Determination of the quasi-static behaviour of viscoelastic materials and spectral analysis

DETERMINATION OF THE QUASI-STATIC BEHAVIOUR OF VISCOELASTIC MATERIALS AND SPECTRAL ANALYSIS

With an Introduction to Linear Theory of Viscoelasticity

Cumhur Alhan

Eindhoven University Of Technology
Department Of Fundamental Mechanics

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In the course of my student life on the Eindhoven University of Technology, what struck me most was a mode of reasoning, which essentially being absurd, nevertheless worked out fruitfully.
This absurd reasoning is perhaps at its best described by Fyodor Dostoevsky:
If Stavrogin believes, he does not think he believes.
If he does not believe, he does not think he does not believe.

an excerpt from 'The Possessed'.

CONTENTS

INTRODUCTION

The mathematical background of the theories of viscoelasticity is the same as that of the linear integral theory which applies to discrete systems and all hereditary phenomena in general. In this respect, linear viscoelasticity forms the mechanical analogon of the linear network theory, while the description of the viscoelastic effects may equally well apply to the description of such hereditary phenomena as the dielectric behaviour and the magnetic after-effect. Mechanically, the integral theory of viscoelasticity may be conceived as a generalization of the classical theory of elasticity, for which the interest centers primarily on the time-dependent behaviour of materials. In the present paper we shall discuss the linear theory of viscoelasticity and the characterization of the time-invariant viscoelastic effects in the short end of the time scale. In principle, by using the linear theory the same information on the viscoelastic effects can be obtained from any time-dependent loading pattern - such as stress-relaxation or creep under resp. homogeneous strain and stress, quasi-static tests in which the stress and strain vary periodically with time; or other types of tests in which the stress or strain is increased at a constant rate, or even a random stress or

strain is applied. Through the transformation equations given in Chapter 1, the results of one type of experiment can be translated into those of another. It is however important to realize that the range of these experiments varies considerably, depending on the type of the test. The quasi-static experiments yield information about the viscoelastic effects in between 10^{-2} - 10^{-6} seconds while the range of the static tests extends from a few seconds up to 40 years. Partly due to this and partly due to the eventual shortcomings of the theory, it is found that the results from various types of experiments do not agree well with another. Keeping this in mind, we will confine our attention mainly on the description of the short-time behaviour, which we will assume to be time-invariant. Obviously, the mechanical properties need not necessarily be time-invariant at long times- as is the case of aging materials- and a more general theory accounting for this fact is discussed elsewhere [1].

In Chapter 2 we will briefly discuss the characterization of viscoelastic effects and the type of experiments needed to determine them. In the 'applications' which form the remaining part of the present paper we will discuss the possible combination of the most common method to determine the quasi-static behaviour - the so-called direct method - with the digital signal analysis techniques. In this way information on the frequency composition of

the material properties can be obtained for a whole range of frequencies at once, that is without having to reproduce the experiment for each frequency of interest. The computation of data can further be utilized through the application of fast Fourier transforms (FFT). Also, digital data can be used to calculate the response of a given material to any type of stress or strain history. Chapter 3 is concerned with the discussion of these techniques; the theoretical background given there is indispensable for the use of these techniques in performing the required experiments. Finally in Chapter 4 we will discuss the possible applications of the above mentioned method to non-linear characterization.

Throughout the text we will simply ignore the temperature dependence of the viscoelastic effects. This, however, is not an essential simplification, since by virtue of the time-temperature superposition principle the data obtained at a given temperature can be transformed to another temperature by a simple transformation of the time-scale [2]-[7]. To some extent, it is also possible to transform the results from static tests to both shorter and longer times by the aid of this principle. The theory given here takes also no notice of such microscopic effects as the molecular constitution, average molecular weight, anisotrophy and inhomogenities. Some of the references given at the end of this paper discuss the influence of microscopic effects to the mechanical behaviour, in particular for high polymeric materials and soft biological tissues.

Here emphasis is laid on providing a theoretical basis for the practical use of the theory, rather than extending the discussion to the evaluation of the eventual success or failure of the theory in predicting the short-time viscoelastic behaviour of real viscoelastic materials. We suggest that this be the subject of future investigations.

I

LINEAR THEORY OF VISCOELASTICITY

1. BOLTZMANN SUPERPOSITION PRINCIPLE

Generally speaking, a material is said to be linearly viscoelastic when

$$\sigma\{\epsilon_1(t) + \epsilon_2(t,\tau)\} = \sigma\{\epsilon_1(t)\} + \sigma\{\epsilon_2(t,\tau)\} \qquad \tau \leq t$$

or in other words, when the stress output due to two arbitrary strain inputs superposed upon one another, i.e. $\sigma\{\epsilon_1(t) + \epsilon_2(t,\tau) \mid \tau \leq t\}$, is equal to the sum of the stress outputs resulting from the strain inputs ϵ_1 and ϵ_2 each acting separately. If, further, the mechanical properties of a given viscoelastic material are presupposed to be time-invariant, then the above requirement can be reformulated as

$$\sigma\{\epsilon_1(t) + \epsilon_2(t-\tau)\} = \sigma\{\epsilon_1(t)\} + \sigma\{\epsilon_2(t-\tau)\} \qquad (1.1)$$

where $\tau \leq t$. This latter requirement, illustrated in Fig. 1.1. below, is called the Boltzmann superposition principle after Boltzmann [8], who first formulated the principle of heredity. Denoting the response function for a step input of strain by $G(t)$, one has for the first input of strain $\epsilon_1 \cdot U(t)$ (Fig. 1.1)

$$\sigma_1(t) = \epsilon_1 G(t) \cdot U(t)$$

i.e. the stress is proportional to the applied strain due to the assumed linearity. In the same way, the response to the second step-change of strain is obtained as

$$\sigma_2(t) = \epsilon_2\, G(t-t_1)\, U(t-t_1)$$

According to the Boltzmann superposition principle, the response to the combination of these two strain inputs is equal to the sum of the responses given above.

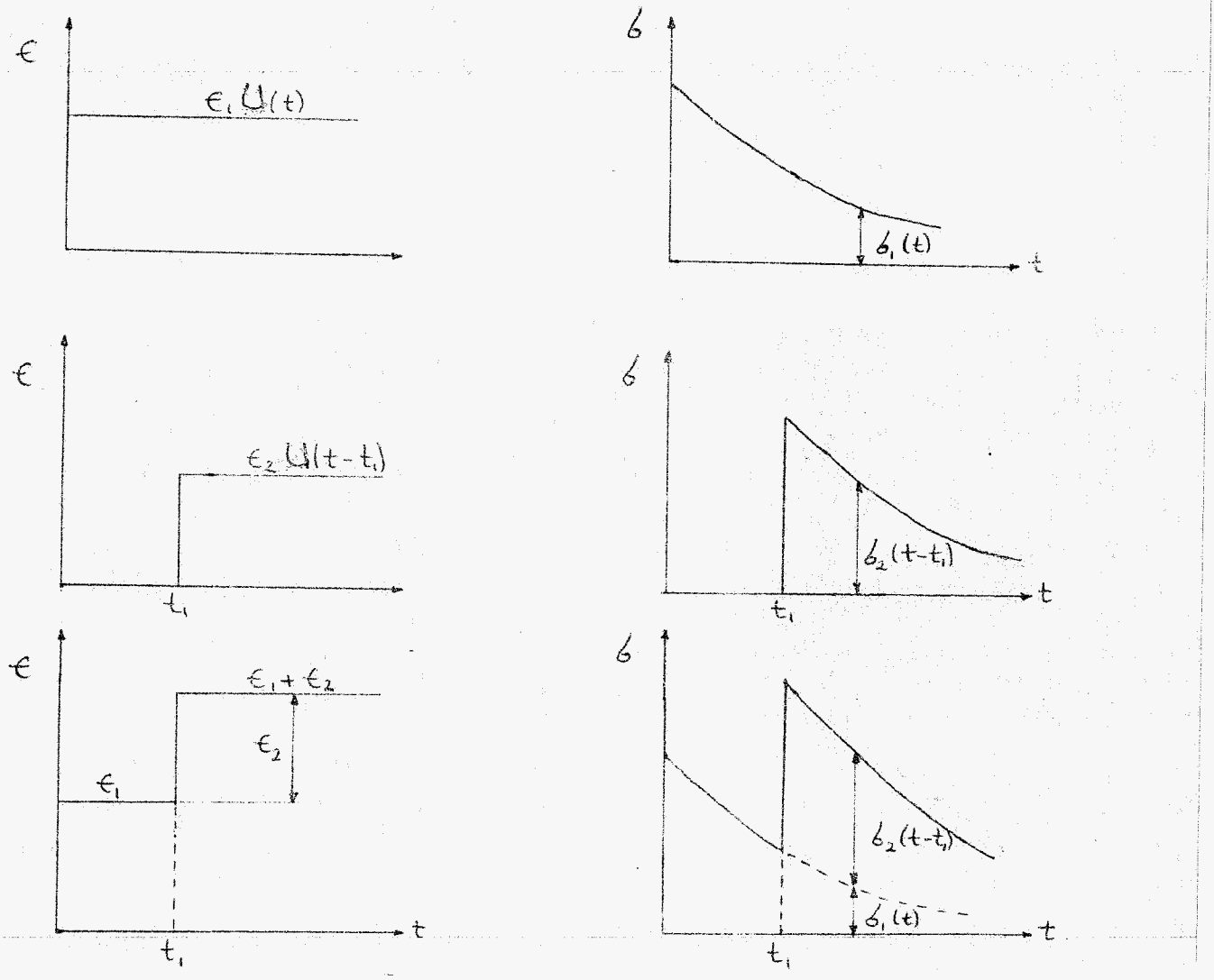

Fig. 1.1 Boltzmann superposition principle

Let us now investigate the situation for an arbitrary strain input $\epsilon(t)$. As Fig. 1.2 shows, an arbitrary strain function can be approximated by

the sum of N step-inputs of strain, i.e.

