Analyticity spaces and trajectory spaces based on a pair of commuting holomorphic semigroups with applications to continuous linear mappings

ANALYTICITY SPACES AND TRAJECTORY SPACES BASED ON A PAIR OF COMMUTING HOLOMORPHIC SEMIGROUPS WITH APPLICATIONS TO CONTINUOUS LINEAR MAPPINGS

by

S.J.L. van Eijndhoven

The investigations were supported by the Netherlands Foundation for Mathematics (SMC) with financial aid from the Netherlands Organization for the Advancement of Pure Research (ZWO).

TECHNISCHE HOGESCHOOL EINDHOVEN

NEDERLAND

ONDERAFDELING DER WISKUNDE

EN INFORMATICA

EINDHOVEN UNIVERSITY OF TECHNOLOGY

THE NETHERLANDS

DEPARTMENT OF MATHEMATICS

AND COMPUTING SCIENCE

Analyticity spaces and trajectory spaces based on a pair of commuting holomorphic semigroups with applications to continuous linear mappings

by

S.J.L. van Eijndhoven

EUT-Report 82-WSK-06

December 1982

Contents Page

Abstract

The theory of generalized functions as introduced by De Graaf, [G], is based on the triplet $S_{X,A} \subset X \subset T_{X,A}$. This triplet is fixed by a Hilbert space X and a non-negative, unbounded self-adjoint operator A in X. Besides a thorough investigation of the spaces $S_{X,A}$ and $T_{X,A}$, four types of continuous linear mappings are discussed in [G]. Moreover, there are brought up so-called Kernel theorems for each of these types. We remark that a Kernel theorem gives conditions such that all linear mappings of a specific type arise from kernels out of a suitable topological tensor product.

In order to obtain these Kernel theorems, De Graaf has introduced the topological tensor products Σ'_A,Σ'_B and Σ_A,Σ_B. In the first part of this paper we shall discuss two general types of spaces, which are determined by a Hilbert space Z and by two commuting, non-negative, unbounded self-adjoint operators in Z. The spaces Σ'_A,Σ'_B and Σ_A,Σ_B are of these types. For the newly introduced spaces we shall give topologies, a pairing and characterizations of their intersections.

In the second part of this paper we shall apply the obtained results to continuous linear mappings. It will lead to a fifth Kernel theorem, and further, to a study of the algebras of continuous linear mappings from $S_{X,A}$ into itself cq. from $T_{X,A}$ into itself, and of extendable linear mappings. The latter mentioned algebra may serve as a model for quantum statistics.

Finally, we shall discuss infinite matrices. It is possible to characterize the continuous linear mappings on a nuclear $S_{X,A}$ space completely by means

of their associated matrices. This characterization provides easy construction of examples. Here we mention the so-called weighted shift operators, which occur in one of the sections. Last but not least, the matrix calculus leads to a construction of nuclear spaces $S_{X,A}$ on which a finite number of given operators in X act continuously.

Introduction

In his paper, [G], De Graaf gives a detailed discussion of the two types of spaces $S_{X,A}$ and $T_{X,A}$, with the intention to describe distribution theory on a general, functional analytic level. As observed in [GE], the space $S_{X,A}$ which may serve as a test space, consists of all analytic vectors of the non-negative, self-adjoint operator A in the Hilbert space X. Therefore, spaces of type $S_{X,A}$ are called 'analyticity spaces'. The elements of the space $T_{X,A}$, which can be considered as a space of generalized functions, are mappings F from $(0,\infty)$ into X with the trajectory property

$$F(t+\tau) = e^{-\tau A} F(t) \ , \quad t,\tau > 0.$$

Consequently, spaces of type $T_{X,A}$ are called 'trajectory spaces'.
In [G], ch.V, topological tensor products of the spaces $S_{X,A}$, $S_{Y,B}$, $T_{X,A}$ and $T_{Y,B}$ are described. For a completion of the algebraic tensor product $S_{X,A} \otimes_a S_{Y,B}$ there can be taken an analyticity space and, similarly, for a completion of $T_{X,A} \otimes_a T_{Y,B}$ a trajectory space. These completions, $S_{X\otimes Y, A\boxplus B}$ and $T_{X\otimes Y, A\boxplus B}$ can be regarded as spaces of continuous linear mappings from $T_{X,A}$ into $S_{Y,B}$ resp. from $S_{X,A}$ into $T_{Y,B}$. For analogous results with respect to the algebraic tensor products $T_{X,A} \otimes_a S_{Y,B}$ and $S_{X,A} \otimes_a T_{Y,B}$ one has to go beyond the common analyticity and trajectory spaces. De Graaf solves this problem by introducing the spaces Σ'_A and Σ'_B, which seem to be outsiders in the theory. However, they are the needed topological tensor products. For instance, each element of Σ'_A corresponds to a continuous linear mapping from $S_{X,A}$ into $S_{Y,B}$.

In this paper we are interested in the structures of the spaces Σ_A' and Σ_B'. In order to understand their topological structure we introduce two new types of topological vector spaces. The spaces Σ_A' and Σ_B' are of these types. But they include the spaces $S_{X,A}$, $S_{Y,B}$ and $T_{X,A}$, $T_{Y,B}$ as well. So it yields a genuine extension of the notions of analyticity space and of trajectory space.

This paper consists of two independent parts, $[E_1]$ and $[E_2]$. Both $[E_1]$ and $[E_2]$ have their own introduction, to which the reader is referred for a more technical discussion of the respective contents.

The first part $[E_1]$ is devoted to the introduction of two general types of spaces, $S(T_{Z,C},D)$ and $T(S_{Z,C},D)$. Here C and D are two commuting, non-negative, self-adjoint operators in a Hilbert space Z. We shall give topologies and a pairing for these types of spaces. We note that for $D = 0$ $S(T_{Z,C},D) = T_{Z,C}$ and $T(S_{Z,C},D) = S_{Z,C}$. Further, we shall describe the intersection of the spaces $T(S_{Z,C},D)$ and $T(S_{Z,D},C)$. It will lead to a fifth Kernel theorem.

In $[E_2]$ we discuss operator theory for analyticity and trajectory spaces, where we feel inspired by operator theory for Hilbert spaces. Because of the Kernel theorems the spaces Σ_A' and Σ_B' can be considered as operator spaces. In our discussion we involve the algebraic structure, the topological structure and their interrelation. Of course Σ_A' and Σ_B' have become much more tractable by the results in $[E_1]$. Further, it is worth mentioning that there has been constructed a matrix calculus for continuous linear mappings on nuclear analyticity spaces. This calculus provides a large variety of examples.

I. Analyticity spaces and trajectory spaces based on a pair of commuting, holomorphic semigroups

Introduction

A main result in the theory on analyticity and trajectory spaces is the validity of four Kernel theorems for four types of continuous linear mappings which appear in this theory. A Kernel theorem provides conditions such that all linear mappings of a specific kind arise from the elements (kernels) out of a suitable topological tensor product. In this connection we recall that $T_{X\otimes Y,A\boxplus B}$ is a topological tensor product of $T_{X,A}$ and $T_{Y,B}$, and to each element of $T_{X\otimes Y,A\boxplus B}$ there corresponds a continuous linear mapping from $S_{X,A}$ into $T_{Y,B}$. Then by [G], ch.VI, $T_{X\otimes Y,A\boxplus B}$ comprises all continuous linear mappings from $S_{X,A}$ into $T_{Y,B}$ if one of the spaces $T_{X,A}$ or $T_{Y,B}$ is nuclear. If X = Y and $A = B$ the condition of nuclearity is even necessary.

In order to prove a Kernel theorem for the continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$, resp. from $T_{X,A}$ into $T_{Y,B}$ the rather curious spaces Σ_A' and Σ_B' are brought up in [G]. The space Σ_A' is a topological tensor product of $T_{X,A}$ and $S_{Y,B}$ and the space Σ_B' of $S_{X,A}$ and $T_{Y,B}$.

In the second part of this paper we shall explicitly formulate the mentioned Kernel theorems within the framework of a thorough discussion of continuous linear mappings on analyticity and trajectory spaces.
During the investigations which led to the second part of this paper, [E_2], we needed a clearer view on those remarkable spaces Σ_A' and Σ_B'.
To this end we studied two new types of spaces, namely $S(T_{Z,C},D)$ and $T(S_{Z,C},D)$ with C and D commuting, non-negative, self-adjoint operators

in a Hilbert space Z. We shall present them here. Up to now these spaces have no other than an abstract use. However, the space $S(T_{Z,C},\mathcal{D})$ can be regarded as the 'analyticity domain' of the operator $\mathcal{D}$ in $T_{Z,C}$ Cf.[GE], Section 7. The space $T(S_{Z,C},\mathcal{D})$ contains all trajectories of $T_{Z,\mathcal{D}}$ through $S_{Z,C}$. We mention the following relations

$$\Sigma'_A = T(S_{X\otimes Y, I\otimes B}, A\otimes I) \quad , \quad \Sigma_A = S(T_{X\otimes Y, I\otimes B}, A\otimes I),$$

$$\Sigma'_B = T(S_{X\otimes Y, A\otimes I}, I\otimes B) \quad , \quad \Sigma_B = S(T_{X\otimes Y, A\otimes I}, I\otimes B).$$

The first section is concerned with the analyticity space $S(T_{Z,C},\mathcal{D})$. This space is a countable union of Frêchet spaces

$$S(T_{Z,C},\mathcal{D}) = \bigcup_{s>0} e^{-s\mathcal{D}}(T_{Z,C}) = \bigcup_{s>0} T_{e^{-s\mathcal{D}}(Z),C} \ .$$

For the strong topology we take the inductive limit topology . We shall produce an explicit system of seminorms which generates this topology, and characterize the elements of $S(T_{Z,C},\mathcal{D})$. We looked for a characterization of null-sequences, bounded subsets and compact subsets of $S(T_{Z,C},\mathcal{D})$ and for the proof of its completeness; however, without success. The second section is devoted to the trajectory space $T(S_{Z,C},\mathcal{D})$. With the introduction of a 'natural' topology, the space $T(S_{Z,C},\mathcal{D})$ becomes a complete topological vector space. Here we have been more successful. The elements, the bounded and the compact subsets, and the null-sequences of $T(S_{Z,C},\mathcal{D})$ will be described completely. Since $T_{X,A}$ is a special $T(S_{Z,C},\mathcal{D})$-space the latter results extend the theory on the topological structure of $T_{X,A}$. Cf.[G], ch.II. In Section 3 we shall introduce a pairing between $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$. With this pairing they can be regarded as each other's strong dual spaces. Further we note that for both spaces a Banach-Steinhaus theorem will be proved.

The extendable linear mappings establish a fifth type of mappings in the theory. They are continuous from $S_{X,A}$ into $S_{Y,B}$, and can be 'extended' to continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$. In order to describe the class of extendable linear mappings it is natural to look for a description of the intersection of Σ'_A and Σ'_B, or, more generally, of $T(S_{Z,C},\mathcal{D})$ and $T(S_{Z,\mathcal{D}},C)$. Therefore in Section 4 we introduce the nonnegative, self-adjoint operators $C \wedge \mathcal{D} = \max(C,\mathcal{D})$ and $C \vee \mathcal{D} = \min(C,\mathcal{D})$. To these both the theory in [G] and the theory of Sections 1-3 apply. The operators $C \wedge \mathcal{D}$ and $C \vee \mathcal{D}$ enable us to represent intersections and algebraic sums of the spaces $S_{Z,C}$, $S_{Z,\mathcal{D}}$, $T_{Z,C}$, $T_{Z,\mathcal{D}}$, $S(T_{Z,C},\mathcal{D})$, etc., as spaces of one of our types. It will lead to a fifth Kernel theorem in $[E_2]$.

The spaces which appear in our theory are ordered by inclusion. In the final section we discuss the inclusion scheme. Since each space can be considered as a space of continuous linear mappings of a specific kind the scheme illustrates the interdependence of these types.

1. The space $S(T_{Z,C},\mathcal{D})$

Let C and $\mathcal{D}$ denote two commuting, non-negative, self-adjoint operators in a Hilbert space Z. We take them fixed throughout this part of the paper. Suppose $C,\mathcal{D}$ admit spectral resolutions $(G_\lambda)_{\lambda\in\mathbb{R}}$ and $(H_\mu)_{\mu\in\mathbb{R}}$, such that

$$\mathcal{C} = \int_{\mathbb{R}} \lambda \, d\, G_\lambda \quad , \quad \mathcal{D} = \int_{\mathbb{R}} \mu \, d\, H_\mu .$$

Then for every pair of Borel sets Δ_1, Δ_2 in $\mathbb{R}$

$$G(\Delta_1)\; H(\Delta_2) = H(\Delta_2)\; G(\Delta_1) \; .$$

Since the operators $e^{-s\mathcal{D}}$, $s > 0$, and $e^{-t\mathcal{C}}$, $t > 0$, consequently commute, for each fixed $s > 0$ the linear mapping $e^{-s\mathcal{D}}$ is continuous on the trajectory space $\mathcal{T}_{Z,\mathcal{C}}$ (Cf.[GE], Section 4). We now introduce the space $S(\mathcal{T}_{Z,\mathcal{C}},\mathcal{D})$ as follows.

(1.1) Definition

$$S(\mathcal{T}_{Z,\mathcal{C}},\mathcal{D}) = \bigcup_{s>0} e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}}) = \bigcup_{n\in\mathbb{N}} e^{-\frac{1}{n}\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}}) \; .$$

We note that $e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}}) \subset e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ for $0 < \sigma < s$. Since the operator $e^{-s\mathcal{D}}$ is injective on $S_{Z,\mathcal{C}}$, the space $e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ is dense in $\mathcal{T}_{Z,\mathcal{C}}$ by duality. Hence $S(\mathcal{T}_{Z,\mathcal{C}},\mathcal{D})$ is a dense subspace of $\mathcal{T}_{Z,\mathcal{C}}$. In the space $e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}}) = \mathcal{T}_{e^{-s\mathcal{D}}(Z),\mathcal{C}}$, the strong topology is the topology generated by the seminorms $q_{s,n}$, $n \in \mathbb{N}$,

$$q_{s,n}(h) = \| e^{s\mathcal{D}} \, h(\tfrac{1}{n}) \|_Z \quad , \quad h \in e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$$

We remark that $e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ is a Frêchet space.

(1.2) Definition

The strong topology on $S(\mathcal{T}_{Z,\mathcal{C}},\mathcal{D})$ is the inductive limit topology, i.e.

the finest locally convex topology for which all injections

$$i_s : e^{-s\mathcal{D}}(T_{Z,C}) \to S(T_{Z,C},\mathcal{D})$$

are continuous.

Note that the inductive limit is not strict!

A subset $\Omega \subset S(T_{Z,C},\mathcal{D})$ is open if and only if the intersection $\Omega \cap e^{-s\mathcal{D}}(T_{Z,C})$ is open in $e^{-s\mathcal{D}}(T_{Z,C})$ for each $s > 0$.
In this section we shall produce a system of seminorms in $S(T_{Z,C},\mathcal{D})$ which induces a locally convex topology equivalent to the strong topology of (1.2). Therefore we introduce the set of functions $F(\mathbb{R}^2)$.

(1.3) <u>Definition</u>

Let θ be an everywhere finite Borel function on $\mathbb{R}^2$. Then $\theta \in F(\mathbb{R}^2)$ if and only if

$$\forall_{s>0}\ \exists_{t>0}\ \sup_{\substack{\lambda\geq 0\\ \mu\geq 0}} \left(|\theta(\lambda,\mu)|\, e^{-\mu s}\, e^{\lambda t}\right) < \infty.$$

Further, $F_+(\mathbb{R}^2)$ denotes the subset of all functions $F(\mathbb{R}^2)$ which are positive on $\{(\lambda,\mu)\,|\,\lambda \geq 0,\ \mu \geq 0\}$.

For $\theta \in F(\mathbb{R}^2)$ the operator $\theta(C,\mathcal{D})$ in X is defined by

$$\theta(C,\mathcal{D}) = \iint_{\mathbb{R}^2} \theta(\lambda,\mu)\ dG_\lambda H_\mu .$$

Here $dG_\lambda H_\mu$ denotes the operator-valued measure on the Borel subsets of $\mathbb{R}^2$ related to the spectral projections of C and $\mathcal{D}$. On the domain

$$D(\theta(C,\mathcal{D})) = \{w \in \mathbf{Z} \mid \iint_{\mathbb{R}^2} |\theta(\lambda,\mu)|^2 \, d(G_\lambda H_\mu w,w) < \infty\}$$

$\theta(C,\mathcal{D})$ is self-adjoint.

The operators $\theta(C,\mathcal{D})$, $\theta \in F(\mathbb{R}^2)$, are continuous linear mappings from $S(T_{Z,C},\mathcal{D})$ into Z. This can be seen as follows. Let $h \in S(T_{Z,C},\mathcal{D})$. Then define

$$\theta(C,\mathcal{D})h = (e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}})e^{s\mathcal{D}}(h(t)).$$

Since there exists $s > 0$ such, that $e^{s\mathcal{D}}\, h(t) \in Z$ for all $t > 0$, and since for each $s > 0$ there exists $t > 0$ such, that the operator $e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}}$ is bounded on Z (cf. Definition (1.3)), the vector $\theta(C,\mathcal{D})h$ is in Z. Hence the following definition makes sense.

(1.4) Definition

For each $\theta \in F_+(\mathbb{R}^2)$ the seminorm p_θ is defined by

$$p_\theta(h) = \|\theta(C,\mathcal{D})h\|_Z\,, \qquad h \in S(T_{Z,C},\mathcal{D})\;.$$

and the set $U_{\theta,\varepsilon}$, $\varepsilon > 0$, by

$$U_{\theta,\varepsilon} = \{h \in S(T_{Z,C},\mathcal{D}) \mid \|\theta(C,\mathcal{D})h\|_Z < \varepsilon\}.$$

The next theorem is the generalization of Theorem (1.4) in [G] to the type of space $S(T_{Z,C},\mathcal{D})$.

(1.5) Theorem

I. For each $\theta \in F_+(\mathbb{R}^2)$ the seminorm p_θ is continuous in the strong topology of $S(T_{Z,C},\mathcal{D})$.

II. Let a convex set $\Omega \subset S(T_{Z,C},\mathcal{D})$ have the property that for each $s > 0$ the set $\Omega \cap e^{-s\mathcal{D}}(T_{Z,C})$ contains a neighbourhood of 0 in $e^{-s\mathcal{D}}(T_{Z,C})$. Then Ω contains a set $U_{\theta,\varepsilon}$ for well-chosen $\theta \in F_+(\mathbb{R}^2)$ and $\varepsilon > 0$. Hence the strong topology in $S(T_{Z,C},\mathcal{D})$ is induced by the seminorms p_θ.

Proof.

I. In order to prove that p_θ is a continuous seminorm on $S(T_{Z,C},\mathcal{D})$ we have to show that $\theta(C,\mathcal{D})$ is a continuous linear mapping from $S(T_{Z,C},\mathcal{D})$ into Z. Therefore, let $s > 0$. Then there is $t > 0$ such that $\|e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}}\| < \infty$. So $\theta(C,\mathcal{D})$ is continuous on $e^{-s\mathcal{D}}(T_{Z,C})$ (cf. [GE] Section 4). Since $s > 0$ is arbitrarily taken, it implies that $\theta(C,\mathcal{D})$ is continuous on $S(T_{Z,C},\mathcal{D})$.

II. We introduce the projections P_{nm}, $n,m \in \mathbb{N}$,

$$P_{nm} = \int_{n-1}^{n}\int_{m-1}^{m} dG_\lambda H_\mu .$$

Then $P_{nm}(\Omega)$ contains an open neighbourhood of 0 in $P_{nm}(Z)$. (We note that $P_{nm}(S(T_{Z,C},\mathcal{D})) \subset P_{nm}(Z)$.) So the following definition makes sense,

$$r_{nm} = \sup\{\rho \mid (h \in P_{nm}(Z) \wedge \|P_{nm}h\| < \rho) \Rightarrow h \in P_{nm}(\Omega)\} .$$

Next we define the function θ as follows

$$\theta(\lambda,\mu) = \frac{n^2m^2}{r_{nm}}\ , \qquad \lambda \in (n-1,n]\ ,\ \mu \in (m-1,m],$$

$$\theta(\lambda,0) = (\lambda,\tfrac{1}{2}), \qquad \lambda > 0\ ,$$

$$\theta(0,\mu) = (\tfrac{1}{2},\mu), \qquad \mu > 0\ ,$$

$$\theta(\lambda,\mu) = 0 \qquad , \qquad \lambda < 0 \vee \mu < 0\ .$$

We shall prove that $\theta \in F(\mathbb{R}^2)$. To this end, let $s > 0$. Then there are $t > 0$ and $\varepsilon > 0$ such that

$$\{h \mid \int_0^\infty \int_0^\infty e^{\mu s} d(G_\lambda H_\mu h(t),h(t)) < \varepsilon^2\} \subset \Omega \cap e^{-\frac{1}{2}s\mathcal{D}}(\mathcal{T}_{Z,C}).$$

because $\Omega \cap e^{-\frac{1}{2}s\mathcal{D}}(\mathcal{T}_{Z,C})$ contains an open neighbourhood of 0 by assumption. So we derive

$$r_{nm} > \varepsilon\ e^{nt}\ e^{-\frac{1}{2}(m-1)s}\ ,\ n,m \in \mathbb{N}\ .$$

With $\lambda \in (n-1,n]$, $\mu \in (m-1,m]$ it follows that

$$\theta(\lambda,\mu)\ e^{\frac{1}{2}\lambda t}\ e^{-\mu s} < \frac{n^2m^2}{r_{nm}}\ e^{\frac{1}{2}nt}\ e^{-(m-1)s}$$

$$\leq \frac{m^2m^2}{\varepsilon}\ e^{-\frac{1}{2}nt}\ e^{-\frac{1}{2}(m-1)s}\ .$$

So $$\sup_{\substack{\lambda\geq 0\\ \mu\geq 0}} (e^{\frac{1}{2}\lambda t}\ e^{-\mu s}\ \theta(\lambda,\mu) < \infty.$$

We claim that

(*) $$\|\theta(C,\mathcal{D})h\| < 1 \Rightarrow h \in \Omega.$$

Suppose $h \in e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ for some $s > 0$. Then for all $t > 0$

$$\sum_{n,m} \| e^{s\mathcal{D}} e^{-t\mathcal{C}} \mathcal{P}_{nm} h \|^2 < \infty$$

and for σ, $0 < \sigma < s$, fixed and every $\tau > t$

$$(**) \qquad \| e^{\sigma\mathcal{D}} e^{-\tau\mathcal{C}} \mathcal{P}_{nm} h \| \le e^{-(m-1)s-\sigma)} e^{-(n-1)(\tau-t)} \| e^{s\mathcal{D}} e^{-t\mathcal{C}} \mathcal{P}_{nm} h \| .$$

Because of assumption $(*)$

$$\| \mathcal{P}_{nm} h \| < (n^2 m^2)^{-1} \, r_{nm} .$$

Hence $n^2 m^2 \mathcal{P}_{nm} h \in \Omega \cap e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ for every $n,m \in \mathbb{N}$. In $e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ we represent h by

$$h = \sum_{n,m}^{N,M} \frac{1}{n^2 m^2} (n^2 m^2 \mathcal{P}_{nm} h) + \Big(\sum_{(n>N)\vee(m>M)} \frac{1}{n^2 m^2} \Big) h_{NM}$$

where

$$h_{NM} = \Big(\sum_{(j>N)\vee(i>M)} \frac{1}{i^2 j^2} \Big)^{-1} \Big(\sum_{(n>N)\vee(m>M)} \mathcal{P}_{nm} h \Big) .$$

With $(**)$ we calculate

$$\| e^{\sigma\mathcal{D}} e^{-\tau\mathcal{C}} h_{NM} \|^2 \le$$

$$\le \Big(N^4 \sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} + M^4 \sum_{n=1}^{\infty} \sum_{m=M+1}^{\infty} \Big) (\| e^{\sigma\mathcal{D}} e^{-\tau\mathcal{C}} \mathcal{P}_{nm} h \|^2)$$

$$\le \Big(N^4 e^{-2N(\tau-t)} + M^4 e^{-2M(s-\sigma)} \Big) \| e^{s\mathcal{D}} e^{-t\mathcal{C}} h \|^2 .$$

Hence $h_{NM} \to 0$ in $e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,C})$ because both $t > 0$ and $\tau > t$ are taken arbitrarily. So for sufficiently large N,M we have $h_{NM} \in [\Omega \cap e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,C})]$. Since h is a sub-convex combination of elements in the convex set $\Omega \cap e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,C})$ the result $h \in \Omega$ follows. □

Similar to [GE], Section 1, we should like to characterize bounded subsets, compact subsets, and sequential convergence in $S(\mathcal{T}_{Z,C},\mathcal{D})$. However, we think that this requires a method of constructing functions in $F_+(\mathbb{R}^2)$ similar to the construction of functions in $B_+(\mathbb{R})$ in the proofs of the characterizations given in [G], Ch.I. Up to now, our attempts to solve this problem were not successful.

Remark. As in [GE] the set $B_+(\mathbb{R})$ consists of all everywhere finite Borel function φ on $\mathbb{R}$ which are strictly positive and satisfy

$$\forall_{\varepsilon>0} : \sup_{x>0} (\varphi(x)e^{-\varepsilon x}) < \infty.$$

Finally, we characterize the elements of $S(\mathcal{T}_{Z,C},\mathcal{D})$.

(1.6) Lemma

$h \in S(\mathcal{T}_{Z,C},\mathcal{D})$ iff there are $\psi \in B_+(\mathbb{R})$, $w \in Z$ and $s > 0$ such that $h = e^{-s\mathcal{D}}\psi(C)w$.

Proof. The proof is an immediate consequence of the following equivalence:

$$F \in \mathcal{T}_{Z,C} \Leftrightarrow \exists_{\psi\in B_+(\mathbb{R})} \exists_{w\in Z} : F = \psi(C)w$$ □

As in [G], Ch.I,it can be proved that $S(\mathcal{T}_{Z,C},\mathcal{D})$ is bornological and barreled.

2. The space $\mathcal{T}(S_{Z,C},\mathcal{D})$

The elements of $\mathcal{T}_{Z,\mathcal{D}}$ are called trajectories, i.e. functions F from $(0,\infty)$ into Z with the following property:

$$\forall_{s>0}\ \forall_{\sigma>0}\ :\ F(s+\sigma) = e^{-\sigma\mathcal{D}}\, F(s)\ .$$

Now the subspace $\mathcal{T}(S_{Z,C},\mathcal{D})$ of $\mathcal{T}_{Z,\mathcal{D}}$ is defined as follows:

(2.1) Definition

The space $\mathcal{T}(S_{Z,C},\mathcal{D})$ contains all elements $G \in \mathcal{T}_{Z,\mathcal{D}}$ which satisfy

$$\forall_{s>0}\ :\ G(s) \in S_{Z,C}\ .$$

Remark. $\mathcal{T}(S_{Z,C},\mathcal{D})$ consists of trajectories of $\mathcal{T}_{Z,\mathcal{D}}$ through $S_{Z,C}$. The space $\mathcal{T}(S_{Z,C},\mathcal{D})$ is not trivial. The embedding of Z into $\mathcal{T}_{Z,\mathcal{D}}$ maps $S_{Z,C}$ into $\mathcal{T}(S_{Z,C},\mathcal{D})$, because the bounded operators $e^{-s\mathcal{D}}$, $s > 0$ and $e^{-t\mathcal{D}}$, $t > 0$, commute.

