim(orbit)} = \mathrm{dim(centralizer)} .$$

We are now able to prove the following fundamental theorem of Arnold. We reproduce the proof in some detail because it is the guideline for further investigations. □

3.6. Theorem (Arnold). Equivalence of versality and transversality. A deformation $A(\lambda)$ of A_0 is versal iff the mapping $A(\lambda)$ is transversal to the orbit of A_0 at $\lambda = 0$.

Proof. Versality implies transversality. Let $A(\lambda)$ be a versal deformation of A_0. If $B(\mu)$ is any deformation of A_0, then by the versality of A, we have

$$B(\mu) = C(\mu)A(\varphi(\mu))C^{-1}(\mu) .$$

Taking the derivative at $\mu = 0$, of both sides, we get

3.6.1. $\quad \forall_{\lambda\in T_0\Lambda}: (D_0B)\lambda = [(D_0C)\lambda, A_0] + (D_0A)(D_0\varphi)\lambda .$

Since 3.6.1 holds for every B, and each vector in $T_{A_0}\mathbb{C}^{n\times n}$ can be written as $(D_0B)\lambda$ for a suitable B; each vector is the sum of a vector in the tangent space to the orbit of A_0 and a vector in the image of D_0A; this is exactly the transversality of the map $A(\lambda)$ at $\lambda = 0$.

Transversality implies versality.

This is more complicated. Suppose A is a transversal mapping. Let N denote the orbit of A_0 and Λ the base of the deformation $A(\lambda)$. By the transversality we have

3.6.2. $\quad T_{A_0}\mathbb{C}^{n\times n} = T_{A_0}N + (D_0A)T_0\Lambda .$

Without loss of generality we may assume that the sum is a direct sum (i.e. $T_{A_0}N$ and $(D_0A)T_0\Lambda$ are linearly independent), for, if the dimension of $T_0\Lambda$ is greater than the codimension of $T_{A_0}N$ we replace Λ by a submanifold $\Lambda_0 \subset \Lambda$ such that the restriction of A to Λ_0 is still transversal to the orbit and

the new sum is a direct sum. If it is proved that a restriction of A is versal, then A itself is certainly versal.
Next we choose a submanifold V in G such that $e \in V$ and V is minimal transversal (see Def. B6) to the centralizer of A_0, so

3.6.3. $$T_eV \oplus T_eZ(A_0) = T_eG \ (= \mathbb{C}^{n\times n}) \ .$$

(The submanifold V can be chosen of the form $e + (\mathbb{B} \cap W)$, where $\mathbb{B}$ is the open unit ball in $\mathbb{C}^{n\times n}$ and W is a complement of $Z(A_0)$ in $\mathbb{C}^{n\times n}$.) Define the map $\beta\colon V \times \Lambda \to \mathbb{C}^{n\times n}$ by

$$\beta(v,\lambda) := vA(\lambda)v^{-1} \ .$$

Then β is a holomorphic mapping in a neighbourhood of $(e,0)$ (considered as a function of $\mathbb{C}^{\dim V+\dim\Lambda}$ into $\mathbb{C}^{n\times n}$) and the derivative at $(e,0)$

$$\beta_* := D_{(e,0)}\beta\colon T_eV \times T_0\Lambda \to T_{A_0}\mathbb{C}^{n\times n}$$

is given by

$$\beta_*(w,\zeta) = [w,A_0] + (D_0A)\zeta; \ w \in T_eV, \ \zeta \in T_0\Lambda \ .$$

From 3.6.1 and 3.6.2 it follows that Ker β_* is trivial and hence Ran $\beta_* = T_{A_0}\mathbb{C}^{n\times n}$. Hence β_* is an invertible linear operator. Applying the inverse function theorem we may conclude that β is a holomorphic diffeomorphism from a neighbourhood of $(e,0)$ of $V \times \Lambda$ onto an open set in $\mathbb{C}^{n\times n}$ containing A_0. Hence, if $B(\mu)$ is any deformation of A_0 and μ is sufficiently small we have

$$B(\mu) = \beta(v,\lambda)$$

for some $v \in V$ and $\lambda \in \Lambda$. Define

$$C(\mu) := \pi_1\beta^{-1}(B(\mu))$$

$$\varphi(\mu) := \pi_2\beta^{-1}(B(\mu))$$

(where π_1 and π_2 are the projections of $V \times \Lambda$ onto V respectively Λ) then for small μ

$$B(\mu) = C(\mu)A(\varphi(\mu))C^{-1}(\mu)$$

which proves the versality of the family A. □

3.7. Remark. Note that, if $f: \mathbb{C}^p \to E_1 \times E_2$ is holomorphic, where E_1 and E_2 are Banach spaces and π_i is bounded projection of $E_1 \times E_2$ onto E_i, then $\pi_i f: \mathbb{C}^p \to E_i$ is holomorphic.

§ 4. Construction of versal deformations

It follows from theorem 3.6 that constructing versal deformations is the same thing as constructing transversal deformations. To do this, the space $\mathbb{C}^{n\times n}$ is equipped with the usual innerproduct.

4.1. Definition. If A and B $\in \mathbb{C}^{n\times n}$ we define

$$(A,B) := \mathrm{trace}(AB^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\bar{b}_{ij}$$

(,) has three useful properties

4.1.1. $(A,A) = \|A\|_E^2$

4.1.2. $(A,B) = (B^*,A^*)$

4.1.3. $(XA,B) = (A,X^*B)$

where $X \in \mathbb{C}^{n\times n}$ and $\|A\|_E$ is the Euclidean norm on $\mathbb{C}^{n\times n}$.

4.2. Lemma. Let $A_0 \in \mathbb{C}^{n\times n}$. The orthogonal complement (with respect to the innerproduct just defined) of the tangent space to the orbit of A_0 is the adjoint of the centralizer of A_0, which is equal to $Z(A_0^*)$.

Proof. For the proof we refer to [ARN I]. It is a special case of theorem 5.7 in chapter II of this paper. □

Note that this lemma constitutes a different proof for codim(orbit) = = dim(centralizer). Since every versal deformation is transversal to the tangent space to the orbit, the minimum number of parameters equals the codimension of the orbit which is the dimension of the centralizer =: d. Hence every matrix has a versal deformation with minimum number of parameters equal to d. It can be chosen in the following way

$$A_0 + B(\lambda)$$

where A_0 is the matrix and $B(\lambda)$ is a family (orthogonal) transversal to the tangent space of the orbit (in the adjoint of the centralizer, $Z(A_0^*)$). For an explicit computation of $Z(A_0^*)$ (where A_0 is a Jordan normal form) and explicit examples of versal deformations we refer to [ARN I]. A way to find new versal families from given versal families is described in the next section.

§ 5. Functions of versal families

Let $T \in \mathbb{C}^{n\times n}$ then $F(T)$ denotes the class of all functions of a complex variable which are locally holomorphic in some open set containing $\sigma(T)$. The open set need not to be connected and depends on $f \in F(T)$. If $f \in F(T)$ one can define $f(T)$ which is again an element of $\mathbb{C}^{n\times n}$ (see [DUN I], Ch. VII, § 1).

5.1. Theorem. Suppose $A(\lambda)$ is a versal deformation of A_0 with base Λ. Let $f \in F(A_0)$, with

5.1.1. f 1-1 on $\sigma(A_0)$

5.1.2. $f'(\lambda) \neq 0$ if $\lambda \in \sigma(A_0)$.

Then $f(A(\lambda))$ is a versal deformation of $f(A_0)$ with base Λ.

Proof. Note that $f \in F(A(\lambda))$, if λ is small enough, and hence $f(A(\lambda))$ is well defined for small λ. Let $\sigma(A_0) = \{\lambda_1,\dots,\lambda_p\}$. From the spectral mapping theorem it (see [DUN I], Ch. VII, § 3, Th. 11) follows that

$$\sigma(f(A_0)) = \{f(\lambda_1),\dots,f(\lambda_p)\} .$$

Since $f \in F(A_0)$ there are disjoint open sets $\Omega_1,\dots,\Omega_p$ in $\mathbb{C}$ such that $\lambda_i \in \Omega_i$ $i = 1,\dots,p$ and f is locally holomorphic on $\cup_{i=1}^{p} \Omega_i$. Since $f'(\lambda_i) \neq 0$ and f is 1-1, it follows from the inverse function theorem for holomorphic functions that we can also find disjoint open sets $\omega_1,\dots,\omega_p$ such that $f(\lambda_i) \in \omega_i$ and $f^{-1}: \bigcup_{i=1}^{p} \omega_i \to \bigcup_{i=1}^{p} \Omega_i$ is locally holomorphic and satisfies

$$(f^{-1}\circ f)(z) = z \text{ if } z \in \Omega_1 \cup \dots \cup \Omega_p .$$

(One could also use the Bührman Lagrange theorem - applied p-times - to prove this.)

Hence $f^{-1} \in F(f(A_0))$ because f^{-1} is holomorphic on an open neighbourhood of $\sigma(f(A_0))$.

Now let $B(\mu)$ be any deformation of $f(A_0)$ with base Γ. If μ is small we have $f^{-1} \in \mathcal{F}(B(\mu))$ and then $f^{-1}(B(\mu))$ is well defined and is a holomorphic function of μ with $f^{-1}(B(0)) = A_0$. From the versality of A it follows that there is a deformation $C(\mu)$ of the identity matrix and a map $\varphi\colon \Gamma \to \Lambda$ with $\varphi(0) = 0$ such that

$$f^{-1}(B(\mu)) = C(\mu)A(\varphi(\mu))C^{-1}(\mu) \ .$$

Applying f to both sides we obtain

$$B(\mu) = C(\mu)f(A(\varphi(\mu)))C^{-1}(\mu)$$

and therefore $f(A(\lambda))$ is a versal deformation of $f(A_0)$ with base Λ. (If $A = CBC^{-1}$ then $f \in \mathcal{F}(A)$ if $f \in \mathcal{F}(B)$; $f(A) = Cf(B)C^{-1}$ in that case). □

<u>5.2. Remark</u>. Condition 5.1.1 and 5.1.2 are both necessary. Take $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $f_1(z) = (z-1)(z-2)$. Then D has a 2-parameter versal deformation, but $f_1(D) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and therefore any versal deformation of $f_1(D)$ depends at least on 4-parameters. This proves that condition 5.1.1 is necessary. Taking

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } f_2(z) = z^2$$

we see that condition 5.1.2 is also necessary.

II. On Orbits and Centralizers of Operators

§ 0. Introduction

It is our aim to study deformations of Hilbert-Schmidt operators on an infinite dimensional separable Hilbert space H. The reader of chapter I may expect that the orbit and the centralizer of an operator must be studied in some detail first.

Only a part of the results seems to be new, many of them are quite standard (e.g. § 4, § 7).

We conclude this chapter with a heuristic approach to the topic of chapter III. The appendix is devoted to some isolated results on projectors in Hilbert space.

§ 1. The orbit and the centralizer in a Banach algebra

The definitions in this section are generalisations of the corresponding definitions in Chapter I, § 3. Let B be a complex Banach algebra with identity e (see [LAR]). The group G of non-singular elements in B, is open in B and contains e.

1.1. Remark. If $B = \mathcal{L}(H)$ then the set G is connected, even if H is infinite dimensional (see [KUI]).

1.2. Definition. If $a \in B$ is fixed we define the map $\alpha_a : G \to B$ by

$$\alpha_a(g) := gag^{-1} .$$

$\alpha_a(G)$ is called the orbit of $a \in B$ under the action of the group G.

α_a is a C^∞-map (the proof is similar to the proof of lemma 5.5 of this chapter) and the derivative at the identity e: $D_e\alpha_a =: Ad_a$ is a linear map from B into B. (Since G is open in B, the tangent space at the identity T_eG is B itself). Note that, in contrast with α_a, the map Ad_a can also be defined in a Banach algebra without identity by putting $Ad_a(g) := ga - ag$ for $g \in B$. Clearly Ad_a is a bounded linear operator whose norm in $\mathcal{L}(B)$ does not exceed $2\|a\|$.

1.3. Definition. $\text{Ker}(\text{Ad}_a)$ is a closed subalgebra of B which will be called the centralizer of a in B, notation $Z_B(a)$. $Z_B(a)$ consists of all elements in B which commute with a. If no ambiguity is caused we sometimes write Z(a) instead of $Z_B(a)$.

If $B = \mathbb{C}^{n\times n}$ the linear manifold $\text{Ad}_a(\mathbb{C}^{n\times n})$ is necessarily closed and it is the tangent space to the orbit of a. However, if B is infinite dimensional, for example $B = \mathcal{L}(H)$, $\text{Ad}_a(B)$ is not necessarily closed (see § 6). We shall only consider the special cases $B = \mathcal{L}(H)$ and B = HS.

§ 2. The centralizer of an operator in $\mathcal{L}(H)$

The main result of this section is our theorem 2.5 which states that for every operator $A \in \mathcal{L}(H)$ the centralizer is infinite dimensional. As a preparation we start with some well known facts about minimal polynomials of operators.

2.1. Lemma. If $A \in \mathcal{L}(H)$ and ψ is a polynomial with complex coefficients of degree $n \geq 1$ such that $\psi(A) = 0$, then there is a unique polynomial φ_0 of degree $k \geq 1$ such that

2.1.1. $\varphi_0(A) = 0$.

2.1.2. There is no polynomial with $1 \leq$ degree $< k$ that annihilates A.

2.1.3. The coefficient of z^k in φ_0 equals 1.

Proof. Suppose ψ annihilates A. Obviously there is a polynomial φ of minimal degree $k \geq 1$ such that $\varphi(A) = 0$. Multiply φ by a complex constant $\neq 0$ such that in the resulting polynomial φ_0, the coefficient of z^k equals 1. Now φ_0 clearly satisfies 2.1.1, 2.1.2 and 2.1.3.

The only thing left to prove is the uniqueness. Suppose φ_1 is a polynomial of degree k such that 2.1.1, 2.1.2 and 2.1.3 are satisfied. Then $\varphi_0 - \varphi_1$ still annihilates A and degree $(\varphi_0 - \varphi_1) \leq k-1$. Since k is minimal it follows that $\varphi_0 - \varphi_1 \equiv 0$ and hence $\varphi_0 \equiv \varphi_1$. □

φ_0 is called the minimal polynomial of the operator A. Unlike finite matrices, most operators on H do not have a minimal polynomial.
This follows from:

2.2. Lemma. If $A \in \mathcal{L}(H)$ and has minimal polynomial φ_0, then the spectrum of A, $\sigma(A)$, consists exactly of the zero's of φ_0.