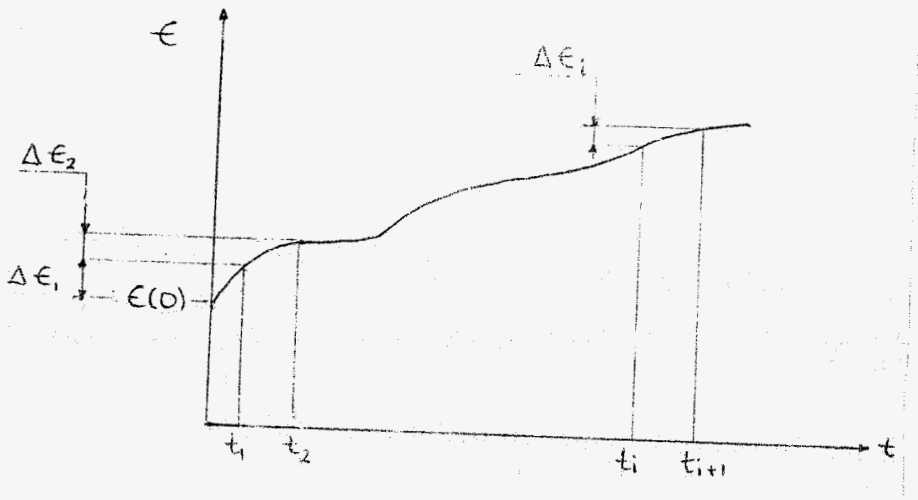

Fig. 1.2 Approximation of a variable strain function by a sum of step-inputs of strain

$$\epsilon(t) = \epsilon(0) + \sum_{i=1}^{N} \Delta\epsilon_i \, U(t-t_i)$$

where $t_{i+1} > t_i$. The stress response to this input can be obtained from (1) as

$$\sigma(t) = \epsilon(0)\, G(t) + \sum_{i=1}^{N} G(t-t_i)\, U(t-t_i)\, \Delta\epsilon_i$$

When the number of steps N tends to infinity, replacing t_i by τ and writing for the strain increment $d\epsilon$ from time τ to $\tau+d\tau$ $\{d\epsilon(\tau)/d\tau\}_{t=\tau} \cdot d\tau$ we obtain

$$\sigma(t) = \epsilon(0)\, G(t) + \int_0^t G(t-\tau)\, \frac{d\epsilon(\tau)}{d\tau}\, d\tau$$

$$= \int_{0^-}^{t} G(t-\tau)\, \frac{d\epsilon(\tau)}{d\tau}\, d\tau$$

In the last step the term $\epsilon(0)\,G(t)$ is included in the integral, thereby shifting the lower limit of integration from 0^+ to 0^-, since we have at this point of discontinuity $\epsilon'(0^+) = \epsilon(0^+)\,\delta(\tau)$. More generally, put the lower limit $-\infty$, then

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)\,\frac{d\epsilon(\tau)}{d\tau}\,d\tau \qquad (1.2)$$

Interchanging the roles of σ and ϵ we could also obtain a similar relation for strain in terms of applied stress

$$\epsilon(t) = \int_{-\infty}^{t} J(t-\tau)\,\frac{d\sigma(\tau)}{d\tau}\,d\tau \qquad (1.3)$$

The material functions $G(t)$ and $J(t)$ in the above relations are respectively called the 'relaxation function' and the 'creep function'. As it is evident from the foregoing discussion, these functions describe the viscoelastic effects under constant strain and stress. In the following we shall sometimes refer to them briefly as the static material functions.

The above way of deriving the hereditary laws of viscoelasticity given by the equations (1.2) and (1.3) is far from being rigorous. Still, this method is often preferred in the literature to make it plausible that the stress at a given time is the sum of the responses to the increments of strain over the whole strain history or vice versa. The Boltzmann superposition principle is presented in the literature as a heuristic law obtained by

analysing the response of mechanical models consisting of linear springs and dashpots, which result into relaxation and creep functions expressed as a finite sum of exponential functions. (When the number of springs and dashpots in these models increase indefinitely we obtain an integral law, but one for which the relaxation and creep functions - or more precisely, the kernels - can in the limit be expressed as Laplace integrals. More general relaxation and creep functions which are not necessarily finite or infinite sums of exponential functions should therefore be reserved for the integral theories of viscoelasticity which contain the phenomenological theory as a special case.

By many authors the Boltzmann superposition principle is accepted as the principle of the linearity of the viscoelastic effects, although Volterra's [9] formulation of the linear hereditary elasticity allows a more general analysis of the linear viscoelastic effects which are not necessarily time-invariant. In the following we shall confine ourselves to the time-invariant hereditary laws given by eq:s (1.2) and (1.3), and it is immaterial to our discussion whether one names it the principle of hereditary elasticity or the Boltzmann superposition principle. The more general linear integral theory

$$\sigma(t) = A\epsilon(t) - \int_a^t g(t,\tau)\,\epsilon(\tau)\,d\tau$$

where the kernel $g(t,\tau)$ can be decomposed into two

parts describing the non-superposable static and quasi-static viscoelastic effects is discussed elsewhere [1].

Perhaps it is useful to make a remark about the terminology employed here. By the term 'quasi-static' we mean that the inertia effects of the material particles are systematically neglected. Also, when we say linear theory we mean explicitly that our interest is primarily aimed at the description of the short-time behaviour of viscoelastic materials (in the order of at most some minutes). The practical results suggest that the complex nature of the viscoelastic effects often requires a separate analysis for various types of loading and deformation histories. A systematical empirical analysis will reveal whether the linear viscoelasticity theory can satisfactorily predict the behaviour of a given material at sufficiently long times.

2. RELAXATION AND CREEP KERNELS

Since the relaxation and creep functions describe the step-response resp. to a step-input of strain and stress, they should be related to one another. Their relation is obtained by applying the Laplace transform to eq.'s (1.2) and (1.3). We obtain thus in the Laplace domain

$$\sigma(s) = s\, G(s)\, \varepsilon(s)$$
$$\varepsilon(s) = s\, J(s)\, \sigma(s)$$

Eliminating σ and ε in the above equations yields

$$G(s)\, J(s) = \frac{1}{s^2} \tag{1.4}$$

An expression in the time domain can be obtained by evaluating the inverse Laplace transform of (1.6). The result is

$$\int_{-\infty}^{t} G(t-\tau)\, J(\tau)\, d\tau = t \quad . \tag{1.5}$$

It is very useful in the theory of viscoelasticity to work with the relaxation and creep kernels defined by the relations

$$g(t) = -\frac{dG}{dt} \tag{1.6}$$

$$h(t) = \frac{dJ}{dt} \tag{1.7}$$

The above definitions suggest that the kernel functions are the equivalent of impulse response functions in the theory of linear systems. Substituting eq.'s (1.6) and (1.7) in (1.2) and (1.3) respectively, and with the initial conditions $\sigma(t) = \epsilon(t) = 0$ for $t<0$ we arrive at the alternative formulations of the hereditary laws

$$\sigma(t) = A\epsilon(t) - \int_0^t g(t-\tau)\epsilon(\tau)d\tau \qquad (1.8)$$

$$\epsilon(t) = B\sigma(t) + \int_0^t h(t-\tau)\sigma(\tau)d\tau \qquad (1.9)$$

While obtaining these relations we have partially integrated eq.'s (1.2) and (1.3) with respect to τ and put $A = G(0)$, $B = J(0)$. Clearly, $A\ (= 1/B)$ gives the instantaneous elasticity of a given material and we shall term it as the 'instantaneous modulus' or the 'impact modulus'. The hereditary law (1.8) was formulated by Volterra as follows. The stress in a given viscoelastic material is equal to the instantaneous response proportional to the strain applied at a given instant decreased by an amount depending on the memory of the material for the past states of strain. The resolvent equation (1.9) may also be subjected to a similar interpretation.

The relation between the relaxation and creep kernels can easily be obtained by applying the Fourier transform to eq.'s (1.8) and (1.9) Or alter

natively, the Laplace transform may be applied. Eliminating σ and ε from the transformed relations then yields

$$h(s) = \frac{g(s)}{1-g(s)} \qquad (1.10\text{-a})$$

or,

$$g(s) = \frac{h(s)}{1+h(s)} \qquad (1.10\text{-b})$$

Also expressions in the time domain can be obtained through the inversion of the above relations. When the kernels are arbitrary functions of their arguments t, τ the Fourier and Laplace transforms can no longer be applied. The relations between kernels of this form are rather complicated and discussed in [1]. The integral equations containing these kernels can be solved by an iterative scheme, known as the method of successive approximations. Here we will restrict ourselves to convolution type kernels describing the time-invariant behaviour of viscoelastic materials. It is assumed that the kernels as well as the functions σ and ε are bounded and continuous, so that the integrals

$$\int_a^b |g(t)|\,dt \;;\; \int_a^b |h(t)|\,dt$$

$$\int_a^b |\sigma(t)|\,dt \;;\; \int_a^b |\varepsilon(t)|\,dt$$

exist in the finite interval (a,b) in which these

functions are defined. This is consistent with the hypothesis that the materials have a fading memory for the more distant events. The interval of integration may be finite or infinite. for the more general case where the given functions are piecewise continuous, sufficiently smooth and absolutely integrable.

