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A THEORY OF GENERALIZED FUNCTIONS BASED ON

HOLOMORPHIC SEMI-GROUPS

Part A: Introduction and survey

Abstract

In a Hilbert space X consider the evolution equation

$$\frac{du}{dt} = - Au$$

with A a nonnegative unbounded self-adjoint operator. A is the infinitesimal generator of a holomorphic semi-group. Solutions $u(\cdot) : (0,\infty) \to X$ of this equation are called trajectories. Such a trajectory may or may not correspond to an "initial condition at $t = 0$" in X. The set of trajectories is considered as a space of generalized functions. The test function space is defined to be $S_{X,A} = \underset{t>0}{U} \ e^{-tA}(X)$.

For the spaces $S_{X,A}$, $T_{X,A}$ I discuss a pairing, topologies, morphisms, tensor products and kernel theorems. Examples are given.

CHAPTER 0. <u>Introduction, survey and examples</u>

In a very inspiring paper, De Bruijn [B] has proposed a theory of generalized functions based on a specific one-parameter semi-group of smoothing operators. This semi-group constitutes what is known as a holomorphic semi-group of bounded self-adjoint operators on $L_2(\mathbb{R})$. This observation has enabled me to generalized De Bruijn's theory and to place it in a wider context of functional analysis.

The present paper contains a theory of two types of topological vector spaces: The analyticity space $S_{X,A}$ and the trajectory space $T_{X,A}$. These spaces are characterized by a one-parameter holomorphic semi-group e^{-tA} of bounded self-adjoint operators on a Hilbert space X. The infinitesimal generator A is a nonnegative unbounded self-adjoint operator in X. If we take a suitable operator A in a Hilbert space $X = L_2(M,\mu)$, then the elements of $S_{X,A}$ can be regarded as test functions on the measure space M and the elements of $T_{X,A}$ can be regarded as generalized functions on M.

The functional analytic approach of the present paper makes it possible to transfer large pieces of Hilbert space theory to distribution theory. This has led to a detailed exposition of continuous linear mappings, of topological tensor products and of four kernel theorems. Systematic considerations on continuous linear mappings are not current in distribution theory!

Topologically our theory of generalized functions can be compared with the usual theories as follows. For simplicity consider generalized functions on $\mathbb{R}^n$. Suppose these generalized functions are described by a Gelfand triple: $S \subset L_2(\mathbb{R}^n) \subset S'$. Three types can be distinguished: (a) Neither S nor S' is metrizable, (b) S is metrizable, S' is not, (c) S' is metrizable, S is not. Examples of cases (a) and (b) are furnished by Schwartz' theory of ordinary, respectively tempered distributions. Our theory is always of type (c).

Next I mention some possibilities, advantages and applications of the present theory.

- Our generalized functions are represented by trajectories, a concept which is very close to the physical intuition of what a generalized function should be.

- Test function spaces can be constructed that are invariant under a set of given operators. We can always do this in an abstract way. However, the

characterization of thus obtained test spaces in terms of classical analysis may be a hard job. For results in this direction see [B], [E$_1$], [ETh], [EG], [EGP], [GZ].

- Many of the test spaces of Gelfand -Shilov are special examples of $S_{X,A}$-spaces. So our general theory applies to them. See [EGP].
- Hyperfunctions of fixed bounded support can be represented by trajectories. So our general theory also applies here. A paper on this subject is in preparation.
- A matrix calculus for continuous operators on nuclear $S_{X,A}$ and $T_{X,A}$ spaces has been developed. See [ETh].
- If $T_{X,A}$ is a nuclear space and B is an arbitrary self-adjoint operator in X, then there is a "Dirac basis" of $T_{X,A}$ consisting of eigenvectors of B. See [ETh], [EG$_2$].
- With the aid of trajectory spaces a mathematical rigorization of Dirac's formalism has been given which goes much beyond the traditional (attempts to) rigorizations. See [ETh], [EG$_1$], [EG$_2$]. Also the CCR and CAR relations in the quantum theory of free fields have been given a mathematical interpretation which comes very close to physical usage. A paper on this subject is in preparation.
- A functional analytic model for quantum statistical mechanics with unbounded observables has been constructed. See [ETh].