Proof. Write $\varphi_0(z) = \prod_{i=1}^{p} (z - \lambda_i)^{h_i}$ with $h_i \geq 1$; λ_i's complex and distinct. From the spectral mapping theorem (see [DUN I], Ch. VII, § 3, Th. 11) it follows that

$$\varphi_0(\sigma(A)) = \sigma(\varphi_0(A)) = \{0\}.$$

Hence $\sigma(A) \subset \{\lambda_1, \dots, \lambda_p\}$.
On the other hand if $1 \leq i \leq p$ the operator $A - \lambda_i I$ must be singular, since if $A - \lambda_i I$ is non-singular, $\varphi_1 := \prod_{j \neq i}^{p} (z - \lambda_j)^{h_j}$ still annihilates A which contradicts the minimality of φ_0.
Hence $\lambda_i \in \sigma(A)$, which completes the proof. □

2.3. Corollary. If A is quasinilpotent (that is $\sigma(A) = \{0\}$) and has an annihilating polynomial, then A is nilpotent.

Lemma 2.2 and corollary 2.3 enable us to prove that for every $A \in \mathcal{L}(H)$ the dimension of the centralizer is infinite. We shall first prove this for a nilpotent operator.

2.4. Lemma. If $A \in \mathcal{L}(H)$ is nilpotent then $\dim Z(A) = \infty$.

Proof. Let $p \in \mathbb{N}$ be the smallest number for which $A^p = 0$, and define

$$N_j := \mathrm{Ker}(A^j); \quad j = 1,2,\dots,p .$$

Every N_j is a subspace of H and $N_1 \subset N_2 \subset \dots \subset N_p = H$.
It is easily seen that $\dim(N_j) = \infty$; $j = 1,2,\dots,p$. For, if $\dim(N_1)$ is finite then it follows that $\dim(N_p)$ is finite but this contradicts $N_p = H$. Since $N_{j-1} \neq N_j$ for $j = 2,\dots,p$ there are non-trivial subspaces $M_1,\dots,M_p \subset H$ such that

$$N_j = M_1 \oplus \dots \oplus M_j; \quad j = 1,2,\dots,p$$

and

$$A(M_1) = \{0\} \text{ and } A(M_j) \subset M_{j-1} \text{ for } j \geq 2 .$$

We now define a subspace $M \subset \mathcal{L}(H)$ as follows:

2.4.1. $$C \in M \text{ iff } \begin{cases} C \in \mathcal{L}(H) , \\ C = 0 \text{ on } N_{p-1} , \\ C(M_p) \subset N_1 . \end{cases}$$

M is the intersection of two closed subsets of $\mathcal{L}(H)$ and therefore is closed in $\mathcal{L}(H)$. Since M_p is non-trivial (this follows from the minimality of p) and N_1 has infinite dimension, M is infinite dimensional. Each operator in M commutes with A. If $C \in M$ we have AC = CA = 0. To prove this write $x = x_1 + x_2 +..+ x_p$ with $x_j \in M_j$ and compute ACx and CAx. Hence $M \subset Z(A)$ and therefore dim $Z(A) = \infty$. □

2.5. Theorem. For every $A \in \mathcal{L}(H)$: dim $Z(A) = \infty$.

Proof. Let $A \in \mathcal{L}(H)$ and suppose dim Z(A) is finite. Since Z(A) contains all powers A^k of A (k = 0,1,2,...) we can find $n \geq 1$ and $\alpha_0,\ldots,\alpha_{n-1} \in \mathbb{C}$ such that

$$A^n + \alpha_{n-1}A^{n-1} +\ldots+ \alpha_0 I = 0 .$$

Hence, by lemma 2.1, A has a minimal polynomial φ_0 of degree $\leq n$. From lemma 2.2 it follows that $\sigma(A)$ consists of the zero's of φ_0 and hence is a finite set say $\sigma(A) = \{\lambda_1,\ldots,\lambda_p\}$.

Define the operators E_j, j = 1,...,p by

2.5.1. $$E_j := \frac{1}{2\pi i} \int_{(\lambda_j)} R(\zeta,A)\,d\zeta$$

where (λ_j) is a small circle centered at λ_j. Then E_j is the projection operator on the invariant subspace $X_j = E_j(H)$ corresponding to the spectral point λ_j. The space H is the direct sum of p subspaces invariant under A:

$$H = X_1 \oplus X_2 \oplus\ldots\oplus X_p \quad .$$

Since H is infinite dimensional there is at least one j with $\dim(X_j) = \infty$. Let A_j denote the restriction of A to the invariant subspace X_j, $j = 1,2,..,p$. Every $x \in H$ has a unique representation $x = x_1 +\ldots+ x_p$ with $x_i \in X_i$.

If P is a polynomial we have

2.5.2. $$P(A)x = P(A_1)x_1 +\ldots+ P(A_p)x_p$$

(because $A^\ell x = A_1^\ell x_1 +\ldots+ A_p^\ell x_p$ for $\ell \in \mathbb{N}$). Taking $P = \varphi_0$ in 2.5.2 it follows that every A_j has a minimal polynomial of degree less than degree(φ_0).

From now on we fix j such that $\dim(X_j) = \infty$ and we shall prove that $\dim_{\mathcal{L}(X_j)} Z(A_j) = \infty$. The spectrum $\sigma(A_j)$ consists of exactly one point λ_j (see [DUN I], Ch. VII, § 3, Th. 20). It is no loss of generality assuming $\lambda_j = 0$, because $Z_{\mathcal{L}(X_j)}(A_j - \lambda_j I_j) = Z_{\mathcal{L}(X_j)}(A_j)$, where I_j is the restriction of I to X_j. According to this assumption the operator A_j is quasinilpotent. Since A_j also has a minimal polynomial it follows from corollary 2.3 that A_j is nilpotent and hence by lemma 2.4 $\dim Z_{\mathcal{L}(X_j)}(A_j) = \infty$.

If $B \in Z_{\mathcal{L}(X_j)}(A_j)$ then $BE_j \in Z_{\mathcal{L}(H)}(A)$, since for arbitrary $x \in H$

$$BE_jAx = BAE_jx = BA_jE_jx = A_jBE_jx = ABE_jx .$$

This contradicts the assumption that $\dim Z_{\mathcal{L}(H)}(A)$ is finite. Hence $\dim Z_{\mathcal{L}(H)}(A) = \infty$. □

2.6. Remark. Obviously, for every $A \in \mathcal{L}(H)$ we have $Z(A^*) = (Z(A))^*$. If A is normal then $Z(A^*) = Z(A)$. The last result is a theorem of Fuglede (see [FUG]), which has a short and elegant proof in [ROS].

§ 3. The centralizer of a normal compact operator

In general it is difficult to compute the centralizer of an operator (in the finite dimensional case, for matrices, the computation can be found in [GAN], Ch. VIII, § 2). For a certain class of operators, however, it is rather easy. This class includes the normal compact operators. We shall describe the centralizer of a normal compact operator and prove that it splits in $\mathcal{L}(H)$. We first quote some standard results on normal compact operators.

Suppose A is normal and compact. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration of $\sigma(A)\setminus\{0\}$ such that $|\lambda_1| \geq |\lambda_2| \geq \ldots$. Define $X_0 := \mathrm{Ker}(A)$ and $X_j = E_j(H)$; $j \geq 1$, where E_j is the projector defined as in 2.5.1:

$$E_j = \frac{1}{2\pi i} \int_{(\lambda_j)} R(\zeta, A)\, d\zeta .$$

Since A is normal the projections E_j are orthogonal and therefore self-adjoint (see [DUN I], Ch. VI, § 3). It is well known that the space H is the direct sum of the orthogonal eigenspaces X_j which reduce A:

$$H = X_0 \oplus X_1 \oplus \ldots$$

and that the operator A has the spectral decomposition

$$A = \sum_{j=1}^{\infty} \lambda_j E_j + 0.E_0$$

where E_0 is the projector on Ker(A). The subspaces X_j are mutually orthogonal and

$$\mathrm{Ker}(A) = \bigcap_{i=1}^{\infty} X_i^{\perp}$$

($X_i^{\perp}$ is the orthogonal complement of X_i).
For arbitrary $x \in \mathrm{Ker}(A)$ and $j \in \mathbb{N}$ we have $x = x_1 + x_2$ with $x_1 \in X_j$ and $x_2 \in X_j^{\perp}$. Hence

$$0 = Ax = Ax_1 + Ax_2 = \lambda_j x_1 + Ax_2 .$$

Since $Ax_2 \in X_j^{\perp}$ (A is normal) we have $x_1 = 0$ because $\lambda_j \neq 0$. Hence $x \in X_j^{\perp}$.

$$\mathrm{Ker}(A) \subset \bigcap_{i=1}^{\infty} X_i^{\perp} .$$

But, since $H = X_0 \oplus X_1 \oplus \ldots$ we have

$$X_0 = \mathrm{Ker}(A) = \bigcap_{i=1}^{\infty} X_i^{\perp} .$$

3.1. Lemma. The centralizer of the normal compact operator A is the subspace

$$\{C \in \mathcal{L}(H) \mid CE_j = E_jC;\ j = 0,1,2,\ldots\} .$$

This lemma is a direct corollary of the preceeding results of this chapter. It is a special case of a result in [HAL II].

Lemma 3.1 enables us to prove our theorem 3.2.

3.2. Theorem. If A is normal and compact in $\mathcal{L}(H)$, then Z(A) splits in $\mathcal{L}(H)$.

Proof. To prove this we give a closed complement of Z(A) in $\mathcal{L}(H)$. Define $V \subset \mathcal{L}(H)$ by

$$V := \{D \in \mathcal{L}(H) \mid D(X_j) \subset X_j^{\perp};\ j = 0,1,2,\ldots\} ,$$

then we shall show that V satisfies

3.2.1. V is a subspace of $\mathcal{L}(H)$.

3.2.2. $V \cap Z(A) = \{0\}$.

3.2.3. $V \oplus Z(A) = \mathcal{L}(H)$.

ad 3.2.1. It is obvious that V is a linear space. Suppose $(D_n)_{n \in \mathbb{N}}$ is a sequence in V with $\lim_{n\to\infty} D_n = D \in \mathcal{L}(H)$ in the uniform topology. If $x \in X_j$ and $y \in X_j$ we have

$$(Dx,y) = \lim_{n\to\infty} (D_n x,y) = 0 \ .$$

Hence $Dx \in X_j^{\perp}$ and therefore $D(X_j) \subset X_j^{\perp}$ and hence V is closed.

ad 3.2.2. Suppose $T \in V \cap Z(A)$. Choose $x \in X_j$ then $T \in V$ implies $Tx \in X_j^{\perp}$ and $T \in Z(A)$ implies $Tx \in X_j$ (lemma 3.1) hence $Tx = 0$. Hence $T = 0$.

ad 3.2.3. Let $T \in \mathcal{L}(H)$. For $h \in H$ we define

$$Ch := \sum_{k=0}^{\infty} (E_k T E_k) h$$

where the E_k's are the projections in the spectral decomposition of A. This definition makes sense because

$$\| \sum_{k=0}^{n} E_k T E_k h \|^2 = \sum_{k=0}^{n} \| E_k T E_k h \|^2 \leq \sum_{k=0}^{n} \| T E_k h \|^2 \leq \| T \|^2 \sum_{k=0}^{n} \| E_k h \|^2 \leq$$
$$\leq \| T \|^2 . \| h \|^2 \ .$$

The sequence $h_n := \sum_{k=0}^{n} (E_k T E_k) h$, $n \in \mathbb{N}$, is a Cauchy-sequence in H and therefore convergent with limit $Ch \in H$. Clearly C is linear and its norm does not exceed $\| T \|$. Hence, $C \in \mathcal{L}(H)$. We now prove $C \in Z(A)$. For $x \in H$ we have

$$CE_j x = \sum_{k=0}^{\infty} E_k T E_k E_j x = E_j T E_j x$$

and

$$E_j C x = \sum_{k=0}^{\infty} E_j E_k T E_k x = E_j T E_j x \ .$$

Hence by lemma 3.1 $C \in Z(A)$.

Define $D := T - C$ then $D \in \mathcal{L}(H)$ and $T = D + C$. $D \in V$ because for $x \in X_j$ and $y \in X_j$ we have

$$(Dx,y) = (Tx - Cx,y) = (Tx - E_jTx,y) =$$
$$= (Tx,y) - (Tx,E_j^*y) = (Tx,y) - (Tx,E_jy) = 0 .$$

(Note that $E_j^* = E_j$).
Hence $DX_j \subset X_j^{\perp}$ and therefore $D \in V$ which completes the proof. □

§ 4. Commutators of operators

If $A \in \mathcal{L}(H)$ the set $Ad_A(\mathcal{L}(H))$ consists of commutators of the form $CA - AC$ where $C \in \mathcal{L}(H)$. If the space H is infinite dimensional, it is an interesting question whether a given operator can be written as a commutator or not. Commutators have been investigated by Halmos, Putnam, Brown and Pearcy in [HAL II], [PUT]and [BRO]. The most important result in this direction is that an operator $A \in \mathcal{L}(H)$ can be written as a commutator iff $A \neq \lambda I + C$, $\lambda \neq 0$ and C compact (see [BRO]).
In § 8 we shall use a theorem which can be deduced from the Kleinecke-Shirokov theorem.

4.1. Theorem (Kleinecke-Shirokov). If $C = PQ - QP$ and $CP = PC$ then $\sigma(C) = \{0\}$.

Proof. See [HAL I], problem 184. □

4.2. Theorem (Putnam). If $C = PQ - QP$; $CP = PC$ and P is normal then $C = 0$.

Proof. See [PUT]. □

4.3. Corollary. If $A \in \mathcal{L}(H)$ is normal then

$$Z(A) \cap \operatorname{Ran} Ad_A = \{0\} .$$

§ 5. Hilbert-Schmidt operators

In this section we quote some standard results from the theory of Hilbert-Schmidt operators. Furthermore we shall prove some new theorems, which will be useful later on. Our theorem 5.7 is a straightforward generalization of lemma 4.2 in chapter I.

5.1. Definition. An operator $A \in \mathcal{L}(H)$ is a Hilbert-Schmidt operator iff there is an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ (which is fixed from now on) for the space H such that

$$|||A||| := \left(\sum_{k=1}^{\infty} \|Ae_k\|^2\right)^{\frac{1}{2}} < \infty .$$

The sum, which is independent of the chosen basis, is called the double-norm of A. The set of all Hilbert-Schmidt operators on H is denoted by HS. For properties of Hilbert-Schmidt operators see [SCH], Ch. II or [DUN II], Ch. XI, § 6.