In $\mathcal{T}(S_{Z,C},\mathcal{D})$ we introduce the seminorms $p_{\psi,s}$, $\psi \in B_+(\mathbb{R})$, $s > 0$, by

(2.2) $$p_{\psi,s} = \|\psi(C)\,F(s)\|_Z\ \ ,\ F \in \mathcal{T}(S_{Z,C},\mathcal{D})\ .$$

The strong topology in $\mathcal{T}(S_{Z,C},\mathcal{D})$ is the locally convex topology induced by the seminorms $p_{\psi,s}$.
The bounded subsets of $\mathcal{T}(S_{Z,C},\mathcal{D})$ can be fully characterized with the aid of the function algebra $F_+(\mathbb{R}^2)$. To this end we first prove the following lemma.

(2.3) Lemma

The subset B in $T(S_{Z,C},\mathcal{D})$ is bounded iff for each $s > 0$ there exists $t > 0$ such that the set $\{F(s) \mid F \in B\}$ is bounded in the Hilbert space $e^{-tC}(Z)$.

Proof. B is bounded in $T(S_{Z,C},\mathcal{D})$ iff each seminorm $p_{\psi,s}$ is bounded on B iff the set $\{F(s) \mid F \in B\}$ is bounded in $S_{Z,C}$ for each $s > 0$. From [GE], Section 1, the assertion follows.

Because of Definition (1.3) for every $\theta \in F_+(\mathbb{R}^2)$ and each $w \in Z$ the vector $\theta(C,\mathcal{D})e^{-s\mathcal{D}}w$ is in $S_{Z,C}$. So the trajectory $s \mapsto \theta(C,\mathcal{D})e^{-s\mathcal{D}}w$ is an element of $T(S_{Z,C},\mathcal{D})$ and it will be denoted by $\theta(C,\mathcal{D})w$.

(2.4) Theorem

The set $B \subset T(S_{Z,C},\mathcal{D})$ is bounded iff there exists $\theta \in F_+(\mathbb{R}^2)$ and a bounded subset V of Z such that $B = \theta(C,\mathcal{D})(V)$

Proof.

$\Leftarrow$) Let $s > 0$. Then there exists $t > 0$ such that

$$\| e^{tC} (C,\mathcal{D}) e^{-s\mathcal{D}} w \| \leq \| e^{tC} \theta(C,\mathcal{D}) e^{-s\mathcal{D}} \| \| w \| .$$

Hence B is a bounded subset by Lemma (2.3).

$\Rightarrow$) Let $n,m \in \mathbb{N}$. Define

$$P_{nm} = \int_{n-1}^{n} \int_{m-1}^{m} d\, G_\lambda H_\mu ,$$

and put $r_{nm} = \sup_{G \in B} (\| P_{nm} G \|)$. Let $s > 0$. Then there are $t > 0$ and $K_{s,t} > 0$ such that

$$r_{nm}^2 = \sup_{G\in B} \left(\int_{n-1}^{n} \int_{m-1}^{m} d(G_\lambda H_\mu\ G,G) \right) \le$$

$$\le e^{2ms}\ e^{-2(n-1)t} \sup_{G\in B} \left(\int_{n-1}^{n} \int_{m-1}^{m} e^{-2\mu s} e^{2\lambda t} d(G_\lambda H_\mu\ G,G) \right) \le$$

$$e^{2ms}\ e^{-2(n-1)t}\ \| e^{tC}\ G(s) \|^2 \le e^{2ms}\ e^{-2nt}\ K_{s,t}^2 .$$

Thus we obtain the following

$$\forall_{s>0}\ \exists_{t>0}\ \exists_{K>0}\ \forall_{n,m\in \mathbb{N}}\ :\ nm\ r_{nm}\ e^{-ms}\ e^{nt} \le K .$$

Define θ on $\mathbb{R}^2$ by

$$\theta(\lambda,\mu) = nm\ r_{nm} \text{ if } r_{nm} \neq 0,\ n-1 \le \lambda < n,\ m-1 \le \mu < m,$$

$$\theta(\lambda,\mu) = e^{-n} \qquad \text{if } r_{nm} = 0 ,$$

$$\theta(\lambda,\mu) = 0 \qquad \text{if } \lambda < 0 \text{ or } \mu < 0 .$$

Then $\theta \in F_+(\mathbb{R}^2)$. To show this, let $s > 0$. Then there are $0 < t < 1$ and $K > 0$ such that for all $\lambda \in [n-1,n)$ and $\mu \in [m-1,m)$

$$\theta(\lambda,\mu)\ e^{-\lambda t}\ e^{-\mu s} \le nm\ r_{nm}\ e^{nt}\ e^{-(m-1)s} \le e^{s}\ K_{s,t}$$

if $r_{nm} \neq 0$, and if $r_{nm} = 0$,

$$\theta(\lambda,\mu)\ e^{\lambda t}\ e^{-\mu s} \le e^{-n}\ e^{nt} < 1 .$$

For each $G \in B$ define W by

$$W = \theta(C,D)^{-1} G = \sum_{r_{nm}\neq 0} \left(\frac{r_{nm}^{-1}}{nm}\ P_{nm}\ G \right) .$$

Then we calculate as follows

$$\|\omega\|_Z^2 = \sum_{r_{nm}\neq 0} n^{-2}m^{-2}(r_{nm}^{-2}\|P_{nm}\,G\|^2) < \sum_{n,m} n^{-2}m^{-2} = (\frac{\pi^2}{6})^2$$

Hence $\omega \in Z$ with $\|\omega\| < \frac{\pi^2}{6}$, and the set $V = \theta(C,\mathcal{D})^{-1}(B)$ is bounded in Z. □

Since $T_{X,A}$ is a special $T(S_{Z,C},\mathcal{D})$ space, Theorem (2.4) yields a characterization of the bounded subsets of $T_{X,A}$.

(2.5) Corollary

Let $B \subset T_{X,A}$. Then B is bounded iff there exists $\psi \in B_+(\mathbb{R})$ and a bounded subset V in X such that $B = \psi(A)(V)$.

Special bounded subsets of $T(S_{Z,C},\mathcal{D})$ are the sets consisting of one single point. This observation leads to the following.

(2.6) Corollary

Let $H \in T(S_{Z,C},\mathcal{D})$. Then there are $\omega \in Z$ and $\theta \in F_+(\mathbb{R}^2)$ such that $H = \theta(C,\mathcal{D})\omega$. (Cf.[GE], Section 2).

Similar to Lemma (2.3) strong convergence in $T(S_{Z,C},\mathcal{D})$ can be characterized.

(2.7) Lemma

Let (H_ℓ) be a sequence in $T(S_{Z,C},\mathcal{D})$. Then $H_\ell \to 0$ in $T(S_{Z,C},\mathcal{D})$ iff $\forall_{s>0}\,\exists_{t>0} : \|e^{tC}\,F_\ell(s)\| \to 0$.

Proof. (H_ℓ) is a null sequence in $\mathcal{T}(S_{Z,C},\mathcal{D})$ iff $(H_\ell(s)$ is a null sequence in $S_{Z,C}$ for each $s > 0$. From [GE], Section 1 the assertion follows. □

(2.8) Theorem

(H_ℓ) is a null sequence in $\mathcal{T}(S_{Z,C},\mathcal{D})$ iff there exists a null sequence (w_ℓ) in Z and $\theta \in F_+(\mathbb{R}^2)$ such that $H_\ell = \theta(C,\mathcal{D})w_\ell$.

Proof. The sequence (H_ℓ) is bounded in $\mathcal{T}(S_{Z,C},\mathcal{D})$. Then construct $\theta \in F_+(\mathbb{R}^2)$ as in Theorem (2.4):

$$\theta(\lambda,\mu) = nm\, r_{nm} \quad \text{if } r_{nm} \neq 0,\ n-1 \le \lambda < n,\ m-1 \le \mu < m\ ,$$

$$\theta(\lambda,\mu) = e^{-n} \quad \text{if } r_{nm} = 0\ ,$$

$$\theta(\lambda,\mu) = 0 \quad \text{if } \lambda < 0 \text{ or } \mu < 0$$

where $r_{nm} = \max_{\ell\in\mathbb{N}} (\| P_{nm}\, H_\ell \|)$.

Let $\varepsilon > 0$. Then there are $N,M \in \mathbb{N}$ such that

$$\sum_{(n>N)\vee(m>M)} \frac{1}{n^2m^2} < (\varepsilon/2)^2.$$

Define $w_\ell = \theta(C,\mathcal{D})^{-1}\, H_\ell = \sum_{r_{nm}\neq 0} \frac{r_{nm}^{-1}}{nm} P_{nm}\, H_\ell$, $\ell \in \mathbb{N}$. Then for all $\ell \in \mathbb{N}$

$$(*) \qquad \sum_{(n>N)\vee(m>M)} n^{-2}m^{-2}\left(r_{nm}^{-2}\, \| P_{nm}H_\ell \|^2\right) < (\varepsilon/2)^2.$$

Further, there exist $t > 0$ and $\ell_0 \in \mathbb{N}$ such that for all $\ell > \ell_0$

$$(**) \qquad \sum_{(n\le N)\wedge(m\le M)\wedge r_{nm}\neq 0} (n^{-2}m^{-2}\ r_{nm}^{-2}\, \| P_{nm}\ H_\ell \|^2) \le$$

$$\le e^{2M} \max_{(n\le N)\wedge(m\le M)\wedge r_{nm}\neq 0} \left[(r_{nm}^{-2}) \| e^{tC}\ H_\ell(1) \|^2 \right] < (\varepsilon/2)^2\ .$$

A combination of (*) and (**) yields the result

$$\|w_\ell\| < \varepsilon \quad \text{for all } \ell > \ell_0$$ □

Since the choice of $\theta \in F_+(\mathbb{R}^2)$ in the proof of the previous theorem has to do only with the boundedness of the sequence (H_ℓ) in $T(S_{Z,C},\mathcal{D})$, Theorem (2.8) implies the following.

(2.9) Corollary

(F_ℓ) is a Cauchy sequence in $T(S_{Z,C},\mathcal{D})$ iff there exists $\theta \in F_+(\mathbb{R}^2)$ and a Cauchy sequence (w_ℓ) in Z such that $F_\ell = \theta(C,\mathcal{D})w_\ell$, $\ell \in \mathbb{N}$. Hence every Cauchy sequence in $T(S_{Z,C},\mathcal{D})$ converges to a limit point.

Further, we have the following extension of the theory in [C].

(2.10) Corollary

(F_ℓ) is a null (Cauchy) sequence in $T_{X,A}$ if there exists a null (Cauchy) sequence (w_ℓ) in X and $\psi \in B_+(\mathbb{R})$ with $F_\ell = \psi(A)w_\ell$, $\ell \in \mathbb{N}$.

Finally we characterize the compact subsets of $T(S_{Z,C},\mathcal{D})$.

(2.11) Theorem

Let $K \subset T(S_{Z,C},\mathcal{D})$. Then K is compact iff there exists $\theta \in F_+(\mathbb{R}^2)$ and a compact subset $W \subset Z$ such that $K = \theta(C,\mathcal{D})(W)$.

Proof.

⇒) Since K is compact, K is bounded in $T(S_{Z,C},\mathcal{D})$. So construct $\theta \in F_+(\mathbb{R}^2)$ and the bounded subset W of Z as in the proof of Theorem (2.4). We shall prove that W is compact. Let (w_ℓ) be a sequence in W. Then $(\theta(C,\mathcal{D})w_\ell)$ is a sequence in K. Since K is compact there exists a subsequence (w_{ℓ_k}) and $w \in Z$ such that

$$\theta(C,\mathcal{D})(w_{\ell_k}-w) \to 0 \quad \text{in } T(S_{Z,C},\mathcal{D}) .$$

The same arguments which led to Theorem (2.8) yield $w_{\ell_k} \to w$ in Z. Hence W is compact in Z.

⇐ Since $\theta(C,\mathcal{D}) : Z \to T(S_{Z,C},\mathcal{D})$ is continuous for each $\theta \in F_+(\mathbb{R}^2)$, the compact set $W \subset Z$ has a compact image $\theta(C,\mathcal{D})(W)$ in $T(S_{Z,C},\mathcal{D})$ for each $\theta \in F_+(\mathbb{R}^2)$ □

(2.12) Corollary

$K \subset T(S_{Z,C},\mathcal{D})$ is compact iff K is sequentially compact.

(2.13) Corollary

$K \subset T_{X,A}$ is compact iff there exists a compact $W \subset X$ and $\psi \in B_+(\mathbb{R})$ such, that $K = \psi(A)(W)$.

(2.14) Theorem

$T(S_{Z,C},\mathcal{D})$ is complete.

Proof. Let (F_α) be a Cauchy net in $T(S_{Z,C},\mathcal{D})$. Then for each $s > 0$ the net $(F_\alpha(s))$ is Cauchy in $S_{Z,C}$. Completeness of $S_{Z,C}$ yields $F(s) \in S_{Z,C}$ with $F_\alpha(s) \to F(s)$. Since $(e^{-s\mathcal{D}})_{s\geq 0}$ is a semigroup of continuous linear mappings on $S_{Z,C}$, the function $s \mapsto F(s)$ is a trajectory of $T(S_{Z,C},\mathcal{D})$. □

Finally,we prove the following result.

(2.15) Lemma

$S_{Z,C}$ is sequentially dense in $\mathcal{T}(S_{Z,C},\mathcal{D})$.

Proof. Let $H \in \mathcal{T}(S_{Z,C},\mathcal{D})$. Then $H(\frac{1}{n}) \in S_{Z,C}$, $n \in \mathbb{N}$ and $H(\frac{1}{n}) \to H$ in $\mathcal{T}(S_{Z,C},\mathcal{D})$. □

3. The pairing of $\mathcal{S}(\mathcal{T}_{Z,C},\mathcal{D})$ and $\mathcal{T}(S_{Z,C},\mathcal{D})$

In this section we introduce a pairing of $\mathcal{S}(\mathcal{T}_{Z,C},\mathcal{D})$ and $\mathcal{T}(S_{Z,C},\mathcal{D})$. It is shown that $\mathcal{S}(\mathcal{T}_{Z,C},\mathcal{D})$ and $\mathcal{T}(S_{Z,C},\mathcal{D})$ can be regarded as each other's strong dual spaces.

(3.1) Definition

Let $h \in \mathcal{S}(\mathcal{T}_{Z,C},\mathcal{D})$ and let $F \in \mathcal{T}(S_{Z,C},\mathcal{D})$. Then the number $\langle\!\langle h,F\rangle\!\rangle$ is defined by

$$\langle\!\langle h,F\rangle\!\rangle = \overline{\langle F(s),\ e^{s\mathcal{D}} h\rangle}\ .$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual pairing of $S_{Z,C}$ and $\mathcal{T}_{Z,C}$.

We note that the above definition makes sense for $s > 0$ sufficiently small and that it does not depend on the choice of $s > 0$ because of the trajectory property of F.

(3.2) Theorem

I. Let $F \in \mathcal{T}(S_{Z,C},\mathcal{D})$. Then the functional

$$h \mapsto \langle\!\langle h,F\rangle\!\rangle$$

is continuous on $\mathcal{S}(\mathcal{T}_{Z,C},\mathcal{D})$.

II. Let ℓ be a continuous linear functional on $\mathcal{S}(\mathcal{T}_{Z,C},\mathcal{D})$. Then there exists $G \in \mathcal{T}(S_{Z,C},\mathcal{D})$ such that

$$\ell(h) = \langle\!\langle h,G \rangle\!\rangle \ , \quad h \in S(\mathcal{T}_{Z,C},\mathcal{D})$$

III. Let $h \in S(\mathcal{T}_{Z,C},\mathcal{D})$. Then the functional

$$F \mapsto \overline{\langle\!\langle h,F \rangle\!\rangle}$$

is continuous on $\mathcal{T}(S_{Z,C},\mathcal{D})$.

IV. Let m be a continuous linear functional on $\mathcal{T}(S_{Z,C},\mathcal{D})$. Then there exists $g \in S(\mathcal{T}_{Z,C},\mathcal{D})$ such that

$$m(F) = \overline{\langle\!\langle g,F \rangle\!\rangle} \ , \quad F \in \mathcal{T}(S_{Z,C},\mathcal{D}).$$

Proof.

I. For every $W \in \mathcal{T}_{Z,C}$ and every $s > 0$

$$\langle\!\langle e^{-s\mathcal{B}}W,F \rangle\!\rangle = \langle F(s),W \rangle \ ,$$

and $W_n \to 0$ in $\mathcal{T}_{Z,C}$ implies $\langle F(s),W_n \rangle \to 0$. Hence the functional $h \to \langle\!\langle h,F \rangle\!\rangle$ is strongly continuous on $S(\mathcal{T}_{Z,C},\mathcal{D})$.

II. Because of the definition of inductive limit topology, each linear functional $\ell \circ e^{-s\mathcal{D}}$ is continuous on $\mathcal{T}_{Z,C}$. So there exists $G(s) \in S_{Z,C}$ with $(\ell \circ e^{-s\mathcal{D}})(W) = \overline{\langle G(s),W \rangle}$, $W \in \mathcal{T}_{Z,C}$, $s > 0$. Since $(e^{-s\mathcal{D}})_{s \geq 0}$ is a semigroup of continuous linear mappings on $S_{Z,C}$ it follows that

$$G(s+\sigma) = e^{-\sigma\mathcal{D}}G(s) \ , \quad s,\sigma \geq 0 \ .$$

So $s \to G(s)$ is in $\mathcal{T}(S_{Z,C},\mathcal{D})$ and

$$\ell(h) = \overline{\langle G(s),e^{s\mathcal{D}}h \rangle} = \langle\!\langle h,G \rangle\!\rangle \ , \ h \in S(\mathcal{T}_{Z,C},\mathcal{D}).$$

III. Following Lemma (1.6), there are $w \in Z$, $s > 0$ and $\psi \in B_+(\mathbb{R})$ with $h = e^{-s\mathcal{D}}\psi(C)w$. Hence the inequality

$$|\langle\!\langle h,F\rangle\!\rangle| = |\langle w,\psi(C)F(t)\rangle| \leq \|w\|\|\psi(C)F(t)\|$$

the continuity follows.

IV. The strong topology in $T(S_{Z,C},\mathcal{D})$ is generated by the seminorms $p_{\psi,s}$ where $s > 0$ and $\varphi \in B_+(\mathbb{R})$. Since m is strongly continuous on $T(S_{Z,C},\mathcal{D})$ there are $\sigma > 0$ and $\varphi \in B_+(\mathbb{R})$ such that

$$|m(F)| \leq p_{\varphi,\sigma}(F) = \|\varphi(C)F(\sigma)\|, \quad F \in T(S_{Z,C},\mathcal{D}).$$

So the linear functional $m \circ \varphi(C)^{-1}e^{\sigma\mathcal{D}}$ is norm continuous on the dense linear subspace $\varphi(C)e^{-\sigma\mathcal{D}}(T(S_{Z,C},\mathcal{D})) \subset Z$. It therefore can be extended to a continuous linear functional on Z. So there exists $w \in Z$ with

$$(m \circ \varphi(C)^{-1}e^{\sigma\mathcal{D}})(\varphi(C)F(\sigma)) = (\varphi(C)F(\sigma),w).$$

Put $g = \varphi(C)e^{-\sigma\mathcal{D}}w \in S(T_{Z,C},\mathcal{D})$. □

Definition

The weak topology on $S(T_{Z,C},\mathcal{D})$ is the topology generated by the seminorms $u_F(h) = |\langle\!\langle h,F\rangle\!\rangle|$, $h \in S(T_{Z,C},\mathcal{D})$.
The weak topology on $S(T_{Z,C},\mathcal{D})$ is the topology generated by the seminorms $u_h(F) = |\langle\!\langle h,F\rangle\!\rangle|$, $F \in T(S_{Z,C},\mathcal{D})$.

A standard argument [Ch], II,§22 shows that the weakly continuous linear functionals on $S(T_{Z,C},\mathcal{D})$ are all obtained by pairing with elements of $T(S_{Z,C},\mathcal{D})$ and vice versa. So it follows that $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$ are reflexive both in the strong and the weak topology.

(3.4) Theorem (Banach-Steinhaus)

I. Let $W \subset T(S_{Z,C},\mathcal{D})$ be weakly bounded. Then W is strongly bounded.

II. Let $V \subset S(T_{Z,C},\mathcal{D})$ be weakly bounded. Then V is strongly bounded.

Proof.

I. Let $s > 0$, and let $\psi \in B_+(\mathbb{R})$. Then following Lemma (1.6) $e^{-s\mathcal{D}}\psi(C)w \in S(T_{Z,C},\mathcal{D})$ for each $w \in Z$ and by assumption there exists $N_w > 0$ such that $|\langle\langle e^{-s\mathcal{D}}\psi(C)w,F\rangle\rangle| = |(w,\psi(C)F(s))| \le N_w$, $F \in W$.

By the Banach-Steinhaus theorem for Hilbert spaces there exists $\alpha_{s,\psi} > 0$ such that

$$\|\psi(C)F(s)\| < \alpha_{s,\psi} .$$

With Lemma (2.3) the proof is finished.

II. Let $\theta \in F_+(\mathbb{R}^2)$. Then for each $w \in Z$, $\theta(C,\mathcal{D})w \in T(S_{Z,C},\mathcal{D})$.

By assumption there exists $M_w > 0$ such that

$$|(\theta(C,\mathcal{D})h,w)| \le M_w$$

for each $w \in Z$. Hence for all $h \in V$

$$\|\theta(C,\mathcal{D})h\| \le \alpha_\theta$$

for some $\alpha_\theta > 0$. □

The next theorem characterizes weakly converging sequences in $T(S_{Z,C},\mathcal{D})$.

(3.5) Theorem

$F_\ell \to 0$ in the weak topology of $T(S_{Z,C},\mathcal{D})$ iff there exists a sequence (w_ℓ) in Z with $w_\ell \to 0$ weakly in Z, and a function $\theta \in F_+(\mathbb{R}^2)$ such that $F_\ell = \theta(C,\mathcal{D})w_\ell$, $\ell \in \mathbb{N}$.

Proof

⇐) Trivial

⇒) The null sequence (F_ℓ) is weakly bounded. So by Theorem (3.4) it is a strongly bounded sequence in Z. As in Theorem (2.8) define r_{nm} for $n,m \in \mathbb{N}$ by

$$r_{nm} = \sup_{\ell \in \mathbb{N}} (\| P_{nm} F_\ell \|) \ .$$

Then $\forall_{s>0} \exists_{t>0} : \sup_{n,m} (nm\, r_{nm}\, e^{-ms} e^{nt}) < \infty$, and the function θ defined by

$$\theta(\lambda,\mu) = n\ m\ r_{nm} \quad \text{if } r_{nm} \neq 0,\ n-1 \le \lambda < n,\ m-1 \le \mu < m\ ,$$

$$\theta(\lambda,\mu) = e^{-n} \quad \text{if } r_{nm} = 0\ ,$$

$$\theta(\lambda,\mu) = 0 \quad \text{elsewhere}\ ,$$

is in $F_+(\mathbb{R}^2)$. Put $w_\ell = \theta(C,D)^{-1} F_\ell = \sum_{r_{nm} \neq 0} n^{-1} m^{-1} r_{nm}^{-1} P_{nm} F_\ell$, $\ell \in \mathbb{N}$.

Let $u \in Z$, and let $\varepsilon > 0$ and $N,M \in \mathbb{N}$ so large that

$$\sum_{(n>N) \vee (n>M)} (n^{-2} m^{-2}) < (\varepsilon/2)^2 \ .$$

Then

$$\Big| \sum_{\substack{(n>N) \vee (m>M) \\ r_{nm} \neq 0}} (u, P_{nm} w_\ell) \Big| \le \|u\| \Big(\sum_{\substack{(n>N \vee (m>M) \\ r_{nm} \neq 0}} \| P_{nm} w_\ell \|^2 \Big)^{\frac{1}{2}}$$

$$\le \|u\| \Big(\sum_{\substack{(n>N) \vee (m>M) \\ r_{nm} \neq 0}} n^{-2} m^{-2} (r_{nm}^{-2} \| P_{nm} F_\ell \|^2) \Big)^{\frac{1}{2}} \ .$$

$$< \varepsilon/2 \|u\| \ .$$

Further, since $P_{nm}u \in S(T_{Z,C},\mathcal{D})$ for all $n,m \in \mathbb{N}$, there exists $\ell_0 \in \mathbb{N}$ such that for all $\ell > \ell_0$

$$\Big|\sum_{\substack{(n\le N)\wedge(m\le M),\\ r_{nm}\ne 0}} (u,P_{nm}w_\ell)\Big| \le \Big|\Big\langle\!\Big\langle \Big\{\sum_{\substack{(n\le N)\wedge(m\ M),\\ r_{nm}\ne 0}} n^{-1}m^{-1}r_{nm}^{-1}P_{nm}u\Big\},F_\ell\Big\rangle\!\Big\rangle\Big| < \varepsilon/2 .$$

Hence, for each $\varepsilon > 0$ and $u \in Z$ there exists $\ell_0 \in \mathbb{N}$ such that for all $\ell > \ell_0$

$$|(u,w_\ell)| \le \Big|\sum_{\substack{(n>N)\vee(m>M),\\ r_{nm}\ne 0}} (u,P_{nm}w_\ell)\Big| + \Big|\sum_{\substack{(n\le N)\wedge(m\le M),\\ r_{nm}\ne 0}} (u,P_{nm}w_\ell)\Big| < \varepsilon.$$

Thus we have proved that $w_\ell \to 0$ weakly in Z, and

$$F_\ell = \theta(C,\mathcal{D})w_\ell. \qquad \square$$

(3.6) <u>Corollary</u>

I. Strong convergence of a sequence in $T(S_{Z,C},\mathcal{D})$ implies its weak convergence.

II. Any bounded sequence in $T(S_{Z,C},\mathcal{D})$ has a weakly converging subsequence.

(3.7) <u>Corollary</u>

(F_ℓ) is a weakly converging null sequence in $T_{X,A}$ iff there exists a weakly converging null sequence (w_ℓ) in X and a function $\psi \in B_+(\mathbb{R})$ such that $F_\ell = \psi(A)w_\ell$, $\ell \in \mathbb{N}$.

<u>Remark</u>: From Theorem (2.4) and Definition (3.2) it follows that the strong topology in $S(T_{Z,C},\mathcal{D})$ equals the so-called Mackey topology (Cf.[Tr],p.369).

4. Spaces related to the operators $C \vee D$ and $C \wedge D$

As in the previous sections, $(G_\lambda)_{\lambda\in\mathbb{R}}$ and $(H_\mu)_{\mu\in\mathbb{R}}$ denote the spectral resolutions of C and D. The orthogonal projection P, defined by

$$P = \iint_{\lambda\geq\mu} dG_\lambda H_\mu$$

commutes with C as well as D.