The parts B and C of the present paper contain the basic functional analytic theory of $S_{X,A}$ spaces, $T_{X,A}$ spaces and linear operators between them. The present part A contains a partial survey of this general theory and a number of examples of analyticity spaces which have been characterized in "classical" analytical terms.

The subdivision of the survey corresponds to the chapters I-VI of parts B and C.

I. The analyticity space $S_{H,A}$

Let A be an unbounded nonnegative self-adjoint operator with domain $D(A)$ in a Hilbert space H with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Using spectral theory the operator e^{zA} can be defined for each complex z. All operators e^{zA} are normal, e^{zA} is unbounded if Re $z > 0$ and bounded if Re $z \le 0$, the operator e^{zA} is self-adjoint if $z \in \mathbb{R}$.

$f \in H$ is called an *analytic vector* for A if

$$f \in D(A^\infty) = \bigcap_{n=1}^{\infty} D(A^n)$$

and

$$\|A^n f\| \le ab^n n! \, , \quad n = 0,1,2,\ldots$$

for some fixed constants a, b only dependent on f.

The *analyticity space* $S_{H,A}$ is the set of all analytic vectors for A. $S_{H,A}$ is a dense linear subspace of H and we can write

$$S_{H,A} = \bigcup_{t>0} e^{-tA}(H) = \bigcup_{t>0} D(e^{tA}) \, .$$

See [Ne]. In other words, for each $f \in S_{H,A}$ there exists $\tau > 0$ such that $e^{\tau A} f \in H$. For τ small enough one even has $e^{\tau A} f \in S_{H,A}$. The domain $D(e^{tA})$, $t \ge 0$, can be made into a Hilbert space by introducing the norm $\|\cdot\|_t = \|e^{tA}\cdot\|$ and corresponding inner product on it.

Since $D(e^{tA}) \supset D(e^{\tau A})$ if $0 < t < \tau$ it is possible to introduce the inductive limit topology on $S_{H,A}$. An explicit complete system of semi-norms for this topology can be constructed in the following way.

Let B_+ denote the set of real valued Borel functions ψ on $\mathbb{R}$ such that

- $\psi(x) \ge \varepsilon > 0$, $\varepsilon \in \mathbb{R}$,
- for all $t > 0$ the function $x \mapsto \psi(x)e^{-tx}$ is bounded on $[0,\infty)$.

By the spectral theorem for self-adjoint operators, the operators $\psi(A)$, $\psi \in B_+$ are well defined and the operators $\psi(A)^{-tA}$, $t > 0$, are all bounded in H. Therefore on $S_{H,A}$ the semi-norms p_ψ are well defined by $p_\psi(f) = \|\psi(A)f\|$. These semi-norms generate the inductive limit topology on $S_{H,A}$. Exploiting these seminorms the following topological results are obtained:

- A subset $U \subset S_{H,A}$ is bounded iff there is a $t > 0$ such that $U \subset D(e^{tA})$ and $e^{tA}(U)$ is a bounded set in H.
- A subset $K \subset S_{H,A}$ is compact iff it is bounded and $e^{tA}(K)$ is a compact set in H for some $t > 0$.
- A sequence $(f_n) \subset S_{H,A}$ is Cauchy iff there is $t > 0$ such that $(e^{tA} f_n)$ is a Cauchy sequence in H.
- $S_{H,A}$ is complete, bornological and barreled.
- $S_{H,A}$ is Montel iff for every $t > 0$ the operator e^{-tA} is compact on H.
- $S_{H,A}$ is nuclear iff for every $t > 0$ the operator e^{-tA} is Hilbert-Schmidt on H.

II. The trajectory space $T_{H,A}$

In H consider the evolution equation

(1) $\qquad \dfrac{du}{dt} = - Au , \quad t > 0 .$

A solution F of this equation is called a *trajectory* if F satisfies

(2.i) $\qquad \forall_{t>0} \; F(t) \in H$

(2.ii) $\qquad \forall_{t>0} \; \forall_{\tau>0} \; e^{-\tau A} F(t) = F(t + \tau) .$

It is emphasized that $\lim_{t \downarrow 0} F(t)$ does not necessarily exist in H-sense. Take e.g. $F(t) = Ae^{-tA} x$, $x \notin D(A)$. The complex vector space of all trajectories is named *trajectory space* and denoted by $T_{H,A}$. The space $T_{H,A}$ is considered as a space of generalized functions. Heuristically speaking, the initial condition of $F(t)$, which is not necessarily an element of H, is a "generalized function".