5.2. Remark. There is a one-to-one correspondence between the set HS and the set of all 2-sided infinite complex matrices $\{\alpha_{ij}\}$ with

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_{ij}|^2 < \infty$$

(with respect to the basis $(e_n)_{n\in\mathbb{N}}$).
If $A \in$ HS the corresponding matrix $\{\alpha_{ij}\}$ has entries $\alpha_{ij} = (Ae_j, e_i)$. Since $A \in$ HS, $(\sum_i \sum_j |\alpha_{ij}|^2)^{\frac{1}{2}}$ converges and equals $|||A|||$. To the matrix $\{\alpha_{ij}\}$ corresponds the Hilbert-Schmidt operator A defined by $Ae_j = \sum_{i=1}^{\infty} \alpha_{ij} e_i$. The correspondence from operators to matrices has the usual algebraic properties.

5.3. Definition. In HS we define an innerproduct as follows

$$(A,B) := \sum_{n=1}^{\infty} (Ae_n, Be_n) .$$

The innerproduct is independent of the chosen basis and makes HS into a Hilbert space (see [SCH], Ch. II). The innerproduct has three useful properties:

5.3.1. $(A,A) = |||A|||^2$,

5.3.2. $(A,B) = \overline{(A^*,B^*)}$,

5.3.3. $(XA,B) = (A,X^*B)$,

for all A, B and $X \in$ HS. From 5.3.2 and 5.3.3 we can deduce

$$(AX,B) = \overline{(X^*A^*,B^*)} = \overline{(A^*,XB^*)} = (A,BX^*) .$$

The innerproduct is a generalization of the innerproduct defined in Chapter I, definition 4.1. For, if $A \in HS$ then AB^* is in the trace class (see [SCH]) and

$$(A,B) = \mathrm{trace}(AB^*) = \sum_{n=1}^{\infty} (Ae_n, Be_n) .$$

5.4. Remark. Since HS provided with double-norm is a B-algebra without identity (with involution) the map α_A, in the sense of definition 1.2, is not defined (it is easily seen that there are no invertible elements in HS). However, if $A \in HS$, the map $\alpha_A\colon G \to \mathcal{L}(H)$, defined in definition 1.2, can be considered as a map from G into HS. This follows from the fact that HS is a two-sided ideal in $\mathcal{L}(H)$.

5.5. Lemma. Let $A \in HS$. The map $\alpha_A\colon G \to HS$ defined by $\alpha_A(g) := gAg^{-1}$ is a C^∞-map considered as a map from $(G, \|\ \|)$ into $(HS, |||\ |||)$. The derivative at the identity operator $I \in \mathcal{L}(H)$ is the map $Ad_A\colon \mathcal{L}(H) \to HS$ which maps g into $[g,A]$.

Proof. We shall first prove that α_A is differentiable at I with derivative Ad_A. If $\|h\| < 1$ we have

$$\alpha_A(I+h) = (I+h)A(I+h)^{-1} = (I+h)A \sum_{n=0}^{\infty} (-1)^n h^n =$$

$$= A + hA - Ah + A \sum_{n=2}^{\infty} (-1)^n h^n + hA \sum_{n=1}^{\infty} (-1)^n h^n .$$

Hence

$$|||\alpha_A(I+h) - \alpha_A(I) - Ad_A(h)||| = |||A \sum_{n=2}^{\infty} (-1)^n h^n + hA \sum_{n=1}^{\infty} (-1)^n h^n||| \le$$

$$\le 2|||A|||.\|h\|^2 \quad \text{if } \|h\| < \tfrac{1}{2} .$$

Hence

$$\lim_{\|h\|\to 0} \frac{|||\alpha_A(I+h) - \alpha_A(I) - Ad_A(h)|||}{\|h\|} = 0 ,$$

and therefore α_A is differentiable at I with derivative Ad_A. With an analogous computation it can be shown that α_A is differentiable at any point $g_0 \in G$ with derivative

$$(D_{g_0}\alpha_A)(h) = g_0[g_0^{-1}h, A]g_0^{-1} .$$

It is obvious that $g_0 \to D_{g_0}\alpha_A$ is C^∞ and hence α_A is C^∞. □

The first part of the lemma has a shorter proof as follows. Note that the map $g \to g^{-1}$ is a C^{∞}-map from G onto G and the mappings $g \to gA$ and $g \to Ag$ are C^{∞}-mappings from $(G, \| \ \|)$ into $(HS, ||| \ |||)$. Hence α_A is a C^{∞}-map.

The map Ad_A can also be considered as a map from HS into HS and then Ker Ad_A is a $||| \ |||$-(closed) subspace of HS which is the centralizer of A in HS: $Z_{HS}(A)$. Thus by $Z_{HS}(A)$ is always meant the set

$$\{C \in HS \mid CA = AC\} .$$

5.6. Remark. If $A \in HS$, the centralizer of A in HS, $Z_{HS}(A)$, is infinite dimensional. To prove this consider A as an operator in $\mathcal{L}(H)$ and copy the proof of theorem 2.5. The only modification that has to be made is the following: replace the subspace $M \subset \mathcal{L}(H)$ defined in 2.4.1 by a subspace $M' \subset HS$ having exactly the same properties as M except that M' consists only of Hilbert-Schmidt operators.

We now prove a generalization of lemma 4.2 of Chapter I.

5.7. Theorem. Let $A \in HS$. The orthogonal complement of $Ad_A(HS)$ in HS is the centralizer of the adjoint of A: $Z_{HS}(A^*)$.

Proof. Let $X \in (Ad_A(HS))^{\perp}$ then for all $Y \in HS$ we have $(X, Ad_A(Y)) = 0$ and hence

$$(X,AY) - (X,YA) = 0 .$$

If we use 5.3.2 and 5.3.3 we obtain

$$(A^*X - XA^*, Y) = 0 \quad \text{for } Y \in HS$$

and hence $X \in Z_{HS}(A^*)$.
If $X \in Z_{HS}(A^*)$ the proof goes the other way around. Hence $(Ad_A(HS))^{\perp} = Z_{HS}(A^*)$. □

5.8. Remark. The proof of theorem 5.7 can be formulated in another way. Note that the map Ad_{A^*} is the adjoint of Ad_A in $\mathcal{L}(HS)$. (Just as in the given proof this is a direct consequence of 5.3.2.) Hence

$$(Ran(Ad_A))^{\perp} = (Ad_A(HS))^{\perp} = Ker(Ad_{A^*}) = Z_{HS}(A^*) .$$

Theorem 5.7 plays an important role in the construction of weakly versal deformations of Hilbert-Schmidt operators (see Ch. III, § 5).

The map Ad_A can also be considered as a map from $\mathcal{L}(H)$ into HS. Obviously, we have

$$Ad_A(\mathcal{L}(H)) \supset Ad_A(HS) \quad .$$

In the next theorem we shall prove that $Ad_A(HS)$ is double-norm dense in $Ad_A(\mathcal{L}(H))$.

5.9. Theorem. Let $A \in HS$. Then

$$\overline{\overline{Ad_A(HS)}} = \overline{\overline{Ad_A(\mathcal{L}(H))}} \quad .$$

(The double bar denotes the double-norm closure.)

Proof. Suppose A has the matrix $\{\alpha_{ij}\}$, with respect to the basis $(e_n)_{n \in \mathbb{N}}$, then A^* has the matrix $\{\bar{\alpha}_{ji}\}$. Hence, for $i \in \mathbb{N}$ we have

$$Ae_i = \sum_{k=1}^{\infty} \alpha_{ki} e_k \quad \text{and} \quad A^* e_i = \sum_{k=1}^{\infty} \bar{\alpha}_{ik} e_k \quad .$$

If $B \in Z_{HS}(A^*)$ and $g \in \mathcal{L}(H)$ we have

$$(Ad_A(g), B) = \sum_{k=1}^{\infty} (gAe_k, Be_k) - \sum_{k=1}^{\infty} (Age_k, Be_k) =$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{jk} (ge_j, Be_k) - \sum_{k=1}^{\infty} (ge_k, A^* Be_k) \quad .$$

Since $BA^* = A^*B$ we obtain

$$(Ad_A(g), B) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{jk} (ge_j, Be_k) - \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{kj} (ge_k, Be_j) \quad .$$

We now prove that both sums are absolutely convergent. Both proofs are alike, so we give only one of them. Applying the Cauchy-Schwarz inequality to the first sum we find:

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_{jk}||(ge_j,Be_k)| \le (\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_{jk}|^2)^{\frac{1}{2}}(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |(ge_j,Be_k)|^2)^{\frac{1}{2}} =$$

$$= |||A|||.|||g^*B||| .$$

For,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |(ge_j,Be_k)|^2 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |(e_j,g^*Be_k)|^2 =$$

$$= \sum_{k=1}^{\infty} \|g^*Be_k\|^2 = |||g^*B|||^2 .$$

Hence both sums are absolutely convergent and therefore we may change the order of summation. Hence

$$(Ad_A(g),B) = 0 .$$

Since $g \in \mathcal{L}(H)$ was arbitrary this proves $B \in (Ad_A(\mathcal{L}(H)))^{\perp}$. Since $B \in Z_{HS}(A^*)$ was arbitrary, we may conclude

$$Z_{HS}(A^*) \subset (Ad_A(\mathcal{L}(H)))^{\perp} .$$

In theorem 5.7 we already have proved that $Z_{HS}(A^*) = (Ad_A(HS))^{\perp}$. Hence

$$Z_{HS}(A^*) = (Ad_A(HS))^{\perp} \supset (Ad_A(\mathcal{L}(H)))^{\perp} \supset Z_{HS}(A^*) ,$$

and therefore $(Ad_A(HS))^{\perp} = (Ad_A(\mathcal{L}(H)))^{\perp}$. Hence

$$\overline{\overline{Ad_A(HS)}} = \overline{\overline{Ad_A(\mathcal{L}(H))}} .$$ □

<u>5.10. Remark</u>. Theorem 5.9 shows that for our purpose, which will become clear in chaper III, we can disregard the difference between $Ad_A(HS)$ and $Ad_A(\mathcal{L}(H))$.

<u>5.11. Remark</u>. In many examples the linear manifolds $Ad_A(HS)$ and $Ad_A(\mathcal{L}(H))$ are not closed. We shall give two examples in the next section.

§ 6. Examples

In chapter I HS and $\mathbb{C}^{n\times n}$ coincide, so $Ad_A(HS) = Ad_A(\mathbb{C}^{n\times n})$ and these finite dimensional linear manifolds are necessarily closed. In the infinite dimensional case there are many examples in which $Ad_A(HS)$ and $Ad_A(\mathcal{L}(H))$ are not closed. This fact forms an additional complication to the theory in chapter III.

6.1. Remark. If $A \in HS$ both $Ad_A(HS)$ and $Ad_A(\mathcal{L}(H))$ are subsets of HS satisfying:

$$Ad_A(HS) \subset Ad_A(\mathcal{L}(H))$$

and

$$\overline{\overline{Ad_A(HS)}} = \overline{\overline{Ad_A(\mathcal{L}(H))}} \quad \text{(see theorem 5.9)} .$$

If $Ad_A(HS)$ is $|||\ |||$-closed, $Ad_A(\mathcal{L}(H))$ must be $|||\ |||$-closed. On the other hand if $Ad_A(\mathcal{L}(H))$, considered as a subset of $\mathcal{L}(H)$, is $\|\ \|$-closed it is necessarily $|||\ |||$-closed, since every $\|\ \|$-closed set in HS is $|||\ |||$-closed ($\|A\| \le |||A|||$).

6.2. Corollary. If $Ad_A(\mathcal{L}(H))$ is not $|||\ |||$-closed then

$Ad_A(HS)$ is not $|||\ |||$-closed

$Ad_A(\mathcal{L}(H))$ is not $\|\ \|$-closed .

We shall give two examples of an operator $A \in HS$ for which $Ad_A(HS)$ and $Ad_A(\mathcal{L}(H))$ are not closed (in $\|\ \|$ or $|||\ |||$). The first example deals with a diagonal operator, the second with a monotone ℓ_2-shift. In the first example we shall give two different proofs to show that $Ad_A(HS)$ is not $|||\ |||$-closed. In both examples we are able to compute the centralizers $Z_{HS}(A)$. The examples are described with respect to the basis $(e_n)_{n\in\mathbb{N}}$.

6.3. Example. Let D be a diagonal operator in HS

$$D = \text{diag}\{\lambda_1, \lambda_2, \ldots\}$$

with λ_i's distinct and $\sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$. It follows from an easy computation that $Z_{\mathcal{L}(H)}(D)$ consists of all diagonal operators in $\mathcal{L}(H)$ and therefore $Z_{HS}(D)$ consists of all diagonal Hilbert-Schmidt operators (with respect to the basis $(e_n)_{n\in\mathbb{N}}$). To compute $\overline{\overline{Ad_D(HS)}}$ we use theorem 5.7, which implies $\overline{\overline{Ad_D(HS)}} = (Z_{HS}(D))^{\perp}$. (Note that $Z_{HS}(D^*) = Z_{HS}(D)$). Suppose $X \in (Z_{HS}(D))^{\perp}$ then for all diagonal operators $\Lambda \in HS$ we have $(\Lambda, X) = 0$ and therefore $\forall_{i\in\mathbb{N}}\ (Xe_i, e_i) = 0$. On the other hand if $\forall_{i\in\mathbb{N}}\ (Xe_i, e_i) = 0$ it follows that $X \in (Z_{HS}(D))^{\perp}$. Hence

$$\overline{\overline{Ad_D(HS)}} = \{X \in HS \mid \forall_{i\in\mathbb{N}}\ (Xe_i, e_i) = 0\} .$$

We now show that there is an operator $F \in \overline{\overline{\mathrm{Ad}_D(HS)}} \setminus \mathrm{Ad}_D(HS)$ and therefore $\mathrm{Ad}_D(HS)$ cannot be $|\!|\!|\ |\!|\!|$-closed in HS. Let $F \in HS$ be the shift operator defined by $Fe_i = \mu_i e_{i+1}$ with $\sum_{i=1}^{\infty} |\mu_i|^2 < \infty$. Obviously, $F \in \overline{\overline{\mathrm{Ad}_D(HS)}}$. Suppose $F \in \mathrm{Ad}_D(HS)$ then for some $C \in HS$ we have

$$CD - DC = F \ .$$

This implies

$$\forall_{i \in \mathbb{N}} \forall_{j \in \mathbb{N}} \quad ([C,D]e_i, e_j) = (Fe_i, e_j) \ .$$

Hence

$$\forall_{i \in \mathbb{N}} \ (Ce_i, e_{i+1}) = \frac{\mu_i}{\lambda_i - \lambda_{i+1}} \ .$$

Hence

6.3.1. $$|\!|\!| C |\!|\!| \geq \| C \| \geq \| Ce_i \| = \Big(\sum_{j=1}^{\infty} |(Ce_i, e_j)|^2 \Big)^{\frac{1}{2}} \geq |(Ce_i, e_{i+1})| = \frac{|\mu_i|}{|\lambda_i - \lambda_{i+1}|} \ .$$

Since $\lim_{i \to \infty} |\lambda_i - \lambda_{i+1}| = 0$ we can find a subsequence $(\lambda_{i_k})_{k \in \mathbb{N}}$ of $(\lambda_i)_{i \in \mathbb{N}}$ such that

$$\forall_{k \in \mathbb{N}} \ |\lambda_{i_k} - \lambda_{i_k+1}| \leq 2^{-k} \ .$$

We now make a special choice for the weights $(\mu_j)_{j \in \mathbb{N}}$ of the shift F. Take

$$\mu_{i_k} = 2^{-\frac{1}{2}k}$$

$$\mu_i = 0 \text{ if } i \neq i_k \text{ for all } k \ .$$

Then $\sum_{i=1}^{\infty} |\mu_i|^2 < \infty$ and

$$\lim_{k \to \infty} \frac{|\mu_{i_k}|}{|\lambda_{i_k} - \lambda_{i_k+1}|} = \infty \ .$$

Hence, by 6.3.1, there is no operator $C \in HS$ (neither in $\mathcal{L}(H)$) such that $CD - DC = F$ and therefore $F \notin \mathrm{Ad}_D(HS)$. Hence $\mathrm{Ad}_D(HS)$ is not $|\!|\!|\ |\!|\!|$-closed in HS. The same arguments prove that $\mathrm{Ad}_D(\mathcal{L}(H))$ is not $|\!|\!|\ |\!|\!|$-closed.