3. Complex Modulus and Complex Compliance

We now wish to investigate the quasi-static behaviour of viscoelastic materials under oscillating strain and stress. Let a harmonical strain history be given by

$$\epsilon(t) = \epsilon_0 e^{j\omega t} \cdot 1(t)$$

where ϵ_0 is the amplitude and ω is the frequency of oscillation. Following the usual convention we mean here the real part of the signal, i.e. $\epsilon(t) = \epsilon_0 \cos\omega t$. Substituting this expression in the hereditary law (1.8) yields

$$\sigma(t) = A\epsilon_0 e^{j\omega t} - \epsilon_0 e^{j\omega t} \int_0^t g(\tau) e^{-j\omega\tau} d\tau$$

Put

$$E(j\omega) = A - \int_0^\infty g(\tau) e^{-j\omega\tau} d\tau = E' + jE'' \qquad (1.11)$$

Then

$$\sigma = E(j\omega)\epsilon \qquad t \to \infty$$

It follows that at sufficiently large times, when a steady state is achieved, the stress response to a harmonical input of strain applied at $t=0$ will also be a harmonical function of time, but one which is out of phase with the input since

$$\sigma = E(j\omega)\,\epsilon_0 e^{j\omega t} = \sigma_0 e^{j(\omega t+\phi)} \tag{1.12}$$

where

$$\sigma_0 = \epsilon_0 \cdot \|E(j\omega)\| = \sqrt{(E'^2 + E''^2)}\,\epsilon_0$$

and

$$\phi = \tan^{-1}(E''/E') \qquad E'' > 0$$

The frequency response function $E(j\omega)$ is a complex function of frequency and it is called the 'complex modulus'. Its real and imaginary components E' and E'' are called the 'storage modulus' and the 'loss modulus', respectively. They are related to one another through the Hilbert transforms*

$$E'(\omega) = A + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{E''(u)}{\omega - u}\,du$$

$$E''(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{E'(u)}{\omega - u}\,du .$$

Equation (1.11) expresses the fact that the relaxation modulus and the complex modulus form a (generalized) Fourier transform pair, which we denote by the notation

$$g \longleftrightarrow A - E ,$$

that is,

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [A - E(j\omega)]\, e^{j\omega t}\,d\omega \tag{1.13-a}$$

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*This is when the considered system is causal, i.e. $\sigma(t) = \epsilon(t) = 0$ for $t < 0$.

or

$$A\,\delta(t) - g(t) = \frac{1}{2\pi}\int_0^{\infty} E(j\omega)\,e^{j\omega t}\,d\omega \qquad (1.13\text{-}b)$$

We may decompose (1.11) into real and imaginary parts to express the quantities E' and E'' in terms of the relaxation kernel. In this way we obtain the Fourier cosine- and sine transforms

$$E'(\omega) = A - \int_0^{\infty} g(\tau)\cos\omega\tau\,d\tau$$

$$E''(\omega) = \int_0^{\infty} g(\tau)\sin\omega\tau\,d\tau \qquad (1.13\text{-}c)$$

The inversion formula is of course

$$A\,\delta(t) - g(t) = \frac{1}{2\pi}\int_0^{\infty} [E'(\omega)\cos\omega t - E''(\omega)\sin\omega t]\,d\omega$$

Equations (1.13-c) may serve as a check for the suitability or otherwise unsuitability of the time-invariant hereditary law for the description of the quasi-static behaviour of a given material.
In the definition of the complex modulus function we have assumed that the material is strain and stress-free at the instant the harmonical strain is applied. Such materials are causal. This assumption is quite natural as all physical systems in nature are causal. However, the assumption of causality brings together with it the steady-state assumptions for the description of the obtained response. To explain this, suppose that we

apply a strain history $\epsilon(t) = \epsilon_0 e^{j\omega t}$ at $t = -\infty$. Whence,

$$\sigma(t) = A\epsilon_0 e^{j\omega t} - \epsilon_0 \int_{-\infty}^{t} e^{j\omega\tau} g(t-\tau)\, d\tau .$$

Replacing the variable of integration by $u = t - \tau$ one gets

$$\sigma(t) = A\epsilon_0 e^{j\omega t} + \epsilon_0 \int_{\infty}^{0} e^{j\omega(t-u)} g(u)\, du$$

$$= A\epsilon_0 e^{j\omega t} - \epsilon_0 e^{j\omega t} \int_{0}^{\infty} g(u)\, e^{-j\omega u}\, du$$

$$= E(j\omega)\epsilon$$

Apparantly, the steady state response to an input applied at $t=0$ is equivalent to the response to an input applied at $t = -\infty$.

In a similar way we can investigate the quasi-static behaviour when the input and the output are interchanged. For a harmonical stress history given by

$$\sigma(t) = \sigma_0 e^{j\omega t} U(t)$$

the corresponding steady-state output may be expressed as

$$\epsilon(t) = H(j\omega)\sigma_0 e^{j\omega t} U(t)$$

$$= (H' + jH'')\sigma_0 e^{j\omega t} = \epsilon_0 e^{j(\omega t - \phi)} \qquad t \to \infty \qquad (1.14)$$

with

$$\epsilon_0 = \sigma_0 \cdot \| H(j\omega) \| = \sqrt{(H'^2 + H''^2)} \cdot \sigma_0$$

and

$$\phi = \tan^{-1}\left(H''/H'\right) \qquad H'' > 0.$$

Here $H(j\omega)$ is called the 'complex compliance', and its real and imaginary parts are respectively termed as the 'storage compliance' and the 'loss compliance'. For a causal material H' and H'' are related through the Hilbert transforms

$$H'(\omega) = B - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H''(u)}{\omega - u}\,du$$

$$H''(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H'(u)}{\omega - u}\,du .$$

It will be of interest to obtain the relations between the complex compliance and the creep kernel. Inserting the expression $\sigma(t) = \sigma_0 e^{j\omega t}\cdot U(t)$ in the hereditary law (1.9) we get

$$H(j\omega) = B + \int_0^{\infty} h(\tau)\, e^{-j\omega\tau}\,d\tau \qquad (1.15)$$

and

$$h(t) = \frac{1}{2\pi}\int_0^{\infty} [H(j\omega) - B]\, e^{j\omega t}\,d\omega \qquad (1.16\text{-a})$$

or

$$h(t) + B\,\delta(t) = \frac{1}{2\pi}\int_0^{\infty} H(j\omega)\, e^{j\omega t}\,d\omega \qquad (1.16\text{-b})$$

The above relations can also be shortly denoted by the one-sided (generalized) Fourier transform pair $h \longleftrightarrow H - B$. Eq.'s (1.16-a,b) can be decomposed into real and imaginary parts to obtain the relations between

the storage and loss compliances and the creep kernel. Thus

$$H'(\omega) = B + \int_0^\infty h(\tau)\cos\omega\tau\, d\tau$$

$$H''(\omega) = -\int_0^\infty h(\tau)\sin\omega\tau\, d\tau \qquad (1.16\text{-}c)$$

and

$$h(t) + B\delta(t) = \frac{1}{\pi}\int_0^\infty [H'(\omega)\cos\omega t - H''(\omega)\sin\omega t]\, d\omega.$$

Inspection of eq.'s (1.12) and (1.14) shows that the relation between the complex modulus and the complex compliance is given by

$$H(j\omega) = \frac{1}{E(j\omega)} = \frac{\epsilon_0}{\sigma_0} e^{-j\phi} \qquad (1.17)$$

The complex modulus and the complex compliance functions are specially useful for the description of the quasi-static viscoelastic effects and for this reason we shall also refer to them as the quasi-static material functions. They are the equivalent of the frequency response functions in the theory of linear systems. As their derivation suggests, they can easily be determined from quasi-static tests by measuring the amplitudes of the input and the output signals and the corresponding phase-lag. Experimental procedures involving Fast Fourier Transforms may efficiently be used to determine

these material functions over a desired frequency range, provided that the inertia effects are negligible within the considered range. Also, by using the data obtained from the quasi-static tests on the composition of the quasi-static functions, other material functions such as the kernels or the static functions can be identified. To this end, numerical procedures will be developed later in this work.

In the subsequent sections we shall give the relations among the various material functions in as much as they are not already derived. The derivation of these relations are straightforward and can be found in [1].

4. Distinction Between Viscoelastic Solids and Fluids. Relations Between Static and Quasi-Static Functions

So far we have not distinguished in our considerations between viscoelastic solids and fluids and the implications this has for the material functions. In fact, the distinction between viscoelastic solids and fluids is not a trivial matter as it seems, since viscoelastic materials exhibit a combination of solid and fluid properties. We may nevertheless characterize a material as fluid when the stress under maintained strain ultimately decays to zero. From this definition, for a viscoelastic material to be a fluid it is necessary that

$$\lim_{t\to\infty} G(t) = G_\infty \longrightarrow 0 \tag{1.18}$$

Also, a viscoelastic fluid, when subjected to a simple shear state of stress will ultimately creep at a constant rate. Thus, in addition to the previously defined creep function, a viscoelastic fluid must have an additional component under constant stress, which is proportional to time. Denoting the proportionality constant by $1/\eta_0$ we can express the creep function of a viscoelastic fluid as

$$J_f(t) = J(t) + \frac{t}{\eta_0} \tag{1.19}$$

η_0, known as 'the zero shear rate viscosity coefficient', is given by the formula [10], [11]

$$\eta_0 = \int_0^\infty G(t)\,dt = \int_0^\infty \int_0^t g(\xi)\,d\xi\,dt \tag{1.20}$$

The relation between the relaxation function and the complex modulus can be derived as

$$A - G(t) \longleftrightarrow \frac{A - E(j\omega)}{j\omega} + \pi f(\omega) G_\infty \tag{1.21}$$

which reduces for a viscoelastic fluid into

$$G(t) \longleftrightarrow \frac{E(j\omega)}{j\omega} \tag{1.22}$$

Separating this last expression into real and imaginary parts one obtains

$$\begin{aligned} E'(\omega) &= \omega \int_0^\infty G(t) \sin\omega t\,dt \\ E''(\omega) &= \omega \int_0^\infty G(t) \cos\omega t\,dt \end{aligned} \tag{1.23}$$

where the inversion formulae are

$$\begin{aligned} G(t) &= \frac{2}{\pi} \int_0^\infty \frac{E'(\omega)}{\omega} \sin\omega t\,d\omega \\ G(t) &= \frac{2}{\pi} \int_0^\infty \frac{E''(\omega)}{\omega} \cos\omega t\,d\omega \end{aligned} \tag{1.24}$$

Again, these equations may be used to check the

applicability of the linear viscoelasticity theory for a given material. In these expressions the term $\eta = \frac{E''(\omega)}{\omega}$ is called the 'dynamic viscosity'. Substituting the second of (1.24) in the first of (1.23) we obtain the Hilbert transforms

$$E'(\omega) = \frac{2}{\pi} \int_0^\infty \eta(u) \frac{\omega^2}{\omega^2 - u^2} \, du$$

$$\eta(\omega) = \frac{2}{\pi} \int_0^\infty E'(u) \frac{1}{u^2 - \omega^2} \, du \qquad (1.25)$$