H can be embedded in $T_{H,A}$ by means of the linear mapping emb:

$$\text{emb}: H \to T_{H,A} : (\text{emb } f)(t) = e^{-tA} f , \quad f \in H , \quad t > 0 .$$

Thus

$$S_{H,A} \subset H \subset T_{H,A} .$$

In $T_{H,A}$ a topology is introduced by the semi-norms

$$F \mapsto \|F(t)\| , \quad t > 0 .$$

This topology makes $T_{H,A}$ into a Fréchet space.

The elements of $T_{H,A}$ can be characterized as follows. Let $F \in T_{H,A}$. Then there exists $w \in H$ and $\psi \in B_+$ such that $F(t) = \psi(A)e^{-tA} w$, $t > 0$.

The following topological results are obtained:

- A subset $U \subseteq T_{H,A}$ is bounded iff each of the sets $\{F(t) \mid F \in U\}$, $t > 0$, is bounded in H.

- A subset $K \subseteq T_{H,A}$ is compact iff each of the set $\{F(t) \mid F \in K\}$, $t > 0$, is compact in H.

- A sequence $(F_n) \subseteq T_{H,A}$ is Cauchy iff for each $t > 0$ $(F_n(t))$ is a Cauchy sequence in H.

- $T_{H,A}$ is complete, bornological and barreled.

- $T_{H,A}$ is Montel iff for every $t > 0$ the operator e^{-tA} is compact on H.

- $T_{H,A}$ is nuclear iff for every $t > 0$ the operator e^{-tA} is Hilbert-Schmidt on H.

III. The pairing of $S_{H,A}$ and $T_{H,A}$

The pairing $\langle \cdot , \cdot \rangle$ between $S_{H,A}$ and $T_{H,A}$ is defined by

(3) $\qquad \langle g,F \rangle = (e^{\tau A} g, F(\tau)) , \quad g \in S_{H,A} , \quad F \in T_{H,A} .$

This definition makes sense for $\tau > 0$ sufficiently small and due to the trajectory property (2.ii) it does not depend on the choice of τ.

By means of the duality (3), weak topologies are introduced on $S_{H,A}$ and $T_{H,A}$. It turns out that both spaces are reflexive in these weak topologies and in the strong topologies of Chapters I and II. Besides a Banach-Steinhaus theorem and a characterization of weak convergence of sequences we prove equivalence of the following statements:

- For each $t > 0$ e^{-tA} is compact on H.

- Each weakly convergent sequence in $S_{H,A}$ $(T_{H,A})$ converges strongly in $S_{H,A}$ $(T_{H,A})$.

IV. <u>Characterization of continuous linear mappings between the spaces</u> $S_{H,A}$ <u>and</u> $T_{H,A}$

The following four types of continuous linear mappings will be studied in detail:

$$S_{H,A} \to T_{H,A} \, , \quad T_{H,A} \to S_{H,A} \, , \quad T_{H,A} \to T_{H,A} \, , \quad S_{H,A} \to S_{H,A} \, .$$

In addition we shall characterize the continuous linear mappings from $S_{H,A}$ into $S_{H,A}$ which can be extended to continuous linear mappings from $T_{H,A}$ into $T_{H,A}$.

Here I only mention the following simple result. Suppose $P : S_{H,A} \to S_{H,A}$ is a continuous linear mapping. P can be extended to

$$\bar{P} : T_{H,A} \to T_{H,A}$$

iff the H-adjoint P^* of P satisfies

$$D(P^*) \supset S_{H,A} \quad \text{and} \quad P^*(S_{H,A}) \subset S_{H,A} \, .$$

Of course one has

$$< P^* f, F > \; = \; < f, \bar{P}F >$$

for all $f \in S_{H,A}$, $F \in T_{H,A}$.