There is another way of proving that $\mathrm{Ad}_D(\mathcal{L}(H))$ is not $|\!|\!|\ |\!|\!|$-closed. Suppose $\mathrm{Ad}_D(\mathcal{L}(H))$ is closed in $|\!|\!|\ |\!|\!|$. Define

$$V := \{X \in \mathcal{L}(H) \mid (Xe_i, e_i) = 0, \ i \in \mathbb{N}\} \ .$$

It follows from the proof of theorem 3.2 that V is a closed complement of $Z_{\mathcal{L}(H)}(D)$ in $\mathcal{L}(H)$:

$$V \oplus Z_{\mathcal{L}(H)}(D) = \mathcal{L}(H) \quad .$$

Consider the restriction Δ of the map Ad_D to the subspace V. Then Δ is a bounded linear operator from the Banach space $(V, \| \ \|)$ onto the Banach space $(Ad_D(\mathcal{L}(H)), ||| \ |||)$. (The norm of Δ does not exceed $2|||D|||$). $\mathrm{Ker}(\Delta) = \mathrm{Ker}(Ad_D) \cap V = Z_{\mathcal{L}(H)}(D) \cap V = \{0\}$ and hence Δ is 1-1. By the closed graph theorem Δ is invertible with bounded inverse, so there is a $\delta > 0$ such that

6.3.2. $\quad \forall_{X \in V} \ |||\Delta(X)||| \geq \delta \|X\|$.

Define the sequence $(X_n)_{n \in \mathbb{N}} \subset V$ as follows:

$$X_n e_j = 0 \text{ if } j \neq n+1; \ X_n e_{n+1} = e_n$$

extend X_n linearly to the whole space H. Now $\|X_n\| = 1$ and

$$\lim_{n\to\infty} |||\Delta(X_n)||| = \lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0 \ .$$

This contradicts 6.3.2 and therefore $Ad_D(\mathcal{L}(H))$ is not $||| \ |||$-closed.

<u>6.4. Remark</u>. If $A \in HS$ is normal (not necessarily diagonal) the same arguments (theorem 3.2 and the closed graph theorem) show that $Ad_A(\mathcal{L}(H))$ is not closed.

<u>6.5. Example</u>. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ such that

6.5.1. $\quad \alpha_1 > \alpha_2 > \ldots > 0$.

6.5.2. $\quad \sum_{i=1}^{\infty} \alpha_i^2 < \infty$.

Let the operator $U \in HS$ be defined by

$$Ue_n = \alpha_n e_{n+1}, \quad n \in \mathbb{N} \ .$$

U is called a monotone ℓ_2-shift with weights $(\alpha_n)_{n \in \mathbb{N}}$. U^* satisfies

$$U^* e_1 = 0$$

and

$$U^* e_n = \alpha_{n-1} e_{n-1}, \ n \geq 2 \ .$$

Note that $UU^*e_1 = 0$ and $U^*Ue_1 = \alpha_1 U^*e_2 = \alpha_1^2 e_1$ and therefore U is not normal. We first compute $Z_{HS}(U^*)$. Note that the only non-trivial invariant subspaces under U^* are

$$M_n := \text{span}\{e_1,\dots,e_n\}; \quad n \in \mathbb{N}$$

(see [HAL I], problem 151).
Suppose $U^*R = RU^*$ then

$$U^*R(M_n) = RU^*(M_n) \subset R(M_n) .$$

Hence $R(M_n)$ is an invariant subspace under U^*, and therefore there is an integer $k \in \mathbb{N}$ such that

$$R(M_n) = M_k \quad (\text{or } R(M_n) = \{0\}) .$$

Clearly $\dim(R(M_n)) \le n$ and hence $k \le n$. This proves that M_n is an invariant subspace under R. Hence every $R \in Z_{HS}(U^*)$ must be uppertriangular (with respect to the basis $(e_n)_{n\in\mathbb{N}}$). We now make a special choice for the weights α_n. Let α_n be given by $\alpha_n = \alpha^n$ where $0 < \alpha < 1$.
Further computation shows that $R \in Z_{HS}(U^*)$ iff

6.5.3. R is uppertriangular

6.5.4. $R_{i,i+k} = \beta_k . \alpha^{k(i-1)}$; $i \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$

6.5.5. $\sum_{k=0}^{\infty} \sum_{i=1}^{\infty} |R_{i,i+k}|^2 < \infty$ where $\beta_k \in \mathbb{C}$ for $k = 0,1,2,\dots$.

Condition 6.5.5 implies $\beta_0 = 0$. Combining 6.5.4, 6.5.5 and $\beta_0 = 0$ we obtain

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |R_{i,i+k}|^2 = \sum_{k=1}^{\infty} |\beta_k|^2 \sum_{i=1}^{\infty} \alpha^{2k(i-1)} = \sum_{k=1}^{\infty} \frac{|\beta_k|^2}{1-\alpha^{2k}} < \infty .$$

Note that for all $k \in \mathbb{N}$ we have

$$|\beta_k|^2 \le \frac{|\beta_k|^2}{1-\alpha^{2k}} \le \frac{|\beta_k|^2}{1-\alpha^2} .$$

Hence

$$\sum_{k=1}^{\infty} \frac{|\beta_k|^2}{1-\alpha^{2k}} < \infty \text{ iff } \sum_{k=1}^{\infty} |\beta_k|^2 < \infty$$

and therefore $R \in Z_{HS}(U^*)$ iff

6.5.6. R is uppertriangular

6.5.7. $R_{i,i+k} = \beta_k \alpha^{k(i-1)}$, $i \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$.

6.5.8. $\beta_0 = 0$ and $\sum_{k=1}^{\infty} |\beta_k|^2 < \infty$.

The double norm of an operator $R \in Z_{HS}(U^*)$ is given by

$$|||R||| = \left(\sum_{k=1}^{\infty} \frac{|\beta_k|^2}{1 - \alpha^{2k}} \right)^{\frac{1}{2}}$$

and R has the matrix

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\ & 0 & \alpha\beta_1 & \alpha^2\beta_2 & \alpha^3\beta_3 & \cdots \\ & & 0 & \alpha^2\beta_1 & \alpha^4\beta_2 & \cdots \\ & & & 0 & \alpha^3\beta_1 & \cdots \\ & & & & 0 & \\ & & & & & \end{pmatrix}$$

We are now able to compute $Z_{\mathcal{L}(H)}(U^*)$. Condition 6.5.6 and 6.5.7 still hold if $R \in Z_{\mathcal{L}(H)}(U^*)$, and if $R \in \mathcal{L}(H)$ we have

$$\sum_{k=1}^{\infty} |R_{1k}|^2 = \| R^* e_1 \| = \sum_{k=0}^{\infty} |\beta_k|^2$$

so also $R \in Z_{\mathcal{L}(H)}(U^*)$ implies

$$\sum_{k=0}^{\infty} |\beta_k|^2 < \infty \ .$$

The only difference between $Z_{\mathcal{L}(H)}(U^*)$ and $Z_{HS}(U^*)$ is the condition $\beta_0 = 0$. Hence

$$Z_{\mathcal{L}(H)}(U^*) = \{\lambda I + R \mid \lambda \in \mathbb{C},\ R \in Z_{HS}(U^*)\}$$

and therefore

$$Z_{\mathcal{L}(H)}(U) = \{\lambda I + R^* \mid \lambda \in \mathbb{C},\ R \in Z_{HS}(U^*)\} \ .$$

(In this example $Z(U) \neq Z(U^*)$.)

$Z_{\mathcal{L}(H)}(U)$ splits in $\mathcal{L}(H)$ and a complement is given by the subspace

$$V := \{X \in \mathcal{L}(H) \mid Xe_1 = 0\} .$$

For $B \in V \cap Z_{\mathcal{L}(H)}(U)$ implies $B = 0$ and if $B \in \mathcal{L}(H)$ we have

$$B = (B_{11}I + C) + (B - B_{11}I - C)$$

where $B_{11} = (Be_1, e_1)$ and C is an operator in $Z_{HS}(U)$ with first column equal to the first column of B except $C_{11} = 0$. Thus $C_{k1} = B_{k1}$, $k \geq 2$ and $C_{11} = 0$. Then $B_{11}I + C \in Z_{\mathcal{L}(H)}(U)$ and $B - B_{11}I - C \in V$ hence any operator in $\mathcal{L}(H)$ is the sum of an operator in $Z_{\mathcal{L}(H)}(U)$ and an operator in V.

Suppose now $Ad_U(\mathcal{L}(H))$ is $|||\ |||$-closed in HS. Exactly the same arguments as in example 6.3 show that this implies

6.5.9. $\quad \exists_{\delta>0} \forall_{X \in V} \quad |||XU - UX||| \geq \delta \| X \| .$

Let $X_n \in \mathcal{L}(H)$ be given by

$$X_n = \text{diag}(0,\ldots,0,\underset{\substack{\uparrow \\ n^{th} \text{ component}}}{1},0,\ldots)\ ; \ n \geq 2 .$$

Then $X_n \in V$, $n \geq 2$, $\|X_n\| = 1$ and

$$|||X_n U - UX_n||| = (\alpha^{2(n-1)} + \alpha^{2n})^{\frac{1}{2}} .$$

Hence $\quad \lim_{n\to\infty} |||Ad_U(X_n)||| = 0 .$

This contradicts 6.5.9 and therefore $Ad_U(\mathcal{L}(H))$ is not $|||\ |||$-closed. From corollary 6.2 it follows that also $Ad_U(HS)$ is not $|||\ |||$-closed.

§ 7. The embedding of HS in HS^+

In the previous section we have seen that the space of Hilbert-Schmidt operators equipped with $|||\ |||$ is a Banach algebra without identity. In this section we "adjoin" an identity element and describe the standard embedding of HS in the extended space HS^+ (see [DUN II], Ch. XI, § 6).

7.1. Definition.

$$HS^+ := \{\langle\alpha, A\rangle \mid \alpha \in \mathbb{C},\ A \in HS\} ,$$

and the operations on HS^+ are the following:

addition	: $\langle\alpha,A\rangle + \langle\beta,B\rangle := \langle\alpha + \beta, A + B\rangle$.
scalar multiplication:	$\lambda\langle\alpha,A\rangle := \langle\lambda\alpha,\lambda A\rangle$.
multiplication	: $\langle\alpha,A\rangle.\langle\beta,B\rangle := \langle\alpha\beta,\alpha B + \beta A + AB\rangle$.
involution	: $(\langle\alpha,A\rangle)^* := \langle\bar{\alpha},A^*\rangle$.
innerproduct	: $(\langle\alpha,A\rangle,\langle\beta,B\rangle) := \alpha\bar{\beta} + (A,B)$.
1-norm	: $\Vert\langle\alpha,A\rangle\Vert_1 := \vert\alpha\vert + \vert\vert\vert A\vert\vert\vert$.
2-norm	: $\Vert\langle\alpha,A\rangle\Vert_2 := (\vert\alpha\vert^2 + \vert\vert\vert A\vert\vert\vert^2)^{\frac{1}{2}}$.

The following lemma holds

7.2. Lemma. HS^+ provided with the defined algebraic operations and $\Vert\ \Vert_1$ is a Banach algebra with identity $e = \langle 1,\mathcal{O}\rangle$ and involution. HS^+ equipped with $\Vert\ \Vert_2$ is a Hilbert space and the norms $\Vert\ \Vert_1$ and $\Vert\ \Vert_2$ are equivalent on HS^+.

Proof. The first part of the lemma is a standard result (see [DUN II], Ch. XI, § 6). We only prove the equivalence of $\Vert\ \Vert_1$ and $\Vert\ \Vert_2$. If $\langle\alpha,A\rangle \in HS^+$ we have

$$\vert\alpha\vert^2 + \vert\vert\vert A\vert\vert\vert^2 \le (\vert\alpha\vert + \vert\vert\vert A\vert\vert\vert)^2 \le 2(\vert\alpha\vert^2 + \vert\vert\vert A\vert\vert\vert^2)$$

and hence

$$\Vert\langle\alpha,A\rangle\Vert_2 \le \Vert\langle\alpha,A\rangle\Vert_1 \le \sqrt{2}\Vert\langle\alpha,A\rangle\Vert_2 \ .$$ □

7.3. Corollary. Lemma 7.2 shows that any $\Vert\ \Vert_1$-open (closed) set is a $\Vert\ \Vert_2$-open (closed) set and vice versa, and therefore every subspace ($\Vert\ \Vert_1$ or $\Vert\ \Vert_2$-closed) has a closed complement, namely the orthogonal complement in the space HS^+, and this complement is also $\Vert\ \Vert_1$-closed.