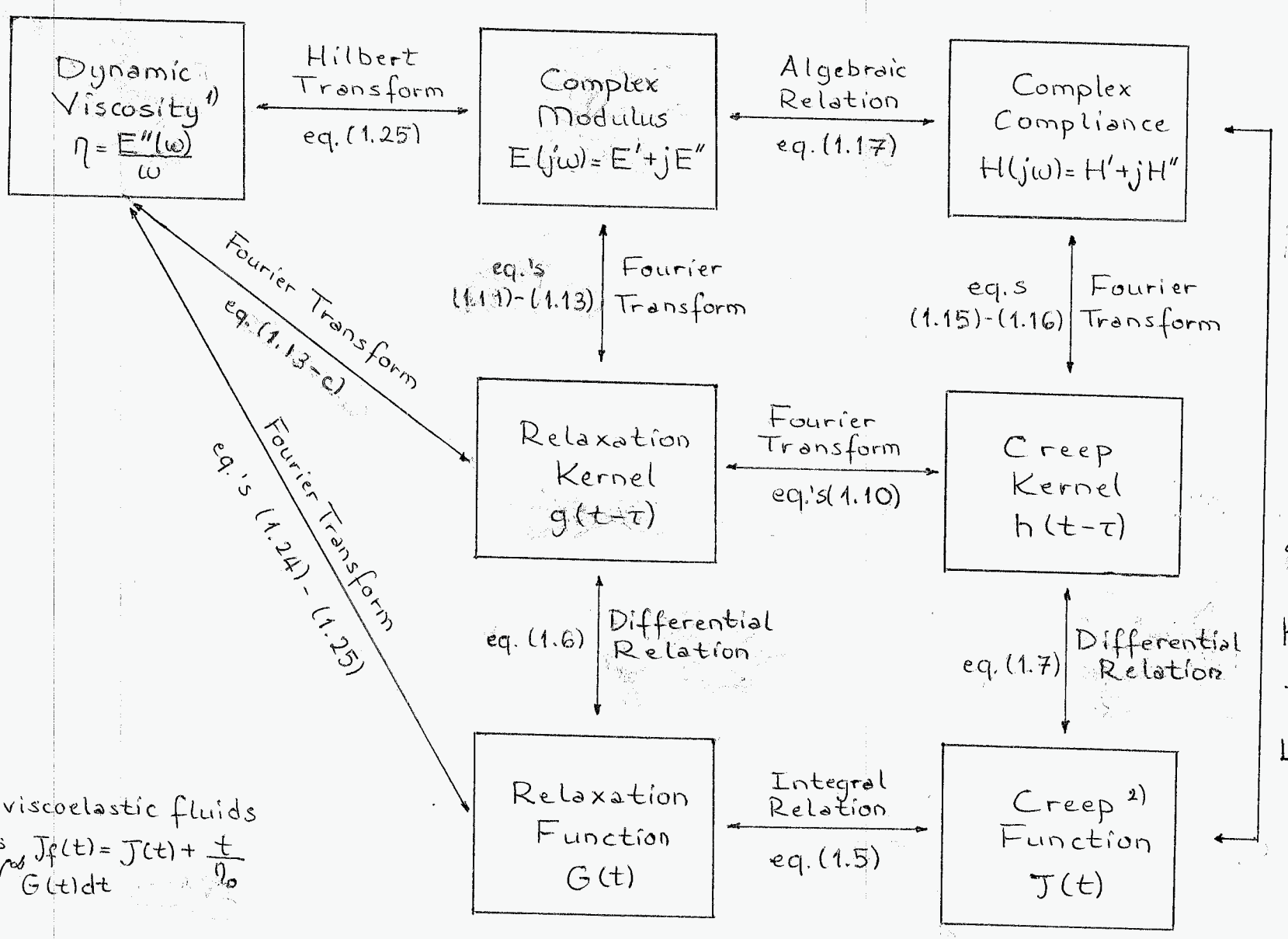
The evaluation of these integrals is discussed in [19], [20]. Finally, the relation between the creep function and the complex compliance is given by

$$J(t) - B \longleftrightarrow \frac{H(j\omega) - B}{j\omega} + \pi \delta(\omega) J_\infty . \qquad (1.26)$$

5. STRUCTURE OF THE LINEAR THEORY

The foregoing results are summarized in Fig. 1.3 showing the structure of the linear theory of viscoelasticity Also reference is made to the equations relating the various material functions to one another. In this figure one may distinguish three different levels both for relaxation and creep effects. The bottom and the top levels contain respectively the static and quasi-static material functions, whereas the middle-level is reserved for the kernels. In the linear theory of viscoelasticity, the kernels describe the transient response of viscoelastic materials. When the kernels are expressed as a sum of exponential functions the linear viscoelasticity theory reduces to the phenomenological theory of viscoelasticity which we will briefly discuss in the next section. In this case the kernels – or their spectra – describe the static behaviour of viscoelastic materials.

It is clear from fig. 1.3 that there is more than one way of obtaining a certain material function from another. The discrete forms of the relations between these material functions appropriate to determine them from quasi-static tests using either periodic or random histories will be developed later. In this way, the Fast Fourier Transforms, combined with digital spectral analysis



1) Only for viscoelastic fluids
2) For fluids $J_f(t) = J(t) + \frac{t}{\eta_0}$ with $\eta_0 = \int_0^\infty G(t)dt$

Fig. 1.3 Diagram showing the structure of the linear theory

techniques will provide an efficient means to investigate the viscoelastic effects and to determine the relevant material functions within a desired frequency range.
It is not required to have analytical expressions for the material functions for the practical applications of the linear theory of viscoelasticity. The digital data obtained from the quasi-static tests on the composition of the material functions will be sufficient to calculate the response of a given material to various strain or stress histories. If, for instance, the response to a periodical strain history is required, then the complex modulus can be determined at various frequencies (a periodical function can be represented by an appropriate Fourier series containing a sufficient number of harmonics) by the method discussed in section 3 above. Then we can easily calculate the Fourier-transformed stress response through $\sigma(\omega) = E(j\omega)\,\epsilon(\omega)$, after which the FFT inversion will give the required stress response in the time domain. The response to an arbitrary history can be obtained in a similar way, where the role of the Fourier series is replaced by the Fourier transform. Also random histories may be treated by taking the necessary statistical precautions. The digital signal analysis techniques speed up these procedures considerably, since one can obtain information about the frequency composition of the

material functions within a desired frequency range and to a desired degree of accuracy at once, i.e., without having to repeat the experiments for each frequency of interest.

6. KERNELS AS LAPLACE INTEGRALS. VISCOELASTIC SPECTRA

For the special choice of the relaxation and creep kernels as Laplace integrals

$$g(t-\tau) = \int_0^{\infty} F(\alpha)\, e^{-\alpha(t-\tau)}\, d\alpha \tag{1.27}$$

$$h(t-\tau) = \int_0^{\infty} S(\beta)\, e^{-\beta(t-\tau)}\, d\beta$$

the hereditary laws (1.8) and (1.9) form the limiting case of the differential equation with constant coefficients [1], [12]

$$P_0\sigma + P_1\frac{d\sigma}{dt} + \cdots + P_n\frac{d^n\sigma}{dt^n} = q_0\epsilon + q_1\frac{d\epsilon}{dt} + \cdots + q_n\frac{d^n\epsilon}{dt} \tag{1.28}$$

as $n \to \infty$. Eq. (1.28) is the constitutive relation corresponding to the phenomenological theory of viscoelasticity and it is referred to as the differential form of the linear viscoelastic law (as opposed to the integral laws). The functions $F(\alpha)$ and $S(\beta)$ are called resp. the relaxation and retardation spectra. These spectra give the distribution of the characteristic constants of the homogeneous equations corresponding to the l.h.s. and r.h.s. of (1.28) respectively.
For the case of line spectra

$$F(\alpha) = \sum_{k=1}^{n} F_k\, \delta(\alpha - \alpha_k) \tag{1.29-a}$$

$$S(\beta) = \sum_{k=1}^{n} S_k \delta(\beta - \beta_k) \qquad (1.29\text{-}b)$$

the kernels (1.27) degenerate into

$$g(t-\tau) = \sum_{k=1}^{n} F_k e^{-\alpha_k (t-\tau)}$$

$$h(t-\tau) = \sum_{k=1}^{n} S_k e^{-\beta_k (t-\tau)} \qquad (1.30)$$

which are the homogeneous solutions corresponding resp. to the l.h.s. and r.h.s. of the equation (1.28). Here α_k, β_k determine the location of the spectral lines, while F_k, S_k give their intensity. It is possible to obtain the coefficients of the differential equation P_k, q_k from the coefficients F_k, α_k or alternatively from S_k, β_k [1].
The relaxation and creep functions corresponding to the kernels of the form (1.27) are given by

$$G(t) = \int_0^\infty A(\alpha) \left[1 - e^{-\alpha t} \right] d\alpha \cdot U(t)$$

$$J(t) = \int_0^\infty B(\beta) \left[1 - e^{-\beta t} \right] d\beta \cdot U(t) \qquad (1.31)$$

where

$$A(\alpha) = \frac{F(\alpha)}{\alpha} \quad ; \quad B(\beta) = \frac{S(\beta)}{\beta} \qquad (1.32)$$

The spectra can be obtained from the kernels (1.27) through the inversion of the Laplace transform. A numerical approximation method which is

often used is Alfrey's Laplace inversion method discussed in Ref. [13], [14]. This involves however a complicated mathematical process. In some cases, the Laplace transform may be replaced by the Fourier transform, since it may be conceived as a generalization of the latter. Specifically, if the spectra are absolutely integrable, i.e. for instance when $\int_{-\infty}^{\infty} |F(\alpha)|\, d\alpha < \infty$, the Laplace transforms (1.27) reduce to the Fourier transforms. A sufficient condition for this to occur is that the spectra vanish as $|t| \longrightarrow \infty$. For viscoelastic fluids it is found that this condition is satisfied, so that the application of Fourier transform does not give rise to any difficulties. For viscoelastic solids, on the other hand, the spectra converge to a non-zero constant which implies that the hereditary action ultimately continues for these materials [6]. It is still possible in this case to employ the Fourier transform in order to obtain the spectra from their corresponding kernels. For instance, if the relaxation spectrum converges to a non-zero constant α_0, say, then the Laplace transform of $F(\alpha)$ will be identical to the Fourier transform of $F(\alpha)e^{-\alpha_0 t}$. Accordingly, the relaxation spectrum can easily be obtained from this last expression after the Fourier transform is applied to invert the first of the equations (1.27).