V. <u>Topological tensor products of</u> $S_{H,A}$ <u>and</u> $T_{H,A}$

On algebraic tensor products of $S_{H,A}$ and $T_{H,A}$ several locally convex topologies can be imposed. In part C, Chapter V, four cases are considered and the completions are fully characterized. As examples I mention the following simple cases:

$$S_{H,A} \otimes S_{H,A} = S_{H\otimes H, A\boxplus A}$$

$$T_{H,A} \otimes T_{H,A} = T_{H\otimes H, A\boxplus A}$$

Here the Hilbert space $H \otimes H$ is the set of Hilbert-Schmidt operators from H into itself and $A \boxplus A = A \otimes I + I \otimes A$

VI. Kernel theorems

The topological tensor products of Chapter V can be put into correspondence with the continuous linear mappings of Chapter IV.

For example, let $K \in T_{H \otimes H, A \boxplus A}$. Then the corresponding operator $\hat{K} : S_{H,A} \to T_{H,A}$ is defined by

$$(\hat{K}\varphi)(t) = e^{-(t-\varepsilon)A} \, \hat{K}(\varepsilon) e^{\varepsilon A} \, \varphi \ .$$

This definition makes sense for $\varepsilon > 0$ sufficiently small and the result does not depend on the choice of ε.

If $T_{H \otimes H, A \boxplus A}$ comprises all continuous linear mappings from $S_{H,A}$ into $T_{H,A}$ we say that a kernel theorem holds. This is precisely the case if $e^{-tA} \in H \otimes H$, i.e. is Hilbert-Schmidt, for all $t > 0$.

Similarly, three more kernel theorems are discussed.

In [ETh] the operator theory of Chapters IV, V and VI is developed further. Amongst others, a kernel theorem is proved for extendable operators.

The program of <u>this</u> paper has also been carried out for general distribution spaces of tempered type by Van Eijndhoven [E_2].

I conclude part A with a series of examples mainly of analyticity spaces. The first example is very simple. It is discussed in some detail for illustrative purposes. The other examples indicate the scope of applications in analysis of the present theory. In the examples I often denote $S_{H,A}$ by S and $T_{H,A}$ by T.

<u>Example 1</u>: $H = L_2(\mathbb{R})$, $A = -\dfrac{d^2}{dx^2}$.

The domain $D(A)$ of A is the Sobolev-space $H^2(\mathbb{R})$. The trajectories are classical solutions of the elementary diffusion equation

$$(4) \qquad \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} \ , \quad x \in \mathbb{R}, \ t > 0 \ ,$$

with the property $\forall_{t>0} \ F(\cdot,t) \in L_2(\mathbb{R})$.

Corresponding to any "initial condition" $f \in L_2(\mathbb{R})$ there is a trajectory given by the well-known formula

$$(5) \qquad F(x,t) = (e^{-tA} f)(x) = \frac{1}{2}(\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) dx, \quad t > 0 .$$

From the strict positivity of the integral operator e^{-tA} it follows that for a given trajectory $F(\cdot,t)$ there exists at most one $g \in L_2(\mathbb{R})$ such that $F(\cdot,t) = e^{-tA} g$. In general there exists no $g \in L_2(\mathbb{R})$ such that $F(\cdot,t) = e^{-tA} g$.

Consider e.g.

$$(6) \qquad G(x,t) = 2(\pi t)^{-\frac{1}{2}} \exp - \frac{(x-a)^2}{4t} .$$

Note that $G(x,t)$ "tends" to the δ-function $\delta(x-a)$ for $t \downarrow 0$. Any derivative of G is again a trajectory.

With (5), its inversion formule (see [BJS]) and some straightforward L_2-estimates it can be shown that $f \in L_2(\mathbb{R})$ belongs to S iff

    (a) f can be extended to an entire analytic function;

$$(7)$$

$$\text{(b)} \quad \exists_{A>0, B>0} \; \forall_{y \in \mathbb{R}} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx \le Ae^{By^2} .$$

Examples of continuous operators on S are e^{zA}, $\operatorname{Re} z \le 0$, R_a, T_b, Z_λ, $\mathcal{D}$ and compositions of these. Here $(R_a f)(x) = e^{iax} f(x)$, $a \in \mathbb{R}$, $(T_b f)(x) = f(x+b)$, $b \in \mathbb{C}$, $(Z_\lambda f)(x) = f(\lambda x)$, $\lambda \in \mathbb{R}$, $\lambda \ne 0$, $(\mathcal{D}f)(x) = df(x)/dx$. These operators can be extended to T.