7.4. Remark. Note that HS^+ provided with $\Vert\ \Vert_1$ is not a B^*-algebra, because in general

$$\Vert\langle\alpha,A\rangle^*\langle\alpha,A\rangle\Vert_1 \neq \Vert\langle\alpha,A\rangle\Vert_1^2 \ .$$

The natural embedding map Emb: $HS \to HS^+$, which maps A into $\langle 0,A\rangle$, is an isometric * isomorphism from HS onto Emb(HS), which is subalgebra of HS^+. For example we have:

$$\text{Emb}(A + B) = \text{Emb}(A) + \text{Emb}(B)$$

$$\text{Emb}(A^*) = (\text{Emb}(A))^*$$

$$(A,B) = (\mathrm{Emb}(A),\mathrm{Emb}(B))$$

$$\|\mathrm{Emb}(A)\|_{1,2} = |||A||| \,.$$

7.5. Definition. Since HS^+ with $\| \ \|_1$ is a Banach algebra with identity $e = \langle 1,\mathcal{O}\rangle$ the set of non-singular elements G^+ is an open set in HS^+ containing e (see § 1). As in § 1 definition 1.2 we can define the mappings $\alpha_a^+ : G^+ \to HS^+$ by $\alpha_a^+(g) = gag^{-1}$ and $Ad_a^+ : HS^+ \to HS^+$, being the derivative of α_a^+ at e. The kernel of Ad_a^+ is the centralizer of a in HS^+; notation $Z_{HS^+}(a)$.

7.6. Remark. If $a = \langle\alpha,A\rangle$, the map Ad_a^+ and the set $Z_{HS^+}(a)$ are closely related to Ad_A respectively $Z_{HS}(A)$. It is easily seen that

7.6.1. $\quad Ad_a^+(HS^+) = \{\langle 0,B\rangle \mid B \in Ad_A(HS)\}$

7.6.2. $\quad Z_{HS^+}(a) = \{\langle\gamma,C\rangle \mid \gamma \in \mathbb{C},\ C \in Z_{HS}(A)\}$.

7.7. Corollary. From 7.6 it follows that $Ad_a^+(HS^+)$ is $\| \ \|_1$-closed iff $Ad_A(HS)$ is $||| \ |||$-closed.

7.8. Theorem. Let $a \in HS^+$. Then

$$(Ad_a^+(HS^+))^\perp = Z_{HS^+}(a^*) \quad .$$

Proof. Use 7.6.1, 7.6.2 and theorem 5.7. □

We now define the map $\theta : HS^+ \to \mathcal{L}(H)$ by

$$\theta(\langle\alpha,A\rangle) := \alpha I + A$$

(see [DUN II], Ch. XI, § 6). Then θ is an injective, continuous, homomorphism from HS^+ into $\mathcal{L}(H)$. We only prove the continuity of θ (the rest of this statement is also easy to verify)

$$\|\theta(\langle\alpha,A\rangle)\| = \|\alpha I + A\| < |\alpha| + \|A\| \le |\alpha| + |||A||| = \|\langle\alpha,A\rangle\|_1 \,.$$

Note that $\langle\alpha,A\rangle \in G^+$ (is invertible in HS^+) iff $\alpha I + A \in G$ (is invertible in $\mathcal{L}(H)$) and

$$\theta(\langle\alpha,A\rangle^{-1}) = (\alpha I + A)^{-1}$$

(see [DUN II], Ch. XI, § 6).

Finally we prove two lemmas which show the relationship between similarity in HS^+ and the induced relation in HS (note that similarity in HS is not yet defined).

7.9. Lemma. Let $<\alpha,A>,<\beta,B> \in HS^+$ and $<\gamma,C> \in G^+$. Then

$$<\beta,B> = <\gamma,C> <\alpha,A> <\gamma,C>^{-1}$$

iff

$$\begin{cases} \alpha = \beta \\ <0,B> = <\gamma,C><0,A><\gamma,C>^{-1} . \end{cases}$$

Proof. Note that $<\gamma,C> \in G^+$ implies $\gamma \neq 0$. The rest of the proof is computation. □

7.10. Lemma. Let $<\alpha,A> \in HS^+$ and $<\gamma,C> \in G^+$, then

$$<\gamma,C><\alpha,A><\gamma,C>^{-1} = <\alpha,(\gamma I + C)A(\gamma I + C)^{-1}> .$$

Proof. Put $<\beta,B> := <\gamma,C><\alpha,A><\gamma,C>^{-1}$. Applying the preceeding lemma we have $\beta = \alpha$ and $<0,B> = <\gamma,C><0,A><\gamma,C>^{-1}$. Hence (using that θ is a homomorphism) we find

$$\theta(<0,B>) = \theta(<\gamma,C>)\theta(<0,A>)\theta(<\gamma,C>^{-1})$$

and therefore

$$B = (\gamma I + C)A(\gamma I + C)^{-1}$$

which completes the proof. □

§ 8. Heuristics

In this section we discuss the possible extension of theorem I.3.6 to deformations of operators defined on an infinite dimensional Hilbert space H. The natural relation with regard to which versality, of deformations of operators is considered is the relation of similarity. If two operators are similar the only difference between them lies in the chosen basis of the underlying Hilbert space H. For example, all spectral properties of two similar operators are the same.

By a deformation of an operator $A_0 \in \mathcal{L}(H)$ we mean a differentiable mapping A from an open neighbourhood U of the origin in a Banach space E into $\mathcal{L}(H)$ with $A(0) = A_0$ and double splitting at 0 (see definition B4). As in definition I.2.1 the space E will be called the base of the deformation. A straightforward generalization of the definition of versal deformation (see definition I.2.4) runs as follows: A deformation A of an operator $A_0 \in \mathcal{L}(H)$ with base E is versal iff for every deformation B of A_0 with base F we have

8.1. $$B(s) = C(s)\, A(\varphi(s)) C^{-1}(s)$$

for small $s \in F$; where C is a deformation of the identity operator $I \in \mathcal{L}(H)$ and φ is a differentiable map from F into E with $\varphi(0) = 0$. Suppose A is a versal deformation of A_0 then by taking the derivatives at t = 0 at both sides of 8.1 we obtain an equation analogous to I.3.6.1:

8.2. $$(D_0B)\zeta = [(D_0C)\zeta, A_0] + (D_0A)(D_0\varphi)\zeta$$

for all $\zeta \in T_0F$.

This implies, just as in the proof of theorem I.3.6, that every operator in $\mathcal{L}(H)$ is the sum of a commutator of the form $[C, A_0]$ and an operator in the image of D_0A. Suppose A_0 is normal. Then by corollary 4.3 we have $Z(A_0) \cap Ad_{A_0}(\mathcal{L}(H)) = \{0\}$. Since by theorem 2.5 $Z(A_0)$ is always infinite dimensional, versality of A implies that $Ran(D_0A)$ is infinite dimensional. It is not difficult to prove, with the aid of the Kleinecke-Shirokov theorem (theorem 4.1) and theorem 2.5, that a complement of $Ad_{A_0}(\mathcal{L}(H))$ is always infinite dimensional (even if A_0 is not normal) and therefore there are no versal deformations with finite dimensional base.

Suppose the original operator A_0 is Hilbert-Schmidt. Let S denote the norm closure of the set $Ad_{A_0}(\mathcal{L}(H))$ in $\mathcal{L}(H)$. Since HS is a two sided ideal in $\mathcal{L}(H)$ every operator in $Ad_{A_0}(\mathcal{L}(H))$ is Hilbert-Schmidt and therefore S is a subset of the set of compact operators on H. Hence versality of A implies that $Ran(D_0A)$ contains at least a complement of the subspace of compact operators in $\mathcal{L}(H)$. For this reason we only study deformations in a smaller class of operators: not in $\mathcal{L}(H)$ but in the space of Hilbert-Schmidt operators which is still a large and important class. So, we shall consider deformations of Hilbert-Schmidt operators in the space HS. In this case we have two possible ways to define similarity and the orbit.

Let $A, B \in$ HS.

i) Similarity induced from $\mathcal{L}(H)$.

$A \sim B$ iff there is a $C \in G \subset \mathcal{L}(H)$ such that $B = CAC^{-1}$.

The corresponding orbit is $N_1 := \alpha_A(G)$ (see remark 5.4).

ii) Similarity induced from HS^+ (see definition 7.5).

$A \sim B$ iff $<0,A> \sim <0,B>$ in HS^+ which by lemma 7.9 and lemma 7.10 is equivalent to

$$B = (\gamma I + C)A(\gamma I + C)^{-1} \quad \text{with } <\gamma,C> \in G^+ .$$

The corresponding orbit is

$$N_2 := \{(\gamma I + C)A(\gamma I + C)^{-1} \mid <\gamma,C> \in G^+\}$$

(the obits N_1 and N_2 need not to be submanifolds of HS).

8.3. Remark. As defined in § 7 of this chapter, the set G^+ is a subset of HS^+. In the heuristic approach of this section, however, we consider G^+ as a subset of G:

$$G^+ = G \cap \{\lambda I + C \mid \lambda \in \mathbb{C},\ C \in HS\} .$$

Of course we want to keep the base of our versal deformations as "small" as possible and therefore the orbits as "large" as possible. Obviously $N_2 \subset N_1$, but by theorem 5.9 we have $\overline{\overline{Ad_{A_0}(\mathcal{L}(H))}} = \overline{\overline{Ad_{A_0}(HS)}}$. ($A_0 + Ad_{A_0}(\mathcal{L}(H))$ and $A_0 + Ad_{A_0}(HS)$ can be considered as linear approximations of N_1 respectively N_2 at A_0). This means that for the "size" of the base of a versal deformation it makes no difference for our theory whether we consider the action of G or G^+ (case i, or case ii)) on HS because we shall prove the equivalence of versality (in fact weak-versality) and transversality to the space $\overline{\overline{Ad_{A_0}(HS)}}$. In case i) (if we consider the action of the group G on HS) it is not guaranteed that there is a submanifold of G minimal transversal to $Z_{\mathcal{L}(H)}(A_0)$ at I because it is not guaranteed that $Z_{\mathcal{L}(H)}(A_0)$ splits in $\mathcal{L}(H)$ (although it does so when A_0 is normal (see theorem 3.2). This submanifold plays an important role in the proof of theorem I.3.6 as well as in the proof of theorem III.4.2). In case ii) we can always find a submanifold of G^+ minimal transversal to $Z_{HS^+}(a_0)$ (where $a_0 = <0,A_0>$) because HS^+ is a Hilbert space and therefore every subspace splits. Therefore we choose case ii).

Suppose A is a versal deformation of $A_0 \in HS$. Then condition 8.2 is still valid for deformations of A_0 in HS. Since, by theorem 5.7 $(Ad_{A_0}(HS))^{\perp} =$ $= Z_{HS}(A_0^*)$ and $\dim Z_{HS}(A_0^*) = \infty$ (see remark 5.6) the subspace $Ran(D_0A) \subset HS$

must be infinite dimensional (under the assumption that A is versal). Hence every versal deformation depends on infinitely many (one dimensional) complex parameters (i.e. the base of the deformation is infinite dimensional). A straightforward generalization of theorem I.3.6 is still impossible. In § 6 we have seen that there are many operators for which $Ad_{A_0}(HS)$ is not $|||\ |||$-closed, (e.g. all normal Hilbert-Schmidt operators). Let A_0 be such an operator. Suppose A is a deformation of A_0 minimal transversal to $\overline{Ad_{A_0}(HS)}$ at 0 that is

$$\mathrm{Ran}(D_0A) \oplus \overline{Ad_{A_0}(HS)} = HS$$

(see definition B5).

Since $Ad_{A_0}(HS)$ is not $|||\ |||$-closed we can choose $X \in \overline{Ad_{A_0}(HS)} \setminus Ad_{A_0}(HS)$ and consider the 1-dimensional deformation B of A_0 defined by

$$B(t) := A_0 + tX; \quad t \in \mathbb{C} .$$

The derivative $D_0B: \mathbb{C} \to HS$ is the linear map $t \to tX$; $t \in \mathbb{C}$, and therefore $(D_0B)(1)$ cannot be written as the sum of an operator in $Ad_{A_0}(HS)$ and an operator in D_0A and hence A is not versal (see 8.2). This means that transversality does not imply versality in the sense defined in this section. In chapter III we shall define weak-versality which is equivalent to transversality.

Appendix.

Some lemmas in Hilbert space

Before starting with chapter III we shall give some standard lemmas on projections in Hilbert space. These lemmas are used in the proof of theorem III.4.2 to get round the difficulties of the infinite dimensional case. Let h denote a Hilbert space.

1. Definition. Let $(V_n)_{n\in\mathbb{N}}$ and V be subspaces of h. We define

$$V_n \overset{s}{\to} V$$

iff

$$\begin{cases} V_1 \subset V_2 \subset \dots \subset V \\ P_n \overset{s}{\to} P \end{cases}$$

where P_n is the orthogonal projection onto V_n and P is the orthogonal projection onto V. $P_n \overset{s}{\to} P$ means convergence in the strong operator topology of $\mathcal{L}(h)$.

2. Lemma. Let $(V_n)_{n\in\mathbb{N}}$ and V be subspaces of h such that $V_1 \subset V_2 \subset \dots \subset V$. Then $V_n \overset{s}{\to} V$ iff for every $x \in V$

$$\lim_{n\to\infty} \min_{v\in V_n} \|x - v\| = 0 .$$

Proof. Only the non-trivial if-part is proved here. Choose $v \in V$ and select a sequence $v_n \in V_n$ with $v_n \to v$, then

$$\|Pv - P_n v\| = \|P(v - v_n) + Pv_n - P_n(v - v_n) - P_n v_n\| \le$$

$$\le (\|P\| + \|P_n\|)\|v - v_n\| + \|Pv_n - P_n v_n\| .$$

Since both P and P_n are orthogonal projections and $V_n \subset V$ we obtain

$$\|Pv - P_n v\| \le 2.\|v - v_n\| .$$

Hence

$$\|Pv - P_n v\| \to 0 \quad \text{if } n \to \infty .$$

If $w \in V^{\perp}$ then $Pw = P_n w = 0$. Hence $P_n \overset{s}{\to} P$. □

3. Lemma. Let $L \in \mathcal{L}(h)$ and $V_n \overset{\subsetneq}{\to} V$ in h. Then $\overline{L(V_n)} \overset{\subsetneq}{\to} \overline{L(V)}$.