(4.1) Definition

The nonnegative, self-adjoint operator $C \wedge D$ is defined by

$$C \wedge D = PCP + (I-P)D(I-P) \ .$$

The nonnegative, self-adjoint operator $C \vee D$ is defined by

$$C \vee D = (I-P)C(I-P) + PDP.$$

Remark: The operators $C \wedge D$ and $C \vee D$ are also given by

$$C \wedge D = \iint_{\mathbb{R}^2} \max(\lambda,\mu) dG_\lambda H_\mu \ , \ C \vee D = \iint_{\mathbb{R}^2} \min(\lambda,\mu) dG_\lambda H_\mu \ .$$

The spaces $S_{Z,C\vee D}$, $S_{Z,C\wedge D}$, $T_{Z,C\vee D}$ and $T_{Z,C\wedge D}$ are well-defined by [GE], Section 1 and 2. With the aid of these spaces sums and intersections of $S_{Z,C}$, $S_{Z,D}$, $T_{Z,C}$, and $T_{Z,D}$ can be described.

(4.2) Theorem

I. $S_{Z,C\wedge D} = S_{Z,C+D} = S_{Z,C} \cap S_{Z,D}$

II. $S_{Z,C\vee D} = S_{Z,C} + S_{Z,D}$

III. $T_{Z,C\wedge D} = T_{Z,C+D} = T_{Z,C} + T_{Z,D}$

IV. $T_{Z,C\vee D} = T_{Z,C} \cap T_{Z,D}$.

(In II, + denotes the usual sum in Z, and in III the usual sum in $T_{Z,C+D}$.)

Proof. From the definition of the projection P we derive easily that for all $t > 0$ the operators $Pe^{-tC}e^{tD}P$ and $(I-P)e^{-tD}e^{tC}(I-P)$ are bounded in Z.

I. Let $f \in S_{Z,C\wedge D}$. Then there are $t > 0$ and $w \in Z$ such that

$$f = e^{-t(C\wedge D)}w = Pe^{-tC}Pw + (I-P)e^{-tD}(I-P)w \ .$$

So $f = e^{-tC}\tilde{w}$ with $\tilde{w} = Pw + (I-P)e^{tC}e^{-tD}(I-P)w \in Z$, and hence $f \in S_{Z,C}$.
Similarly it follows that $f \in S_{Z,D}$.
On the other hand, let $g \in S_{Z,C} \cap S_{Z,D}$. Then for some $w, v \in Z$ and $t > 0$,

$$g = e^{-tC}w \quad \text{and} \quad g = e^{-tD}v \ .$$

So g can be written as

$$g = Pg + (I-P)g = Pe^{-tC}Pw + (I-P)e^{-tD}(I-P)v =$$

$$= e^{-t(C\wedge D)}(Pw + (I-P)v) \in S_{Z,C\wedge D}.$$

Finally, we prove that $S_{Z,C\wedge D} = S_{Z,C+D}$.
Since $C + D \geq C \wedge D$ it is obvious that $S_{Z,C+D} \subset S_{Z,C\wedge D}$.
Now let $f \in S_{Z,C\wedge D}$. Then $f = (Pe^{-tC}P + (I-P)e^{-tD}(I-P)w$ for certain $t > 0$ and $w \in Z$. Thus we find

$$f = e^{-\frac{1}{2}t(C+D)}[Pe^{-\frac{1}{2}tC}e^{\frac{1}{2}tD}P + (I-P)e^{\frac{1}{2}tD}e^{\frac{1}{2}tC}(I-P)]w, \text{ and}$$

$f \in S_{Z,C+D}$.

II. Let $f \in S_{Z,C\vee D}$. Then there are $w \in Z$ and $t > 0$ such that

$$f = e^{-t(C\vee D)}w = Pe^{-tD}Pw + (I-P)e^{-tC}(I-P)w \ .$$

So $f \in S_{Z,C} + S_{Z,D}$. On the other hand let $u,v \in Z$ and let $t > 0$. Put $g = e^{-tC}u + e^{-tD}v$. Then

$$g = e^{-t(C\vee D)}[e^{t(C\vee D)}e^{-tC}u + e^{t(C\vee D)}e^{-tD}v].$$

Since $C \vee D \leq C$ and $C \vee D \leq D$, this yields $g \in S_{Z,C\vee D}$.

III. Let $G \in T_{Z,C\wedge D}$. Then $w \in Z$ and $\varphi \in B_+(\mathbb{R})$ are such that $G = \varphi(C\wedge D)w$. Since $\varphi(C\wedge D) = \varphi(C)P + \varphi(D)(I-P)$,

$$G = \varphi(C)Pw + \varphi(D)(I-P)w \in T_{Z,C} + T_{Z,D}.$$

On the other hand let $\varphi,\psi \in B_+(\mathbb{R})$ and let $u,v \in Z$. Put

$$G = \varphi(C)u + \psi(D)v \ .$$

Since the operators $\varphi(C)e^{-t(C\wedge D)}$ and $\psi(D)e^{-t(C\wedge D)}$, $t > 0$, are bounded on Z, for all $t > 0$

$$e^{-t(C\wedge D)}G = (e^{-t(C\wedge D)}\varphi(C)u + e^{-t(C\wedge D)}\psi(D)v) \in Z \ .$$

Hence $G \in T_{Z,C\wedge D}$. Because $S_{Z,C\wedge D} = S_{Z,C+D}$ also topologically, it is clear that $T_{Z,C\wedge D} = T_{Z,C+D}$.

IV. Let $H \in T_{Z,C} \cap T_{Z,D}$. Then there are $\psi,\chi \in B_+(\mathbb{R})$ and $v,w \in Z$ such that $H = \psi(C)w$ and $H = \chi(D)v$. So H can be written as

$$H = \psi(C)(I-P)w + \chi(D)Pv \ ,$$

and $e^{-t(C\vee D)}H = e^{-tC}\psi(C)(I-P)w + e^{-tD}\chi(D)Pv \in Z$. This implies $H \in T_{Z,C\vee D}$. Since $C \vee D \leq C$ and $C \vee D \leq D$ we have

$$T_{Z,C\vee D} \subset T_{Z,C} \text{ and } T_{Z,C\vee D} \subset T_{Z,D}. \qquad \square$$

It is obvious that the operators $\mathcal{C} \wedge \mathcal{D}$ and $\mathcal{C} \vee \mathcal{D}$ commute. So the spaces $S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C} \vee \mathcal{D})$, $S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{C} \wedge \mathcal{D})$, $T(S_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C} \vee \mathcal{D})$, $T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C} \wedge \mathcal{D})$ are well defined. Here, for convenience, we have omitted the subscript Z. Similar to Theorem (4.2) we shall prove the following.

(4.3) Theorem

I. $S(T_{\mathcal{C}},\mathcal{D}) \cap S(T_{\mathcal{D}},\mathcal{C}) = S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{C} \wedge \mathcal{D})$,

II. $S(T_{\mathcal{C}},\mathcal{D}) + S(T_{\mathcal{D}},\mathcal{C}) = S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C} \vee \mathcal{D})$,

III. $T(S_{\mathcal{C}},\mathcal{D}) \cap T(S_{\mathcal{D}},\mathcal{C}) = T(S_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C} \vee \mathcal{D})$,

IV. $T(S_{\mathcal{C}},\mathcal{D}) + T(S_{\mathcal{D}},\mathcal{C}) = T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C} \wedge \mathcal{D})$.

Proof

I. Let $k \in S(T_{\mathcal{C}},\mathcal{D}) \cap S(T_{\mathcal{D}},\mathcal{C})$. Then there are $\varphi,\psi \in B_+(\mathbb{R})$, $t > 0$ and $u,v \in Z$ such that $k = e^{-t\mathcal{C}}\varphi(\mathcal{D})u$ and $k = e^{-t\mathcal{D}}\psi(\mathcal{C})v$.

Put $\chi = \max(\varphi,\psi)$. Then $\chi \in B_+(\mathbb{R})$ and k is given by

$$k = e^{-t\mathcal{C}}\chi(\mathcal{D})\tilde{u} \text{ and } k = e^{-t\mathcal{D}}\chi(\mathcal{C})\tilde{v}$$

with $\tilde{u} = \chi^{-1}(\mathcal{D})\varphi(\mathcal{D})u \in Z$ and $\tilde{v} = \chi^{-1}(\mathcal{C})\psi(\mathcal{C})v \in Z$. So

$$k = Pk + (I - P)k = Pe^{-t\mathcal{C}}\chi(\mathcal{D})\tilde{u} + (I - P)e^{-t\mathcal{D}}\chi(\mathcal{C})\tilde{v}$$

$$= e^{-t(\mathcal{C}\wedge\mathcal{D})}\chi(\mathcal{C} \vee \mathcal{D})[P\tilde{u} + (I - P)\tilde{v}] .$$

This yields $k \in S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{C} \wedge \mathcal{D})$.

On the other hand, let $\varphi \in B_+(\mathbb{R})$ and let $w \in Z$, $t > 0$. Then for $h = \varphi(\mathcal{C} \vee \mathcal{D})e^{-t(\mathcal{C}\wedge\mathcal{D})}w$,

$$h = \varphi(\mathcal{C})e^{-t\mathcal{D}}(\varphi(\mathcal{C})^{-1}\varphi(\mathcal{C} \vee \mathcal{D})e^{t\mathcal{D}}e^{-t(\mathcal{C}\wedge\mathcal{D})}w).$$

Hence $h \in S(T_{\mathcal{C}},\mathcal{D})$. Similarly it can be shown that $h \in S(T_{\mathcal{D}},\mathcal{C})$.

II. Let $h \in S(T_C,\mathcal{D}) + S(T_\mathcal{D},C)$. Then there are $w,v \in Z$, $t > 0$ and $\chi \in B_+(\mathbb{R})$, such that

$$h = e^{-tC}\chi(\mathcal{D})w + e^{-t\mathcal{D}}\chi(C)v \ .$$

Hence h can be written as

$$h = e^{-t(C\vee\mathcal{D})}\chi(C \wedge \mathcal{D})[e^{t(C\vee\mathcal{D})}e^{-tC}\chi^{-1}(C \wedge \mathcal{D})\chi(\mathcal{D})w +$$

$$+ e^{t(C\vee\mathcal{D})}e^{-t\mathcal{D}}\chi^{-1}(C \wedge \mathcal{D})\chi(C)v] \ .$$

Since $C \vee \mathcal{D} \leq C,\mathcal{D}$ and $C \wedge \mathcal{D} \geq C,\mathcal{D}$, this yields $h \in S(T_{C\wedge\mathcal{D}},C \vee \mathcal{D})$.

In order to prove the other inclusion, assume that $g \in S(T_{C\wedge\mathcal{D}},C \vee \mathcal{D})$. Then there are $w \in Z$, $t > 0$ and $\varphi \in B_+(\mathbb{R})$ such, that

$$g = e^{-t(C\vee\mathcal{D})}\varphi(C \wedge \mathcal{D})w =$$

$$= e^{-t\mathcal{D}}\varphi(C)Pw + e^{-tC}\varphi(\mathcal{D})(I - P)w \in S(T_C,\mathcal{D}) + S(T_\mathcal{D},C).$$

III. Let $Q \in T(S_C,\mathcal{D}) \cap T(S_\mathcal{D},C)$ and let $t > 0$. Then there exists $s > 0$ such, that $e^{sC}e^{-t\mathcal{D}}Q \in Z$ and $e^{s\mathcal{D}}e^{-tC}Q \in Z$.
Hence $Pe^{sC}e^{-t\mathcal{D}}PQ \in Z$ and $(I - P)e^{s\mathcal{D}}e^{-tC}(I - P)Q \in Z$ which implies $e^{s(C\wedge\mathcal{D})}e^{-t(C\vee\mathcal{D})}Q \in Z$.

On the other hand, let $R \in T(S_{C\wedge\mathcal{D}},C \vee \mathcal{D})$, and let $t > 0$. Then take $s > 0$ such, that $e^{s(C\wedge\mathcal{D})}e^{-t(C\vee\mathcal{D})}R \in Z$. This yields

$$e^{s\mathcal{D}}e^{-tC}R = [Pe^{s\mathcal{D}}e^{-tC}P + (I - P)e^{s\mathcal{D}}e^{-tC}(I - P)]R$$

$$= [Pe^{(s+t)\mathcal{D}}e^{-(s+t)C}P + (I - P)][e^{s(C\wedge\mathcal{D})}e^{-t(C\vee\mathcal{D})}]R \ .$$

So R can be seen as an element of $T(S_\mathcal{D},C)$, and similarly as an element of $T(S_C,\mathcal{D})$.

IV. Let $Q \in T(S_C,D) + T(S_D,C)$. Then there are $Q_1 \in T(S_C,D)$ and $Q_2 \in T(S_D,C)$ such that $Q = Q_1 + Q_2$ with the sum understood in T_{C+D}. Let $t > 0$. Then there is $s > 0$ such that

$$e^{sC}e^{-tD}Q_1 \in Z \text{ and } e^{sD}e^{-tC}Q_2 \in Z .$$

Hence $e^{s(C\vee D)}e^{-t(C\wedge D)}Q =$

$$= (Pe^{(t+s)D}e^{-(t+s)C}P + (I-P))e^{sC}e^{-tD}Q_1 +$$

$$+ (P + (I - P)e^{(t+s)C}e^{-(t+s)D}(I-P))e^{sD}e^{-tC}Q_2 ,$$

so that $Q \in T(S_{C\vee D}, C \wedge D)$.

Finally, let $R \in T(S_{C\vee D}, C \wedge D)$ and let $t > 0$. Then there is $s > 0$ with

$$e^{s(C\vee D)}e^{-t(C\wedge D)}R \in Z.$$

Hence $R = PR + (I - P)R$ and $e^{sD}e^{-tC}PR =$

$$= Pe^{s(C\vee D)}e^{-t(C\wedge D)}R \in Z \text{ and similarly } e^{sC}e^{-tD}R \in Z.$$

Thus we have shown $R \in T(S_C,D) + T(S_D,C)$. □

The preceding theorems play a major role in the inclusion scheme which we give in Section 5. The results of Theorem (4.3) will lead to a fifth Kernel theorem in [E2].

5. The inclusion scheme

The spaces which are introduced in [G] and in the previous sections fit into an inclusion scheme. Here we shall give some properties of the spaces in this scheme. The reader may as well skip the proofs. They are added for completeness. Let $\widetilde{C}$ and $\widetilde{D}$ denote two commuting, nonnegative, self-adjoint operators in Z.

(5.1) Lemma

Let $\widetilde{C} \geq \widetilde{D}$. Then

$$S(T_{\widetilde{D}},\widetilde{C}) = S_{\widetilde{C}} \quad \text{and} \quad T(S_{\widetilde{D}},\widetilde{C}) = T_{\widetilde{C}}.$$

Proof. It is clear that $S_{\widetilde{C}} \subset S(T_{\widetilde{D}},\widetilde{C})$ and $T(S_{\widetilde{D}},\widetilde{C}) \subset T_{\widetilde{C}}$.
So let $f \in S(T_{\widetilde{D}},\widetilde{C})$. Then there are $t > 0$ and $\varphi \in B_+(\mathbb{R})$ and $w \in Z$ such that $f = e^{-t\widetilde{C}}\varphi(\widetilde{D})w$. Hence

$$f = e^{-t/2\widetilde{C}}(\varphi(\widetilde{D})e^{-t/2\widetilde{C}}w) \in S_{\widetilde{C}} \ ,$$

because $\varphi(\widetilde{D})e^{-t/2\widetilde{C}}$ is a bounded operator on Z.
Similarly, $T_{\widetilde{C}} \subset T(S_{\widetilde{D}},\widetilde{C})$ can be proved. □

(5.2) Lemma

$S(T_{\widetilde{D}},\widetilde{C}) \subset T(S_{\widetilde{C}},\widetilde{D})$.

Proof. Let $h \in S(T_{\widetilde{D}},\widetilde{C})$. Then h can be written as

$$h = e^{-t\widetilde{C}}\varphi(\widetilde{D})w \quad ,$$

where $t > 0$, $\varphi \in B_+(\mathbb{R})$ and $w \in Z$. Hence, for all $s > 0$,

$$e^{-s\widetilde{D}}e^{t\widetilde{C}}h = \varphi(\widetilde{D})e^{-s\widetilde{D}}w \in Z.$$

With $\text{emb}(h) : s \to e^{-s\widetilde{D}}h$, the proof is complete. □

$$
\begin{array}{ccccccc}
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}\wedge\mathcal{D}) & = & T_{\mathcal{C}\wedge\mathcal{D}} \\
\parallel & & \cup & & \cup & & \cup \\
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{D}) & = & T_{\mathcal{D}} \\
\cup & & \cup & & \cup & & \parallel \\
S_{\mathcal{C}} & \subset & S(T_{\mathcal{D}},\mathcal{C}) & \subset & T(S_{\mathcal{C}},\mathcal{D}) & \subset & T_{\mathcal{D}} \\
\parallel & & \cup & & \cup & & \cup \\
S_{\mathcal{C}} & = & S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}) & \subset & T(S_{\mathcal{C}},\mathcal{C}\vee\mathcal{D}) & \subset & T_{\mathcal{C}\vee\mathcal{D}} \\
\cup & & \cup & & \cup & & \parallel \\
S_{\mathcal{C}\wedge\mathcal{D}} & = & S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}\wedge\mathcal{D}) & \subset & T(S_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T_{\mathcal{C}\vee\mathcal{D}} \\
\cap & & \cap & & \cap & & \parallel \\
S_{\mathcal{D}} & = & S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{D}) & \subset & T(S_{\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T_{\mathcal{C}\vee\mathcal{D}} \\
\parallel & & \cap & & \cap & & \cap \\
S_{\mathcal{D}} & \subset & S(T_{\mathcal{C}},\mathcal{D}) & \subset & T(S_{\mathcal{D}},\mathcal{C}) & \subset & T_{\mathcal{C}} \\
\cap & & \cap & & \cap & & \parallel \\
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{C}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}) & = & T_{\mathcal{C}} \\
\parallel & & \cap & & \cap & & \cap \\
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}\wedge\mathcal{D}) & = & T_{\mathcal{C}\wedge\mathcal{D}}
\end{array}
$$

Fig. (5.3) The inclusion scheme

A row in the inclusion scheme (5.3) is of the form

(5.4) $$S_{\tilde{\mathcal{C}}} \subset S(T_{\tilde{\mathcal{D}}},\tilde{\mathcal{C}}) \subset T(S_{\tilde{\mathcal{C}}},\tilde{\mathcal{D}}) \subset T_{\tilde{\mathcal{D}}}.$$

(5.5) Theorem

In (5.4) all embeddings are continuous and have dense ranges.

Proof. We proceed in three steps.

(i) $S_{\tilde{\mathcal{C}}} \subset S(T_{\tilde{\mathcal{D}}},\tilde{\mathcal{C}})$

Let (w_n) be a null sequence in $S_{\tilde{\mathcal{C}}}$. Then there is $t > 0$ such that

$e^{t\widetilde{C}} w_n \to 0$ in Z. So for all $s > 0$

$$e^{t\widetilde{C}} \mathrm{emb}(w_n)(s) = e^{t\widetilde{C}} e^{-s\widetilde{\mathcal{D}}} w_n \to 0$$

in X. This proves that the embedding emb : $S_{\widetilde{C}} \hookrightarrow S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$ is continuous. To show that $S_{\widetilde{C}}$ is dense in $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$, let $H \in T(S_{\widetilde{\mathcal{D}}},\widetilde{C})$ with $\langle\langle f,H\rangle\rangle = 0$ for all $f \in S_{\widetilde{C}}$. Then $\langle f,H\rangle = 0$ for all $f \in S_{\widetilde{C}}$. So $H = 0$, and $S_{\widetilde{C}}$ is dense in $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$.

(ii) $S(T_{\widetilde{\mathcal{D}}},\widetilde{C}) \subset T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$.

First we remind that in Lemma (5.2) we showed how $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$ can be embedded in $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$. The embedding is continuous. To show this, let $s > 0$ and $\psi \in \mathcal{B}_+(\mathbb{R})$. Then the seminorm

$$h \to \|\psi(\widetilde{C}) e^{-s\widetilde{\mathcal{D}}} h\|$$

is continuous on $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$.

Now let $g \in S(T_{\widetilde{C}},\widetilde{\mathcal{D}})$, the dual of $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$. Then g can be written as $g = \varphi(\widetilde{C})u$ where $u \in S_{\widetilde{\mathcal{D}}}$ and $\varphi \in \mathcal{B}_+(\mathbb{R})$. Suppose

$$\langle\langle g,h\rangle\rangle = 0 \quad , \quad h \in S(T_{\widetilde{\mathcal{D}}},\widetilde{C}).$$

Then for all $f \in S_{\widetilde{C}}$ and all $\chi \in \mathcal{B}_+(\mathbb{R})$

$$(\varphi(\widetilde{C})f \ , \ \chi(\widetilde{\mathcal{D}})u) = 0 \ .$$

Hence $u = 0$, and $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$ is dense in $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$.

(iii) $T(S_{\widetilde{C}},\widetilde{\mathcal{D}}) \subset T_{\widetilde{\mathcal{D}}}$.

The continuity of the embedding follows from the continuity of the seminorms

$$t \to \|H(t)\| \quad , \ t > 0 \ ,$$

on $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$.

Further, let $f \in S_{\tilde{\mathcal{D}}}$ and suppose $\langle f,H\rangle = 0$ for all $H \in \mathcal{T}(S_{\tilde{\mathcal{C}}},\tilde{\mathcal{D}})$. Then $(f,h) = 0$ for all $h \in S_{\tilde{\mathcal{C}}}$. So $f = 0$. □

Consider the inclusion subscheme of (5.3).

(5.6) $$S_{\mathcal{C}\wedge\mathcal{D}} \subset S_{\mathcal{C}} \subset S_{\mathcal{C}\vee\mathcal{D}} \, .$$

Then similar to Theorem (5.5) we show

(5.7) Theorem

In (5.6) all embeddings are continuous and have dense ranges.

Proof. We proceed in two steps.

(i) Let (f_n) be a null sequence in $S_{\mathcal{C}\wedge\mathcal{D}}$. Then there is $t > 0$ such that $\| e^{t(\mathcal{C}\wedge\mathcal{D})} f_n \| \to 0$. Hence

$$\| e^{t\mathcal{C}} f_n \| \le \| e^{t\mathcal{C}} e^{-t(\mathcal{C}\wedge\mathcal{D})} \| \, \| e^{t(\mathcal{C}\wedge\mathcal{D})} f_n \| \to 0 .$$

Further, let $G \in \mathcal{T}_{\mathcal{C}}$ and suppose for all $f \in S_{\mathcal{C}\wedge\mathcal{D}}$,

$$\langle f,G\rangle = 0 \, .$$

So for all $x \in Z$ and $t > 0$, $(x, e^{-t(\mathcal{C}\wedge\mathcal{D})}G) = 0$. This implies $G = 0$, and hence $S_{\mathcal{C}\wedge\mathcal{D}}$ is dense in $S_{\mathcal{C}}$.

(ii) $S_{\mathcal{C}} \subset S_{\mathcal{C}\vee\mathcal{D}}$:

Follows from (i) because $\mathcal{C} = (\mathcal{C} \vee \mathcal{D}) \wedge \mathcal{C}$. □

(5.8) Corollary

In the inclusion scheme

$$\mathcal{T}_{\mathcal{C}\vee\mathcal{D}} \subset \mathcal{T}_{\mathcal{C}} \subset \mathcal{T}_{\mathcal{C}\wedge\mathcal{D}}$$

all embeddings are continuous and have dense ranges.

Proof. Follows from Theorem (5.7) by duality. □

Finally we consider the inclusion subscheme.

(5.9) $$T(S_{C\wedge D}, C \vee D) \subset T(S_C, C \vee D) \subset T(S_C, D) \;.$$

We prove

(5.10) Theorem

In (5.9) all embeddings are continuous and have dense ranges.

Proof. We proceed in two steps.

(i) Since the seminorms

$$F \to \| \varphi(C) e^{-t(C\vee D)} F \| \quad , \quad t > 0, \ \varphi \in B_+(\mathbb{R})$$

are continuous in $T(S_{C\wedge D}, C \vee D)$, the embedding of $T(S_{C\wedge D}, C \vee D)$ in $T(S_C, C \vee D)$ is continuous. Further, $S_{C\wedge D} \subset T(S_{C\wedge D}, C \vee D)$ is dense in S_C, and S_C is dense in $T(S_C, C \vee D)$. So $T(S_{C\wedge D}, C \vee D)$ is dense in $T(S_C, C \vee D)$. (See Lemma (1.16)).

(ii) The seminorms

$$G \to \| \varphi(C) e^{-tD} G \| \quad , \quad t > 0, \ \varphi \in B_+(\mathbb{R}) \quad ,$$

are continuous in $T(S_C, C \vee D)$. So the embedding from $T(S_C, C \vee D)$ into $T(S_C, D)$ is continuous. Further we note that S_C is dense both in $T(S_C, C \vee D)$ and in $T(S_C, D)$ by Theorem (2.15). Hence $T(S_C, C \vee D)$ is dense in $T(S_C, D)$. □

(5.11) Corollary

In the inclusion scheme

$$S(T_{C\wedge D}, C \vee D) \supset S(T_C, C \vee D) \supset S(T_C, D)$$

all embeddings are continuous and have dense ranges.

Finally, the main result of this section will be given.

(5.12) Theorem

In (5.3) all embeddings are continuous and have dense ranges.

Proof. Follows from Theorem (5.5), (5.7) and (5.10), and from Corollary (5.8) and (5.11). □

II. On continuous linear mappings between analyticity and trajectory spaces

Introduction

Here X and Y will denote Hilbert spaces, and A will be a nonnegative self-adjoint operator in X and B a nonnegative self-adjoint operator in Y. In [G], the fourth chapter contains a detailed discussion of the four types of continuous linear mappings:

$S_{X,A} \to S_{Y,B}$, $S_{X,A} \to T_{Y,B}$, $T_{X,A} \to S_{Y,B}$, $T_{X,A} \to T_{Y,B}$. Cf.[GE], Section 4.

In order to prove a Kernel theorem for each of these types, in addition to the topological tensor products $S_{X\otimes Y,A\boxplus B}$ and $T_{X\otimes Y,A\boxplus B}$, the spaces Σ_A' and Σ_B' have been introduced. Σ_A' and Σ_B' are topological tensor products of $T_{X,A}$ and $S_{Y,B}$ and of $S_{X,A}$ and $T_{Y,B}$. Each element of Σ_A' corresponds to a continuous linear mapping from $S_{X,A}$ into $S_{Y,B}$. If every continuous linear mapping from $S_{X,A}$ into $S_{Y,B}$ arises from an element of Σ_A', then, in De Graaf's terminology, the Kernel theorem holds true. Similar statements apply to Σ_B', $S_{X\otimes Y,A\boxplus B}$ and $T_{X\otimes Y,A\boxplus B}$.