Further I want to show that certain strongly divergent Fourier integrals can be interpreted as elements of T.

This interpretation is closely related to the Gauss-Weierstrass summation method. Let g be a measurable function on $\mathbb{R}$ such that for each $\varepsilon > 0$ the function $g(y)e^{-\varepsilon y^2}$ is in $L_2(\mathbb{R})$. The possibly divergent Fourier integral $\int_{-\infty}^{\infty} g(y)e^{iyx} dy$ can be considered as an element of T. Its trajectory G is given by $G(x,t) = \int_{-\infty}^{\infty} g(y)e^{-ty^2} e^{iyx} dy$. This simple illustration of our theory certainly has some elegant features, but in some respects the

illustration is too simple. Since e^{-tA} is not a Hilbert-Schmidt operator there is no kernel theorem in this case. That is to say, there exist continuous linear mappings from S into T which do not arise from a trajectory of the diffusion equation in $L_2(\mathbb{R}^2)$. Cf. Chapter VI, case b.

Example 2: $H = L_2(\mathbb{R})$, $A = \left(- \dfrac{d^2}{dx^2}\right)^{\frac{1}{2}}$.

$D(A)$ is the Sobolev-space $H^1(\mathbb{R})$. In this case

$$(e^{-tA} u)(x) = \int_{-\infty}^{\infty} K_t(x,y) u(y) dy$$

with

$$K_t(x,y) = \frac{1}{\pi} \int_0^{\infty} \exp(-|k|t) \cos k(x-y) dk = \frac{1}{\pi} \frac{t}{t^2 + (x-y)^2}$$

which is just the Poisson-kernel for solving the Dirichlet problem in a half plane $t \geq 0$. This is not surprising, since, at least formally,

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} - \sqrt{-\frac{\partial^2}{\partial x^2}}\right)\left(\frac{\partial}{\partial t} + \sqrt{-\frac{\partial^2}{\partial x^2}}\right).$$

Here S consists of the functions f which are analytic on a strip around the real axis and which satisfy

$$\sup_{-h<y<h} \int_{-\infty}^{\infty} |f(x+iy)|^2 dx < \infty$$

for h sufficiently small. The width of the strip depends on f.

Example 3: $H = L_2([0,2\pi])$, $A = \left(- \dfrac{d^2}{dx^2}\right)^{\alpha}$.

$D(A)$ is the periodic Sobolev space $H^{2\alpha}_{per}([0,2\pi])$. In this case

$$(e^{-tA} u)(x) = \int_0^{2\pi} K_t(\alpha;x,y) u(y) dy$$

with

$$K_t(\alpha;x,y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\{-|n|^{2\alpha} t + in(x-y)\} =$$

$$= \frac{1}{2\pi} \left\{ 1 + 2 \sum_{n=1}^{\infty} \exp\{-n^{2\alpha} t\} \cos n(x-y) \right\}.$$

For $\alpha = 1$:

$$K_t(1;x,y) = \frac{1}{2\pi} \theta_3(x-y,e^{-t}).$$

Here θ_3 is one of Jacobi's theta functions [WW], p. 464.
For $\alpha = \frac{1}{2}$:

$$K_t(\tfrac{1}{2};x,y) = \frac{1}{2\pi} \frac{\sinh t}{\cosh t - \cos(x-y)}.$$

In the latter case the smoothing integral operator is the Poisson kernel for the solution of the Dirichlet problem for the unit disc. For a general discussion of this phenomenon see Example 8.

For each $\alpha > 0$: $K_t(\alpha;\cdot,\cdot) \in L_2([0,2\pi]^2)$. Therefore in this case kernel theorems hold. See Chapter VI.

Certain strongly divergent Fourier series do have a meaning within this theory.

Let $\alpha > 0$ be fixed. If a sequence of complex numbers $\{c_n\}_{n=-\infty}^{\infty}$ is such that $\{|c_n| \exp -\varepsilon|n|^{2\alpha}\}$ is a bounded sequence for each $\varepsilon > 0$, then $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ can be viewed as a generalized function. The corresponding trajectory is given by

$$G(x,t) = \sum_{n=-\infty}^{\infty} c_n \exp\{-|n|^{2\alpha} t + inx\}.$$

For $\alpha = 1$, S consists of periodic entirely analytic functions f with period 2π and growth estimate

$$|f(x+iy)| \leq Ae^{By^2}$$

where A and B depend on f.