Proof. Clearly

$$\overline{L(V_1)} \subset \overline{L(V_2)} \subset \ldots \subset \overline{L(V)} .$$

Let $x \in \overline{L(V)}$. We first prove that if $\varepsilon > 0$ there is a $z \in L(V_{n(\varepsilon)})$ such that

$$\|x - z\| < \varepsilon .$$

To do so select $y = Lv$, $v \in V$ with

$$\|x - y\| < \tfrac{1}{2}\varepsilon$$

next choose $n(\varepsilon)$ and $w \in V_{n(\varepsilon)}$, with

$$\|w - v\| < \frac{\varepsilon}{2(\|L\| + 1)} .$$

Define $z := Lw$, then $z \in L(V_{n(\varepsilon)})$ and

$$\|x - z\| \leq \|x - y\| + \|y - z\| \leq \|x - y\| + \|L\|.\|v - w\| < \varepsilon .$$

This proves

$$\min_{z \in L(V_{n(\varepsilon)})} \|x - z\| < \varepsilon$$

and hence, since $\overline{L(V_1)} \subset \overline{L(V_2)} \subset \ldots \subset \overline{L(V)}$

$$\min_{z \in L(V_n)} \|x - z\| < \varepsilon \quad \text{if } n \geq n(\varepsilon) .$$

So

$$\lim_{n\to\infty} \min_{z \in L(V_n)} \|x - z\| = 0 .$$

Since $x \in \overline{L(V)}$ is arbitrary the previous lemma proves

$$\overline{L(V_n)} \overset{\subsetneq}{\to} \overline{L(V)} .$$ □

We quote a standard result on the sum of subspaces (see [HAL I], problem 8).

4. Lemma. If V and W are subspaces on h with $V \cap W = \{0\}$ and if V has finite dimension, then $V \oplus W$ is closed (equal to $\mathrm{span}(V \cup W)$) and the canonical projection operators

$$P_V : V \oplus W \to V$$
$$P_W : V \oplus W \to W$$

are bounded. (Considered as operators in $\mathcal{L}(V \oplus W)$).

5. Corollary. If V and W satisfy the assumptions of lemma 4 and if $N := (V \oplus W)^{\perp}$ then bounded projections P_V, P_W and P_N (onto V, W and N) exist, such that

$$P_V + P_W + P_N = \mathrm{id}_h$$

$$\mathrm{Ker}(P_V) = W \oplus N$$

$$\mathrm{Ker}(P_W) = V \oplus N$$

$$\mathrm{Ker}(P_N) = V \oplus W \; .$$

Note that all direct sums are equal to the span and hence are closed.

III. Deformations of Hilbert-Schmidt Operators

§ 0. Introduction

In this chapter we shall consider deformations of Hilbert-Schmidt operators and we shall prove the main theorem of this paper (theorem 5.5) which is the extension of theorem I.3.6.

As pointed out in chapter II, § 8, transversality of a deformation to the closure of Ad(HS) does not imply versality in the sense of chapter I. In § 4 we shall define weak-versality, which, as proved in that section, is equivalent to transversality.

The theory is first developed in the Banach algebra HS^+, but with the aid of lemma II.7.9 and lemma II.7.10 the theory can be translated immediately to Hilbert-Schmidt operators (see § 5).

Before starting with § 1 we choose an arbitrary element $x_0 \in HS^+$ which remains fixed throughout the sections 1,2,3,4. In these sections we shall use the shorter notations: S^+ for the $\| \ \|_1$-closure of $Ad^+_{x_0}(HS^+)$, Z^+ for $Z_{HS^+}(x_0)$ and Ad^+ for the map $Ad^+_{x_0}$ (see II, § 7).

§ 1. Slices in G^+

In this section we define submanifolds of $G^+ \subset HS^+$ of a simple form, which are called slices. Note that the set G^+ is a submanifold of HS^+ (proof: G^+ is open).

1.1. Lemma. Suppose V is a finite dimensional subspace of HS^+. Let B^+ denote the $\| \ \|_1$-open unit ball in HS^+:

$$B^+ := \{a \in HS^+ \mid \|a\|_1 < 1\} .$$

Define

$$G^+(V) := e + (B^+ \cap V) := \{e + a \mid a \in B^+ \cap V\}$$

then $G^+(V) \subset G^+$ and $G^+(V)$ is a finite dimensional submanifold of G^+. The tangent space of $G^+(V)$ at x equals V: $T_x G^+(V) = V$.
A submanifold of this type is called a finite dimensional slice.

Proof. Let W denote the orthogonal complement of V in HS^+. Define $B_{e,1} := e + B^+$ and $V_1 := V \cap B^+$ and $W_1 := W \cap B^+$, then $V_1 \subset V$ and $W_1 \subset W$ are open sets in the relative topology, induced by $\| \ \|_1$, of V respectively W.

Let P_1 and P_2 denote orthogonal projection on V respectively W. Define

$$\psi: B_{e,1} \to V_1 \times W_1$$

by

$$\psi(x) := (P_1(x - e), P_2(x - e)) .$$

Then ψ is a C^∞-diffeomorphism from $B_{e,1}$ onto $V_1 \times W_1$ and $\psi(G^+(V)) = V_1 \times \{0\}$. Hence, by definition B2, $G^+(V)$ is a submanifold of G^+ which is diffeomorphic to an open set in $\mathbb{C}^k$, where $k = \dim(V)$. The statement about the tangent space is obvious. □

<u>1.2. Definition</u>. If N_1 and N_2 are (C^p, $p \geq 1$) submanifolds of a manifold M then we say N_1 intersects N_2 at x iff $x \in N_1 \cap N_2$ and $T_xN_1 \cap T_xN_2 = \{0\}$.

<u>1.3. Lemma</u>. Let V be a finite dimensional subspace of HS^+ such that $V \cap Z^+ = \{0\}$. Then the slice $G^+(V)$ intersects Z^+ at e.

<u>Proof</u>. This is a trivial consequence of definition 1.2 and the previous lemma. □

§ 2. <u>Deformations in HS^+. Versality in a submanifold</u>

<u>2.0. Notation</u>. The letters H and K will denote Hilbert spaces and Ω_H, Ω_K will always denote open neighbourhoods of the origin in H respectively K.

<u>2.1. Definition</u>. A deformation of an element $x_0 \in HS^+$ is a map $x \in C^1(\Omega_H \to HS^+)$ such that $x(0) = x_0$ and x is double splitting at 0 (see definition B4). The space H is called the base of the deformation.

<u>2.2. Remark</u>. Since x is a map from an open subset of a Hilbert space into a Hilbert space, double splitting at 0 is equivalent to $\mathrm{Ran}(D_0x)$ is closed (see definition B4).

In the following lemma we introduce a submanifold of HS^+.

<u>2.3. Lemma</u>. Suppose $G^+(V)$ is a finite dimensional slice intersecting Z^+ at e, i.e. $V \cap Z^+ = \{0\}$ (see definition 1.2).
Let x be a deformation of x_0 with base H, transversal to $x_0 + S^+$ at 0.
Assume furthermore that

2.3.1. $x_* := D_0 x$ is injective .

2.3.2. $\text{Ran } x_* \cap Ad^+(V) = \{0\}$.

Define

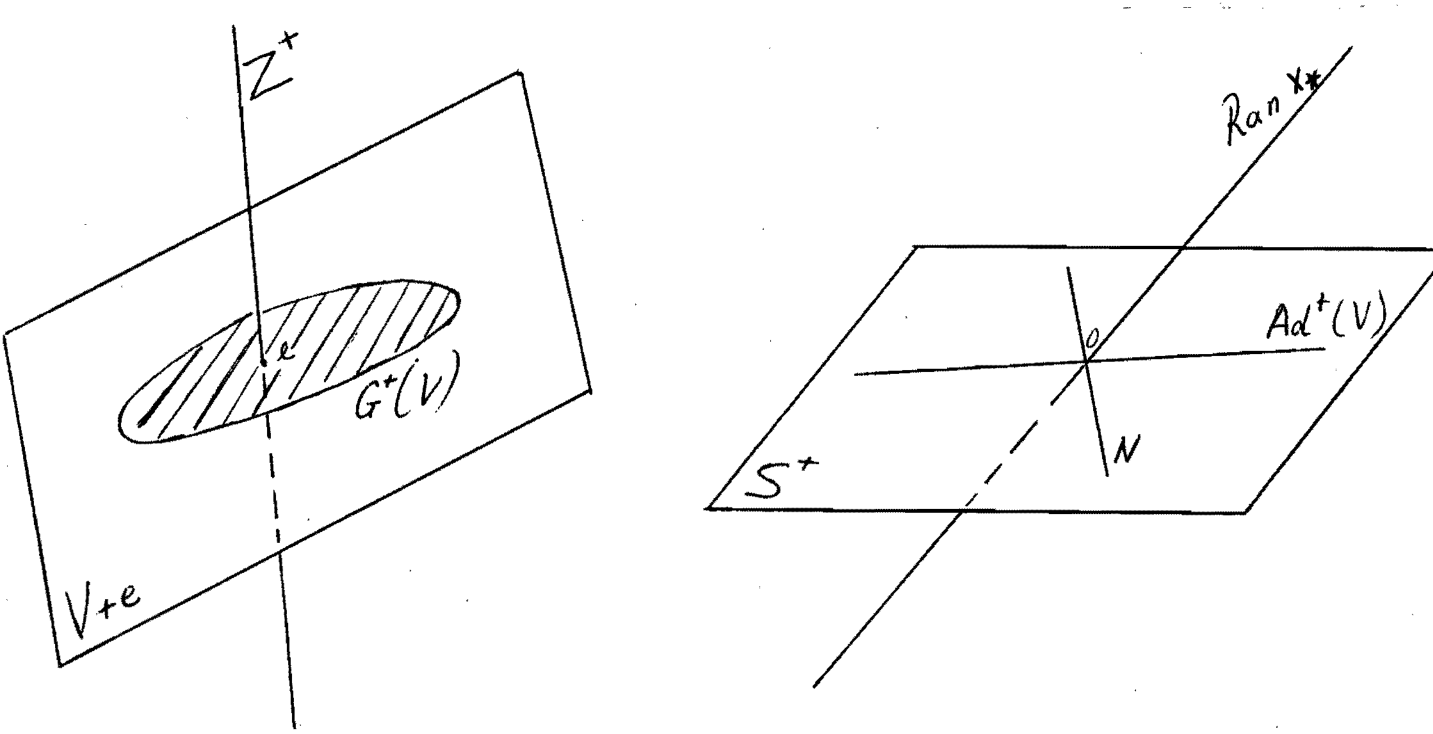
$$M_0 := \{gx(t)g^{-1} \mid g \in G^+(V),\ t \in \Omega_H\}$$

(where Ω_H is the open set on which x is defined).
Then there is an open ball $B_{x_0} \subset HS^+$ centered at x_0 such that

$$M := M_0 \cap B_{x_0}$$

is a submanifold of HS^+.

Proof.



Since $Ad^+(V)$ is finite dimensional and $\text{Ran } x_* \cap S^+$ is closed ($\text{Ran } x_*$ and S^+ are closed) it follows from II, appendix, lemma 4 that the space $(\text{Ran } x_* \cap S^+) \oplus Ad^+(V)$ is closed (note that we may write $\oplus$ since by 2.3.2 $\text{Ran } x_* \cap Ad^+(V) = \{0\}$). Let N denote a closed complement of the subspace $(\text{Ran } x_* \cap S^+) \oplus Ad^+(V)$ in S^+ (e.g. the orthogonal complement in S^+). We shall prove that

$$(0,e,0) \in U_1 \times U_2 \times U_3$$

$$x_0 \in U_0$$

and γ is a diffeomorphism from $U_1 \times U_2 \times U_3$ onto U_0. Now

$$\gamma^{-1}(x_0) = (0,e,0)$$

and

$$\gamma^{-1}(U_0 \cap M_0) = U_1 \times U_2 \times \{0\} .$$

Hence, by definition B2, $M = M_0 \cap B_{x_0}$ is a submanifold of HS^+ if $B_{x_0} \subset U_0$. The tangent space at x_0 is the subspace

$$T_{x_0} M = \text{Ran } x_* \oplus Ad^+(V) .$$

□

The following lemma deals with deformations of x_0 with values in M.

2.4. Lemma. Let $G^+(V)$, x and M be defined as in lemma 2.3. Suppose y is a deformation of x_0 with base K and values in M, i.e.

$$y \in C^1(\Omega_K \to M);$$

y is double splitting at 0 and $y(0) = x_0$.
Then there is an open neighbourhood Ω_K^1 of the origin in K and there are mappings

$$c \in C^1(\Omega_K^1 \to G^+(V))$$

$$\varphi \in C^1(\Omega_K^1 \to \Omega_H)$$

with $c(0) = e$ and $\varphi(0) = 0$ such that

$$y(t) = c(t)x(\varphi(t))c^{-1}(t); \quad t \in \Omega_K^1 .$$

Proof. (The proof of this lemma is analogous to the proof of theorem I.3.6). The set $B^+ \cap V$ is open in the relative topology of V and contained in G^+. Define

$$\beta: (B^+ \cap V) \times \Omega_H \to M$$

by

$$\beta(v,t) := (e + v)x(t)(e + v)^{-1} .$$

This definition makes sense since $e + v \in G^+$ if $v \in B^+ \cap V$. With the same arguments as used in the proofs of theorem I.3.6 and lemma 2.3 it can be

$$\text{Ran } x_* \oplus \text{Ad}^+(V) \oplus N = HS^+ .$$

Let $y \in HS^+$ then, by the transversality of the deformation x to $x_0 + S^+$ we have $y = y_1 + y_2$ with $y_1 \subset \text{Ran } x_*$ and $y_2 \in S^+ \ominus \text{Ran } x_*$. Since

$$(\text{Ran } x_* \cap S^+) + \text{Ad}^+(V) + N = S^+$$

we have

$$y_2 = 0 + y_3 + y_4$$

with $y_3 \subset \text{Ad}^+(V)$ and $y_4 \in N$. Hence $y = y_1 + y_3 + y_4$ with $y_1 \in \text{Ran } x_*$, $y_3 \in \text{Ad}^+(V)$, $y_4 \subset N$. We leave it to the reader to notice that Ran x_*, $\text{Ad}^+(V)$ and N are mutually independent.
We now define $\gamma: \Omega_H \times G^+(V) \times N \to HS^+$ by

$$\gamma(t,g,n) := gx(t)g^{-1} + n .$$

Then $\gamma(0,e,0) = x_0$ and γ is differentiable in $\Omega_H \times G^+(V) \times N$ (see [LAN], Ch. I, § 3, prop. 11). The derivative at (0,e,0)

$$\gamma_* := D_{(0,e,0)}\gamma: H \times V \times N \to HS^+$$

is given by

$$\gamma_*(t,g,n) = x_*(t) + [g,x_0] + n .$$

(The space $H \times V \times N$ becomes a Banach space in one of the usual ways; by defining $\|(t,g,n)\| := \max(\|t\|_H, \|g\|_1, \|n\|_1)$ and then the map γ_* is a bounded linear operator from $H \times V \times N$ into HS^+).
Since

$$\text{Ran } x_* \oplus \text{Ad}^+(V) \oplus N = HS^+$$

and

$$V \cap Z^+ = \{0\}$$

we may conclude

$$\text{Ker } \gamma_* = (0,0,0)$$

and

$$\text{Ran } \gamma_* = HS^+ .$$

Hence, since γ_* is bounded, it follows from the closed graph theorem of Banach that γ_* is invertible as a linear operator. Hence by the inverse function theorem (see [LAN], Ch. I, § 5, Th. 1). γ is a local diffeomorphism at (0,e,0) and therefore there are open sets $U_1 \subset \Omega_H$, $U_2 \subset G^+(V)$, $U_3 \subset N$ and $U_0 \subset HS^+$ such that:

shown that β determines a C^1-diffeomorphism from an open neighbourhood $\Omega_V^0 \times \Omega_H^0$ of $(0,0)$ in $V \times H$ onto an open subset $\Omega_M^0 \subset M$ containing x_0 (Ω_V^0 is open in the relative topology of V and Ω_M^0 is a set of the form $\mathcal{O} \cap M$ where $\mathcal{O}$ is open in HS^+; M is given the relative topology induced by $\| \ \|_1$).