In order to gain a deeper understanding of the topological structure of the spaces Σ_A' and Σ_B', we have introduced the more general type of spaces $T(S_{Z,C},D)$ and $S(T_{Z,C},D)$, where C and D are commuting, nonnegative, self-adjoint operators in the Hilbert space Z. The following relations have been mentioned:

$$\Sigma_A' = T(S_{X\otimes Y,I\otimes B},A\otimes I) \quad , \quad \Sigma_A = S(T_{X\otimes Y,I\otimes B},A\otimes I) \ ,$$

$$\Sigma_B' = T(S_{X\otimes Y,A\otimes I},I\otimes B) \quad , \quad \Sigma_B = S(T_{X\otimes Y,A\otimes I},I\otimes B) \ .$$

So obviously results in [E_1] apply to the spaces Σ_A', Σ_B', Σ_A and Σ_B.

Thus, the intersection of Σ_A' and Σ_B' is a space of type $T(S_{Z,C},\mathcal{D})$. This observation leads to a Kernel theorem for so-called extendable mappings. Cf [GE], Section 4.

Precise formulations of the above-mentioned five Kernel theorems can be found in Section 1. In the remaining sections we consider the case X = Y and $A = B$. Hence, we investigate the spaces

$$T^A = T(S_{X\otimes X,I\otimes A},A\otimes I) \text{ and } T_A = T(S_{X\otimes X,A\otimes I},I\otimes A).$$

In Section 2 we shall prove that T^A and T_A admit an algebraic structure and that they are homeomorphic. The homeomorphism is denoted by ^{c}. The mapping ^{c} is also a homeomorphism from the space $S_A = S(T_{X\otimes X,A\otimes I},I\otimes A)$ onto $S^A = S(T_{X\otimes X,I\otimes A},A\otimes I)$. Put $E_A = T^A \cap T_A$. Then E_A is an algebra and it inherits several properties of the algebras T^A and T_A. The mapping ^{c} is an involution on E_A. The strong dual E_A' equals the algebraic sum $S_A + S^A$. We shall extend ^{c} to E_A' in a natural way.

In the sequel we shall confine our attention to nuclear analyticity spaces $S_{X,A}$. Then, because of the Kernel theorems the space $T^A(T_A)$ comprises all continuous linear mappings from $S_{X,A}(T_{X,A})$ into itself. Inspired by operator theory for Hilbert spaces, we introduce the topology of pointwise and weak pointwise convergence in $T^A(T_A)$. These topologies correspond to the strong and weak operator topology for Von Neumann algebras, while the weak and strong topology of $T^A(T_A)$ correspond to the ultra-weak and uniform operator topology.

In Sections 3 and 4 we study the relations between the algebraic and the topological structure of T^A and T_A. It appears that separate multiplication is continuous in all mentioned topologies. The effects of the results of the previous sections on the algebra E_A and its strong dual E_A' are investigated in Section 5.

In Section 6 we indicate possibilities to interprete parts of quantum statistics by means of the mathematical apparatus developed for the spaces E_A and E_A'. They seem to be more appropriate than any operator algebra on a Hilbert space, because in general E_A contains unbounded, selfadjoint operators. However, we emphasize that we consider it as an Ansatz only. We are not fully aware of all consequences of such redescription.

If the Kernel theorem holds true, each continuous linear mapping from $S_{X,A}$ into itself has a well-defined infinite matrix. Section 7 of this paper is devoted to a thorough description of this kind of matrices. There are manageable, necessary and sufficient conditions on the entries of an infinite matrix, such, that its corresponding linear mapping is continuous on $S_{X,A}$. The thus obtained identification between T^A and a class $M(T^A)$ of well-specified infinite matrices enables us to construct a large variety of elements in T^A. Particularly, we note here that the matrix calculus will be of great importance in a forthcoming paper on one-parameter (semi-)groups of elements of T^A. In Section 8 we treat a subclass of $M(T^A)$, the class of unbounded weighted shifts. Weighted shifts are the simplest, non-trivial operators in T^A.

In the final section our matrix calculus yields the construction of nuclear analyticity spaces on which a prescribed set of linear operators act continuously.

1. Kernel theorems

In this section we shall recall the four Kernel theorems introduced in [G], ch.VI, and we shall add one to them.

The Hilbert space $X\otimes Y$ of all Hilbert-Schmidt operators from X into Y can be regarded as a topological tensor product of X and Y. Let A and B denote nonnegative self-adjoint operators in X and Y. Let $w \in D(A)$. Then for all $v \in Y$, we define

$$A\otimes I\,(w\otimes v) = Aw\otimes v\ .$$

With the aid of linear extension, the operator $A\otimes I$ is well-defined on the algebraic tensor product $D(A)\otimes_a Y$. It can be proved that $A\otimes I$ with domain $D(A)\otimes_a Y$ is nonnegative and essentially self-adjoint. Cf.[W],[G]. Similarly $I\otimes B$ with domain $X\otimes_a D(B)$ is nonnegative and essentially self-adjoint in $X\otimes Y$. Further, the operators $A\otimes I$ and $I\otimes B$ commute, i.e., their spectral projections commute. So the operator $A\boxplus B = A\otimes I + I\otimes B$ with domain

$$\{W \in X\otimes Y \mid \int_{\mathbb{R}^2} (\lambda+\mu)^2 d((E_\lambda\otimes F_\mu)W,W) < \infty\}$$

is self-adjoint and nonnegative. Consequently the spaces $S_{X\otimes Y,A\boxplus B}$ and $T_{X\otimes Y,A\boxplus B}$ are well-defined. In [G] it is proved that $S_{X\otimes Y,A\boxplus B}$ is a topological product of $S_{X,A}$ and $S_{Y,B}$, and $T_{X\otimes Y,A\boxplus B}$ a topological tensor product of $T_{X,A}$ and $T_{Y,B}$. We note that $e^{-t(A\boxplus B)} = e^{-tA}\otimes e^{-tB}$, $t \geq 0$.

Case (a). Continuous linear mappings from $T_{X,A}$ into $S_{Y,B}$.
An element $\theta \in S_{X\otimes Y,A\boxplus B}$ induces a linear mapping $T_{X,A} \to S_{Y,B}$ in the following way. Let $F \in T_{X,A}$. Then θF is defined by

(a) $\theta F = e^{-\varepsilon B}(e^{\varepsilon B}\theta e^{\varepsilon A})F(\varepsilon)$

where $\varepsilon > 0$ has to be taken sufficiently small.

(1.1) Theorem

I. For each $\theta \in S_{X\otimes Y, A\boxplus B}$, the linear operator θ: $T_{X,A} \to S_{Y,B}$ as defined by (a) is continuous.

II. For $\theta \in S_{X\otimes Y, A\boxplus B}$, $F \in T_{X,A}$ and $G \in T_{Y,B}$,

$$\langle \theta F, G\rangle_Y = \langle \theta, F\otimes G\rangle_{X\otimes Y} .$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then $S_{X\otimes Y, A\boxplus B}$ comprises all continuous linear mappings from $T_{X,A}$ into $S_{Y,B}$.

IV. $S_{X\otimes X, A\boxplus B}$ comprises all continuous linear mappings from $T_{X,A}$ into $S_{X,A}$, iff for each $t > 0$ the operator e^{-tA} is Hilbert-Schmidt.

Proof. Cf.[G], Theorem 6.1. □

Case (b). Continuous linear mappings from $S_{X,A}$ into $T_{Y,B}$.
Let $\Phi \in T_{X\otimes Y, A\boxplus B}$. For $f \in S_{X,A}$ we define $\Phi f \in T_{Y,B}$ by

(b) $(\Phi f)(t) = e^{-(t-\varepsilon)B}\Phi(\varepsilon)e^{\varepsilon A}f \quad , \quad t > 0 ,$

where $\varepsilon > 0$ has to be taken sufficiently small.

(1.2) Theorem

I. For each $\Phi \in T_{X\otimes Y, A\boxplus B}$ the linear mapping Φ: $S_{X,A} \to T_{Y,B}$ defined by (b) is continuous.

II. For each $\Phi \in T_{X\otimes Y, A\boxplus B}$, $f \in S_{X,A}$ and $g \in S_{Y,B}$

$$\langle g, \Phi f\rangle_Y = \langle f \otimes g, \Phi\rangle_{X\otimes Y}.$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is H.S. then $T_{X\otimes Y, A\boxplus B}$ comprises all continuous minear mappings from $S_{X,A}$ into $T_{Y,B}$.

IV. $T_{X\otimes X, A\boxplus B}$ comprises all continuous linear mappings from $S_{X,A}$ into $T_{X,A}$ iff for each $t > 0$ the operator e^{-tA} is H.S.

Proof. Cf.[G], Theorem 6.2. □

In [G], Ch.V, the spaces Σ'_A and Σ'_B are introduced as follows.

$$\Sigma'_A = \{P \in T_{X\otimes Y, A\otimes I} \mid \forall_{t>0}: P(t) \in S_{X\otimes Y, A\boxplus B}\} ,$$

$$\Sigma'_B = \{K \in T_{X\otimes Y, I\otimes B} \mid \forall_{t>0}: K(t) \in S_{X\otimes Y, A\boxplus B}\} .$$

It is not hard to prove that Σ'_A equals the space $T(S_{X\otimes Y, I\otimes B}, A\otimes I)$ and Σ'_B the space $T(S_{X\otimes Y, A\otimes I}, I\otimes B)$ both set theoretically and topologically. Cf. [E_1], Section 2; [G], Ch.V.

Let $F \in T_{X,A}$ and $g \in S_{Y,B}$. Then $F\otimes g$ is defined as the trajectory

$$F\otimes g: t \to F(t)\otimes g.$$

Since $F(t)\otimes(e^{\varepsilon B}g) \in X\otimes Y$ for $\varepsilon > 0$ sufficiently small and all $t > 0$, the trajectory $F\otimes g$ is an element of $T(S_{X\otimes Y, I\otimes B}, A\otimes I)$. So the algebraic tensor product of $T_{X,A}$ and $S_{Y,B}$ is contained in $T(S_{X\otimes Y, I\otimes B}, A\otimes I)$. De Graaf proves that $T(S_{X\otimes Y, I\otimes B}, A\otimes I)$ is a complete topological tensor product of $T_{X,A}$ and $S_{Y,B}$. Moreover, for $F \in T_{X,A}$ and $g \in S_{Y,B}$ the tensor product $F\otimes g$ is an element of $S(T_{X\otimes Y, A\otimes I}, I\otimes B)$, because there exists $\varepsilon > 0$ fixed such that

$$(I \otimes e^{\varepsilon B})(F \otimes g) = F \otimes (e^{\varepsilon B} g) \in T_{X\otimes Y, A\otimes I} .$$

So the algebraic tensor product $T_{X,A} \otimes_a S_{Y,B}$ is also contained in $S(T_{X\otimes Y,A\otimes I}, I\otimes B)$. By similar arguments it follows that the space $T(S_{X\otimes Y,A\otimes I}, I\otimes B)$ is a complete topological tensor product of the spaces $S_{X,A}$ and $T_{Y,B}$. The algebraic tensor product $S_{X,A} \otimes_a T_{Y,B}$ is contained in $S(T_{X\otimes Y,I\otimes B}, A\otimes I)$. We note that $S(T_{X\otimes Y,A\otimes I}, I\otimes B)$ is included in $T(S_{X\otimes Y,I\otimes B}, A\otimes I)$, and that $S(T_{X\otimes Y,I\otimes B}, A\otimes I)$ is included in $T(S_{X\otimes Y,A\otimes I}, I\otimes B)$, Cf.[$E_1$], Section 5.

Case c. Continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.
Let $P \in T(S_{X\otimes Y,I\otimes B}, A\otimes I)$. Then for $f \in S_{X,A}$ we define Pf by

$$\text{(c)} \qquad P(f) = P(\varepsilon) e^{\varepsilon A} f,$$

where $\varepsilon > 0$ has to be taken sufficiently small. We note that (c) does not depend on the choice of $\varepsilon > 0$. Since $P(\varepsilon) \in S_{X\otimes Y,I\otimes B}$ we have $Pf \in S_{Y,B}$.

(1.3) Theorem

I. For each $P \in T(S_{X\otimes Y,I\otimes B}, A\otimes I)$ the linear operator $P: S_{X,A} \to S_{Y,B}$ defined by (c) is continuous.

II. For each $P \in T(S_{X\otimes Y,I\otimes B}, A\otimes I)$, $f \in S_{X,A}$ and $G \in T_{Y,B}$

$$\overline{\langle Pf, G \rangle_Y} = \langle\langle f \otimes G, P \rangle\rangle_{X\otimes Y} .$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is H.S. then $T(S_{X\otimes Y,I\otimes B}, A\otimes I)$ comprises all continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.

IV. $T(S_{X\otimes Y,I\otimes A},A\otimes I)$ comprises all continuous linear mappings from $S_{X,A}$ into itself iff for each $t > 0$ the operator e^{-tA} is H.S.

Proof. Cf.[G], Theorem 6.3. □

Case (d). Continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$.

Let $K \in T(S_{X\otimes Y,A\otimes I},I\otimes B)$. For $F \in T_{X,A}$, define $KF \in T_{Y,B}$ by

$$(KF)(t) = K(t)e^{\varepsilon(t)A}F(\varepsilon(t)) \ . \tag{d}$$

This definition makes sense for all $t > 0$ and for $\varepsilon(t) > 0$ sufficiently small. We have $(KF)(t) \in S_{Y,B}$, because $K \in T_{X\otimes Y,I\otimes B}$.

(1.4) Theorem.

I. For each $K \in T(S_{X\otimes Y,A\otimes I},I\otimes B)$, the linear mapping $K: T_{X,A} \to T_{Y,B}$ defined in (d), is continuous.

II. For each $K \in T(S_{X\otimes Y,A\otimes I},I\otimes B)$, $F \in T_{X,A}$, $g \in S_{Y,B}$

$$\langle g,KF\rangle_Y = \langle\!\langle F\otimes g,K\rangle\!\rangle_{X\otimes Y}.$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is H.S., then $T(S_{X\otimes Y,A\otimes I},I\otimes B)$ comprises all continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$.

IV. $T(S_{X\otimes X,A\otimes I},I\otimes A)$ comprises all continuous linear mappings from $T_{X,A}$ into itself iff the operator e^{-tA} is Hilbert-Schmidt for all $t > 0$.

Proof. Cf.[G], Theorem 6.4. □

(1.5) Definition

A continuous linear mapping E from $S_{X,A}$ into $S_{Y,B}$ is called extendable, if E can be extended to a continuous linear mapping from $T_{X,A}$ into $T_{Y,B}$.

In [G], necessary and sufficient conditions are given in order that a linear mapping on $S_{X,A}$ is extendable.
In [E_1] for a pair of commuting, nonnegative, self-adjoint operators we have defined the operator $C \wedge D$ by

$$C \wedge D = \iint_{\mathbb{R}^2} \max(\lambda,\mu)\, dG_\lambda H_\mu ,$$

and the operator $C \vee D$ by

$$C \vee D = \iint_{\mathbb{R}^2} \min(\lambda,\mu)\, dG_\lambda H_\mu .$$

where $(G_\lambda)_{\lambda\in\mathbb{R}}$ and $(H_\mu)_{\mu\in\mathbb{R}}$ are the spectral resolutions of C and D.
Moreover, we have shown that

$$T(S_{Z,C},D) \cap T(S_{Z,D},C) = T(S_{Z,C\wedge D},C\vee D).$$

Applying this result to the spaces $T(S_{X\otimes Y,I\otimes B},A\otimes I)$ and $T(S_{X\otimes Y,A\otimes I},I\otimes B)$, we find that their intersection equals the space $T(S_{X\otimes Y,A\bigotimes B},A\oslash B)$ with

$$A\bigotimes B = (A\otimes I) \wedge (I\otimes B) \text{ and } A\oslash B = (A\otimes I) \vee (I\otimes B).$$

(1.6) Definition

The canonical mapping emb: $S_{X,A}\otimes_a S_{Y,B} \to T(S_{X\otimes Y,A\bigotimes B},A\oslash B)$ is defined by

$$\text{emb}(f\otimes g) : t \to e^{-t(A\oslash B)}(f\otimes g) .$$

It is obvious that $\text{emb}(f\otimes g) \in T(S_{X\otimes Y,A\bigotimes B},A\oslash B)$.
The space $T(S_{X\otimes Y,A\bigotimes B},A\oslash B)$ is a complete topological tensor product of the spaces $S_{X,A}$ and $S_{Y,B}$. By this we mean

(1.7) Theorem

I. $\mathcal{T}(S_{X\otimes Y,A\otimes B}, A\oslash B)$ is complete.

II. The mapping $\otimes : S_{X,A} \times S_{Y,B} \to \mathcal{T}(S_{X\otimes Y,A\otimes B}, A\oslash B)$ is continuous.

III. $S_{X,A} \otimes_a S_{Y,B}$ is dense in $\mathcal{T}(S_{X\otimes Y,A\otimes B}, A\oslash B)$.

Proof.

I. All spaces of this kind are complete. Cf[E_1], Section 2.

II. It is sufficient to check continuity at [0;0]. Let $\psi \in B_+(\mathbb{R})$, and let t > 0. Then

$$\| \psi(A\oslash B) e^{-t(A\oslash B)} (f \otimes g) \|_{X\otimes Y} \le$$

$$\le \| \psi(A) f \|_X \| g \|_Y + \| f \|_X \| \psi(B) g \|_Y < \varepsilon ,$$

as soon as $\| \psi(A) f \|$ and $\| \psi(B) g \|$ are small enough. Cf.[G], Ch.I.

III. Following [G], Ch.V, the space $S_{X,A} \otimes_a S_{Y,B}$ is dense in $S_{X\otimes Y,A\boxplus B}$. From [$E_1$], Section 5, it follows that $S_{X\otimes Y,A\boxplus B}$ is dense in $\mathcal{T}(S_{X\otimes Y,A\otimes B}, A\oslash B)$. □

The strong dual space of $\mathcal{T}(S_{X\otimes Y,A\otimes B}, A\oslash B)$ is equal to the space $S(\mathcal{T}_{X\otimes Y,A\otimes B}, A\oslash B)$, where

$$S(\mathcal{T}_{X\otimes Y,A\otimes B}, A\oslash B) = S(\mathcal{T}_{X\otimes Y,A\otimes I}, I\otimes B) + S(\mathcal{T}_{X\otimes Y,I\otimes B}, A\otimes I) .$$

Hence, for all $f \in S_{X,A}$, $g \in S_{Y,B}$ and all $F \in \mathcal{T}_{X,A}$, $G \in \mathcal{T}_{Y,B}$

$$f \otimes G + F \otimes g \in S(\mathcal{T}_{X\otimes Y,A\otimes B}, A\oslash B) .$$

Case (e). Extendable linear mappings from $S_{X,A}$ into $S_{Y,B}$.

Let $E \in \mathcal{T}(S_{X\otimes Y,A\otimes B}, A\oslash B)$. Then for $f \in S_{X,A}$ we define Ef by

$$(e_1) \qquad Ef = e^{\varepsilon(A\otimes B)}[(e^{-\varepsilon A}\otimes I)(E(\varepsilon))]e^{\varepsilon A}f \ ,$$

where $\varepsilon > 0$ has to be taken sufficiently small. Definition (e_1) does not depend on the choice of ε. Further $Ef \in S_{Y,B}$ because $e^{\tau(A\otimes B)}(e^{-\tau A}\otimes I)$ is a bounded operator on $X\otimes Y$, and because $E(\tau) \in S_{X\otimes Y,A\otimes B} \subset S_{X\otimes Y,I\otimes B}$.
Let $F \in T_{X,A}$. We define the extension $\overline{E}$ on $T_{X,A}$ by

$$(e_2) \qquad (\overline{E}F)(t) = e^{t(A\otimes B)}(I\otimes e^{-tB})(E(t)e^{\varepsilon(t)A})F(\varepsilon(t)) \ , \quad t > 0$$

where each $\varepsilon(t) > 0$ has to be taken sufficiently small. We have $\overline{E}F \in T_{Y,B}$, because the operator $e^{tA\otimes B}(I\otimes e^{-tB})$ is bounded on $X\otimes Y$ for all $t > 0$, and because $E(t) \in S_{X\otimes Y,A\otimes B} \subset S_{X\otimes Y,A\otimes I}$.

<u>Remark</u>: If $E \in T(S_{X\otimes Y,A\otimes B},A\otimes B)$ then E can be embedded in $T(S_{X\otimes Y,I\otimes B},A\otimes I)$ as follows

$$\mathrm{emb}_1(E) : t \to e^{t(A\vee B)}(e^{-tA}\otimes I)(E(t)) \ ,$$

and in $T(S_{X\otimes Y,A\otimes I},I\otimes B)$ as

$$\mathrm{emb}_2(E) : t \to e^{t(A\otimes B)}(I\otimes e^{-tB})(E(t)) \ .$$

Cf.[E_1], Section 4.

The proof of the next theorem will be omitted; it is an immediate corollary of Theorem (1.3) and (1.4).

(1.8) <u>Theorem</u>

I. By (e_1) and (e_2), each element of $T(S_{X\otimes Y,A\otimes B},A\otimes B)$ provides a continuous and extendable linear mapping from $S_{X,A}$ into $S_{Y,B}$.

II. For each $E \in \mathcal{T}(S_{X\otimes Y, A\otimes B}, A\otimes B)$, $f \in S_{X,A}$, $g \in S_{Y,B}$, $F \in \mathcal{T}_{X,A}$ and $G \in \mathcal{T}_{Y,B}$,

$$\langle\langle f \otimes G + F \otimes g, E \rangle\rangle = \langle Ef, G\rangle + \langle g, \overline{E}F\rangle \ .$$

III. If for each $t > 0$ at least one of the operators e^{-tA} or e^{-tB} is Hilbert-Schmidt, then $\mathcal{T}(S_{X\otimes Y, A\otimes B}, A\otimes B)$ comprises all extendable linear mappings from $S_{X,A}$ into $S_{Y,B}$.

IV. $\mathcal{T}(S_{X\otimes X, A\otimes A}, A\otimes A)$ comprises all extendable linear mappings iff the operator e^{-tA} is Hilbert Schmidt for all $t > 0$.

By Theorem (1.8) we have given the space of extendable linear mappings the structure of a space of type $\mathcal{T}(_{Z,C}, \mathcal{D})$, if at least one of the spaces $S_{X,A}$ and $S_{Y,B}$ is nuclear.

2. The algebras $\mathcal{T}^A$, $\mathcal{T}_A$ and $\mathcal{E}_A$

The space $\mathcal{T}^A = \mathcal{T}(S_{X\otimes X, I\otimes A}, A\otimes I)$ comprises all continuous linear mappings from $S_{X,A}$ into itself if and only if the operator e^{-tA} is Hilbert-Schmidt for all $t > 0$. So in this case $\mathcal{T}^A$ admits an algebraic structure. If the space $S_{X,A}$ is not nuclear, then it is less natural that $\mathcal{T}^A$ is an algebra. Yet it is true. To show this, let $P_1, P_2 \in \mathcal{T}^A$. Then by the previous section for each $f \in S_{X,A}$ by definition,

$$P_1(P_2 f) = P_1(\tau_1)e^{\tau_1 A}(P_2(\tau_2)e^{\tau_2 A}f)$$

where $\tau_1, \tau_2 > 0$ have to be taken sufficiently small. Thus to the product P_1P_2 there corresponds the trajectory (P_1P_2) in $\mathcal{T}^A$

$$(P_1P_2): t \to P_1(\tau)e^{\tau A}P_2(t)$$

where for each $t > 0$ we have to take $\tau > 0$ so small that $e^{\tau A}P_2(t) \in X\otimes X$. With the above-derived multiplication $(P_1,P_2) \to (P_1P_2)$, $\mathcal{T}_A$ is an algebra. Similarly, there exists a multiplication operation on $\mathcal{T}_A \times \mathcal{T}_A$, $(K_1,K_2) \to (K_1K_2)$, where

$$(K_1K_2): t \to K_1(t)e^{\tau A}K_2(\tau).$$

(2.1) <u>Definition</u>

The linear mapping ^{c} on $\mathcal{T}_{X\otimes X,A\boxplus A}$ is defined by

$$\Phi^c : t \to \Phi(t)^* \quad , \Phi \in \mathcal{T}_{X\otimes X,A\boxplus A}.$$

Remark: Φ^c is called the adjoint of Φ.

(2.2) <u>Lemma</u>

The mapping ^{c} is a strongly continuous bijection on $\mathcal{T}_{X\otimes X,A\boxplus A}$ with $\Phi^{cc} = \Phi$.

<u>Proof</u>. The lemma is a natural consequence of the definition of ^{c}, and of the strong topology in $\mathcal{T}_{X\otimes X,A\boxplus A}$. □

Since $\mathcal{T}^A, \mathcal{T}_A$ can be seen as subspaces of $\mathcal{T}_{X\otimes X,A\boxplus A}$, the mapping ^{c} is well-defined on $\mathcal{T}^A$ and $\mathcal{T}_A$. It is not difficult to see that for $P \in \mathcal{T}^A$ its adjoint P^c is given by $P^c: t \to P(t)^*$. Here we note that $t \to P(t)$ is a trajectory in $\mathcal{T}^A$.

(2.3) <u>Lemma</u>

The mapping ^{c} is a bijection from $\mathcal{T}^A$ onto $\mathcal{T}_A$.

<u>Proof</u>. Let $t > 0$, and let $P \in \mathcal{T}^A$. Then there is $\tau > 0$ such that

$$e^{\tau A}P(t) \in X\otimes X$$

or, equivalently

$$P(t) \in \mathcal{D}(I \otimes e^{\tau A}) \ .$$

So its adjoint $P(t)^*$ is in $\mathcal{D}(e^{\tau A} \otimes I)$, which yields $P^c \in \mathcal{T}_A$. Similarly for $K \in \mathcal{T}_A$ we derive $K^c \in \mathcal{T}^A$. Hence ^{c} is a bijection. □

(2.4) Theorem

The mapping $^c \colon \mathcal{T}^A \to \mathcal{T}_A$ is a homeomorphism.

Proof. It is clear that ^{c} is a bijection satisfying $(P_1 P_2)^c = P_2^c P_1^c$. Further, each seminorm on $\mathcal{T}^A$ transforms into a seminorm on $\mathcal{T}_A$ by the mapping ^{c}. In particular, for all $P \in \mathcal{T}^A$,

$$\| \psi(A) P(t) \|_{X \otimes X} = \| (I \otimes \psi(A)) P(t) \|_{X \otimes X} = \| (\psi(A) \otimes I) P(t)^* \|_{X \otimes X},$$

where $\psi \in \mathcal{B}_+(\mathbb{R})$ and $t > 0$. Thus the result is established. Cf.[E_1], Section 2. □

(2.5) Corollary

The mapping $^c \colon \mathcal{T}_A \to \mathcal{T}^A$ is a homeomorphism.