For $\alpha = \frac{1}{2}$, S consists of periodic functions which are analytic on a strip around the real axis and have period 2π.

This example can be generalized to analytic test function spaces on n-dimensional tori.

Example 4: $H = L_2(\mathbb{R})$, $A = \log\left(I - \dfrac{d^2}{dx^2}\right)^{\frac{1}{2}}$.

For the domain of $I - d^2/dx^2$ we take again $H^2(\mathbb{R})$. For $t > 0$ the set $e^{-tA}(H)$ equals the usual Sobolev-space $H^t(\mathbb{R})$. In this case we have

$$S = H^{+0} = \bigcup_{t>0} H^t$$

$$T = H^{-0} = \bigcap_{t>0} H^{-t}.$$

So the test functions (generalized functions) are only a tiny little bit smoother (more singular) than the $L_2(\mathbb{R})$-functions.
$H^{+0}(\mathbb{R})$ is an *inductive* limit in contradistinction to $H^\infty(\mathbb{R}) = \bigcap_{t>0} H^t(\mathbb{R})$, a current test space of smooth functions, which is a *projective* limit.
Cf. [GS]. It follows from Chapter I Th. 1.11 and Chapter II Th. 2.11 that $H^{+0}(\mathbb{R})$ and $H^{-0}(\mathbb{R})$ are neither Montel nor nuclear. (Neither is $H^\infty(\mathbb{R})$.)
If we replace $\mathbb{R}$ by some finite interval $[a,b]$ and if we take self-adjoint boundary conditions for the operator d^2/dx^2, then $H^{+0}([a,b])$ and $H^{-0}([a,b])$ are Montel but not nuclear. (The space $H^\infty([a,b])$ is nuclear now!)
This example can be generalized to (open subsets of) $\mathbb{R}^n$ in an obvious way.

Example 5: $H = L_2(\mathbb{R})$, $A = \left(x^2 - \dfrac{d^2}{dx^2}\right)^{1/2\alpha}$, $\alpha \geq \frac{1}{2}$.

We can write

$$(e^{-tA} u)(x) = \int_{-\infty}^{\infty} K_t(\alpha; x, y) u(y) \, dy.$$

An explicit expression for K_t if $\alpha = \frac{1}{2}$ can be found in [B]. In a probably not generally known paper [GZ], the Chinese mathematician Zhang Gong-Zhing

has proved that the Gelfand-Shilov space S_α^α, $\alpha \geq \frac{1}{2}$ consists of precisely those $L_2(\mathbb{R})$-functions which satisfy

$$(f, \psi_n) = \mathcal{O}(\exp(-\tau n^{1/2\alpha}))$$

for some $\tau > 0$. Here ψ_n denotes the nth Hermite function, $n = 0, 1, 2, \ldots$. Since $A\psi_n = (2n+1)^{1/2\alpha} \psi_n$, it follows that $S_\alpha^\alpha = S_{H,A}$

For each $t > 0$, $\alpha \geq \frac{1}{2}$

$$K_t(\alpha; x, y) = \sum_{n=0}^{\infty} e^{-tn^{1/2\alpha}} \psi_n(x) \psi_n(y) \ .$$

This kernel belongs to $L_2(\mathbb{R}^2)$. Therefore all spaces S_α^α, $\alpha \geq \frac{1}{2}$, are nuclear. See Chapter I, Theorem 1.11.

In [GZ] the dual space $(S_\alpha^\alpha)'$ has been described by means of Hermite pansions introduced by Korevaar, [K].