Let π_1 and π_2 denote the canonical projections of $\Omega_V^0 \times \Omega_H^0$ onto Ω_V^0 respectively Ω_H^0. Obviously there is an open set $\Omega_K^1 \subset \Omega_K^0$ such that $y(\Omega_K^1) \subset \Omega_M^0$. Hence if $t \in \Omega_K^1$ we have:

$$y(t) = \beta(w,s)$$

for some $w \in V$ and $s \in H$. Hence

$$y(t) = c(t)x(\varphi(t))c^{-1}(t); \qquad t \in \Omega_K^1$$

where

$$c(t) := e + \pi_1\beta^{-1}(y(t))$$

and

$$\varphi(t) := \pi_2\beta^{-1}(y(t)) \ .$$

Since β^{-1} is C^1 on Ω_M^0 and π_1 and π_2 are both C^∞, it follows from [LAN], Ch. I, § 3, prop. 7 that c and φ are C^1 on Ω_K^1. □

2.5. Remark. As in chapter I the theory in this chapter is essentially local. We do not care how small Ω_K^1 is.

§ 3. An exponential map

Let V, $G^+(V)$ and M be defined as in lemma 2.3 and β as in the proof of lemma 2.4.

3.1. Definition. The mapping EXP: $T_{x_0}M \to M$ is defined by

$$EXP := \beta \circ \beta_*^{-1}$$

where $\beta_* := D_{(0,0)}\beta$.

3.2. Lemma. If $a \in T_{x_0}M$ and $\|a\|_1$ is sufficiently small we have

3.2.1. $\quad \|EXP(a) - (x_0 + a)\|_1 = o(\|a\|_1)$.

Proof. We compute EXP(0) and D_0EXP:

$$\mathrm{EXP}(0) = \beta(\beta_*^{-1}(0)) = \beta(0) = x_0$$

$$D_0\mathrm{EXP} = D_0(\beta \circ \beta_*^{-1}) = \mathrm{id}_{T_{x_0}M}$$

and this implies 3.2.1. □

§ 4. Weakly versal deformations. Weak-versality ⇔ transversality

4.1. Definition. A deformation x of $x_0 \in HS^+$ with base H is weakly versal iff for every deformation y of x_0 with finite dimensional base K there exists a map $\varphi \in C^1(\Omega_K \to \Omega_H)$, with $\varphi(0) = 0$, such that for every $\varepsilon > 0$ there is a deformation c_ε of the identity $e \in HS^+$ with base K such that:

4.1.1. $\quad \| y(s) - c_\varepsilon(s) x(\varphi(s)) c_\varepsilon^{-1}(s) \|_1 \le \varepsilon \| s \|; \quad s \in \Omega_K^\varepsilon$

where Ω_K^ε is open in K and depends on ε. (note that if Ω_K^ε is small enough $c_\varepsilon(s) \in G^+$).

In the next theorem we shall prove the equivalence of weak-versality and transversality to the set $x_0 + S^+$. The proof of the implication weak-versality ⇒ transversality is rather easy. The proof of the implication the other way around is based on the following idea. The map $y - x_0$ splits into two parts (depending on ε) y_1 and y_2, y_1 with values in $T_{x_0}M_\varepsilon$ and y_2 with values in the orthogonal complement of $T_{x_0}M_\varepsilon$. (M_ε is a submanifold of HS^+ of the type described in lemma 2.3). The map $x_0 + y_1$ is close enough to a deformation described in lemma 2.4 and $\| y_2 \|_1$ is small. At the end of the proof we shall see that the transformation of the base, φ_ε, can be chosen independently of ε.

4.2. Theorem. (Weak-versality ⇔ transversality). x is a weakly versal deformation of x_0 iff x is transversal to $x_0 + S^+$ at 0.

Proof. A) weak-versality ⇒ transversality. Suppose $y \in C^1(\Omega_K \to HS^+)$ is an arbitrary deformation of x_0 with base K. Then by the weak-versality of x we have

4.2.1. $\quad \| y(s) - c_\varepsilon(s) x(\varphi(s)) c_\varepsilon^{-1}(s) \|_1 \le \varepsilon \| s \|; \quad s \in \Omega_K^\varepsilon \quad .$

Define

4.2.2. $\quad z_\varepsilon(s) := y(s) - c_\varepsilon(s)x(\varphi(s))c_\varepsilon^{-1}(s)$

for $s \in \Omega_K^\varepsilon$. Then $z_\varepsilon(0) = 0$ and z_ε is C^1 on Ω_K^ε.
The derivative of z_ε at $s = 0$

$$z_{\varepsilon,*} := D_0 z_\varepsilon : K \to HS^+$$

is given by

4.2.3. $\quad z_{\varepsilon,*}(\xi) = y_*(\xi) - ([c_{\varepsilon,*}(\xi), x_0] + x_*\varphi_*(\xi))$

for all $\xi \in K$, where $y_* = D_0 y$, $x_* := D_0 x$, $\varphi_* := D_0\varphi$ and $c_{\varepsilon,*} := D_0 c_\varepsilon$. From 4.2.1 and 4.2.3 we derive $\|z_{\varepsilon,*}\| < 2\varepsilon$ where the norm is the norm of $\mathcal{L}(K \to HS^+)$.
Using 4.2.3 we obtain

$$y_* = x_*\varphi_* + \lim_{\varepsilon\to 0} [c_{\varepsilon,*}, x_0]$$

where the limit is taken in the norm topology of $\mathcal{L}(K \to HS^+)$. Hence any vector in Ran y_* can be written as the sum of a vector in Ran x_* and a vector in S^+. Since y is arbitrary it follows that Ran $x_* + S^+ = HS^+$ and therefore x is transversal to $x_0 + S^+$ (see definition B5).
B) Transversality ⇒ weak-versality. Let x be a deformation of x_0 with base H transversal to $x_0 + S^+$ at 0. Then Ran x_* is closed and contains a complement of S^+ in HS^+.
We shall assume that x_* satisfies the conditions

4.2.4. $\quad x_*$ is injective .

4.2.5. $\quad \text{Ran } x_* \cap S^+ = \{0\}$.

These assumptions imply the conditions 2.3.1 and 2.3.2 of lemma 2.3 for every finite dimensional V. On the other hand these assumptions cause no loss of generality. Since, if x_* is not injective we replace the base H by H' (e.g. the orthogonal complement of Ker x_*) such that the derivative x'_* at 0 of the restriction x' of x to H' is injective and the deformation x' is still transversal to the manifold $x_0 + S^+$ at 0. Obviously weak-versality of x' implies weak-versality of x. Moreover, if condition 4.2.5 is not satisfiedwe can use similar arguments: since S^+ is closed and x_* is continuous $\overset{\leftarrow}{x_*}(S^+)$ is closed in H. Therefore it is possible to replace H by H' (a complement of the space $\overset{\leftarrow}{x_*}(S^+) \cap \text{Ran } x_*$) such that the restriction x' of x to H' satisfies 4.2.5 and is still transversal to $x_0 + S^+$ at 0. If x' is weakly versal then x itself is certainly weakly versal.

Now if these assumptions are fulfilled we choose a sequence of finite dimensional subspaces $(V_n)_{n\in\mathbb{N}}$ with $V_n \subset HS^+$ such that $V_n \overset{s}{\to} V$ where V is the orthogonal complement of Z^+ in HS^+ and $\overset{s}{\to}$ is defined in II, appendix, definition 1. The V_n's can be chosen as follows:

$$V_n := \text{span}\{f_1,\dots,f_n\}$$

where $f_1, f_2, \dots$ is an orthonormalbasis for $V \subset HS^+$. Applying lemma 3 of the appendix of chapter II we may conclude

$$Ad^+(V_n) \overset{s}{\to} \overline{Ad^+(V)} = S^+ .$$

Define M_n as follows

$$M_n := \{gx(t)g^{-1} \mid g \in G^+(V_n),\ t \in \Omega_H\} \cap B_n$$

where B_n is an open ball centered at x_0 and $G^+(V_n)$ is a finite dimensional slice (see § 1). Then by lemma 2.3 M_n is a submanifold of HS^+ if the ball B_n is small enough.

For every n we have

$$T_{x_0}M_n = \text{Ran } x_* \oplus Ad^+(V_n)$$

and this space is closed by II, appendix, lemma 4. Since x satisfies 4.2.5 the sum is a direct sum. Let N_n denote the orthogonal complement of $\text{Ran } x_* \oplus Ad^+(V_n)$ in HS^+. Then

$$\text{Ran } x_* \oplus Ad^+(V_n) \oplus N_n = HS^+$$

(compare the proof of lemma 2.3). From II, appendix, corollary 5 it follows that bounded projections P, Q_n and R_n exist onto $\text{Ran } x_*$, $Ad^+(V_n)$ and N_n respectively such that

$$P + Q_n + R_n = id_{HS^+}$$

$$Ker(P) = Ad^+(V_n) \oplus N_n$$

$$Ker(Q_n) = \text{Ran } x_* \oplus N_n$$

$$Ker(R_n) = \text{Ran } x_* \oplus Ad^+(V_n) .$$

Define

$$L_n := P + Q_n$$

then L_n is the projector onto $\text{Ran } x_* \oplus Ad^+(V_n)$ with kernel N_n. Since $Ad^+(V_n) \overset{s}{\to} S^+$ and $\text{Ran } x_* \oplus S^+ = HS^+$ we have $\text{Ran } x_* \oplus Ad^+(V_n) \overset{s}{\to} HS^+$ and therefore

4.2.6. $\quad L_n \overset{s}{\to} id_{HS^+}$.

Since $R_n + L_n = id_{HS^+}$ we also have

4.2.7. $\quad R_n \overset{s}{\to} 0 \quad (N_n \overset{s}{\to} \{0\})$.

Now let y be any deformation of x_0 with finite dimensional base K. Write

4.2.8. $\quad y(s) - x_0 = y_1(s) + y_2(s)$

with

$$y_1(s) := L_n(y(s) - x_0)) \quad \text{and} \quad y_2(s) := R_n(y(s) - x_0)) \ .$$

Then $y_1 \in C^1(\Omega_K \to T_{x_0}M_n)$ and $y_2 \in C^1(\Omega_K \to N_n)$ (where Ω_K is the open set on which y is defined). Since $y(s) - x_0 = y_*(s) + o(\|s\|)$ (where $y_* := D_0 y$) we have

4.2.9. $\quad y_2(s) = R_n y_*(s) + o(\|s\|)\ ; \qquad s \in \Omega_K$

(note that $\|R_n\| \leq 1$ and therefore the o term is uniform in n).
Suppose dim K = m. Since y_* is linear and bounded the image of the closed unit ball B_K in K is contained in an m-dimensional disc $D \subset HS^+$, that is the intersection of an m-dimensional subspace and a closed ball centered at 0 with radius $\|y_*\|$. By 4.2.7 we have

$$\forall_{f \in D} \lim_{n\to\infty} R_n f = 0 \ .$$

Since D is finite dimensional and R_n is linear we have

$$\lim_{n\to\infty} (\max_{s \in B_K} \|R_n y_*(s)\|_1) = 0 \ .$$

Hence, if $\varepsilon > 0$ is fixed, we can choose n so large that

$$\|R_n y_*(s)\|_1 \leq \frac{\varepsilon}{8} \|s\|; \qquad s \in K \ .$$

Combining this with 4.2.9 we obtain

4.2.10. $\quad \|y_2(s)\|_1 \leq \frac{\varepsilon}{4} \|s\|$

on a sufficiently small subset Ω_K^0 of Ω_K. Since the image of Ω_K^0 under y_1 is contained in $T_{x_0}M_n$ and $y_1(0) = 0$ it is possible to define

$$z(s) := EXP(y_1(s))$$

on an open set in K containing 0 (depending on n), where EXP is defined in definition 3.1.

Now $z(s) \subset M_n$ and $z(0) = EXP(0) = x_0$. Hence z is a deformation of x_0 with values in M_n. By lemma 2.4 there are mappings

$$\varphi_n \in C^1(\mathcal{O}_K^n \to \Omega_H)$$
$$c_n \in C^1(\mathcal{O}_K^n \to G^+(V_n))$$

with $\varphi_n(0) = 0$, $c_n(0) = e$ and

$$z(s) = c_n(s)x(\varphi_n(s))c_n^{-1}(s); \qquad s \in \mathcal{O}_K^n .$$

$\mathcal{O}_K^n$ is open in K.