The definitions (a) - (d) of the preceding section, which indicate how the elements of each of the four tensor products induce continuous linear mappings, lead to the following

(2.6) Lemma

Let $f, g \in \mathcal{S}_{X,A}$, and let $F, G \in \mathcal{T}_{X,A}$. Then

$$\langle f,\Phi g\rangle = \langle g,\Phi^c f\rangle \quad , \quad \Phi \in T_{X\otimes X,A\boxplus A} ,$$

$$\langle Pf,G\rangle = \langle f,P^c G\rangle \quad , \quad P \in T^A ,$$

$$\langle g,KF\rangle = \langle K^c g,F\rangle \quad , \quad K \in T_A ,$$

$$\overline{\langle \theta F,G\rangle} = \langle \theta^c G,F\rangle \quad , \quad \theta \in S_{X\otimes X,A\boxplus A} ,$$

We note that P^c is the representant in T_A of P' and K^c the representant in T^A of K', where P' and K' denote the dual mappings of P and K.

Following [E_1], Section 2, each element $H \in T(S_{Z,C},\mathcal{D})$ can be written as $H = \Theta(C,\mathcal{D})w$, where $w \in Z$ and $\theta \in F_+(\mathbb{R}^2)$, i.e. a function from $\mathbb{R}^2$ into $\mathbb{R}^+$ satisfying

$$\forall_{s>0}\ \exists_{t>0} : \sup_{\lambda\geq 0,\mu\geq 0} (\theta(\lambda,\mu)e^{-t\lambda}e^{s\mu}) < \infty.$$

Applying this result to T^A we can write for $P \in T^A$

$$P = \theta(I\otimes A,A\otimes I)(W) ,$$

for a well-chosen $W \in X\otimes X$ and $\theta \in F_+(\mathbb{R}^2)$. Then it is obvious that

$$P(t)^* = (I \otimes e^{-tA})\theta(A\otimes I,I\otimes A)(W^*) .$$

Hence $P^c = \theta(A\otimes I,I\otimes A)(W^*)$. Similarly for $K \in T_A$, $K = \chi(A\otimes I,I\otimes A)(V)$, where $V \in X\otimes X$ and $\chi \in F_+(\mathbb{R}^2)$,

$$K^c = \chi(I\otimes A,A\otimes I)(V^*).$$

The strong dual spaces S_A of T_A and S^A of T^A are given by

$$S_A = S(T_{X\otimes X, A\otimes I}, I\otimes A)$$

and

$$S^A = S(T_{X\otimes X, A\otimes I}, I\otimes A) \ .$$

As already observed by De Graaf, we have $S_A \subset T^A$ and $S^A \subset T_A$. The mapping ^{c} is a continuous bijection from S_A onto S^A, and even a homeomorphism $S_A \to S^A$ because of the equalities

$$\| \Theta(A\otimes I, I\otimes A)(\theta) \|_{X\otimes X} = \| \Theta(I\otimes A, A\otimes I)(\theta^c) \|_{X\otimes X},$$

for all $\Theta \in F_+(\mathbb{R}^2)$ and for all $\theta \in S_A$. Cf.[E_1], Section 1. The elements S_A and S^A are characterized as follows.

$$\Psi \in S^A \Leftrightarrow \exists_{\psi\in B_+(\mathbb{R})} \exists_{t>0} \exists_{W\in X\otimes X} : \Psi = \psi(A)We^{-tA}$$

$$\Phi \in S_A \Leftrightarrow \exists_{\varphi\in B_+(\mathbb{R})} \exists_{t>0} \exists_{V\in X\otimes X} : \Phi = e^{-tA}V\varphi(A) \ .$$

Thus, it easily follows that

$$\Psi^c = e^{-tA}W^*\psi(A) \in S_A$$

$$\Phi^c = \varphi(A)V^*e^{-tA} \in S^A$$

The weak topology for T^A is the coarsest topology in which all linear functionals on T^A obtained by pairing with elements of S^A are continuous Hence, the weak topology is generated by the seminorms

$$s_\Phi(P) = |\langle\langle \Phi, P\rangle\rangle| \quad , \quad P \in T^A$$

where $\Phi \in S^A$. Similarly the weak topology for T_A is generated by

$$r_\Psi(K) = |\langle\langle \Psi, K\rangle\rangle| \quad , \quad K \in T_A,$$

where $\Psi \in S_A$. The following lemma shows that ^{c} is weakly continuous.

(2.7) Lemma

Let $P \in T^A$ and let $\Phi \in S^A$. Then

$$\overline{\langle\langle \Phi, P\rangle\rangle} = \langle\langle \Phi^c, P^c\rangle\rangle .$$

Proof. There are $W, V \in X \otimes X$, and $\Theta \in F_+(\mathbb{R}^2)$, $\psi \in B_+(\mathbb{R})$ and $t > 0$ such that $P = \Theta(I \otimes A, A \otimes I)(W)$ and $\Phi = \psi(A) V e^{-tA}$. So employing spectral integrals with respect to the spectral resolution $(E_\lambda \otimes E_\mu)_{(\lambda,\mu)\in\mathbb{R}^2}$ of $I \otimes I$, we may write

$$\langle\langle \Phi, P\rangle\rangle = \iint_{\mathbb{R}^2} \Theta(\lambda,\mu) e^{-t\lambda} \psi(\mu) d(E_\mu V E_\lambda, W)_{X\otimes X} .$$

Since $\overline{(E_\mu V E_\lambda, W)_{X\otimes X}} = (E_\lambda V^* E_\mu, W^*)_{X\otimes X}$, we derive

$$\overline{\langle\langle \Phi, P\rangle\rangle} = \iint_{\mathbb{R}^2} \Theta(\mu,\lambda) e^{-t\lambda} \psi(\mu) d(E_\lambda V^* E_\mu, W^*) =$$

$$= \iint_{\mathbb{R}^2} \Theta(\lambda,\mu) e^{-t\mu} \psi(\lambda) d(E_\mu V^* E_\lambda, W^*) =$$

$$= \langle\langle e^{-tA} V^* \psi(A), \ \Theta(A \otimes I, I \otimes A)(W^*)\rangle\rangle =$$

$$= \langle\langle \Phi^c, P^c\rangle\rangle \qquad \square$$

(2.8) Theorem

I. The mapping ^{c}: $T^A \to T_A$ resp. $T_A \to T^A$ is weakly continuous.
II. The mapping ^{c}: $S^A \to S_A$ resp. $S_A \to S^A$ is weakly continuous.

The algebra E_A is defined as $E_A = T^A \cap T_A$; it consists of extendable linear mappings from $S_{X,A}$ into itself. In Section 1 we have shown that

$$E_A = T(S_{X\otimes X, A\otimes A}, A\otimes A) \ .$$

Naturally, the strong topology of E_A is generated by the seminorms

$$s_{\psi,t}(E) = \|\psi(A\otimes A)e^{-t(A\otimes A)}(E)\|_{X\otimes X} \quad , E \in E_A \ .$$

where $t > 0$ and $\psi \in B_+(\mathbb{R})$. The seminorms $s_{\psi,t}$ are equivalent to the seminorms $u_{\psi,t}$ and $v_{\psi,t}$,

$$u_{\psi,t}(E) = \psi(A)Ee^{-tA} \quad , \quad E \in E_A,$$

$$v_{\psi,t}(E) = e^{-tA}E\psi(A) \quad , \quad E \in E_A.$$

So the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous if the spaces carry their strong topology.
The dual space E_A' of E_A is expressed by the algebraic sum

$$E_A' = S^A + S_A \qquad (+ \text{ in } T_{X\otimes X, A\boxplus A}) \ .$$

Hence, the weak topology of E_A is equivalent to the topology induced by the weak topologies of T^A and T_A. Put differently, the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous if the spaces carry their weak topology.
The mapping ^{c} is a continuous bijection from E_A onto itself. Since

$E'_A \subset T_{X\otimes X, A\boxplus A}$, the mapping ^{c} is well defined on E'_A. We should like to write

$$(\Phi + \Psi)^c = \Phi^c + \Psi^c \ , \ \Phi \in S^A, \ \Psi \in S_A .$$

However, the choice of Φ and Ψ is not unique, because $S_A \cap S^A = S_{X\otimes X, A\boxplus A}$. In order to show the independence of the specific choice of Φ and Ψ in the wanted equality, suppose

$$\Phi_1 + \Psi_1 = \Phi_2 + \Psi_2$$

where $\Phi_1, \Phi_2 \in S^A$ and $\Psi_1, \Psi_2 \in S_A$. Then $\Phi_1 - \Phi_2 = \Psi_2 - \Psi_1$. Hence $\Phi_1 - \Phi_2 \in S^A \cap S_A = S_{X\otimes X, A\boxplus A}$. This implies

$$\Phi_1^c - \Phi_2^c = \Psi_2^c - \Psi_1^c \in S_{X\otimes X, A\boxplus A},$$

which yields

$$\Phi_1^c + \Psi_1^c = \Phi_2^c + \Psi_2^c \ .$$

The above-mentioned result leads to the following theorem

(2.9) Theorem

I. The mapping ^{c} is a strongly and weakly continuous linear bijection from E_A onto itself. It satisfies

$$E^{cc} = E \ , \ (E_1E_2)^c = E_2^cE_1^c \ , \ E_1, E_2, E \in E_A .$$

Hence, ^{c} is an involution on E_A.

II. The mapping ^{c} is a strongly and weakly continuous bijection from E'_A onto itself with $\theta^{cc} = \theta$, $\theta \in E'_A$.

III. Let $E \in E_A$. Then $E = \Theta(A \otimes A, A \otimes A)(W)$ for $\Theta \in F_+(\mathbb{R}^2)$ and $W \in X \otimes X$. We have $E^c = \Theta(A \otimes A, A \otimes A)(W^*)$.

IV. For $E \in E_A$ and $\theta \in E'_A$

$$\overline{\langle\langle \theta, E \rangle\rangle} = \langle\langle \theta^c, E^c \rangle\rangle .$$

If the Kernel theorem holds true, the algebra T^A comprises all continuous linear mappings from $S_{X,A}$ into itself. So T^A can be identified with the algebra of all continuous linear mappings from $S_{X,A}$ into itself.
As a space of linear mappings, T^A obtains some natural topologies from its domain space $S_{X,A}$, such as the topology of pointwise convergence and the topology of weak pointwise convergence. Similar constructions exist in the algebras T_A and E_A.
In the following chapters we shall deepen the topological structure of the algebras T^A, T_A and E_A. We shall investigate their affiliation with the respective algebraic structures.

3. The topological structure of the algebra T^A.

In the remaining part of this paper we assume that the space $S_{X,A}$ is nuclear. Equivalently, we assume that T^A comprises all continuous linear mappings from $S_{X,A}$ into itself. Then, besides its weak and its strong topology denoted by τ_s and τ_w in the sequel, we introduce the topologies τ_p and τ_{wp} for T^A.

(3.1) Definition. (The topology of pointwise convergence)

The topology τ_p is the locally convex topology for T^A induced by the seminorms $u_{f,\psi}$,

$$u_{f,\psi} = \| \psi(A) P f \| \quad , \quad P \in T^A,$$

where $f \in S_{X,A}$ and $\psi \in B_+(\mathbb{R})$.

The net (P_α) in T_A is τ_p-convergent if and only if the net $(P_\alpha f)$ in $S_{X,A}$ is strongly convergent for all $f \in S_{X,A}$. The topology τ_p is the coarsest topology for which the linear mappings $T^A \to S_{X,A}$,

$$P \to Pf \ , \ P \in T^A,$$

are strongly continuous for all $f \in S_{X,A}$.

The following result is remarkable. In fact, the strong topology of T^A is not introduced as a specific operator topology. Yet, it is one.

(3.2) Lemma

The topology τ_s is equivalent to the topology of uniform pointwise convergence on bounded subsets of $S_{X,A}$.

Proof. Let (P_α) be a strongly convergent net in T^A with limit P and let B be a bounded subset of $S_{X,A}$. Then there is $t > 0$ so that the set $e^{tA}(B)$ is bounded in X. For all $f \in B$, all $\psi \in B_+(\mathbb{R})$ and all α

$$\| \psi(A)(P_\alpha - P) f \| \leq \| \psi(A)(P_\alpha(t) - P(t) \| \, \| e^{tA} f \| \ .$$

On the other hand, let $\varepsilon > 0$ and let $t > 0$. Suppose

$$P_\alpha f \to Pf$$

strongly in $S_{X,A}$ and uniformly on the bounded subset $\{e^{-tA}w \,|\, \|w\| = 1\}$.
Then for each $\psi \in B_+(\mathbb{R})$ there is α_1, such that

$$\| \psi(A)(P_\alpha(t) - P(t)) w \| < \varepsilon/2 \ ,$$

for all $\alpha > \alpha_1$ and all $\omega \in X$ with $\|\omega\| = 1$. Hence,

$$\|\psi(A)(P_\alpha(t) - P(t))\| \leq \varepsilon/2 < \varepsilon . \qquad \square$$

Remark: In the proof of Lemma (3.2) we employed the norm $\|\cdot\|$ of the Banach algebra $\mathcal{B}(X)$ instead of the Hilbert-Schmidt norm $\|\cdot\|_{X\otimes X}$. However, this is allowed because of the following relation

$$\|P(t)\| \leq \|P(t)\|_{X\otimes X} \leq \|P(t/2)\| \, \|e^{-t/2 A}\|_{X\otimes X} , \; P \in \mathcal{T}^A .$$

(3.3) <u>Definition</u>. (The topology of weak pointwise convergence)

The topology τ_{wp} is the locally convex topology generated by the semi-norm $u_{f,G}$,

$$u_{f,G}(P) = |\langle Pf,G\rangle| \; , \; P \in \mathcal{T}^A ,$$

where $f \in \mathcal{S}_{X,A}$ and $G \in \mathcal{T}_{X,A}$.

The net (P_α) in $\mathcal{T}^A$ converges to $P \in \mathcal{T}^A$ in τ_{wp}-sense if and only if $\langle (P_\alpha - P)f,G\rangle \to 0$ for all $f \in \mathcal{S}_{X,A}$ and $G \in \mathcal{T}_{X,A}$. The topology τ_{wp} is the coarsest topology for which the linear mappings

$$P \to \langle Pf,G\rangle \; , \; P \in \mathcal{T}^A$$

are all continuous. τ_p is the topology of uniform weak pointwise convergence on bounded subsets of $\mathcal{T}_{X,A}$. The latter proposition is an immediate consequence of the characerization of bounded subsets of $\mathcal{T}_{X,A}$. The above introduced topologies for $\mathcal{T}^A$ are ordered as follows

(3.4)
$$\begin{array}{ccccc} & & \tau_s & & \\ & \subset & & \supset & \\ \tau_w & & & & \tau_p \\ & \supset & \tau_{wp} & \subset & \end{array}$$

Here $\subset$ means 'coarser than'.

(3.5) Theorem. (Principle of uniform boundedness)

Let B be a subset of T^A. Then the following statements are equivalent

I. B is τ_s-bounded.

II. B is τ_w-bounded.

III. B is τ_p-bounded

IV. B is τ_{wp}-bounded.

Proof. The equivalence I $\Leftrightarrow$ II follows from [E_1], Section 3. Further, it is clear that I $\Rightarrow$ III $\Rightarrow$ IV.

IV $\Rightarrow$ III: Each weakly bounded set in $S_{X,A}$ is strongly bounded, cf.[GE], Section 3. From this observation the assertion follows.

III $\Rightarrow$ I: For all $\psi \in B_+(\mathbb{R})$, $t > 0$ and $w \in X$, there exists $\alpha(t,\psi,w)$ such that the set $\{\psi(A)Pe^{-tA} | P \in B\}$ is strongly bounded in $B(X)$. Hence, the uniform boundedness for $B(X)$ yields $\alpha(t,\psi) > 0$ with $\|\psi(A)Pe^{-tA}\| \leq \alpha(t,\psi)$. Hence

$$\|\psi(A)Pe^{-tA}\|_{X\otimes X} \leq \alpha(t/_2,\psi)\|e^{-t/_2 A}\|_{X\otimes X},\ P \in B. \qquad \square$$

(3.5) Lemma

Let (P_n) be a sequence in T^A such that $\lim_{n\to\infty} P_n f$ exists in $S_{X,A}$ for each $f \in S_{X,A}$. Then $P : f \to \lim_{n\to\infty} P_n f$ is continuous, i.e., $P \in T^A$.

Proof. By Theorem (3.5) the sequence (P_n) is τ_s-bounded. So for each $t > 0$ there is $\alpha_t > 0$ such that $\|P_n(t)\| \leq \alpha_t$, $n \in \mathbb{N}$. It is obvious that P is a linear mapping from $S_{X,A}$ into itself. Further, for all $w \in X$, $\|w\| = 1$ and for all $t > 0$

$$\| Pe^{-tA}w \| \leq \| (P - P_n)e^{-tA}w \| + \alpha_t \leq \alpha_t + 1$$

for $n \in \mathbb{N}$ sufficiently large. Hence $P \in \mathcal{T}^A$ by [GE], Section 4. □

(3.7) Theorem

$\mathcal{T}^A$ is sequentially τ_p-complete and, similarly, sequentially τ_{wp}-complete

Proof. The proof is an immediate consequence of Lemma (3.5) and the (weak) sequential completeness of $S_{X,A}$. □

In the remaining part of this section we investigate the relation between the topological structure of $\mathcal{T}^A$ and its algebraic structure. First we have the following result.

(3.8) Theorem

Joint multiplication is strongly sequentially continuous in $\mathcal{T}^A$.

Proof. Let (P_n) and (T_n) be two converging sequences in $\mathcal{T}^A$ with $P_n \to P$ and $T_n \to T$. Let $t > 0$, and let $\psi \in \mathcal{B}_+(\mathbb{R})$. Then there exists $\varepsilon > 0$ and $C > 0$ such that

$$\| e^{\varepsilon A} T_n(t) \| < C \; , \; n \in \mathbb{N} \; ,$$

and

$$\| e^{\varepsilon A}(T_n(t) - T(t)) \| \to 0$$

because the sequence $(T_n(t))$ converges to $T(t)$ strongly in $S_{X \otimes X, I \otimes A}$. Hence the inequality

$$\| \psi(A)(P_n T_n - PT)(t) \| \leq$$

$$\leq \|\psi(A)(P_n - P)(\varepsilon)\| \|e^{\varepsilon A} T_n(t)\| + \|\psi(A)P(\varepsilon)\| \|e^{\varepsilon A}(T_n - T)(t)\|$$

for all $n \in \mathbb{N}$, yields the desired result. □

As observed by De Graaf $S_A \subset T^A$, we have the following stronger result.

(3.9) <u>Lemma</u>

S_A is a proper two-sided ideal in T^A.

<u>Proof</u>. From the characterization of the elements of S_A we obtain the equivalence $\Phi \in S_A \Leftrightarrow \Phi$ represents a continuous linear mapping from $S_{X,A}$ into $e^{-tA}(X)$ for some $t > 0$.
Let $P_1, P_2 \in T^A$ and let $\Phi \in S_A$. Then Φ maps $S_{X,A}$ into some $e^{-\alpha A}(X)$ and further P_1 maps $e^{-\alpha A}(X)$ into $e^{-\beta A}(X)$ for some $\beta > 0$ (cf.[GE], Section 4). So $P_1 \Phi P_2$ maps $S_{X,A}$ into $e^{-\beta A}(X)$ continuously, and hence $P_1 \Phi P_2 \in S_A$.
Since $I \notin S_A$, the ideal S_A is proper. □

(3.10) <u>Corollary</u>

S^A is a proper, two-sided ideal in T_A.

<u>Proof</u>. Follows directly from the properties of the adjoint mapping ^{c} and Lemma (3.9). □

(3.11) <u>Corollary</u>

Let $\Phi \in S^A$ and $P \in T^A$. Then

$$\langle\langle \Phi, P \rangle\rangle = \langle\langle P^c \Phi, I \rangle\rangle = \overline{\langle\langle \Phi^c P, I \rangle\rangle}$$

and

$$\langle\!\langle \Phi,P\rangle\!\rangle = \overline{\langle\!\langle \Phi^c,P^c\rangle\!\rangle} = \overline{\langle\!\langle P\Phi^c,I\rangle\!\rangle} = \langle\!\langle \Phi P^c,I\rangle\!\rangle .$$

(Note that $\langle\!\langle \Phi P^c,I\rangle\!\rangle$ = trace (ΦP^c)).

Proof. The proof is an application of Lemma (2.2) and Corollary (3.9). □

(3.12) Definition

The algebra Σ with topology τ is called locally convex, if

- (Σ,τ) is a locally convex, topolocical vector space.
- Separate multiplication is continuous in (Σ,τ).

(3.13) Theorem

The algebra T^A is locally convex if it carries each of the topologies τ_s, τ_w, τ_p and τ_{wp}.

Proof. We shall only prove the continuity of separate multiplication.

I. (T^A,τ_s)

Let $P \in T^A$ be fixed. Then for all $T \in T^A$

$$\|\psi(A)(TP)(t)\|_{X\otimes X} \le \|\psi(A)T(\varepsilon)\|_{X\otimes X}\|e^{\varepsilon A}P(t)\|$$

for $\varepsilon > 0$ sufficiently small. Hence $T \to TP$ is continuous. To show the continuity of $P \to TP$, let $T \in T^A$ be fixed, and let $\varepsilon > 0$. Further, let $t > 0$ and let $\psi \in B_+(\mathbb{R})$. Then there is an open null-neighbourhood Ω in $S_{X,A}$ such, that

$$\|\psi(A)Tf\| < \varepsilon/2$$

as soon as $f \in \Omega$. The existence of Ω follows from the continuity of T. Let (P_α) be a net in T^A that converges strongly to P. Then there exists α_1 such that for all $f \in \{e^{-tA}w \mid \|w\| \leq 1\}$ uniformly

$$(P_\alpha - P)f \in \Omega$$

if $\alpha > \alpha_1$. So α_1 does not depend on the choice of f. (Lemma (3.2)). Hence, if $\alpha > \alpha_1$, then

$$\|\psi(A)T(P_\alpha - P)f\| < \varepsilon/2$$

for all $f \in S_{X,A}$ with $\|e^{tA}f\| \leq 1$. The latter observation leads to the result

$$\|\psi(A)T(P_\alpha - P)(t)\| \leq \varepsilon/2 < \varepsilon$$

if $\alpha > \alpha_1$. This finishes the proof.

II. (T^A, τ_w) .

Let $P_1, P_2 \in T^A$. Then for each $\Phi \in S^A$

$$\langle\!\langle \Phi, P_1TP_2 \rangle\!\rangle = \langle\!\langle P_1^c \Phi P_2^c, T \rangle\!\rangle .$$

Hence

$$P \mapsto |\langle\!\langle \Phi, P_1TP_2 \rangle\!\rangle|$$

is a weakly continuous seminorm on T^A.

III. (T^A, τ_p).

Let $T_\alpha f \to Tf$ for all $f \in S_{X,A}$.

Then $T_\alpha P_2 f \to TP_2 f$ and hence by continuity of P_1, $P_1T_\alpha P_2 f \to P_1TP_2 f$.

This completes the proof.

IV. (T^A, τ_{wp}).

The seminorm

$$T \mapsto |\langle T(P_2 f), P_1^c G \rangle|$$

is τ_{wp}-continuous for each $f \in S_{X,A}$ and each $G \in T_{X,A}$. □

4. The topological structure of the algebra T_A

As we have already assumed in Section 3, T_A comprises all continuous linear mappings from $T_{X,A}$ into itself. The strong topology and the weak topology of T_A will be denoted respectively by σ_w and σ_s. In correspondence with the topologies τ_p and τ_{wp} of T_A we first introduce the topologies σ_p and σ_{wp}.

(4.1) Definition

The topology σ_p is the locally convex topology of T_A induced by the seminorms $v_{F,t}$

$$v_{F,t}(R) = \|(RF)(t)\| \quad , \quad R \in T_A$$

where $F \in T_{X,A}$ and $t > 0$.

The net (R_α) in T_A converges to $R \in T_A$ in σ_p-sense if and only if $R_\alpha F \to RF$ strongly for all $F \in T_{X,A}$. The topology σ_p is the coarsest topology for which the linear mappings $T_A \to T_{X,A}$

$$R \mapsto RF \quad , \quad R \in T_A \ ,$$

are all continuous.

(4.2) Lemma

The topology σ_s is equivalent to the topology of uniform pointwise convergence on bounded subsets of $T_{X,A}$.

<u>Proof</u>. Let (R_α) be a strongly convergent net in T_A with limit R. Let B be a strongly bounded subset of $T_{X,A}$. Then there exists $\psi \in B_+(\mathbb{R})$ and a bounded subset W of X such that $B = \psi(A)(W)$ (Cf.[E_1], Section 2). Hence for all $w \in W$

$$\| e^{-tA}(R_\alpha - R)\psi(A)w \| \leq \| (R_\alpha(t) - R(t))\psi(A)\| \| w\| \quad .$$

On the other hand, let $\varepsilon > 0$ and let $\psi \in B_+(\mathbb{R})$. Suppose $R_\alpha F \to RF$ strongly in $T_{X,A}$ and uniformly for $F \in \{\psi(A)w | \|w\| \leq 1\}$. Then for each $t > 0$ there is α_1 such that

$$\| (R_\alpha(t) - R(t))\psi(A)w \| < \varepsilon/2$$

for all $\alpha \geq \alpha_1$ and all $w \in X$ with $w \leq 1$. Hence

$$\| (R_\alpha(t) - R(t))\psi(A) \| \leq \varepsilon/2 < \varepsilon \quad . \qquad \square$$

(Remember the remark after Lemma (3.2).)

(4.3) <u>Definition</u> (The topology of weak pointwise convergence).

The topology τ_{wp} is the locally convex topology induced by the seminorms

$$v_{G,f}(R) = |\langle f,RG\rangle| \quad , \quad R \in T_A \ ,$$

where $f \in S_{X,A}$ and $G \in T_{X,A}$.

The net (R_α) converges to R in (T_A,τ_{wp}) if and only if $\langle f,(R_\alpha - R)G\rangle \to 0$ for all $f \in S_{X,A}$ and $G \in T_{X,A}$. The topology τ_{wp} is the coarsest topology for which the linear mappings $T_A \to \mathbb{C}$

$$R \mapsto \langle f,RG\rangle \quad , \quad R \in T_A \ ,$$

are all continuous. The topology σ_p is the topology of uniform, weak pointwise convergence on bounded subsets of $S_{X,A}$.

The above introduced topologies are ordered as follows

(4.4)
$$\begin{array}{ccccc} & & \sigma_s & & \\ & \subset & & \supset & \\ \sigma_w & & & & \sigma_p \\ & \supset & & \subset & \\ & & \sigma_{wp} & & \end{array}$$

(4.5) Theorem (Principle of uniform boundedness).

Let B be a subset of T^A. Then the following statements are equivalent

I. B is σ_s-bounded ;

II. B is σ_p-bounded ;

III. B is σ_w-bounded ;

IV. B is σ_{wp}-bounded.

Proof. We shall only prove the implication II $\Rightarrow$ I. The other implications are trivial or easy corollaries of other structure theorems.