The spectra can also be obtained from the quasi-static functions. To derive the relations between

these material functions, we substitute eq.'s (1.27) in (1.11) and (1.15). The relation between the complex modulus and the relaxation spectrum is then obtained as

$$E(j\omega) = A - \int_0^{\infty} \frac{F(\alpha)}{\alpha + j\omega}\, d\alpha \tag{1.33}$$

or, separating into real and imaginary parts

$$E'(\omega) = A - \int_0^{\infty} \frac{\alpha\, F(\alpha)}{\alpha^2 + \omega^2}\, d\alpha$$

$$E''(\omega) = \int_0^{\infty} \frac{\omega\, F(\alpha)}{\alpha^2 + \omega^2}\, d\alpha \tag{1.34}$$

For the relation between the complex compliance and the retardation spectrum one has

$$H(j\omega) = B + \int_0^{\infty} \frac{S(\beta)}{\beta + j\omega}\, d\beta \tag{1.35}$$

or, again, decomposing into real and imaginary parts

$$H'(\omega) = B + \int_0^{\infty} \frac{\beta\, S(\beta)}{\beta^2 + \omega^2}\, d\beta$$

$$H''(\omega) = -\int_0^{\infty} \frac{\omega\, S(\beta)}{\beta^2 + \omega^2}\, d\beta \tag{1.36}$$

Approximate methods of inverting these Stieltjes integrals are discussed in [15]-[17]. A rigorous inversion method is given however in [18], [19]. The inversion formulae for eq.'s (1.33)-(1.34) are

$$F(\omega) = \pm \frac{1}{\pi\omega} \operatorname{Im}\{E(\omega e^{\pm j\pi})\}$$

$$F(\omega) = \pm \frac{2}{\pi\omega} \operatorname{Im}\{E'(\omega e^{\pm j\frac{\pi}{2}})\} \qquad (1.37)$$

$$F(\omega) = \pm \frac{2}{\pi\omega} \operatorname{Re}\{E''(\omega e^{\pm j\frac{\pi}{2}})\}$$

where $E(\omega e^{\pm j\pi})$ is in fact a symbolic way of writing $\lim_{\varepsilon \to 0} E(-\omega \pm j\varepsilon)$ and the symbols Im, Re denote resp. the imaginary and real components of the given expressions.
Similarly, the inversion formulae for eq.'s (1.35)-(1.36) are

$$S(\omega) = \pm \frac{1}{\pi\omega} \operatorname{Im}\{H(\omega e^{\pm j\pi})\}$$

$$S(\omega) = \pm \frac{2}{\pi\omega} \operatorname{Im}\{H'(\omega e^{\pm j\frac{\pi}{2}})\} \qquad (1.38)$$

$$S(\omega) = \pm \frac{2}{\pi\omega} \operatorname{Re}\{H''(\omega e^{\pm j\frac{\pi}{2}})\}$$

It is noted that the analytical representation of the dynamical data is required for the application of this inversion method. Then the variable ω is replaced by $\omega e^{\mp j\frac{\pi}{2}}$ or $j\omega$ by $\omega e^{\mp j\pi}$, and the complex expression thus obtained is decomposed into real and imaginary parts to obtain the desired solution. If the imaginary component vanishes, then the rigorous way should be taken and the limit should be calculated precisely to obtain a non-trivial solution. The result will then be a line in the spectrum. The relations between the relaxation and retardation spectra are obtained by the same method [20].

It is conventional to express the kernels (1.27) in terms of the reciprocals of the variables α and β, i.e. in terms of $1/\alpha$ and $1/\beta$ respectively. Then the role of the spectra is replaced by the so-called 'distribution functions'. Hence

$$g(t-\tau) = \int_0^\infty u\, Y(u)\, e^{-\frac{(t-\tau)}{u}}\, d\left(\frac{1}{u}\right) \qquad u = \frac{1}{\alpha}$$

$$h(t-\tau) = \int_0^\infty v\, Z(v)\, e^{-\frac{(t-\tau)}{v}}\, d\left(\frac{1}{v}\right) \qquad v = \frac{1}{\beta} \tag{1.39}$$

where the functions

$$Y(u) = \alpha F(\alpha) = \alpha^2 A(\alpha)$$
$$Z(v) = \beta S(\beta) = -\beta^2 B(\beta) \tag{1.40}$$

are called the distribution function of the relaxation times and the distribution function of the retardation times respectively. In the case of a discrete distribution one has

$$Y(u) = \sum_{k=1}^{n} A_k\, \delta(u-u_k) \qquad u_k = \frac{1}{\alpha_k}$$

$$Z(v) = \sum_{k=1}^{n} B_k\, \delta(v-v_k) \qquad v_k = \frac{1}{\beta_k} \tag{1.41}$$

where u_k and v_k are the relaxation and the retardation time constants, respectively. When the eq.'s (1.41) are substituted in the Laplace integrals (1.39) then the kernels degenerate into

$$g(t-\tau) = \sum_{k=1}^{n} A_k \alpha_k e^{-\alpha_k (t-\tau)} = \sum_{k=1}^{n} F_k e^{-\alpha_k (t-\tau)}$$

$$h(t-\tau) = \sum_{k=1}^{n} B_k \beta_k e^{-\beta_k (t-\tau)} = \sum_{k=1}^{n} S_k e^{-\beta_k (t-\tau)} \qquad (1.42)$$

The previous discussion about the viscoelastic spectra applies also to the distribution functions. Since the relations between the distribution functions and other material functions are similar to that of the viscoelastic spectra we will avoid here the unnecessary repetition. Lastly, to obtain a complete picture of the theory we give the relation between the relaxation spectrum and the zero shear rate viscosity coefficient. From eq.'s (1.20), (1.31) and (1.32) one obtains easily

$$\eta_0 = \int_0^{\infty} F(\alpha)\, d\alpha \qquad (1.43)$$

The diagram of the previous section showing the structure of the linear theory can be extended to the special case when the kernels are expressable as a Laplace integral of their corresponding spectra. This is given in Fig. 1.4 below, where the relations between the spectra and various material functions are also included. In order to avoid unnecessary complications, the distribution functions are not included in the figure. Of course, the use of this diagram makes sense only when the exponential type kernels are capab

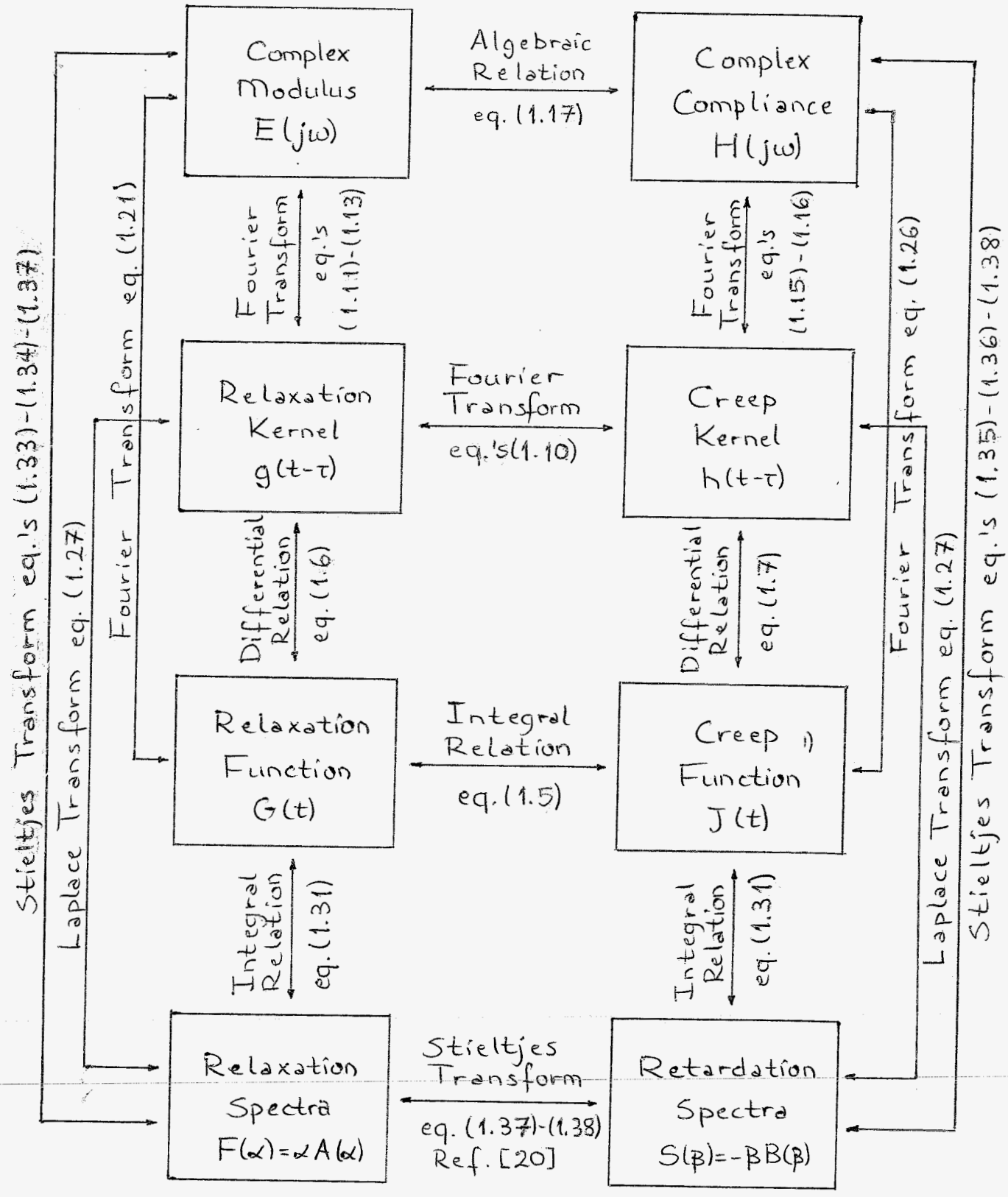


Fig. 1.4 Phenomenological theory with exponential type kernels (') see eq. (1.19) for viscoelastic fluids)

le of predicting both the quasi-static and the static behaviour of a given viscoelastic material, i.e. when it is possible to combine the phenomenological and the linear viscoelasticity theory through the spectrum functions. For the materials tested, it is however expected that such an approach is unlikely to have success; although the spectra can be used to approximate the relaxation and creep curves to any desired degree of accuracy, they are not appropriate to predict the viscoelastic effects for non-homogeneous strain or stress histories.