Since $y_1(s) = EXP^{-1}(z(s))$ it follows from 3.2.1 that

$$y_1(s) = -x_0 + c_n(s)x(\varphi_n(s))c_n^{-1}(s) + o(\|s\|) .$$

Combining this with 4.2.8 and 4.2.10 we obtain

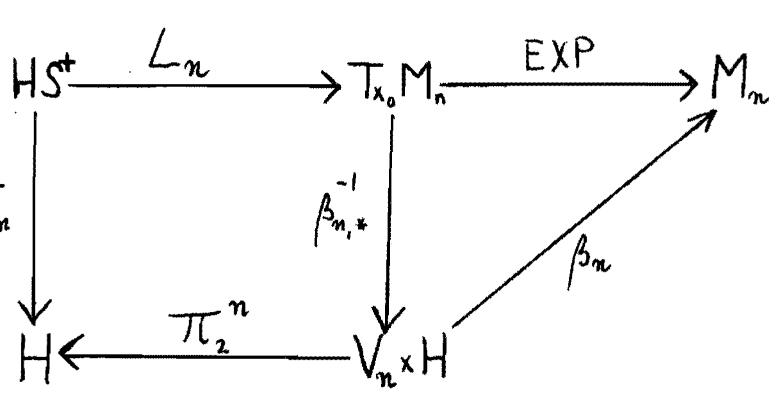
4.2.11. $\quad \| y(s) - c_n(s)x(\varphi_n(s))c_n^{-1}(s)\|_1 \leq \frac{1}{2}\varepsilon\|s\|$

for $s \in \Omega_K^n$, where Ω_K^n is sufficiently small and open. (note that n depends on ε). The only thing left to prove is that φ_n can be chosen independently of n (of ε).

Let $\pi_2^n: V_n \times H \to H$ denote the canonical projection on the second factor. Let β_n denote the diffeomorphism defined in lemma 2.4. Define $\beta_{n,*} = D_{(0,0)}\beta_n$ and

$$T_n := \pi_2^n \circ \beta_{n,*}^{-1} \circ L_n$$

then T_n is a linear map from HS^+ into H and the following diagram commutes.



By lemma 2.4 we have

$$\varphi_n = \pi_2^n(\beta_n^{-1}(z)) = \pi_2^n(\beta_n^{-1}(\mathrm{EXP}(y_1)) =$$

$$= \pi_2^n((\beta_n^{-1} \circ \beta_n \circ \beta_{n,*}^{-1})y_1) =$$

$$= \pi_2^n(\beta_{n,*}^{-1}(y_1)) =$$

$$= \pi_2^n(\beta_{n,*}^{-1}(L_n(y - x_0))) .$$

Hence

4.2.12. $\varphi_n(s) = T_n(y(s) - x_0)$

φ_n is only defined on a small neighbourhood of $0 \in K$, depending on n, but we can extend dom φ_n to dom y by 4.2.12 because dom $T_n = HS^+$. From 4.2.12 we can deduce $\varphi_{n,*} = T_n y_*$ where $\varphi_{n,*} := D_0\varphi_n$. We shall prove that $(\varphi_{n,*})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(K \to H)$. To do this consider first the composition

$$\pi_2^n \circ \beta_{n,*}^{-1}: \mathrm{Ran}\, x_* \oplus \mathrm{Ad}^+(V_n) \to H$$

$$x_*(t) + [v,x_0] \xrightarrow{\beta_{n,*}^{-1}} (v,t) \xrightarrow{\pi_2^n} t .$$

Since x_* is injective and Ran x_* is closed it follows from the closed graph theorem that there is a $\delta > 0$ such that

$$\| x_*(t) \|_1 \geq \delta \| t \|; \qquad t \in H$$

and therefore $\| \pi_2^n \circ \beta_{n,*}^{-1} \|$ is bounded by a constant independent on n say A. If $f \in HS^+$ and $n > m$ we have

$$T_n f - T_m f = \pi_2^n \circ \beta_{n,*}^{-1} \circ L_n(f) - \pi_2^m \circ \beta_{m,*}^{-1} \circ L_m(f) =$$

$$= \pi_2^n \circ \beta_{n,*}^{-1} \circ (L_n - L_m)(f) ,$$

because

$$\pi_2^n = \pi_2^m \text{ on } V_m \times H \quad (n > m)$$

$$\beta_{n,*} = \beta_{m,*} \text{ on } \mathrm{Ran}\, x_* \oplus \mathrm{Ad}^+(V_m) \quad (n > m) .$$

Hence

$$\| T_n f - T_m f \| \leq \| \pi_2^n \circ \beta_{n,*}^{-1} \| \, \| (L_n - L_m) f \| \leq A \| (L_n - L_m) f \| .$$

Since $L_n \overset{s}{\to} id_{HS^+}$ we may conclude

$$\forall_{f \in HS^+} \lim_{\substack{n,m\to\infty \\ (n>m)}} \| T_n f - T_m f \| = 0 \ .$$

Hence

$$\forall_{s \in K} \lim_{\substack{n,m\to\infty \\ (n>m)}} \| \varphi_{n,*}(s) - \varphi_{m,*}(s) \| = 0 \ .$$

Since K is finite dimensional and $\varphi_{n,*}$ is linear we may conclude:

$$\lim_{\substack{n,m\to\infty \\ (n>m)}} \max_{\|s\|=1} \| \varphi_{n,*}(s) - \varphi_{m,*}(s) \| = 0 \ .$$

Hence

$$\lim_{\substack{n,m\to\infty \\ (n>m)}} \| \varphi_{n,*} - \varphi_{m,*} \| = 0$$

and therefore the sequence $(\varphi_{n,*})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{L}(K \to H)$. Define $\psi \in \mathcal{L}(K \to H)$ by

$$\psi := \lim_{n\to\infty} \varphi_{n,*} \ .$$

We shall prove that if n is large enough we may replace φ_n by ψ in 4.2.11 if $\frac{1}{2}\varepsilon$ is replaced by ε. First we choose n so large that

4.2.13. $\| x_*(\psi(s)) - x_*(\varphi_{n,*}(s)) \|_1 \le \frac{1}{24} \varepsilon \| s \|$.

Furthermore, we choose a small open set in K on which

4.2.14. $\| x(\varphi_n(s)) - (x_0 + x_*(\varphi_{n,*}(s))) \|_1 \le \frac{1}{24} \varepsilon \| s \|$

(note that this is possible since $x(0) = x_0$ and $\varphi_n(0) = 0$).
Finally we restrict ourselves to an open set such that

4.2.15. $\| x(\psi(s)) - (x_0 + x_*(\psi(s))) \|_1 \le \frac{1}{24} \varepsilon \| s \|$.

Combining 4.2.13, 4.2.14 and 4.2.15 we obtain

4.2.16. $\| x(\varphi_n(s)) - x(\psi(s)) \|_1 \le \frac{1}{8} \varepsilon \| s \|$

on a (small) open set in K.

Now let $\Omega_K^{\prime n}$ be open in K such that

$$\max(\| c_n(s) \|, \| c_n^{-1}(s) \|) < 2$$

for $s \in \Omega_K^{\prime n}$ and also 4.2.11 and 4.2.16 hold on $\Omega_K^{\prime n}$. Then

$$\| y(s) - c_n(s) x(\psi(s)) c_n^{-1}(s) \|_1 \leq \varepsilon \| s \|$$

for $s \in \Omega_K^{\prime n}$, and the proof is complete. □

§ 5. Deformations of Hilbert-Schmidt operators

5.0. Introduction

In this section we employ the theory developed in § 4 to study deformations of Hilbert-Schmidt operators. For the transition of deformations in HS^+ to deformations in HS we use lemma II.7.9 and lemma II.7.10. For an arbitrary operator $A_0 \in HS$ a minimal weakly versal deformation is constructed in theorem 5.6 (by minimal weakly versal we mean minimal transversal). As an example we shall give a weakly versal deformation of a diagonal operator. From now on $A_0 \in HS$ is fixed and we shall use the shorter notations $Z(A_0^*)$, Ad and S for respectively $Z_{HS}(A_0^*)$, Ad_{A_0} and the $|||\ |||$-closure of $Ad_{A_0}(HS)$.

5.1. Definition. A deformation of an operator $A_0 \in HS$ is a map $A \in C^1(\Omega_H \to HS)$ such that $A(0) = A_0$ and A is double splitting at 0. As usual Ω_H is open in H, the base of the deformation.

5.2. Definition. A deformation of an operator $A_0 \in HS$ with base H is weakly versal iff for every deformation B of A_0 with finite dimensional base K there exists a map $\varphi \in C^1(\Omega_K \to H)$, with $\varphi(0) = 0$, such that for every $\varepsilon > 0$ there is a deformation $C_\varepsilon(s)$ of the idendity operator $I \in \mathcal{L}(H)$ of the form $C_\varepsilon(s) = \gamma_\varepsilon(s) I + D_\varepsilon(s)$, $s \in K$, where $D_\varepsilon \in HS$ is a deformation of $0 \in HS$ and γ_ε is a deformation of $1 \in \mathbb{C}$, such that

5.2.1. $$||| B(s) - C_\varepsilon(s) A(\varphi(s)) C_\varepsilon^{-1}(s) ||| \leq \varepsilon \| s \|; \quad s \in \Omega_K^\varepsilon .$$

The reader may have noticed that definition 5.1 and 5.2 are analogous to definition 2.1 and 4.1.

5.3. Lemma. If A is a deformation of $A_0 \in HS$ with base H and $x_0 := \langle 0, A_0 \rangle$ then the map $x: \mathbb{C} \oplus H \to HS^+$ defined by

$$x(\alpha, t) := \langle \alpha, A(t) \rangle$$

is a deformation of $x_0 \in HS^+$ in the sense of definition 2.1 with base $\mathbb{C} \oplus H$. We shall say that the deformation x corresponds to the deformation A. The proof of this lemma is left to the reader.

5.4. Lemma. Let A be a deformation of A_0 and x the corresponding deformation of $x_0 := \langle 0, A_0 \rangle$ (see 5.3). Then
A is weakly versal iff x is weakly versal and
A is transversal to $A_0 + S$ iff x is transversal to $x_0 + S^+$.

Proof. We shall only prove
A is weakly versal only if x is weakly versal.
A is transversal to $A_0 + S$ if x is transversal to $x_0 + S^+$.
The remainder of the proof is left to the reader. Suppose A is a weakly versal deformation of A_0 with base H and let $y := \langle \beta, Y \rangle$ be any deformation of x_0 with finite dimensional base K. Then Y is a deformation of A_0 and hence, since A is weakly versal, there is a map $\varphi \in C^1(\Omega_K \to H)$ such that for every $\varepsilon > 0$ there is a deformation of I of the form $C_\varepsilon = \gamma_\varepsilon I + D_\varepsilon$ such that

$$|||Y(s) - C_\varepsilon(s) A(\varphi(s)) C_\varepsilon^{-1}(s)||| \le \varepsilon \|s\|; \quad s \in \Omega_K^\varepsilon .$$

Hence by lemma II.7.9 and lemma II.7.10 we have

$$\| \langle \beta(s), Y(s) \rangle - \langle \gamma_\varepsilon(s), D_\varepsilon(s) \rangle \langle \beta(s), A(\varphi(s)) \rangle \langle \gamma_\varepsilon(s), D_\varepsilon(s) \rangle^{-1} \|_1 \le \varepsilon \|s\|$$

$s \in \Omega_K^\varepsilon$.
which can be written as

$$\|y(s) - c_\varepsilon(s) x(\psi(s)) c_\varepsilon^{-1}(s)\|_1 \le \varepsilon \|s\|; \quad s \in \Omega_K^\varepsilon$$

where $c_\varepsilon(s) := \langle \gamma_\varepsilon(s), D_\varepsilon(s) \rangle$ is a deformation of $e \in HS^+$ and $\psi(s) := (\beta(s), \varphi(s)) \in \mathbb{C} \oplus H$ satisfies $\psi(0) = (0,0)$. This proves the weak-versality of the deformation x (see definition 4.1).
Suppose x is transversal to $x_0 + S^+$ at 0. Then Ran x_* contains a closed complement of S^+ in HS^+.

Since $\quad \text{Ran } x_* = \text{Ran}\langle id, A_* \rangle = \mathbb{C} \oplus \text{Ran } A_*$

(by $\langle id, A_* \rangle$ we mean the map $(\alpha, t) \to \langle \alpha, A_*(t) \rangle$) and since $S^+ = \{\langle 0, B \rangle \mid B \in S\}$

(see remark II.7.6) this implies that Ran A_* contains a closed complement of S in HS and hence A is transversal to $A_0 + S$ at 0. □

The following theorem, which is the main theorem of this paper follows from lemma 5.4 and theorem 4.2.

5.5. Theorem. A deformation A of an operator $A_0 \in HS$ is weakly versal iff A is transversal to $A_0 + S$ at 0.

Construction of weakly versal deformations.

5.6. Theorem. Every operator $A_0 \in HS$ has a (minimal) weakly versal deformation. It can be given the following form

$$A(X) := A_0 + X; \quad X \in Z(A_0^*) .$$

The base of this deformation is $Z(A_0^*)$.

Proof. Note that A_* is the linear embedding map from $Z(A_0^*)$ into HS. Hence A is double splitting at 0 and Ran $A_* = Z(A_0^*)$. Since $Z(A_0^*)$ is the orthogonal complement of S in HS (see theorem II.5.7) the deformation A is (orthogonal) transversal to $A_0 + S$ at 0 and hence by theorem 5.5 A is weakly versal. A is minimal weakly versal because A is minimal transversal (see definition B5). □

5.7. Corollary. If B is a deformation of $A_0 \in HS$ with finite dimensional base K then there is a map $\varphi: K \to Z(A_0^*)$ with $\varphi(0) = \mathcal{O}$ such that for every $\varepsilon > 0$ there is a deformation C_ε of $I \in \mathcal{L}(H)$ of the form described in definition 5.2 such that

$$|||B(s) - C_\varepsilon(s)(A_0 + \varphi(s))C_\varepsilon^{-1}(s)||| \le \varepsilon \| s \|; \quad s \in \Omega_K^\varepsilon .$$

5.8. Example. Let D be a diagonal operator in HS

$$D = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots\} \text{ with } \lambda_i\text{'s complex ,}$$

distinct and $\sum_{i=1}^{\infty} |\lambda_i|^2 < \infty$.

Then $Z(D^*) = Z(D)$ is the set of all diagonal operators in HS. It follows from corollary 5.7 that if B is a deformation of D with finite dimensional base K we have

$$|||B(s) - C_{\varepsilon}(s)\Lambda(s)C_{\varepsilon}^{-1}(s)||| \le \varepsilon\|s\|; \qquad s \in \Omega_K^{\varepsilon}$$

where $\Lambda(s)$ is diagonal for all $s \in \Omega_K^{\varepsilon}$ and $\Lambda(0) = D$.

5.9. Remark. If the space H is finite dimensional ($H = \mathbb{C}^n$) then $HS = \mathbb{C}^{n\times n}$ and then theorem 5.5 of this section is equivalent to theorem 3.6 of chapter I. Because a weakly versal deformation A of the matrix A_0 is transversal to orbit of A_0, by theorem 5.5, and hence by theorem I.3.6 A is a versal deformation of the matrix A_0 in the sense of definition I.2.4. So, if H is finite dimensional we have

A is weakly versal $\Leftrightarrow$ A is versal .

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