II $\Rightarrow$ I: For all $t > 0$, $w \in X$ and $\psi \in B_+(\mathbb{R})$, we thus assume that the set

$$\{e^{-tA} R\psi(A)w \mid R \in B\}$$

is strongly bounded in $B(X)$. Hence, the uniform boundedness principle for $B(X)$ yields $\alpha(t,\psi) > 0$ with $\| e^{-tA} R\psi(A)\| \leq \alpha(t,\psi)$, $R \in B$. Hence

$$\| e^{-tA} R\psi(A) \|_{X\otimes X} \leq \alpha(\tfrac{1}{2}t,\psi) \| e^{-\frac{1}{2}tA} \|_{X\otimes X} , \quad R \in B. \qquad \square$$

(4.6) Lemma

Let (R_n) be a sequence in T_A such that $\lim_{n\to\infty} R_n F$ exists in $T_{X,A}$ for each

$F \in T_{X,A}$. Then R: $F \to \lim_{n\to\infty} R_n F$ is continuous, i.e. $R \in T_A$.

Proof. By the preceding theorem the sequence (R_n) is τ_s-bounded. So for each $t > 0$ there exists $\beta_t > 0$ such that $\|R_n(t)\| \le \beta_t$, $n \in \mathbb{N}$. It is clear that R maps $T_{X,A}$ into itself. Further, for all $w \in X$ with $\|w\| = 1$, and for all $t > 0$

$$\|e^{-tA}Rw\| \le \|e^{-tA}(R - R_n)w\| + \beta_t \le \beta_t + 1$$

for $n \in \mathbb{N}$ sufficiently large. Hence $R \in T_A$ by [GE], Section 4. □

(4.7) Theorem

T_A is sequentially σ_p- and σ_{wp}-complete.

In Section 2 we have proved that the mapping ^{c} from T^A onto T_A is $\tau_s \leftrightarrow \sigma_s$ and $\tau_w \leftrightarrow \sigma_w$ continuous, and its inverse ^{c} is $\sigma_s \leftrightarrow \tau_s$ and $\sigma_w \leftrightarrow \tau_w$ continuous. We do not know whether the mapping ^{c} is $\tau_p \leftrightarrow \sigma_p$ continuous and whether its inverse is $\sigma_p \leftrightarrow \tau_p$ continuous. However, for $f \in S_{X,A}$ and $G \in T_{X,A}$,

$$|\langle Pf,G\rangle| = |\langle f,P^cG\rangle| \quad , \quad P \in T^A.$$

So it follows that $P \mapsto P^c$, $P \in T^A$, is $\tau_{wp} \leftrightarrow \sigma_{wp}$ continuous and $R \mapsto R^c$, $R \in T_A$, is $\sigma_{wp} \leftrightarrow \tau_{wp}$ continuous.

With the above observed kinds of continuity of the mapping ^{c} and the mentioned properties of ^{c} the following results are straightforward corollaries of Theorem (3.8) and Theorem (3.13).

(4.8) Theorem

- Joint multiplication is sequentially continuous in T_A.
- The algebra T_A is locally convex if it carries each of the

topologies σ_s, σ_w and σ_{wp}.

Completing this section we prove the following.

(4.9) Theorem

The algebra T^A with topology τ_p is locally convex.

Proof. Let $R_\alpha F \to RF$ for all $F \in T_{X,A}$. Then for $S_1, S_2 \in T_A$, $R_\alpha S_2 F \to RS_2 F$ and hence by continuity of S_1, $S_1 R_\alpha S_2 F \to S_1 RS_2 F$. This completes the proof. □

5. The topological structure of the algebra E_A

Because of the assumption in Section 3 that $S_{X,A}$ is nuclear, E_A comprises all continuous linear mappings from $S_{X,A}$ into itself which are extendable to $T_{X,A}$. In Section 3 we observed that the strong and the weak topology of E_A, denoted by ρ_s and ρ_w in the sequel, admit the following characterizations

- ρ_s is the coarsest locally convex topology on E_A for which the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous with respect to the strong topology of T^A resp. T_A.
- ρ_w is the coarsest locally convex topology on E_A for which the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous with respect to the weak topology of T^A resp. T_A.

Similarly we introduce the topologies ρ_p and ρ_{wp}.

(5.1) Definition

The topology ρ_p is the coarsest locally convex topology on E_A for which the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous with respect to τ_p

resp. σ_p. The net (E_α) in $\mathcal{E}_A$ converges to E if and only if $E_\alpha f \to Ef$ strongly in $\mathcal{S}_{X,A}$ for all $f \in \mathcal{S}_{X,A}$ as well as $E_\alpha G \to EG$ strongly in $\mathcal{T}_{X,A}$ for all $G \in \mathcal{T}_{X,A}$.

(5.2) Lemma

The topology ρ_s is equivalent to the topology of uniform τ_p- and σ_p-convergence on bounded sets in $\mathcal{S}_{X,A}$ resp. $\mathcal{T}_{X,A}$.

Proof. Cf. Lemma (3.2) and (4.2). □

(5.3) Definition

The topology ρ_{wp} is the coarsest locally convex topology on $\mathcal{E}_A$ for which the embeddings $\mathcal{E}_A \hookrightarrow \mathcal{T}^A$ and $\mathcal{E}_A \hookrightarrow \mathcal{T}_A$ are continuous with respect to τ_{wp} resp. σ_{wp}. The net (E_α) in $\mathcal{E}_A$ converges to E if and only if $E_\alpha f \to Ef$ weakly in $\mathcal{S}_{X,A}$ for all $f \in \mathcal{S}_{X,A}$ as well as $E_\alpha G \to EG$ weakly in $\mathcal{T}_{X,A}$ for all $G \in \mathcal{T}_{X,A}$.

The above introduced topologies of $\mathcal{T}^A$ are ordered as follows.

(5.4)
$$\begin{array}{ccccc} & & \rho_s & & \\ & \subset & & \supset & \\ \rho_w & & & & \rho_p \\ & \supset & & \subset & \\ & & \rho_{wp} & & \end{array}$$

(5.5) Theorem (Principle of uniform boundedness)

Let B be a subset of $\mathcal{E}_A$. Then the following statements are equivalent.

I. B is ρ_s-bounded;

II. B is ρ_w-bounded;

III. B is ρ_p-bounded;

IV. B is ρ_{wp}-bounded.

Proof. Cf. Theorem (3.5) and (4.5). □

(5.6) Theorem

E_A is sequentially complete in ρ_p- and ρ_{wp}-sense.

Proof. Cf. Theorem (3.7) and (4.7). □

The adjoint mapping ^{c} becomes an involution on the algebra E_A. From the previous sections it follows that ^{c} is ρ_s-, ρ_w- and ρ_{wp}-continuous. From Theorem (3.13), (4.8) and (4.9) we obtain immediately

(5.7) Theorem

- Joint multiplication is strongly sequentially continuous in E_A.
- Separate multiplication is ρ_s-, ρ_w-, ρ_p- and ρ_{wp}-continuous.

The dual space E_A' of E_A can be represented by the algebraic sum of the spaces S_A and S^A. So every continuous linear functional ℓ on E_A can be written as

$$\ell : E \mapsto \langle\langle K_1, E\rangle\rangle_{S_A} + \langle\langle K_2, E\rangle\rangle_{S^A} ,$$

where $K_1 \in S_A$ and $K_2 \in S^A$. The choice of K_1 and K_2 is not unique because $S_A \cap S^A = S_{X\otimes X, A\boxplus A}$, cf.[E1], Section 4.

(5.8) Proposition

The space $S_{X\otimes X, A\boxplus A}$ is a proper, two-sided ideal in E_A.

Proof. S_A and S^A are proper, two-sided ideals in T^A resp. T_A. Hence $S_{X\otimes X, A\boxplus A} = S_A \cap S^A$ is a proper two-sided ideal in $T_A \cap T^A = E_A$. □

Let $E_1, E_2 \in \mathcal{E}_A$. Then for all $(K_1 + K_2) \in \mathcal{E}'_A$, define

$$E_1(K_1 + K_2)E_2 := E_1K_1E_2 + E_1K_2E_2 \ .$$

Then $E_1(K_1 + K_2)E_2$ is a well-defined element of $\mathcal{E}'_A$ by Lemma (3.9) and Corollary (3.10). In order to prove this, we have to show that the definition of $E_1(K_1 + K_2)E_2$ does not depend on the choice of K_1 and K_2. So let $K_1 + K_2 = 0$. Then $K_1 = -K_2 \in S_A \cap S^A = S_{X\otimes X, A\boxplus A}$. By Proposition (5.8), $E_1K_1E_2 = -E_1K_2E_2 \in S_{X\otimes X, A\boxplus A}$. Hence, $E_1K_1E_2 + E_1K_2E_2 = 0$, which completes the proof.

These observations imply the following.

(5.9) <u>Lemma</u>

Let $K \in \mathcal{E}'_A$ and $E \in \mathcal{E}_A$. Then

$$\langle\langle K,E\rangle\rangle = \overline{\langle\langle K^c,E^c\rangle\rangle}$$

$$\langle\langle K,E\rangle\rangle = \langle\langle E^cK,I\rangle\rangle$$

$\langle\langle EK,I\rangle\rangle = \langle\langle KE,I\rangle\rangle$ or equivalently trace (EK) = trace(KE).

<u>Proof</u>. Cf. Corollary (3.11). □

In a forthcoming paper we shall give a complete description of two subalgebras of $\mathcal{E}_A$, where we no longer assume that $S_{X,A}$ is nuclear. There we shall treat two topological algebras, the commutant of $\{A\}'$ and the double commutant $\{A\}''$. Inspired by the thesis of Pijls [Pij], we have been able to prove that $\{A\}'' \subset \mathcal{E}_A$ is a commutative GW^*-algebra, i.e. a commutative generalized Von Neumann algebra. The notion of GW^*-algebra has been introduced by Allan,[Al].

In the following section, we shall indicate how the theory could provide a mathematical model of quantum statistics. Therefore we introduce the notion of state in E'_A and the notion of positive element in E_A. We realize that the applications in Section 6 probably will raise more questions than they do answer.

6. Applications to quantum statistics

In this section we consider a quantum mechanical system in which the dynamics are determined by a Hamiltonian operator H, i.e. a self-adjoint operator in some appropriate Hilbert space X. We assume the almost inevitable condition that there can be found a nuclear analyticity space $S_{X,A}$ such that H and each of the unitary operators $e^{i\alpha H}$, $\alpha \in \mathbb{R}$, are continuous linear mappings on $S_{X,A}$. Further, for the states of the quantum system we take the one-dimensional subspaces of the trajectory space $T_{X,A}$. In [E_3] we have proved that $T_{X,A}$ contains almost all (generalized) eigenvectors of H.

In this section we adopt the terminology and notation of Dirac. The elements of $T_{X,A}$ are called kets and they are denoted by $|F\rangle$. Conjugate to the kets are the bras, denoted by $\langle F|$. The bra space is also a trajectory space, it has an antilinear structure. In [E_3] we have interpreted Dirac's bracket notion so that the expression

$$\langle F|G\rangle$$

makes sense for arbitrary kets and bras. In fact, $\langle F|G\rangle$ denotes the function

$$\langle F|G\rangle \ : \ s \mapsto \langle\overline{|F\rangle(s), |G\rangle}\rangle$$

The elements of $S_{X,A}$ are called test kets. The bras conjugated to them are called test bras. In this section we shall only consider the bracket of a test bra $\langle g|$ and a ket $|F\rangle$ resp. of a bra $\langle G|$ and a test ket $|f\rangle$. Then for their brackets we may take the ordinary numbers $\langle g|F\rangle(0)$ and $\langle G|f\rangle(0)$.

At a certain instant the dynamical system is supposed to be in one or other of a number of possible states according to some given probability law. Following Dirac, [Di], these states may establish a discrete set, a continuous range or both together. Here we look at the discrete case. Suppose that the possible states are given by normalized test kets $|m\rangle$, $m \in \mathbb{N}$. Let p_m denote the probability that the system is in the m-th state. Then we define the quantum density operator ρ by

(6.1) $$\rho = \sum_{m=1}^{\infty} p_m |m\rangle\langle m| \quad , \quad \sum_{m=1}^{\infty} p_m = 1 \quad , \quad p_m \geq 0 \, ,$$

where, according to Dirac $|m\rangle\langle m| = |m\rangle \otimes |m\rangle$.

In Schrödinger's picture the kets will evoluate in time in accordance with Schrödinger's equation

$$i\hbar \frac{d}{dt}|F\rangle = H|F\rangle$$

and the bras with the hermetian conjugate of this equation. Since without disturbance the system remains in the same state, corresponding to a ket which satisfies Schrödinger's equation, the p_m's are constant in time. We therefore have the following equation

(6.2) $$\begin{aligned} i\hbar\dot{\rho} &= \sum_m p_m \left(H|m\rangle\langle m| - |m\rangle\langle m|H\right) \\ &= H\rho - \rho H = [H,\rho]. \end{aligned}$$

For convenience we shall take $\hbar = 1$ in the sequel.

In our interpretation, the observables of the quantum system are represented by self-adjoint operators in X, which maps $S_{X,A}$ continuously into itself. Or, equivalently, by the symmetric elements of E_A with a self-adjoint extension in X.

If the system is in the m-th state, the expectation value $\langle\beta\rangle$ of any observable β equals

$$\langle\beta\rangle = \langle m|\beta|m\rangle.$$

Hence, if we insert the distribution law of the system corresponding to the above-introduced density operator ρ, then the average expectation value $\langle\beta\rangle$ is given by

$$(6.3) \qquad \langle\beta\rangle = \sum_m p_m \langle m|\beta|m\rangle = \langle\!\langle \rho,\beta \rangle\!\rangle = \mathrm{tr}(\rho\beta),$$

whenever $\rho \in E_A'$. Put $\beta = I$. Then it follows that

$$\langle I\rangle = \sum_m p_m = 1 \ .$$

The solution of equation (5.2) is given by

$$\rho(t) = e^{-itH}\rho_0 e^{itH}, \quad t \geq 0 \ ,$$

where $\rho(0)$ is ρ_0. Since the unitary operators $e^{i\alpha H}$, $\alpha \in \mathbb{R}$, are extendable, and since E_A' remains invariant under right and left multiplication by elements of E_A. (See Lemma (5.2)), we have $\rho(t) \in E_A'$, $t \geq 0$ iff $\rho_0 \in E_A'$.

Let β_0 be any observable. Then the average expectation value at time t equals

$$\langle\beta_0\rangle(t) = \langle\!\langle \rho(t), \beta_0(t) \rangle\!\rangle = \langle\!\langle \rho_0, e^{itH}\beta_0(t)e^{-itH} \rangle\!\rangle$$

where we have written $\beta_0(t)$ to indicate that the observable β_0 can intrinsically depend on t. Put $\beta(t) = e^{itH}\beta_0(t)e^{-itH}$. Then

$$(6.4.a) \qquad \dot{\beta} = i[H,\beta] + \frac{\partial\beta}{\partial t}$$

(6.4.b) $$\frac{d}{dt}(<\beta>) = i<[H,\beta]> + <\frac{\partial\beta}{\partial t}>$$

where $\frac{\partial\beta}{\partial t}(\tau) = e^{i\tau H}\frac{d\beta_0}{dt}(\tau)e^{-i\tau H}$. The differential equations (6.4.a) and (6.4.b) determine the evolution of the observables in the Heisenberg picture.

Now we are in a position to describe a quantum mechanical system in terms of observables out of some suitably chosen space E_A, and 'states' in its corresponding strong dual E_A'. We emphasize that the notion of state will get a meaning different from the one in the beginning of this section.

(6.5) Definition

A symmetric element $P \in E_A$ is called positive if $<f|P|f> \geq 0$ for all test kets $|f>$.

A positive element P of E_A leads to a positive, density defined, symmetric operator $\widetilde{P}$ in X. This operator $\widetilde{P}$ admits a so-called Friedrichs extension P_F in X, cf.[Fa]. The operator P_F is positive and self-adjoint in X. Hence, at least every positive element of E_A is an observable.

(6.6) Definition

Let $\sigma \in E_A'$. Then σ is called real if $\sigma(P) \in \mathbb{R}$ for all $P \in E_A$ with $P = P^c$.

From Section 5 we obtain the following characterization.

(6.7) Theorem

$\sigma \in E'_A$ is real iff $\sigma^c = \sigma$.

Proof. Let $P \in E_A$ be symmetric. Then by Section 5

$$\langle\!\langle \sigma, P \rangle\!\rangle = \overline{\langle\!\langle \sigma^c, P^c \rangle\!\rangle}.$$

This leads to the following equivalences

$$\langle\!\langle \sigma, P \rangle\!\rangle \in \mathbb{R} \text{ for all } P \in E_A \text{ with } P = P^c \Leftrightarrow$$

$$\Leftrightarrow \langle\!\langle \sigma, P \rangle\!\rangle = \langle\!\langle \sigma^c, P \rangle\!\rangle \text{ for all } P \in E_A \text{ with } P = P^c \Leftrightarrow$$

$$\Leftrightarrow \quad \sigma = \sigma^c.$$

The latter equivalence is due to the fact that every $E \in E_A$ is a combination of two symmetric elements, $E = \frac{E + E^c}{2} + i(\frac{E - E^c}{2i})$ □

Remark: Let $\sigma \in E'_A$ with $\sigma = \sigma^c$. Then $\sigma = s_1 + s_2$ with $s_1 \in S^A$ and $s_2 \in S_A$. (Cf. Section 5). Put $s = \frac{s_1 + s_2^c}{2}$. Then $s \in S^A$ and $\sigma = s + s^c$.

(6.8) Definition

Let $\sigma \in E'_A$ be a real functional. Then σ is called a state if

- $\sigma(P) \geq 0$ for all positive $P \in E_A$;
- $\sigma(I) = 1$, i.e. a state is always normalized.

In order to characterize the states in E'_A we prove the following.

(6.9) Lemma

Let $E \in E_A$, and let Π_n denote the orthogonal projection onto the linear span of the first n eigenvectors of A. Then the sequence $\{\Pi_n E \Pi_n\}$ converges to E in $\mathcal{E}_A$.

Proof. Let $t > 0$. Then we can take $\tau > 0$ such, that both

$$\| e^{2\tau A} E e^{-\frac{1}{2}tA} \|_{X \otimes X} < \infty$$

and

$$\| e^{-\frac{1}{2}tA} E e^{2\tau A} \|_{X \otimes X} < \infty.$$

Now we compute as follows

$$\| e^{\tau A}(E - \Pi_n E \Pi_n) e^{-tA} \|_{X \otimes X} \leq$$

$$\leq \| e^{\tau A}(I - \Pi_n) E \Pi_n e^{-tA} \|_{X \otimes X} + \| e^{\tau A} E (I - \Pi_n) e^{-tA} \|_{X \otimes X} \leq$$

$$\leq (\| (I - \Pi_n) e^{-\tau A} \| + \| (I - \Pi_n) e^{-\frac{1}{2}tA} \|) \| e^{2\tau A} E e^{-\frac{1}{2}tA} \|_{X \otimes X} .$$

Hence, $\| e^{\tau A}(E - \Pi_n E \Pi_n) e^{-tA} \|_{X \otimes X} \to 0$ for $n \to \infty$.

Similarly we can prove

$$\| e^{-tA}(E - \Pi_n E \Pi_n) e^{\tau A} \|_{X \otimes X} \to 0 \text{ for } n \to \infty.$$

So the assertion has been shown. □

Remark: Let $P \in \mathcal{E}_A$ be positive. Then for each $n \in \mathbb{N}$, the operator $\Pi_n P \Pi_n$ is an element of $\mathcal{E}_A$. In fact $\Pi_n P \Pi_n$ is a positive self-adjoint Hilbert-Schmidt operator. So there exists $f_j^{(n)} \in \Pi_n(X)$, $j = 1,\ldots,n$, such that

$$\Pi_n P \Pi_n = \sum_{j=1}^{n} \mu_j^{(n)} | f_j^{(n)} \rangle \langle f_j^{(n)} |$$

with $\mu_j \geq 0$. It leads to the following characterization.

(6.10) Theorem

Let $\sigma \in \mathcal{E}'_A$ be real. Then σ is a state iff

$$\langle\langle \sigma, |f\rangle\langle f| \rangle\rangle \geq 0$$

for all test kets $|f\rangle$.

Proof

$\Rightarrow$) Trivial. The projections $P_{|f\rangle} = |f\rangle\langle f|$ are elements of $\mathcal{E}_A$ and positive, for all test kets $|f\rangle$.

$\Leftarrow$) Let $P \in \mathcal{E}_A$ be positive. Let the projection Π_n, $n \in \mathbb{N}$, be as in Lemma (5.9). The functional $E \mapsto \langle\langle \sigma, E \rangle\rangle$ is strongly continuous on $\mathcal{E}_A$. Hence

$$\langle\langle \sigma, P \rangle\rangle = \lim_{n\to\infty} \langle\langle \sigma, \Pi_n P \Pi_n \rangle\rangle .$$

With the above remark it can be easily seen that for all $n \in \mathbb{N}$ $\langle\langle \sigma, \Pi_n P \Pi_n \rangle\rangle \geq 0$. Hence $\langle\langle \sigma, P \rangle\rangle \geq 0$.

Thus we have shown that σ is a state. □

Remark: Since $\sigma \in \mathcal{E}'_A \subset \mathcal{T}_{X\otimes X, A\boxplus A}$, and $|f\rangle\langle f| \in \mathcal{S}_{X\otimes X, A\boxplus A}$ we derive $\langle\langle \sigma, |f\rangle\langle f| \rangle\rangle = \langle f|\sigma|f\rangle$. (See [Di]).

Special elements of $\mathcal{E}'_A$ are the pure states. Here is the definition.

(6.11) Definition

A state ρ is called pure if there exists a normalized test ket $|f\rangle$ with $\rho = |f\rangle\langle f|$.

Of course, one might wonder why we don't take normalizable kets in Definition (5.11), i.e. kets in the Hilbert space X. The following lemma shows the answer.

(6.12) Lemma

Let $|w\rangle$ be a ket. Then

$$|w\rangle\langle w| \in E'_A \Leftrightarrow |w\rangle \text{ is a test ket.}$$

Proof

$\Rightarrow$) Suppose $|w\rangle \notin S_{X,A}$. Then there exists $\psi \in B_+(\mathbb{R})$ such that $|w\rangle \notin D(\psi(A))$. The operator $\psi(A)^2$ is in E_A, but

$$\langle\langle |w\rangle\langle w|, \psi(A)^2 \rangle\rangle = \infty .$$

Hence $|w\rangle\langle w| \notin E'_A$.

$\Leftarrow$) Trivial. □

The pure states admit the folllowing characterization.

(6.13) Theorem

A state ρ is pure if and only if $\rho \in S^A$ (or S_A) with $\rho^2 = \rho$.

Proof. If ρ is pure, $\rho = |f\rangle\langle f|$ for some test ket $|f\rangle$. Hence $\rho \in S_{X\otimes X, A\boxplus A} = S^A \cap S_A$, and ρ is a projection. On the other hand, $\rho \in S^A$ and ρ is a state yield $\rho = \rho^c \in S_A$. Hence $\rho \in S_{X\otimes X, A\boxplus A}$; ρ is a Hilbert-Schmidt projection with $tr(\rho) = 1$. So there exists a normalized $|f\rangle \in X$ with $\rho = |f\rangle\langle f|$. By Lemma (5.11) $|f\rangle$ is a test ket. □

(6.14) Theorem

Every pure state in E'_A is an extreme point in the set of states.

Proof. Let $|f\rangle$ be a normalized test ket, and Π_n, $n \in \mathbb{N}$, denote the projection as introduced in Lemma (6.9). Suppose there exist states $\sigma, \sigma_2 \in E'_A$ and $0 < \alpha < 1$ such that

$$|f\rangle\langle f| = \alpha\sigma_1 + (1-\alpha)\sigma_2 .$$

Then for all $n \in \mathbb{N}$ with $\Pi_n|f\rangle \neq 0$

$$\frac{\Pi_n|f\rangle\langle f|\Pi_n}{\|\Pi_n|f\rangle\|^2} = \frac{\alpha\,\sigma_1(\Pi_n)}{\|\Pi_n|f\rangle\|^2}\left[\frac{\Pi_n\sigma_1\Pi_n}{\sigma_1(\Pi_n)}\right] + \frac{(1-\alpha)\,\sigma_2(\Pi_n)}{\|\Pi_n|f\rangle\|^2}\left[\frac{\Pi_n\sigma_2\Pi_n}{\sigma_2(\Pi_n)}\right]$$

Take $k \in \mathbb{N}$ fixed, with $\Pi_k|f\rangle\langle f|\Pi_k \neq 0$. Then $\dfrac{\Pi_k|f\rangle\langle f|\Pi_k}{\|\Pi_k|f\rangle\|^2}$ is an extreme point of the unit ball of $\Pi_k(X)\otimes\Pi_k(X)$. Hence, we may assume

$$\Pi_k|f\rangle\langle f|\Pi_k = \Pi_k\sigma_1\Pi_k .$$

Since $\Pi_k\Pi_\ell = \Pi_k$ for all $\ell \geq k$ we derive

$$\forall_{n\in\mathbb{N}} : \Pi_n|f\rangle\langle f|\Pi_n = \Pi_n\sigma_1\Pi_n .$$

By Lemma (6.9) the sequences $\{\Pi_n|f\rangle\langle f|\Pi_n\}$ and $\{\Pi_n\sigma_1\Pi_n\}$ converge to $|f\rangle\langle f|$ resp. σ_1 weakly. Hence $\sigma_1 = |f\rangle\langle f|$. □

In the following theorem we prove that the pure states are the only extreme points in the set of states.

(6.15) Theorem

Let ρ be an extreme point in the set of states. Then ρ is a pure state

Proof. Since $\rho \neq 0$, there exists a normalized test ket $|f\rangle$ such that

$$\rho(|f\rangle\langle f|) \neq 0 .$$

Remark: The following implication can be shown rather easily:

$$\left(\forall_{|f>\in S_{X,A}} : \rho(|f><f|) = 0\right) \Rightarrow (\rho = 0) \ .$$

Put $P_{|f>} = |f><f|$. Then ρ can be written as

$$\rho = \rho \circ P_{|f>} + \rho \circ (I - P_{|f>})$$

where $(\rho \circ P_{|f>})(E) = \rho(P_{|f>}E)$, $E \in \mathcal{E}_A$. So $(\rho \circ P_{|f>})(I) = \rho(P_{|f>}) \neq 0$.