Gong-Zhing has proved that the dual space $(S_\alpha^\alpha)'$ can be identified with Hermite pansions $\Sigma a_n \psi_n$ which satisfy

$$\forall_{t>0} \ a_n = \mathcal{O}(\exp(tn^{1/2\alpha})) \ .$$

It will be clear that Hermite pansions of this type correspond to trajectories F in the following way:

$$F: (0, \infty) \to L_2(\mathbb{R}) : t \mapsto \sum_{n=0}^{\infty} a_n e^{-tn^{1/2\alpha}} \psi_n \ .$$

The spaces $S_{\frac{1}{2}}^{\frac{1}{2}}$ and S_1^1 deserve some special attention. The test space, introduced and studied in detail by De Bruijn in [B] is in fact $S_{\frac{1}{2}}^{\frac{1}{2}}$. It contains entire analytic functions f of growth behaviour

$$|f(x + iy)| \leq Me^{-Ax^2 + By^2} \ ,$$

M, A and B are dependent on f. For his description of the dual $(S_{\frac{1}{2}}^{\frac{1}{2}})'$ De Bruijn introduces the concept that is called a trajectory in the present paper.

Further, $S_{\frac{1}{2}}^{\frac{1}{2}}$ is the analyticity domain of a unitary representation of the Schrödinger group in $L_2(\mathbb{R})$. Cf. Example 9.

The functions $f \in S_1^1$ are analytic on a strip around the real axis and obey the estimate

$$\sup_{-h \le y \le h} |f(x+iy)| \le A e^{-B|x|} \ .$$

A, B and h are dependent on f.

S_1^1 is the analyticity domain of a unitary representation of the Heisenberg group in $L_2(\mathbb{R})$. Cf. Example 9.

Example 6: $H = L_2(\mathbb{R})$, $A = \left(-\dfrac{d^2}{dx^2} + x^{2k}\right)^{k+1/2k}$, $k \in \mathbb{N}$.

$$H = L_2(\mathbb{R}) , \quad A = \left((-1)^k \frac{d^{2k}}{dx^{2k}} + x^2\right)^{k+1/2k} , \quad k \in \mathbb{N} .$$

The S-spaces are the respective Gelfand-Shilov spaces $S_{1/k+1}^{k/k+1}$, $S_{k/k+1}^{1/k+1}$. These spaces are all nuclear. For the proofs see [EGP].

Example 7: $H = L_2(0,\infty)$, $A = -\dfrac{d^2}{dx^2} + x^2 + \dfrac{\alpha^2 - \frac{1}{4}}{x^2} - 2\alpha$, $\alpha > -1$.

For each α the S-space consists of functions f such that $x^{-(\alpha+\frac{1}{2})} f(x)$ is an even function in the Gelfand-Shilov space $S_{\frac{1}{2}}^{\frac{1}{2}}$. For the proof see [EG], [E$_1$]. For each α the S-space is nuclear and invariant under the Hankel transform

$$(\mathbb{H}_\alpha f)(x) = \int_0^\infty J_\alpha(xy)\sqrt{xy} \, f(y) dy \ .$$

The preceding examples can all be generalized to n-dimensions in a direct obvious way. These generalizations can also be obtained by application of the theory of Chapter V where topological tensor products of the spaces S and T are formed.

Our type of distributions can be introduced on any, not too bad, differentiable manifold M by taking for A a positive elliptic differential operator which is self-adjoint in an L_2-space over M. On a compact Riemannian manifold

one may take the Laplace-Beltrami operator. This can be done on the q-dimensional unit sphere S^q in $\mathbb{R}^{q+1}$ for example. In the latter case however a very nice semi-group of smoothing operators can be chosen which is closely related to the Poisson-integral solution of the Dirichlet-problem for the unit ball in $\mathbb{R}^{q+1}$. This is the subject of the next example.

Example 8: $H = L_2(S^q)$, $A = -\frac{1}{2}(q-1)I + \sqrt{\frac{1}{4}(q-1)^2 I - \Delta_{LB}}$.

Here Δ_{LB} denotes the Laplace-Beltrami operator, I denotes the identity operator.

After introducing orthogonal spherical coordinates $x_i = rF_i(\theta_1, \ldots, \theta_q)$, $1 \le i \le q+1$ in $\mathbb{R}^{q+1}$ we obtain for the Laplacian

(8)
$$\Delta = \sum_{i=1}^{q+1} \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial r^2} + \frac{q}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{LB} .$$

From this it follows, see [M] p. 4, that an m-th order spherical harmonic is an eigenvector of Δ_{LB} with eigenvalue equal to $- m(m+q-1)$.

A simple calculation shows that each m-th order spherical harmonic is an eigenvector with eigenvalue m of our operator A.