II

CHARACTERIZATION OF VISCOELASTIC EFFECTS

1. CHARACTERIZATION OF STATIC BEHAVIOUR

In studying the viscoelastic effects, with a few exceptional cases, almost invariably one-dimensional experiments are performed to determine the material functions. In this chapter we shall restrict our discussion to the study of one-dimensional behaviour, the more because the consideration of a multi-dimensional case may be due when one has difficulties in describing the one-dimensional behaviour. One-dimensional tests can be employed both for shear and extension / compression and we will use the same symbols for the material functions in both of the two cases.
The static behaviour of a viscoelastic material can be determined through a stress-relaxation test or a creep test; or through a combination of these tests known as a creep-and-recovery test. These three types of tests will briefly be discussed below. For a more extensive treatment see Ref. [2], [5] and [21].

1.1 Stress-relaxation

In a stress-relaxation experiment, the stress is determined as a function of time after a step function of strain is applied. (Fig. 2.1). The stress may decay to an equilibrium value or zero, depending on whether the sample material is a viscoelastic solid or fluid. Ideally, i.e. when a true step-chan-

ge could be realized, the stress response is given by the formula

$$\sigma(t) = \epsilon_0 \, G(t) \cdot 1(t) \tag{2.1}$$

It is however practically impossible to introduce an infinite change of strain corresponding to the block curves in Fig. 2.1 and the practical situation will be similar to that given by the dotted-curves in the figure. Thus the impact modulus will hardly be accessible and the measurements will be reliable only after a certain time has passed.

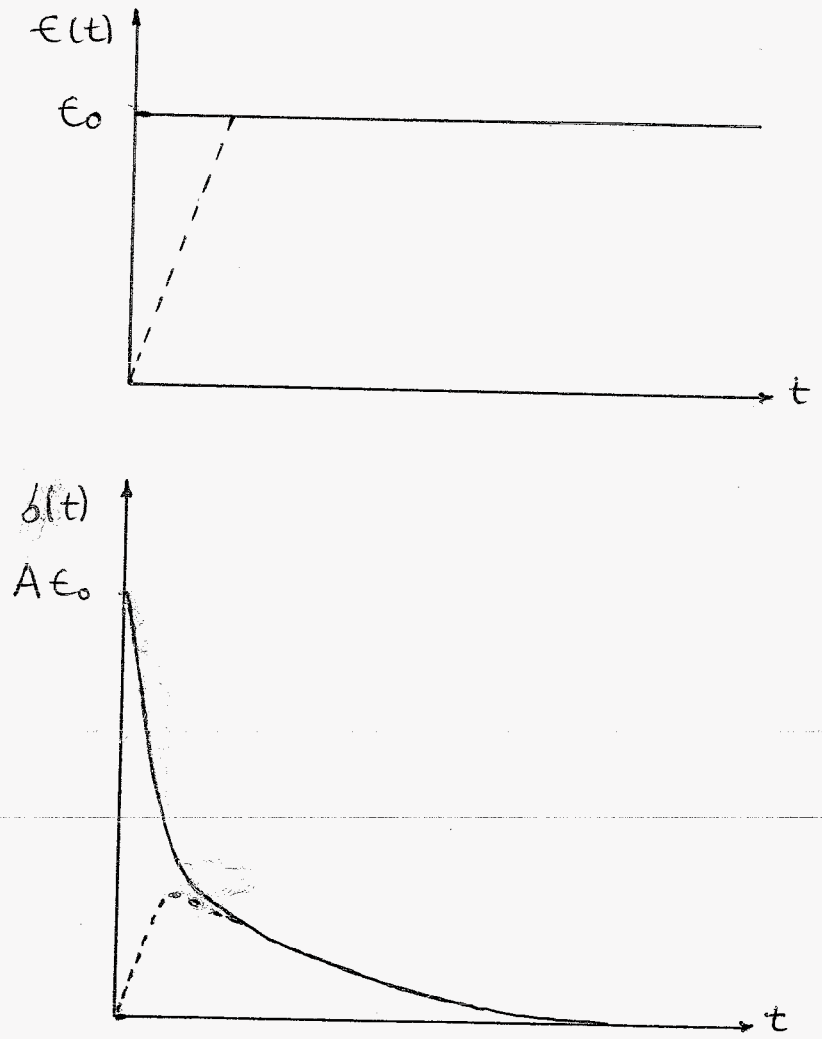


Fig. 2.1 Comparison of finite and infinite rate of strain application

Theoretically, this process of loading at an infinitely high rate of strain due to the sudden application of strain can be considered as a dynamical process and information about the short-time behaviour of a given material can be obtained from quasi-static or dynamical tests. It is noted that for many viscoelastic materials the major part of the stress-relaxation takes place immediately after the strain is applied, whence the importance of the determination of the short-time behaviour, especially for dynamical applications, is clear.

1.2 Creep

In a creep test, the strain is determined as a function of time after a step-change of stress is applied. (Fig. 2.2). The strain may approach a finite equilibrium value or increase indefinitely, depending on whether the test material is a viscoelastic solid or fluid. The strain response to a step-change of stress is given by

$$\epsilon(t) = \sigma_0 J(t) U(t) + \frac{t}{\eta_0} \sigma_0 \qquad (2.2)$$

where $1/\eta_0 = 0$ for a viscoelastic solid.

Here, too, it is practically impossible to produce a true step-function of stress and, as is shown in Fig. 2.2, if special notice is not taken of this fact the results obtained may be misleading.

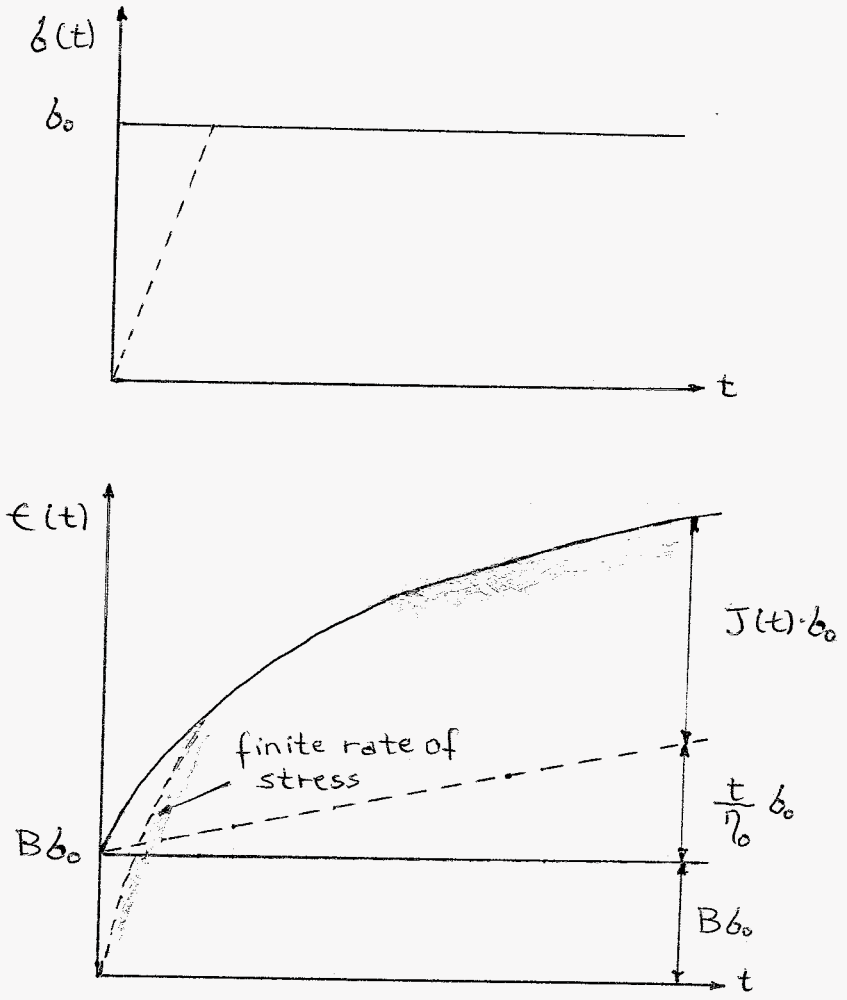


Fig. 2.2 Comparison of finite and infinite rate of stress application

For viscoelastic fluids, the analysis of creep data is more complicated than that of the stress-relaxation because of the necessity of separating the hereditary and Newtonian-flow components of creep.