1) Suppose $\rho \circ (I - P_{|f>}) \neq 0$, and consequently $\rho(I - P_{|f>}) \neq 0$. Then we can write $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$, where

$$\rho_1 = \frac{\rho \circ P_{|f>}}{\rho(P_{|f>})} \ , \quad \rho_2 = \frac{\rho \circ (I - P_{|f>}}{1 - \rho(P_{|f>})} \ ,$$

$$\alpha = \rho(P_{|f>}).$$

The functionals ρ_1 and ρ_2 are states. This can be seen as follows

$$\rho_1(I) = \frac{\rho(P_{|f>})}{\rho(P_{|f>})} = 1 \ ,$$

and

$$\rho_1(E) = (\rho(P_{|f>}))^{-1}\rho(P_{|f>}E) = (\rho(P_{|f>}))^{-1}\rho(P_{|f>}EP_{|f>}).$$

For the latter equality see Lemma (5.9) and observe that $P^2_{|f>} = P_{|f>}$. Thus we derive $\rho_1(E) \in \mathbb{R}$ for all $E \in \mathcal{E}_A$ with $E = E^c$ and $\rho_1(E) \geq 0$ for all positive $E \in \mathcal{E}_A$. Similarly, ρ_2 is a state. But now we have got a contradiction, because ρ is extreme. Hence $\rho \circ (I - P_{|f>}) = 0$, and consequently $\rho = \rho \circ P_{|f>}$ and $\rho(P_{|f>}) = 1$. Further, it easily follows that for all test kets $|g>$

$$\rho(|g\rangle\langle g|) = |\langle f|g\rangle|^2 .$$

Employing the projections Π_n, $n \in \mathbb{N}$, as introduced in Lemma (5.9), we find that for each symmetric $E \in \mathcal{E}_A$ and for each $n \in \mathbb{N}$ there exists $\mu_j^{(n)} \in \mathbb{R}$ and $|f_j^{(n)}\rangle \in \Pi_n(X)$ such that

$$\Pi_n E \Pi_n = \sum_{j=1}^{n} \mu_j^{(n)} |f_j^{(n)}\rangle\langle f_j^{(n)}|$$

and

$$\rho(\Pi_n E \Pi_n) = \rho\left(\sum_{j=1}^{n} \mu_j^{(n)} |f_j^{(n)}\rangle\langle f_j^{(n)}|\right) =$$

$$= \sum_{j=1}^{n} \mu_j^{(n)} |\langle f|f_j^{(n)}\rangle|^2 =$$

$$= \langle f|\Pi_n E \Pi_n|f\rangle .$$

Letting $n \to \infty$, by Lemma (5.9) we obtain

$$\rho(\Pi_n E \Pi_n) \to \rho(E)$$

and

$$\langle f|\Pi_n E \Pi_n|f\rangle \to \langle f|E|f\rangle .$$

Hence for all symmetric $E \in \mathcal{E}_A$, $\rho(E) = \langle f|E|f\rangle$.

This yields $\rho = |f\rangle\langle f|$. □

1) Remark: Let $\rho \in \mathcal{E}_A'$ be a real positive functional, i.e. $\rho(P) \geq 0$ for all positive $P \in \mathcal{E}_A$. Let $n \in \mathbb{N}$, and let $E \in \mathcal{E}_A$. Then the following inequality is immediate from the finite-dimensional case

$$|\rho(\Pi_n E \Pi_n)|^2 \leq \rho(\Pi_n)\rho(\Pi_n E^c E \Pi_n) \ .$$

So the limit $n \to \infty$

$$|\rho(E)|^2 \leq \rho(I)\rho(E^c E).$$

Consequently $\rho(I) = 0 \Leftrightarrow \rho = 0$.

(6.16) Theorem

The linear span of the pure states is dense in E_A'.

Proof. We assume that $P \in E_A$ and $\langle f|P|f\rangle = 0$ for all test kets $|f\rangle$. Then $\langle f+g|P|f+g\rangle$ and $\langle f+ig|P|f+ig\rangle = 0$, and hence, $\mathrm{Re}(\langle f|P|g\rangle) = 0$ and $\mathrm{Im}(\langle f|P|g\rangle) = 0$ for all test kets $|f\rangle$ and test bras $\langle g|$. So $P = 0$. □

Finally we shall characterize the state in S^A (or S_A) or equivalently the states in $S_{X\otimes X, A\boxplus A}$.

(6.17) Theorem

Let $\rho \in S_{X\otimes X, A\boxplus A}$. Then the following statements are equivalent.

(1) ρ is a state.

(2) ρ is positive and self-adjoint with $\mathrm{tr}(\rho) = 1$.

(3) There exist normalized $|j\rangle \in S_{X,A}$ and positive numbers p_j satisfying

$$\exists_{s>0} \sum_{j=1}^{\infty} p_j^2 \| e^{sA} |j\rangle \|^2 < \infty \ ,$$

and $\sum_j p_j = 1$ such that

$$\rho = \sum_j p_j |j\rangle\langle j| .$$

Proof. The proof proceeds as follows: (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2):

From Theorem (6.10) it follows that ρ is a positive operator on $S_{X,A}$. Since ρ is Hilbert Schmidt and $\rho^c = \rho$, ρ is a positive, self-adjoint operator on X with $tr(\rho) = 1$.

(2) ⇒ (3):

By definition, there exists $s > 0$ such that $\rho = e^{-sA}We^{-sA}$ for some $W \in X\otimes X$ with $W \geq 0$. Since $\rho \in X\otimes X$ and $\rho \geq 0$, there exists an orthonormal basis $(|j\rangle)$ in X, and positive numbers p_j such that

$$\rho = \sum_j p_j |j\rangle\langle j| \text{ with } \sum_j p_j = 1 .$$

Further, since We^{-sA} is Hilbert Schmidt and $We^{-sA}|j\rangle = p_j e^{sA}|j\rangle$,

$$\sum_{j=1}^{\infty} \| We^{-sA}|j\rangle \|^2 = \sum_{j=1}^{\infty} p_j^2 \| e^{sA}|j\rangle \|^2 < \infty .$$

(3) ⇒ (1)

Note first that $\langle\langle \rho, I \rangle\rangle = \sum_{j=1}^{\infty} p_j \langle j|j\rangle = \sum_{j=1}^{\infty} p_j = 1.$

Let $s > 0$ as indicated. Then

$$\rho|j\rangle = p_j|j\rangle .$$

Put $W = e^{sA}\rho e^{sA}$. Then $We^{-sA}|j\rangle = p_j e^{sA}|j\rangle$.

Hence We^{-sA} is Hilbert-Schmidt and thus we find that

$$\rho = e^{-sA}We^{-sA} \in S_{X\otimes X, A\boxplus A} .$$

If $E \in \mathcal{E}_A$ is symmetric then $\langle j|E|j\rangle \in \mathbb{R}$ and hence $\rho(E) \in \mathbb{R}$. If $E \in \mathcal{E}_A$ is positive, then $\langle j|E|j\rangle \geq 0$ and hence $\rho(E) \geq 0$. Thus it is clear that ρ is a state. □

As a rule the dynamical state of a quantum system at a certain instant cannot be represented by one single ket, but we have a statistical mixture of kets. Therefore, in the beginning of this section we introduced the quantum density ρ (cf.(5.1)). According to the probability law determined by ρ, the quantum system is in one or other of a number of possible states. So it makes sense to define ρ to be the state of the quantum system at a given time.

If at $t = 0$ the quantum system is in the state ρ_0, at $t = \tau$ the system is in the state $\rho(\tau)$ with

$$\rho(\tau) = e^{-i\tau H}\rho_0 e^{i\tau H}.$$

So ρ satisfies the evolution equation (cf. (5.2))

$$\dot{\rho} = -i[H,\rho] .$$

In order to arrive at a mathematical rigorous theory, we only consider $\rho_0 \in \mathcal{E}_A'$. Then for every $t > 0$, $\rho(t) \in \mathcal{E}_A'$, because $e^{itH} \in \mathcal{E}_A$ for all $t \in \mathbb{R}$. (See Section 4). At every time t we can compute the expectation value $\langle\beta\rangle$ with respect to ρ of the observable $\beta \in \mathcal{E}_A$,

$$\langle\beta\rangle(t) = \langle\!\langle \rho(t),\beta \rangle\!\rangle,$$

where for convenience we have assumed that β is constant in time.

Now in general we shall assume that any state in $\mathcal{E}_A'$ as defined in Definition (5.8) represents an initial state of the quantum system in the above indicated way. A state σ_0 evoluates in time according to

$$e^{-itH}\sigma_0 e^{itH}, \ t > 0 .$$

So the statistical mixture determined by the quantum density operator ρ is a particular kind of state; states such as ρ have an immediate physical interpretation. From (5.14) we obtain that every state $\rho_0 \in S_{X \otimes X, A \boxplus A}$ induces a statistical mixture. The pure states are special types of statistical mixtures; one knows with certainty that the system is in a state determined by one test ket.

We conclude this section with a short discussion of the three possible types of dynamical quantum systems.

(1) <u>The Hamiltonian operator H admits a purely discrete spectrum</u>

This case is the easiest one to treat and it probably contains the most promising results.

Let H be a Hamiltonian operator in X with eigenvalues $E_1 \le E_2 \le \ldots$, and corresponding normalized eigenkets $|E_1>, |E_2>, \ldots$. Then the eigenkets $|E_i>$ of H establish a complete orthonormal basis for X. Define the positive numbers λ_n, $n \in \mathbb{N}$, as follows

$$\lambda_1 = E_1 \quad \lambda_n = \max(\lambda_{n-1} + 1, |E_n|), \ n > 1 ,$$

and the self-adjoint operator A by

$$A|E_n> = \lambda_n |E_n>$$

followed by linear extension and unique self-adjoint extension to X. Then the analyticity space $S_{X,A}$ is nuclear because $\sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty$ for all $t > 0$.

Further, H is continuous on $S_{X,A}$ because $\sup_{n\in\mathbb{N}} (|E_n| e^{-\lambda_n t}) < \infty$. Hence, $H \in E_A$. Similarly if follows that the unitary operators $e^{i\alpha H}$, $\alpha \in \mathbb{R}$, are elements of E_A. So the space $S_{X,A}$ satisfies the required conditions.

An important example of a statistical mixture is given by the state

$$\rho_0 = \sum_{n=1}^{\infty} p_n \; |E_n\rangle\langle E_n| \; , \; p_n \geq 0, \; \sum_{n=1}^{\infty} p_n = 1 \; .$$

Then ρ is represented by a diagonal matrix, and seen as a bounded operator on X, ρ clearly commutes with A and H. Since $\rho \in E_A'$, it satisfies

$$\exists_{\alpha>0} \forall_{a>0} \exists_{M>0} \forall_{n\in\mathbb{N}} (p_n e^{-a\lambda_n} \; e^{\alpha\lambda_n}) < M \; .$$

Hence $p_n = O(e^{-\lambda_n \alpha})$, and $\rho \in S_{X\otimes X, A\boxplus A}$. It is obvious that without disturbance the state ρ does not depend on the time t. We note that it is obvious that every term $|E_n\rangle\langle E_n|$ of the series does not depend on t, i.e. the system remains in a stationary state as long as disturbances do not occur.

In general a state ρ is given by

$$\rho = \sum_{n,m} \rho_{nm} |E_n\rangle\langle E_m| \; .$$

However, in many physically realistic cases the non-diagonal elements can be neglected.

An example for class (1) is given by the one dimensional harmonic oscillator where $H = \frac{1}{2}(\frac{-d^2}{dx} + x^2 + 1)$. Then H is self-adjoint in $L_2(\mathbb{R})$

with $E_n = n$, $n \in \mathbb{N}$ as its eigenvalues and the Hermite functions as its eigenfunctions. Hence, we can take $A = H$. We note that the space $S_{L_2(\mathbb{R}),H}$ is equal to the space $S^{\frac{1}{2}}_{\frac{1}{2}}$ of Gelfand-Shilov. Well-defined observables are the momentum operator $i\frac{d}{dx}$ and the position operator x.

(2) The Hamilton operator H admits a purely continuous spectrum

This is a harder case. We are able to construct a nuclear analyticity space $S_{X,A}$ such that H is continuous on $S_{X,A}$ (cf. Section 9). Then to almost every point in the spectrum of H there corresponds on eigenket in the trajectory space $T_{X,A}$. However, it is not clear whether the unitary operators $e^{i\alpha H}$, $\alpha \in \mathbb{R}$, are continuous on $S_{X,A}$, and this problem has not been solved yet. Of course, we could weaken the conditions on $S_{X,A}$ and skip nuclearity. Then the analyticity space $S_{X,|H|}$ with$(H)= (H^2)^{\frac{1}{2}}$ would be ideal. But nuclearity seems to play an essential role both in the discussions of this section and in our interpretation of Dirac's formalism.

There is another approach. Sometimes iH is one of the skew-adjoint generators of a unitary Lie group representation on X with nuclear analyticity space. We shall explain this to some extent. Let G be a finite dimensional Lie group with Lie algebra $A(G)$. Let U be a representation of G into the space of unitary operators on X, and ∂U the corresponding infinitesimal representation of $A(G)$ in X. Then for every $a \in A(G)$ the operator $\partial U(a)$ is skew-adjoint in X, by Stone's theorem.

Our first assertion is the following one.

- There exists $a_1 \in A(G)$ such that $iH = \partial U(a_1)$.

Since G has dimension $d < \infty$ there are $a_2,\dots,a_d \in A(G)$ such that $\{a_1,\dots,a_d\}$ generates the Lie group G in the usual way. Following Nelson, [Ne], the analyticity space corresponding to the unitary representation U is equal to

$$S_{X,\Delta^{\frac{1}{2}}}$$

where $\Delta = 1 - ((\partial U(a_1))^2 + (\partial U(a_2))^2 + \dots + (\partial U(a_d))^2)$.

Then our second assumption is

- $S_{X,\Delta^{\frac{1}{2}}}$ is nuclear.

In [GE], Section 7, we have given several cases of unitary representations of Lie groups G with a nuclear analyticity space $S_{X,\Delta^{\frac{1}{2}}}$. Moreover, we have proved that both the unitary operators $U(g)$, $g \in G$ and the skew-adjoint operators $\partial U(a_j)$, $j = 1,\dots,d$, are all continuous on $S_{X,\Delta^{\frac{1}{2}}}$. So under the above-mentioned assumptions the nuclear analyticity space $S_{X,\Delta^{\frac{1}{2}}}$ has the desired properties.

An example for this type of operators is the Hamiltonian operator of the free particle in one dimension,

$$H = -\frac{d^2}{dx^2} .$$

An appropriate algebra is the six-dimensional algebra generated by

$$i\frac{d^2}{dx^2} , \; i(\frac{x}{dx}x + x\frac{d}{dx}), \; ix^2, \; ix, \; \frac{d}{dx} , \; i .$$

It corresponds to the infinitesimal representation belonging to the unitary representation of the Schrödinger groups on $L_2(\mathbb{R})$. The Schrödinger group is obtained as a semidirect product of $SL(2,\mathbb{R})$ and of W_1, the Weyl group. We note that the Schrödinger group is the symmetry group of the Schrödinger equation of the free particle (see [Mi]).

(3) The Hamiltonian operator H admits a discrete/continuous spectrum

In many applications the intersting part of the spectrum of H is the discrete one. So we split X into the direct sum $X = X_d \oplus X_c$ such that H_d, the restriction of H to X_d, acts invariantly in X_d and H_d is a self-adjoint operator in X_d with discrete spectrum,and such that H_c, the restriction of H to X_c, acts invariantly in X_c and H_c is a self-adjoint operator in X_c with a purely continuous spectrum.
An example for this case is the Hamiltonian operator of the hydrogen atom.

7. The matrices of the elements of T_A and T^A

As in Section 3 we still assume that $S_{X,A}$ is a nuclear space. So in $S_{X,A}$ there exists an orthonormal basis (v_j) for X consisting of eigenvectors of A with eigenvalues λ_j, $\lambda_1 \le \lambda_2 \le \dots$ satisfying

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} < \infty$$

for all $t > 0$. Then the space T^A contains all linear mappings from $S_{X,A}$ into itself, and T_A all linear mappings from $T_{X,A}$ into itself. Let $L \in T^A$. Then to L there can be associated the well-defined matrix (L_{ij}) as follows

$$L_{ij} = (Lv_j, v_i), \quad i,j = 1,2\dots \quad .$$

This section is devoted to the kind of infinite matrices which arises in this way. We shall produce necessary and sufficient conditions on a matrix (Q_{ij}) in order that its associated linear operator Q is a continuous linear mapping on $S_{X,A}$. We emphasize that there are no elegant nor applicable conditions on infinite matrices which imply boundedness of its associated operator in X (see [Ha], Ch.IV).

Since the linear mapping L is continuous on $S_{X,A}$, it satisfies

$$\forall_{t>0} \exists_{s>0} \exists_{C>0} : \| e^{sA} L e^{-tA} \|_{X \otimes X} \le C$$

where $\| \cdot \|_{X\otimes X}$ denotes the norm in $X \otimes X$. This implies that the columns Lv_j, $j \in \mathbb{N}$, of the matrix (L_{ij}) satisfy

(7.1) $$\forall_{t>0}\exists_{s>0}\exists_{C>0}\forall_{i\in\mathbb{N}}: \| e^{sA} L v_i \|_X \le C e^{\lambda_i t} .$$

Put $b_i = Lv_i$, $i \in \mathbb{N}$. Then the vectors b_i span the range $L(S_{X,A})$ and from (7.1) it follows that there exists $s > 0$ such that $b_i \in e^{-sA}(X)$, $i \in \mathbb{N}$. Define the trajectory $\hat{L} : (0,\infty) \to X\otimes X$ by

$$\hat{L}(t) = \sum_{i=1}^{\infty} e^{-\lambda_i t}(v_i \otimes b_i), \quad t > 0.$$

Then $\hat{L}(t) \in S_{X\otimes X, I\otimes A}$. To show this let $0 < t_1 < t$, and choose $s > 0$ and $C > 0$ such that

$$\| e^{sA} b_i \| \le C e^{\lambda_i t_1} , \; i \in \mathbb{N} .$$

Then

$$\| e^{sA}\hat{L}(t)\|_{X\otimes X} = \| \sum_{i=1}^{\infty} e^{-\lambda_i t} v_i \otimes (e^{sA} b_i)\|_{X\otimes X} \le$$

$$\le \sum_{i=1}^{\infty} e^{-\lambda_i t} \| e^{sA} b_i \|_X \le C \sum_{i=1}^{\infty} e^{-\lambda_i (t-t_1)} < \infty .$$

Hence $\hat{L}(t) \in S_{X\otimes X, I\otimes A}$. It is obvious that

$$\hat{L}(t_1 + t_2) = (e^{-t_1 A} \otimes I)\hat{L}(t_2) \; , \; t_1, t_2 > 0 .$$

So $\hat{L} \in T^{A}$. Since for all $f \in S_{X,A}$

$$\hat{L}f = \sum_{i=1}^{\infty} (f,v_i) b_i = \sum_{i=1}^{\infty} (f,v_i) L v_i = Lf,$$

the linear mapping L is represented by the series

$$\sum_{i=1}^{\infty} v_i \otimes b_i$$

with convergence in T^A.

On the other hand, let there be given $b_1, b_2, \ldots$ in $S_{X,A}$ satisfying

(7.2) $$\forall_{t>0} \exists_{\tau>0} \exists_{C>0} \forall_{i\in\mathbb{N}} : \| e^{\tau A} b_i \| \leq C e^{\lambda_i t} \ .$$

Then it is obvious that the series $\sum_{i=1}^{\infty} v_i \otimes b_i$ converges in T^A, and represents the linear mapping

$$f \mapsto \sum_{i=1}^{\infty} (f, v_i) b_i \ , \ f \in S_{X,A}.$$

So the following characterization holds true.

(7.3) <u>Characterization</u> (the columns)

Let W be a linear operator in X with domain containing the linear span $\langle v_1, v_2, \ldots \rangle$. Then W maps $S_{X,A}$ continuously into itself iff the Wv_i, $i \in \mathbb{N}$, satisfy condition (7.2). W is represented in T^A by the series $\sum_{i=1}^{\infty} v_i \otimes (Wv_i)$.

The conjugate L^c of L is an element of T_A. Hence, as a continuous linear mapping from $T_{X,A}$ into itself L^c satisfies the following condition

$$\forall_{t>0} \exists_{s>0} \exists_{C>0} : \| e^{-tA} L^c e^{sA} \|_{X\otimes X} \leq C.$$

Put $B_j = L^c v_j \in T_{X,A}$. Then they satisfy

(7.4) $$\forall_{t>0} \exists_{s>0} \exists_{C>0} \forall_{j\in\mathbb{N}} : \| B_j(t) \|_X \leq C e^{-s\lambda_j} \ .$$

The trajectories B_j span $L^c(T_{X,A})$, and

$$B_j = \sum_{i=1}^{\infty} \bar{L}_{ji} v_i, \quad j \in \mathbb{N}$$

where the series converges in $T_{X,A}$. Hence B_j represents the j-th row of the matrix (L_{ij}). Define the trajectory $\tilde{L}$ by

$$\tilde{L}(t) = \sum_{j=1}^{\infty} B_j(t) \times v_j \quad , \; t > 0.$$

Then for each $t > 0$, $s_0 > 0$ can be chosen such that

$$\| B_j(t) \|_X \le C e^{-\lambda_j s_0} \quad , \; j \in \mathbb{N} \, ,$$

and for $0 < s < s_0$,

$$\| e^{sA} \tilde{L}(t) \|_{X \otimes X} \le \| \sum_{j=1}^{\infty} B_j(t) \otimes (e^{sA} v_i) \|_{X \otimes X} \le$$

$$\le C \sum_{j=1}^{\infty} e^{-\lambda_j (s_0 - s)} < \infty \, .$$

Hence, $\tilde{L}(t) \in S_{X \otimes X, I \otimes A}$, $t > 0$, and $\tilde{L} \in T^A$. Since

$$\tilde{L} f = \sum_{j=1}^{\infty} < f, B_j > v_j = \sum_{j=1}^{\infty} (Lf, v_j) v_j = Lf, \; f \in S_{X,A},$$

the mapping L is represented by the series $\sum_{j=1}^{\infty} B_j \otimes v_j$ with convergence in T^A.

On the other hand, let there be given B_1, B_2,... satisfying condition (7.4), then similarly it can be shown that the series $\sum_{j=1}^{\infty} B_j \otimes v_j$ represents the linear mapping

$$f \mapsto \sum_{j=1}^{\infty} \langle f, B_j \rangle v_j \ , \quad f \in S_{X,A}$$

in T_A. Thus we obtain a second characterization of the elements in T^A.

(7.5) Characterization (the rows)

Let W be a linear operator in X with domain containing the linear span $\langle v_1, v_2, \ldots \rangle$, and put $B_j = \sum_{i=1}^{\infty} (\overline{Wv_i, v_j}) v_i$. Then W is continuous on $S_{X,A}$ iff $B_j \in T_{X,A}$, $j \in \mathbb{N}$, with

$$\forall_{t>0} \exists_{s>0} \exists_{C>0} \forall_{j \in \mathbb{N}} : \| B_j(t) \|_X \le C e^{-\lambda_j s} \ .$$

We have $W = \sum_{j=1}^{\infty} B_j \otimes v_j$.

A complete characterization of the rows and columns of the matrices of elements in T^A is quite something, however, a characterization of the entries is much more useful. The following theorem characterizes the entries.

(7.6) Theorem

Let the infinite matrix (L_{ij}) satisfy

(7.7) $$\forall_{t>0} \exists_{s>0} : \sup_{i,j \in \mathbb{N}} (e^{-\lambda_j t} e^{\lambda_i s} | L_{ij} |) < \infty \ .$$

Then L defined by

$$L = \sum_{i,j} L_{ij} v_j \otimes v_i$$

is in $\mathcal{T}^A$, and conversely.

Proof.

$\Rightarrow$) Let $t > 0$. Then there are $s > 0$ and $C > 0$ such that

$$(e^{-\frac{1}{2}\lambda_j t} e^{\frac{3}{2}\lambda_i s} |L_{ij}|) < C, \quad i,j \in \mathbb{N}.$$

This yields the following estimate

$$\| e^{sA} L e^{-tA} \|^2_{X \otimes X} = \sum_{i,j} e^{-2\lambda_j t} e^{2\lambda_i s} |L_{ij}|^2 \leq$$

$$\leq C^2 \sum_{i,j} e^{-\lambda_j t} e^{-\lambda_i s} < \infty.$$

Since $t > 0$ has been taken arbitrarily, the result $L \in \mathcal{T}^A$ follows.

$\Leftarrow$) Let $L \in \mathcal{T}^A$. Then $\forall_{t>0} \exists_{s>0}$:

$$\sup_{i,j} (e^{-\lambda_j t} e^{\lambda_i s} |L_{ij}|) \leq \| e^{sA} \; e^{-tA} \|_{X \otimes X} < \infty$$

where $L_{ij} = (L v_j, v_i)$. □

We shall often employ condition (7.7). It is of great help in the construction of examples and counterexamples. In the sequel, we shall identify the space $\mathcal{T}^A$ with the space $M(\mathcal{T}^A)$ of infinite matrices which satisfy condition (7.7).

The following lemma shows that the product in $\mathcal{T}^A$ corresponds to the matrix product in $M(\mathcal{T}^A)$.

(7.8) <u>Lemma</u>

Let $R, S \in T^A$. Then the matrix of $R \circ S$ is given by

$$(R \circ S)_{ij} = \sum_{\ell=1}^{\infty} R_{i\ell} S_{\ell j} \quad , \quad i,j \in \mathbb{N}$$

where each of the series converges absolutely.

<u>Proof</u>. Let $t > 0$, $i,j \in \mathbb{N}$. Following Theorem (6.6) there are $s, s_0 > 0$ such that

$$S_{\ell j} \leq C_S \, e^{\lambda_j t} e^{-\lambda_\ell s_0}$$

and

$$R_{i\ell} \leq C_R \, e^{\frac{1}{2}\lambda_\ell s_0} \, e^{-\lambda_i s}$$

for some $C_S, C_R > 0$. This leads to the following estimate

$$\left| e^{\lambda_i s} \left(\sum_{\ell=1}^{\infty} R_{i\ell} S_{\ell j} \right) e^{-\lambda_j t} \right| \leq$$

$$\leq \sum_{\ell=1}^{\infty} \left(\left| e^{\lambda_i s} R_{i\ell} \, e^{-\frac{1}{2}\lambda_\ell s_0} \right| \left| e^{\lambda_\ell s_0} S_{\ell j} e^{-\lambda_j t} \right| e^{-\frac{1}{2}\lambda_\ell s_0} \right)$$

$$\leq C_S \, C_R \left(\sum_{\ell=1}^{\infty} e^{-\frac{1}{2}\lambda_\ell s_0} \right) .$$

Thus $\left(\sum_{\ell=1}^{\infty} R_{i\ell} S_{\ell j} \right)$ is an element of $M(T^A)$. Finally we have

$$\sum_{i,j} \left(\sum_{\ell} R_{i\ell} S_{\ell j} \right) v_j \otimes v_i =$$

$$= \sum_{i,j} \left(\sum_{\ell,k} R_{i\ell} S_{kj} (v_k, v_\ell) \right) v_j \otimes v_i$$

$$= \Big(\sum_{i,\ell} R_{i\ell}\; v_\ell \otimes v_i\Big)\cdot\Big(\sum_{j,k} S_{kj}\; v_j \otimes v_k\Big)$$

$$= R \circ S.$$

The conjugation $^c : T^A \to T_A$ induces a conjugation on $M(T^A)$. The precise result is given in the following lemma.

(7.9) Lemma

Let $L \in T^A$. Then $L^c \in T_A$, and

$$L^c = \sum_{i,j} \bar{L}_{ji} (v_j \otimes v_i)$$

where convergence of the series is in T_A.