Introduction of $r = e^{-t}$ in (8) transforms the Laplacian into

$$\Delta = e^{2t}\left\{\frac{\partial^2}{\partial t^2} - (q-1)\frac{\partial}{\partial t} + \Delta_{LB}\right\} .$$

The expression between { } can be factored into two evolution equations. Thus

$$\Delta = e^{2t}\left[\frac{\partial}{\partial t} + \frac{1}{2}(q-1)I + \{\frac{1}{2}(q-1)^2 I - \Delta_{LB}\}^{\frac{1}{2}}\right] \cdot$$

$$\cdot \left[\frac{\partial}{\partial t} - \frac{1}{2}(q-1)I + \{\frac{1}{4}(q-1)^2 I - \Delta_{LB}\}^{\frac{1}{2}}\right] .$$

The second factor can be written as $\partial/\partial t + A$.

From these considerations it follows that substitution of $r = e^{-t}$ in the Poisson integral formula, cf. [M] p. 41, leads to an integral expression for e^{-tA}

$$(9) \qquad (e^{-tA} u)(\xi) = \frac{1}{\omega_q} \int_{S^q} \frac{(2^{-1} e^t)^{\frac{1}{2}(q-1)} \sinh t}{(\cosh t + \xi \cdot \eta)^{\frac{1}{2}(q+1)}} u(\eta) d\omega_q(\eta) \ .$$

Here ξ and η denote elements of S^q, i.e. unit vectors in $\mathbb{R}^{q+1}$. The "surface measure" of S^q is denoted by $d\omega_q$, while $\xi \cdot \eta$ denotes the inner product of ξ and η.

For fixed η the kernel in (9) denotes the trajectory of the δ-function centered at η. For a representation of the kernel involving zonal harmonics, see [GSe]. Here the operator e^{-tA} is obviously Hilbert-Schmidt. So kernel theorems are available. Similarly to Example 3 certain strongly divergent sequences of spherical harmonics can be "summed".

Example 9: Analytic vectors.

In [Ne], Nelson introduced the notion analytic vector. He uses the notation $C^\omega(A)$ for $S_{H,A}$.

The notion analytic vector was also introduced for unitary representations of Lie groups (see [Ne], [Wa], [Go] and [Na]):

Let G be a finite dimensional Lie group. A unitary representation U of G is a mapping

$$g \to U(g) \ , \quad g \in G$$

from G into the unitary operators on some Hilbert space H.

A vector $f \in H$ is called an analytic vector for the representation U, if the mapping

$$g \mapsto U(g) f$$

is analytic on G. We denote the space of analytic vectors for U by $C^\omega(U)$. Let A(G) denote the Lie algebra of the Lie group G, and let $\{p_1, \dots, p_d\}$ be a basis for A(G). Then for every $p \in A(G)$

$$s \mapsto U(\exp(sp))$$

is a one-parameter group of unitary operators on H. By Stone's theorem its infinitesimal generator, denoted by $\partial U(p)$, is skew-adjoint. Thus the Lie algebra A(G) is represented by skew-adjoint operators in H. Put

$$\Delta = I - \sum_{k=1}^{d} (\partial U(p_k))^2 \ .$$

Nelson, [Ne], has proved that the operator Δ can be uniquely extended to a positive, self-adjoint operator in H. Denote its extension by Δ, also. Then we have, with [Ne], [Go], $C^{\omega}(U) = S_{H,\Delta^{\frac{1}{2}}}$.

Following [Go], proposition 2.1, an operator $\partial U(p)$ maps $S_{H,\Delta^{\frac{1}{2}}}$ into itself. Since $\partial U(p)$ is skew-adjoint, continuity follows from Chapter IV Theorem 4.2.

In several cases the space $S_{H,\Delta^{\frac{1}{2}}}$ is nuclear. We mention the following cases.

$S_{H,\Delta^{\frac{1}{2}}}$ is nuclear if U is an irreducible unitary representation of G on H and one of the following conditions is satisfied:

(i)   G is semi-simple with finite center.

(ii)   G is the semi-direct product $A \otimes K$ where A is an abelian invariant subgroup and K is a compact subgroup, e.g. the Euclidean groups.

(iii) G is nilpotent.

For a proof see [Na], possibly other cases can be found in [Wa].
This example has been taken from [ETh].

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