1.3 Creep and recovery

In a creep and recovery test a square-wave of stress is applied and the corresponding strain response is determined (Fig. 2.3). Hence for a stress history given by

$$\sigma(t) = \begin{cases} 0 & t<0 \\ \sigma_0 & 0 \le t < t_0 \\ 0 & t \ge t_0 \end{cases} \qquad (2.3)$$

eq. (1.4) yields for the strain response

$$\begin{aligned} \epsilon(t) &= \epsilon_c(t) = \sigma_0 \cdot J(t) \qquad 0 \le t < t_0 \\ \epsilon(t) &= \epsilon_c(t) - \sigma_0 J(t-t_0) \\ &= \epsilon_c(t) - \epsilon_c(t-t_0) \qquad t \ge t_0 \end{aligned} \qquad (2.4)$$

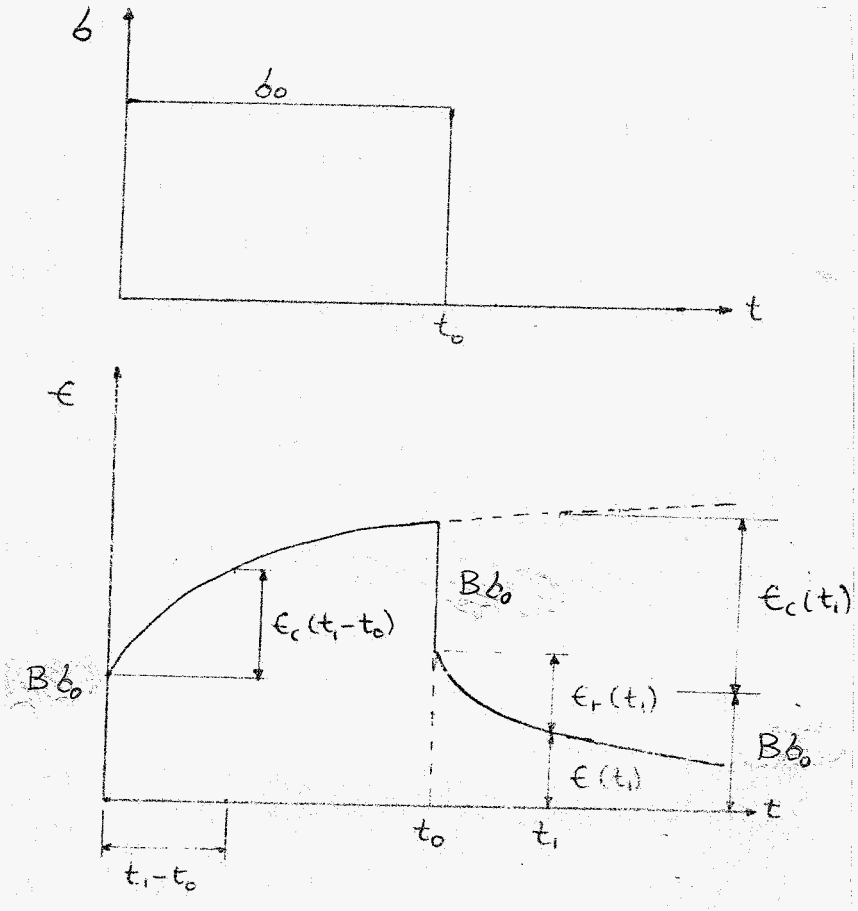


Fig. 2.3 Creep and recovery. For linear materials it should hold that $\epsilon_c(t_1-t_0) = \epsilon_r(t_1)$, or in other words, recovery is identical with the transformed creep.

It follows from the second of (2.4) that the recovery is given by

$$\epsilon_r(t) = \epsilon_c(t-t_0) = \sigma_0 J(t-t_0) \qquad (2.5)$$

i.e. the creep recovery ϵ_r is identical with the transformed creep curve. The curves in Fig. 2.3 describe the ideal situation in which a perfect square wave of stress is applied, but practically only a saw-tooth like application of stress can be achieved.

For real viscoelastic materials, data obtained from relaxation and creep tests do not succesfully predict the creep and recovery behaviour or other similar loading situations in which a multiple of step-changes of stress and strain is applied. Consequently, viscoelastic materials are sometimes modelled separately for various loading and unloading programs. It seems that a certain interaction occurs between events occuring at different times, which may be attributed to the nonlinearity of the material or at least to the more complicated material behaviour than the Boltzmann superposition principle prevails. In our opinion, it should be possible to gain more insight into the matter by a systematical empirical analysis. The digital spectral analysis methods discussed presently may prove to be especially useful for this purpose.

2. CHARACTERIZATION OF DYNAMIC BEHAVIOUR

Static tests do not suffice to obtain complete information about the mechanical behaviour of viscoelastic materials. As it has previously been mentioned, for most materials the major part of the stress-relaxation, for instance, takes place immediately after the load is removed and a dynamical analysis is indispensible in order to investigate the short-time behaviour of viscoelastic materials. Further, the stress or strain histories consisting of a multiple of step-functions produce viscoelastic effects which are not succesfully predicted by the data from static tests. Dynamic tests provide twice as much direct experimental information as static tests because two independent measurements (usually the amplitude ratio and the phase angle) are made at each frequency of interest. In fact, the curves for the static functions are the mirror-images of the corresponding quasi-static function curves under appropriate conditions such as time-invariance and linearity. If the functions $E'(\omega)$ and $H'(\omega)$ are plotted against $\log(\frac{1}{\omega})$, then this has the effect of reflecting the curves in the axis of $\log\omega=0$. If, also, the corresponding static function curves are plotted against $\log t$ in the same scale, then they will be found to trace the same path with resp. $E'(\omega)$ and $H'(\omega)$. Of course, the curves for $E''(\omega)$ and $H''(\omega)$ can also be brought into the same

form as $G(t)$ and $J(t)$ since they are not independent of the functions $E'(\omega)$ and $H'(\omega)$ under the conditions of assumed linearity. By the aid of this method the static tests extending to large values of time can be combined with dynamical tests for the short-time behaviour in order to obtain the material response over a wide range of time [22], [23].

The dynamical methods to determine the viscoelastic effects can be divided into two cathegories, depending on whether the sample size, consistency and frequency are such that the inertia of the sample material can be neglected. (When the characteristic length of the sample l (length in the direction of stretch in simple extension and thickness perpendicular to the direction of slide in simple shear) is small when compared with the wavelength of elastic waves propagated at the frequency of measurement, i.e. $l \ll (E'/\rho)^{1/2}/\omega$, the inertia of the sample can be disregarded and in this case we are considering the quasi-static tests [24]. Otherwise, we are considering the dynamical tests.
The most common quasi-static method is the so-called direct method in which the applied harmonical function of strain or stress and the in-phase and out-of-phase components of the response are directly measured for a given frequency. A method of quickening this procedure such that the quasi-static functions can be measured for periodical, absolutely integrable or even random histories for

a whole range of frequencies is the following. From section 1.3 we have for the response to a harmonical strain history $\epsilon(t) = \epsilon_0 e^{j\omega t}$

$$\sigma = E(j\omega_0) e^{j\omega_0 t}$$

where ω_0 is the frequency of oscillation. If a periodical strain history is applied instead of a harmonical history, then it may be represented by the Fourier series comprising of infinitely many harmonics

$$\epsilon(t) = \sum_{n=-\infty}^{\infty} \epsilon_n e^{jn\omega_0 t}$$

where the coefficients $\epsilon_n = |\epsilon_n| e^{j\phi_n}$ are given by

$$\epsilon_n = \frac{1}{T} \int_0^T \epsilon(t) e^{-jn\omega_0 t} dt .$$

Here T is the period of the applied strain history. The stress response can also be expressed as

$$\sigma(t) = \sum_{n=-\infty}^{\infty} \sigma_n e^{jn\omega_0 t}$$

with the coefficients

$$\sigma_n = \frac{1}{T} \int_0^T \sigma(t) e^{-jn\omega_0 t} dt$$

Now the complex modulus can be determined from

$$E(jn\omega_0) = \frac{\sigma_n}{\epsilon_n} \qquad (2.6)$$

Similarly, one has for the complex compliance

$$H(jn\omega_0) = \frac{\epsilon_n}{\sigma_n} \qquad (2.7)$$

Since the functions $\epsilon(t)$ and $\delta(t)$ represented by their corresponding Fourier series are defined only for the integer values of n, the coefficients b_n and ϵ_n form line spectra as shown in Fig. 2.4. The quantities $|b_n|$, $|\epsilon_n|$ and ϕ_n can also be plotted as a function of the discrete frequency $\omega = n\omega_0$ and they are called resp. the amplitude and phase spectra.

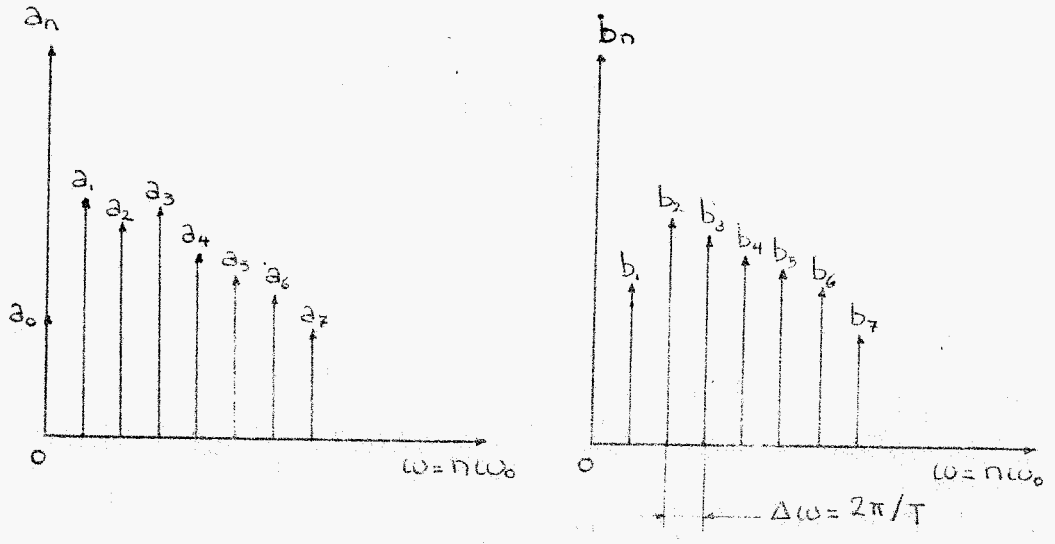


Fig. 2.4 The frequency spectrum of a signal $\epsilon(t)$ which may be represented by its coefficients $\epsilon_n = a_n - jb_n$ spaced $2\pi/T$ apart.