Proof. From Theorem (7.8) we obtain

$$L(t) = \sum_{i,j} e^{-\lambda_j t} L_{ij} v_j \otimes v_i \ , \ t > 0 \ ,$$

with convergence in $S_{X\otimes X, I\otimes A}$ for each $t > 0$. Hence we find

$$L(t)^* = \sum_{i,j} e^{-\lambda_j t} \bar{L}_{ji}\; v_i \otimes v_j =$$

$$= \sum_{i,j} e^{-\lambda_i t} \bar{L}_{ji}\; v_j \otimes v_i \ , \ t > 0$$

with convergence in $S_{X\otimes X, A\otimes I}$ for each $t > 0$. □

If $L \in T^A$, then the matrix elements $\bar{L}_{ji}$ satisfy $\forall_{t>0} \exists_{s>0}$:

$$\sup_{i,j} \left(e^{-\lambda_i t} e^{\lambda_j s} |\bar{L}_{ji}|\right) < \infty \ .$$

Conversely, if the matrix (Q_{ij}) satisfies, $\forall_{t>0}\exists_{s>0}$:

$$\sup_{i,j} e^{-\lambda_i t} e^{\lambda_j s} |Q_{ij}| < \infty ,$$

then $(\bar{Q}_{ji})$ is the matrix of an elements in T^A.

Thus we arrive at the following theorem.

(7.10) Theorem

Let (Q_{ij}) be an infinite matrix. Then

$$Q = \sum_{i,j} Q_{ij}\; v_j \otimes v_i$$

is an element of T_A iff the matrix elements Q_{ij}, $i,j \in \mathbb{N}$, satisfy

(7.11) $$\forall_{t>0}\exists_{s>0} : \sup_{i,j} (e^{-\lambda_i t} e^{\lambda_j s} |Q_{ij}|) < \infty .$$

We note that $Q_{ij} = \overline{\langle v_i, Qv_j\rangle}$.

As a corollary of Theorem (7.6) and (7.10) we derive the following

(7.12) Corollary

The matrix (E_{ij}) represents an element of E_A if and only if it satisfies the condition (7.7) and (7.11).

In the following section we introduce the class of weighted shift operators. This kind of operators plays an important role in a lot of computations in mathematical physics (cf. the annihilation- and creation operator in a suitable representation). Further, because of their simple structure, the above-mentioned class provides the necessary illustrations of the theory.

8. The class of weighted shifts

For convenience we first introduce a set $\mathcal{D}_A$ of diagonal operators. A diagonal operator D is a linear operator in X which is well-defined on the linear span $\langle v_1, v_2, \ldots \rangle$, and which operates on this span as follows:

$$D v_j = \delta_j v_j, \quad j \in \mathbb{N},$$

with $\delta_j \in \mathbb{C}$. Hence, the matrix of D is diagonal. Following Theorem (7.6), $D \in T^A$ if and only if

$$\forall_{t>0}: \sup_j \left(|\delta_j| e^{-\lambda_j t} \right) < \infty .$$

Hence, D^c is also in T^A, and D is extendable.

(8.1) Definition

$\mathcal{D}_A \subset E_A$ denotes the set of diagonal operators D in X which satisfy

$$\forall_{t>0}: \sup_{j \in \mathbb{N}} |\delta_j| e^{-\lambda_j t} < \infty$$

where δ_j, $j \in \mathbb{N}$, are the diagonal entries of the matrix of D.

This section contains a first investigation of the special class of elements of T^A established by the weighted shift operators or, shortly, weighted shifts. A weighted shift W is a linear operator in X which is well defined on the linear span $\langle v_1, v_2, \ldots \rangle$, and which operates as follows

$$W v_j = \omega_j v_{j+1}, \quad j \in \mathbb{N},$$

with $\omega_j \in \mathbb{C}$, $j \in \mathbb{N}$. Hence, W is uniquely determined by its matrix with respect to the basis (v_j) given by

$$W_{ij} = \omega_j \delta_{i,j+1} \ , \ i,j \in \mathbb{N} ,$$

where $\delta_{\ell,k}$ denotes Kronecker's delta. Then following Theorem (7.6) the linear mapping $W \in T^A$ if and only if

$$\forall_{t>0} \exists_{s>0}: \sup_j \left(|\omega_j| e^{-\lambda_j t} e^{\lambda_{j+1} s} \right) < \infty \tag{8.2}$$

and $W^c \in T^A$ if and only if

$$\forall_{t>0} \exists_{s>0}: \sup_{j>1} \left(|\omega_{j-1}| e^{-\lambda_j t} e^{\lambda_{j-1} s} \right) < \infty .$$

Since $\lambda_{j-1} \le \lambda_j$ it is clear that continuity of W implies continuity of W^c. Hence, a continuous weighted shift is extendable.

Condition (8.2) can be rewritten into

$$\forall_{t>0} \exists_{s>0}: \sup_{j \in \mathbb{N}} |\alpha_j| \exp\left\{ -\lambda_j t \left(1 - \frac{\lambda_{j+1}}{\lambda_j} \right) \right\} < \infty .$$

In the remaining part of this section we impose the following condition on the eigenvalues of A.

$$\exists_M \forall_{j \in \mathbb{N}} : \frac{\lambda_{j+1}}{\lambda_j} \le M . \tag{8.3}$$

This condition is not very severe; they imply the following order estimate, $\lambda_j = O(M^j)$. Less severe conditions restrict the number of weigted shifts in T^A. If condition (8.3) is dropped, then $\frac{\lambda_{j+1}}{\lambda_j} \to \infty$, $j \to \infty$. Let U be the unilateral shift given by $U v_j = v_{j+1}$, $j \in \mathbb{N}$. So U is a bounded operator on X. Suppose $U \in T^A$. Then there should be $s > 0$ such that

$$\sup_{j\in\mathbb{N}} (e^{\lambda_{j+1}\tau-\lambda_j}) = \sup_{j\in\mathbb{N}} e^{\lambda_{j+1}\left(\tau-\frac{\lambda_j}{\lambda_{j+1}}\right)} < \infty .$$

Since $\lambda_j \to \infty$ and $\frac{\lambda_j}{\lambda_{j+1}} \to 0$, the assumption $U \in T^A$ yields a contradiction. Hence $U \notin T^A$. If the eigenvalues λ_j do not satisfy condition (8.3), there only occur Hilbert-Schmidt operators in E_A.

Because of condition (8.3) it follows that (8.2) reduces to

(8.4) $$\forall_{t>0} \sup_{j\in\mathbb{N}} (|\omega_j| e^{-\lambda_j t}) < \infty .$$

So the following characterization is an immediate consequence of Definition (8.1) and (8.4).

(8.5) Characterization

Let W be a weighted shift. Then $W \in T^A$ iff there exists a $D \in \mathcal{D}_A$ such that $W = UD$.

The following definition generalizes the notion of weighted shifts.

(8.6) Definition

A linear operator $W^{(n)}$ in X is called a weighted n-shift, $n \in \mathbb{N} \cup \{0\}$ if $W^{(n)}$ satisfies

$$W^{(n)} v_j = \omega_j^{(n)} v_{j+n} , \quad n \in \mathbb{N}$$

with $\omega_j^{(n)} \in \mathbb{C}$.

Hence, a weighted 0-shift is a diagonal operator, a weighted 1-shift is an ordinary weighted shift. Let $W^{(n)}$ be a weighted n-shift with weight sequence $(\gamma_j^{(n)})$. Then $W^{(n)} \in T^A$ if and only if

(8.7) $$\forall_{t>0}\exists_{s>0}: \sup_{j\in\mathbb{N}} (|\gamma_j^{(n)}| e^{-\lambda_j t} e^{\lambda_{j+n} s}) < \infty .$$

Because of (8.3) there exists M > 0 such that

$$\frac{\lambda_{j+n}}{\lambda_j} \leq M^n , \quad j \in \mathbb{N} .$$

So (8.7) is equivalent to

(8.8) $$\forall_{t>0}: \sup_{j\in\mathbb{N}} (|\gamma_j^{(n)}| e^{-\lambda_j t}) < \infty .$$

This yields the following characterization.

(8.9) Characterization

Let $W^{(n)}$ be a weighted n-shift, $n \in \mathbb{N} \cup \{0\}$. Then $W^{(n)} \in T^A$ iff there exists $D \in \mathcal{D}_A$ such that $W^{(n)} = U^n D$.

Since $U \in E_A$ and $D \in E_A$ for all $D \in \mathcal{D}_A$, from (8.9) we derive that every weigthed n-shift, $n \in \mathbb{N} \cup \{0\}$, is extendable.

(8.10) Definition

The operator $W^{(-n)}$, $n \in \mathbb{N}$, is called a weighted (-n)-shift if

$$W^{(-n)} v_j = \omega_{j-n}^{(-n)} v_{j-n} , \quad j > n, \ j \in \mathbb{N}$$

with $\omega_j^{(-n)} \in \mathbb{C}$.

If the linear mapping $W^{(-n)} \in T^A$ then it satisfies

$$\forall_{t>0}\exists_{s>0}: \sup_{\substack{j\in\mathbb{N}\\ j>n}} (|\omega_{j-n}^{(-n)}| e^{-\lambda_j t} e^{\lambda_{j-n} s}) < \infty ,$$

or equivalently

(8.11) $$\forall_{t>0}: \sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_{j+n} t}) < \infty ,$$

since $\lambda_{j-n} < \lambda_j$ for $j \geq n$, $j \in \mathbb{N}$. The latter condition is equivalent to

(8.12) $$\forall_{t>0}: \sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_j t}) < \infty .$$

The implication (8.12) ⇒ (8.11) is trivial. In order to prove that (8.11) implies (8.12), let $t > 0$. Then

$$\sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_j t}) = \sup_{j\in\mathbb{N}} \left(|\omega_j^{(-n)}| e^{-(\lambda_j/\lambda_{j+1} \cdots \lambda_{j+n-1}/\lambda_{j+n})\lambda_{j+n} t}\right)$$

$$\leq \sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_{j+n} t M^{-n}}) < \infty ,$$

with $M > 0$ such that $\frac{\lambda_{j+1}}{\lambda_j} < M$, $j \in \mathbb{N}$.
So similar to (8.9) the weighted (-n)-shifts in $\mathcal{T}^A$ are characterized by

(8.13) <u>Characterization</u>

Let $W^{(-n)}$ be a weighted (-n)-shift. Then $W^{(-n)} \in \mathcal{T}^A$ iff there exists $D \in \mathcal{D}_A$ such that $W^{(-n)} = D(U^*)^n$.

Since U^* and $D \in \mathcal{D}_A$ both are extendable, each $W^{(-n)}$ is extendable. Further, the product $W^{(k_1)}W^{(k_2)}$ with $k_1, k_2 \in \mathbb{Z}$ is a weighted (k_1+k_2)-shift and the conjugate $(W^{(k_1)c})$ is a $(-k_1)$-shift. So the weighted k-shifts $k \in \mathbb{Z}$, establish an involutive semi-group in E_A.

The weighted k-shifts, $k \in \mathbb{Z}$, span the algebra T^A in a very special way.

(8.14) Theorem

Let $L \in T^A$ with matrix (L_{ij}). Define the weighted k-shifts $W^{(k)}$ by

$$W^{(k)} v_j = L_{j+k,j} \; v_j \;, \; j > \max\{0,-k\}, \quad j \in \mathbb{N},$$

where $k \in \mathbb{Z}$. Then $W^{(k)} \in E_A$ and $\sum_{k \in \mathbb{Z}} W^{(k)}$ represents L. This series converges absolutely.

Proof. The eigenvalues λ_j of A satisfy the following estimates

For $n \in \mathbb{N} \cup \{0\}$,

$$(*) \qquad e^{\lambda_{j+n} s} \le e^{-\lambda_n (s_0 - s)} e^{\lambda_{j+n} s_0}$$

with $j \in \mathbb{N}$, $s_0 > 0$, and $0 < s < s_0$. For $n \in \mathbb{N}$,

$$(**) \qquad e^{-\lambda_j t} \le e^{-\lambda_n (t - t_0)} e^{-\lambda_j t_0}$$

with $j \in \mathbb{N}$, $j > n$, $t_0 > 0$ and $t > t_0$.

First note that it is obvious that each $W^{(k)}$, $k \in \mathbb{Z}$, is continuous and hence extendable (cf.(8.9) and (8.13)). So we only prove the second assertion. Let $t > 0$. Then there exists $s > 0$ such that

$$\| e^{2sA} L e^{-\frac{1}{2} tA} \|_{X \otimes X} < \infty.$$

For $n \in \mathbb{N} \cup \{0\}$ by $(*)$ we have

$$\| e^{sA}{}_W(n) e^{-tA} \|_{X\otimes X} \leq e^{-\lambda_n s}\Big(\sum_{j=1}^{\infty} |e^{2s\lambda_{n+j}} L_{n+j,j} e^{-t\lambda_j}|^2\Big)^{1/2}$$

$$\leq e^{-\lambda_n s} \| e^{2sA} L e^{-\frac{1}{2}tA} \|_{X\otimes X} .$$

For $n \in \mathbb{N}$ by (**) we have

$$e^{sA}{}_W(-n) e^{-tA} \;{}_{X\otimes X} \leq e^{-\frac{1}{2}\lambda_n t}\Big(\sum_{j=n+1}^{\infty} |e^{s\lambda_{j-n}} L_{j-n,j} e^{-\frac{1}{2}t\lambda_j}|^2\Big)^{1/2}$$

$$\leq e^{-\frac{1}{2}\lambda_n t} \| e^{2sA} \; e^{-\frac{1}{2}tA} \|_{X\otimes X} .$$

A combination of the above results yields for all $N_1, N_2 \in \mathbb{N}$

$$\sum_{k=-N_1}^{N_2} \| e^{sA}{}_W(k) e^{-tA} \|_{X\otimes X} \leq$$

$$\leq \| e^{2sA} L e^{-\frac{1}{2}tA} \|_{X\otimes X}\Big(\sum_{n=1}^{N_1} e^{-\frac{1}{2}\lambda_n t} + \sum_{n=0}^{N_2} e^{-\lambda_n s}\Big)$$

Hence, the series $\sum_{k\in\mathbb{Z}} e^{sA}{}_W(k) e^{-tA}$ converges absolutely in $X\otimes X$.
Since $X\otimes X$ is a Hilbert space absolute convergence implies convergence and therefore

$$e^{sA} L e^{-tA} = \sum_{k\in\mathbb{Z}} e^{sA}{}_W(k) e^{-tA}.$$

Thus we have proved the second assertion. □

Since all weighted k-shifts, $k \in \mathbb{Z}$, are extendable, the following corollary is immediate.

(8.15) Corollary

The space T^A in Theorem (8.14) can be replaced by T_A.

For the weighted k-shifts $W^{(k)}$ spectral properties can be discussed in detail and eigenvectors in $T_{X,A}$ and $S_{X,A}$ can be constructed. This may be a subject for further investigation.

9. Construction of an analyticity space $S_{X,A}$ for some given operators in X

Given a finite number of linear operators in a Hilbert space X, the question arises whether there can be constructed nuclear analyticity spaces on which these operators are continuous linear mappings. In this section we shall show that for a finite number of bounded operators on X, resp. for a finite number of commuting self-adjoint operators in X, such a construction is indeed possible. The proof of the results of this section is closely related to the theory on matrices of elements in T^A (cf. Section 7).

Let P be a bounded, self-adjoint operator on X. Following [Ha], p.201, P can be represented by a Jacobi matrix, i.e. there exists an orthonormal basis (e_r) in X such that the matrix of P satisfies

$$(Pe_r,e_j) = 0 \text{ if } |r-j| < 1 \ , \ r,j \in \mathbb{N} \, ;$$

If we define the positive self-adjoint operator A in X by

$$Ae_j = je_j \ , \ j \in \mathbb{N} \, ,$$

followed by linear and unique self-adjoint extension, then we have the following result.

(9.1) Lemma

The self-adjoint operator P is an element of T_A.

Proof. Following Theorem (7.6) we have to show

$$\forall_{t>0}\exists_{s>0}: \sup_{,j} (e^{-jt}e^{rs}|(Pe_j,e_r)|) < \infty.$$

Let $t > 0$, and let $0 < s < t$. Then

$$\sup_{r,j} e^{-jt}e^{rs} \; |(Pe_j,e_r)| \leq \|P\|e^{-jt+(j+1)s} < e^s\|P\| ,$$

where $\|P\|$ denotes the norm of P in $B(X)$ □

With the aid of Lemma (9.1) the more general case of an unbounded self-adjoint operator T can be solved. To this end let $(F_\lambda)_{\lambda\in\mathbb{R}}$ denote the spectral resolution of the identity for T and Π_ℓ, $\ell \in \mathbb{N}$, the spectral projection

$$\Pi_\ell = \left(\int_{\ell-1}^{\ell} + \int_{-\ell}^{-\ell+1}\right)dF_\lambda \; .$$

Then X is decomposed into

$$X = \bigoplus_{i=1}^{\infty} \Pi_\ell(X)$$

where in each invariant subspace $\Pi_\ell(X)$ the estimate

$$\|Tf_\ell\| \leq \ell\|f_\ell\| \; , \; f_\ell \in \Pi_\ell(X) \; ,$$

holds true. So if we put $T_\ell = \Pi_\ell T\Pi_\ell$, then T_ℓ is bounded on X , and there exists an orthonormal basis $(e_j^{(\ell)})$ such that $\left(\left(T_\ell e_j^{(\ell)},e_r^{(\ell)}\right)\right)$ is a Jacobi matrix.

Define the positive self-adjoint operator A by

$$Ae_j^{(\ell)} = (j+\ell)e_j^{(\ell)}, \; j \in \mathbb{N}, \; \ell \in \mathbb{N}$$

followed by linear and unique self-adjoint extension. Then the eigenvalues of A are the numbers $\lambda_n = n+1$ with multiplicity n, $n \in \mathbb{N}$.
So all the operators e^{-tA}, $t > 0$, are Hilbert-Schmidt and the analiticity space $S_{X,A}$ is nuclear.
Put $f_j^{(n)} = e_j^{(n+1-j)}$, $j = 1,\dots,n$. Then the vectors $f_j^{(n)}$ are the eigenvectors of A with eigenvalue λ_n. Enumerating the $f_j^{(n)}$'s in the usual way, we have constructed a complete orthonormal basis (g_k) for X, which yields the following theorem.

(9.2) Theorem

The operator T maps $S_{X,A}$ continuously into itself.

Proof. Let $t > 0$, and let $0 < s < t$. Then

$$\sup_{\ell,k} |(e^{sA}Te^{-tA}g_\ell,g_k)| =$$

$$= \sup_{r,n}\sup_{j,m} \left\{e^{(r+n)s}e^{-(j+m)t}|(Te_j^{(m)},e_r^{(n)})|\right\} =$$

$$= \sup_m (e^{-m(t-s)}\sup_{r,j} (e^{rs}e^{-jt}|(T_m e_j^{(m)},e_r^{(m)})|)) \le$$

$$\le \sup_m (me^{-(t-s)}) \sup_{|r-j|\le 1} (e^{rs}e^{-jt}) < \infty \; .$$ □

In order to establish a similar result for N bounded operators $B_1,B_2,\dots,B_N$ on X, we shall construct an orthonormal basis in X such that the matrix of each B_ν, $\nu = 1,\dots,N$, is column finite, i.e. for every $j \in \mathbb{N}$ there exists $r_0 \in \mathbb{N}$ such that

$$(B_\nu)_{rj} = 0 \text{ for } r > r_0.$$

To this end, let (f_r) be an orthonormal basis in X. Put $e_1 = f_1$. There exists an orthonormal set $\{e_2, e_3, \dots, e_{k_1}\} \perp \{e_1\}$ with $k_1 \le (n+1) + 1$, such that

$$B_\nu e_1 \in \langle e_1, \dots, e_{k_1} \rangle, \quad \nu = 1,\dots,N$$

and

$$f_2 \in \langle e_1, \dots, e_{k_1} \rangle .$$

Similarly, there exists an orthonormal set $\{e_{k_1+1}, \dots, e_{k_2}\} \perp \{e_1, \dots, e_{k_1}\}$, $k_2 \le 2(n+1) + 1$, such that

$$B_\nu e_2 \in \langle e_1, \dots, e_{k_2} \rangle , \quad \nu = 1,\dots,N$$

and

$$f_3 \in \langle e_1, \dots, e_{k_2} \rangle.$$

Continuing in this way we derive sets $\{e_{k_{\ell-1}+1}, \dots, e_{k_\ell}\}$ with $k_\ell \le \ell(n+1) + 1$ and with $\{e_{k_{\ell-1}+1}, \dots, e_{k_\ell}\} \perp \{e_1, \dots, e_{k_{\ell-1}}\}$ such that

$$B_\nu e_\ell \in \langle e_1, \dots, e_{k_\ell} \rangle , \quad \nu = 1,\dots,N$$

and

$$f_{\ell+1} \in \langle e_1, \dots, e_{k_\ell} \rangle .$$

Thus we obtain an orthonormal basis (e_r) in X. This basis is complete because $f_\ell \in \langle e_1, e_2, \ldots, e_{k_{\ell+1}} \rangle$, $\ell \in \mathbb{N}$. The matrix of each B_ν, $1 \leq \nu \leq N$, is column finite, because

$$(B_\nu e_j, e_r) = 0 \text{ if } r > j(N+1) + 1.$$

Now define the positive self-adjoint operator A by

$$Ae_j = je_j \ , \ j \in \mathbb{N} \ ,$$

followed by linear and unique self-adjoint extension. Then

(9.3) Theorem

The linear operators $B_1, \ldots, B_N$ map the nuclear analyticity space $S_{X,A}$ continuously into itself.

Proof. Let $\nu \in \{1, \ldots, N\}$, and let $t > 0$, $s > 0$ with $0 < s < \frac{t}{N+1}$. Then

$$\sup_{r,j} |(B_\nu e_j, e_r)| e^{-jt} e^{rs} =$$

$$= \sup_{1 \leq r \leq j(n+1)+1} (|(B_\nu e_j, e_r)| e^{-jt} e^{rs}) \leq$$

$$\leq \|B_\nu\| e^s \sup_{j \in \mathbb{N}} e^{-j(t-(N+1)s)} \leq e^s \|B_\nu\| \ . \qquad \square$$

With the aid of Theorem (9.3) we can extend the result of Theorem (9.2) to hold true for a finite number of commuting self-adjoint operators in X. Let $T_1, T_2, \ldots, T_N$ be N commuting self-adjoint operators in X with resolutions of identity $(F_\lambda^{(\nu)})$, $\nu = 1, \ldots, N$. So their spectral projections commute, i.e. $F^{(\nu)}(\Delta_\nu) F^{(\mu)}(\Delta_\mu) = F^{(\mu)}(\Delta_\mu) F^{(\nu)}(\Delta_\nu)$ where Δ_ν, Δ_μ denote Borel sets in $\mathbb{R}$. Let Π_ℓ, $\ell \in \mathbb{N}^N$, denote the projection

$$\Pi_\ell = F^{(1)}(\ell_1 - 1 \le |\lambda| < \ell_1) \circ\circ\circ F^{(N)}(\ell_N - 1 \le |\lambda| < \ell_N) .$$

Then for all $f_\ell \in \Pi_\ell(X)$, $T_\nu f_\ell \in \Pi_\ell(X)$ and $\|T_\nu f_\ell\| \le \ell_\nu \|f_\ell\|$.

Further, $X = \bigoplus_{\ell \in \mathbb{N}^N} \Pi_\ell(X)$.

Since each operator $T_\nu|\Pi_\ell(X)$ is bounded, there exists an orthormal basis $(e_j^{(\ell)})$ in $\Pi_\ell(X)$ such that for all $\nu = 1,\dots,N$,

$$(T_\nu e_j^{(\ell)}, e_r^{(\ell)}) = 0 \text{ if } r > j(N+1) + 1.$$

Define the positive,self-adjoint operator A in X by

$$A e_j^{(\ell)} = (j + |\ell|) e_j^{(\ell)} \text{ , } j \in \mathbb{N} \text{ , } \ell \in \mathbb{N}^N \text{ ,}$$

followed by the usual extensions (Note that $|\ell| = \ell_1 + \dots + \ell_N$). Then the eigenvalues of A are the numbers $\lambda_p = N+p$, $p \in \mathbb{N}$, with multiplicity $\binom{N+p-1}{N}$. Hence, the analyticity space $S_{X,A}$ is nuclear.

Renumerating the orthonormal basis $(e_j^{(\ell)})$ yields an orthonormal basis $(g_n)_{n \in \mathbb{N}}$ for X. We have

(9.4) <u>Theorem</u>

Each of the operators T_ν, $\nu = 1,\dots,N$ is a continuous linear mapping from $S_{X,A}$ into itself.

<u>Proof.</u> Let $\nu = 1,\dots,N$, and let $0 < s < \frac{t}{N+1}$. Then

$$\sup_{n,m} |(e^{sA} T_\nu e^{-tA} g_m, g_n)| =$$

$$= \sup_{r,j\in\mathbb{N}} \sup_{k,\ell\in\mathbb{N}^N}\left(e^{-(|\ell|+j)t}e^{(|k|+r)s}\left|\left(T_\nu e_j^{(\ell)}, e_r^{(k)}\right)\right|\right) \leq$$

$$\leq e^r \sup_{\ell\in\mathbb{N}^n}\left(\ell_\nu e^{-|\ell|(t-s)}\right) \sup_{j\in\mathbb{N}}\left(e^{-j(t-(N+1)s)}\right) < \infty . \qquad \square$$

ACKNOWLEDGMENT

I wish to thank prof. J. de Graaf for inspiring discussions, helpful suggestions and critical reading of the manuscript.

REFERENCES

[Al] Allan, G.R.,On a class of locally convex algebras, Proc. London Math. Soc. 17 (1967), p. 91-114.

[Ch] Choquet, G., Lectures on analysis, Vol.II, W.A. Benjamin Inc., New York, 1969.

[Di] Dirac, P.A.M., The principles of quantum mechanics, 1958, Clarendon Press, Oxford.

[E_3] Eijndhoven, S.J.L. van, Generalized eigenfunctions with applications to Dirac's formalism, EUT-Report 82-WSK-03, Eindhoven University of Technology, 1982.

[Fa] Faris, W.G., Self-adjoint operators, Lect. notes in mathematics, Springer,Berlin, 1974, no. 433

[G] Graaf, J.de, A theory of generalized functions based on holomorphic semigroups. TH-Report 79-WSK-02, Eindhoven University of Technology, 1979.

[GE] Graaf, J.de, Eijndhoven, S.J.L. van, Analyticity spaces, trajectory spaces and linear mappings between them, Memorandum 1982-09, Eindhoven University of Technology, 1982, Preprint.

[Ha] Halmos, P.R., A Hilbert space problem book, Springer, New York, 1974.

[Mi] Miller, W., Symmetry and separation of variables, Addison-Wesley, Massachusetts, 1977.

[Ne] Nelson, E., Analytic vectors, Ann. Math, 70 (1959), p. 572-615.

[Pij] Pijls, H.G.J., Locally convex algebras in spectral theory and eigenfunction expansions, Mathematical centre tracts 66, Amsterdam, 1976.

[W] Weidmann, J., Linear operators in Hilbert spaces, G.T.M., vol 68, Springer, New York, 1980.

[Tr] Trêves, F., Topological vector spaces, distribution and kernels, Academic Press, New York, 1967.