When the applied history is not a periodic function it can not be represented by an appropriate Fourier series. Yet, if it satisfies the condition

$$\int_{-\infty}^{\infty} |\epsilon(t)|\,dt < \infty$$

then it may be thought as having a period $T \longrightarrow \infty$ and it can be represented by the Fourier integral

$$\epsilon(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon(\omega)\, e^{j\omega t}\, d\omega$$

i.e. the integral sum of the harmonics $\epsilon(\omega)e^{j\omega t}$. Since it is assumed that $T \to \infty$, $\Delta\omega \to 0$ in Fig. 2.4 and the spectral lines merge together to form a continuous frequency spectrum. The function is defined by the Fourier transform

$$\epsilon(\omega) = \int_{-\infty}^{\infty} \epsilon(t)\, e^{-j\omega t}\, dt$$

such that $\epsilon(t) \longleftrightarrow \epsilon(\omega)$. Similarly, one has for the response $\sigma(t) \longleftrightarrow \sigma(\omega)$. Thus, if the strain and stress histories are absolutely integrable, then the quasi-static material functions can easily be determined from the ratio's of the Fourier transforms of the stress and strain

$$E(j\omega) = \frac{1}{H(j\omega)} = \frac{\sigma(\omega)}{\epsilon(\omega)} . \qquad (2.8)$$

The situation for random histories will be discussed in the next chapter.

An alternative way of specifying the quasi-static material functions by the direct method is to measure the ratio of voltage to current rather than the ratio of stress to strain after converting the obtained mechanical signals into electrical ones by means of an electromechanical transducer. The advantage of this method is contained in its precision when the displacements are kept exceedingly small. A third quasi-static method to be mentioned here is that of a resonant system with added inertia, where a large mass M is attached to the specimen whose inertia is negligibly small

compared to that of the added mass M. Then the system is forced to vibrate at the resonance frequency wherefore the dynamical properties can be measured and the quasi-static material functions can easily be calculated. Alternatively the system may be brought to vibrate freely and the logarithmic decrement ν is obtained as the natural logarithm of the ratio of two successive oscillations. Then the quasi-static properties are obtained from the simple relation $E''/E' = \nu/\pi$ at the natural frequency of the system. In these experiments the sample material is treated as a special spring whose proportionality modulus is a complex function of frequency, which of course is an oversimplification of the linear theory. Another major disadvantage of these tests is the fact that the frequency of measurement is discrete and it can be altered only by replacing the added inertia or changing the sample geometry. A discussion about these methods as well as the required apparatus can be found in [24]-[29]. As for the dynamic methods for which the inertia of the sample material is not negligible, we mention the case which is similar to the last-mentioned method, but where the characteristic length of the specimen is of the same order as that of the wavelength of the elastic waves such that resonant vibrations can be excited in the sample. In this case no inertia need be added since the resonance is provided by the sample itself. Naturally, this method has the same disadvantages as the one with added inertia.

A second dynamical method, the impulse response method, involves the measurement of the transient response due to an impulsive input of stress. This method has been used for compression of soft viscoelastic solids and shear of viscoelastic fluids [24]. Other dynamical methods to be mentioned here involve the wave propagation techniques. When the critical dimension of the sample is large compared with the wavelength of elastic waves propagated at the frequency of interest, the inertia of the sample can not be neglected and there is no simple relation between the applied load and the deformation. The mechanical properties of the sample material can however be derived from the velocity and the attenuation of the elastic waves propagated instead of observing the stress and strain. If, further, the amplitude of the wave decays to zero before it reaches the opposite boundary, then one need not observe the wave at all; it is sufficient to measure the complex ratio of force to velocity at the surface from which the wave is propagated in order to determine the complex modulus. Although the latter method is easier, the measurements are restricted to the special choices of frequency. The wave propagation techniques are frequently used, especially because they enable one to obtain information on the viscoelastic effects at shorter times than it is possible by the aid of the previously mentioned methods. The wave propagation techniques are treated in [6], [11], [30]-[32].

To allow a comparison, the ranges of the quasi-static and dynamic methods are given in Table 2.1 below. These ranges are not absolute limitations and represent the time intervals which have usually been covered experimentally.

Table 2.1 (taken from [24])

Comparisons of various quasi-static and dynamic methods

Class	Type	Time range (sec.)
Quasi-static	1. Direct method	$10^{-2} - 10^{6}$
	2. Direct method with electro-mechanical transducers	$10^{-4} - 10^{-1}$
	3. Resonance measurements with added inertia	$10^{-5} - 10^{2}$
Dynamic	4. Resonance measurements without added inertia	$10^{-4} - 1$
	5. Impulse response	2×10^{-7}
	6. Wave propagation	$10^{-2} - 4 \times 10^{-4}$ (bulk waves to 10^{-7})
	7. Wave attenuation	$10^{-3} - 10^{-7}$

In addition to the quasi-static and dynamic methods discussed above, the short-time behaviour of visco-elastic materials can be determined from experiments in which the sample material is loaded or stretched at a constant rate. We shall proceed to discuss them in the next section.

3. CONSTANT RATE TESTS

The short-time behaviour of viscoelastic materials can also be determined from constant rate tests in which either the strain is increased at a constant rate and the resulting stress is measured or, less commonly, the stress is increased uniformly and the corresponding strain is measured. For a linear viscoelastic material, this yields the same information as the other time-dependent stress and strain histories discussed above [33]. To make this plausible, consider an experiment in which the specimen is stretched uniformly from its original length to a certain value of strain ϵ_0, which is held constant afterwards. The stretching process takes place at a sufficiently large (constant) rate of strain (Fig. 2.5).

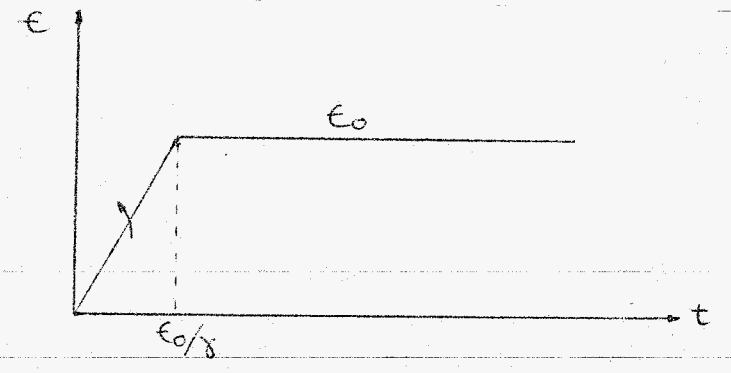


Fig. 2.5 constant strain rate test

If we substitute $g(t-\tau) = -\frac{dG(t-\tau)}{d(t-\tau)} = \frac{dG(t-\tau)}{d\tau}$ in eq. (1.8) we obtain with $A = G(0)$

$$\sigma(t) = G(0)\epsilon(t) - \int_{0^+}^{t} \frac{dG(t-\tau)}{d\tau}\epsilon(\tau)d\tau$$

Integrating this expression partially with respect to τ yields

$$\sigma(t) = G(0)\epsilon(t) - \left[G(t-\tau)\epsilon(\tau)\right]_{0^+}^{t} + \int_{0^+}^{t} G(t-\tau)\frac{d\epsilon(\tau)}{d\tau}d\tau$$

or, after simplification

$$\sigma(t) = \epsilon(0)\,G(t) + \int_{0^+}^{t} G(t-\tau)\,\frac{d\epsilon(\tau)}{d\tau}\,d\tau \qquad (2.9)$$

Now, inserting the strain history illustrated in Fig. 2.5

$$\epsilon(t) = \begin{cases} \gamma t & 0 \le t < \epsilon_0/\gamma \\ \epsilon_0 & t \ge \epsilon_0/\gamma \end{cases}$$

in eq. (2.9), one obtains simply

$$\sigma(t) = \gamma \int_{0}^{\epsilon_0/\gamma} G(t-\tau)\,d\tau \qquad (2.10)$$

since $\epsilon(0) = 0$. Thus the stress measured in a constant rate of strain test is related to the integral of the relaxation function by eq. (2.10) and the relaxation function can be determined by differentiating the obtained experimental curve. If the strain rate is sufficiently high, then this corresponds to the short-time behaviour since the effect of increasing the strain rate is the same as compressing the time scale. For, at $t = \epsilon_0/\gamma$,

eq.(2.10) gives

$$\sigma\left(\frac{\epsilon_0}{\gamma}\right) = \int_0^{\epsilon_0/\gamma} G\left(\frac{\epsilon_0}{\gamma} - \tau\right) d\tau \ .$$

Replacing the variable of integration τ by η/γ, the above expression becomes

$$\sigma\left(\frac{\epsilon_0}{\gamma}\right) = \int_0^{\epsilon_0} \frac{1}{\gamma} \cdot G\left(\frac{\epsilon_0 - \eta}{\gamma}\right) d\eta$$

As γ increases indefinitely, it is clear that

$$\sigma\left(\epsilon_0/\gamma\right) = G(0)\frac{\epsilon_0}{\gamma} \quad ; \quad \gamma \longrightarrow \infty \ . \tag{2.11}$$

Completely consistent with this result, discontinuity in strain causes a discontinuity in stress proportional to the impact modulus $G(0)$ or, in other words, for an indefinitely high rate of strain the viscoelastic materials behave elastically. Thus, it is possible to some extent to obtain information about the impact behaviour of viscoelastic materials if the rate involved is sufficiently high. It is preferable to use the data obtained from constant rate tests to predict the material behaviour at substantially lower strain or stress rates, since the relaxation or creep rates are in general considerably high at very short-times and the results obtained from lower rates may be misleading.

Experimental observations show that the data obtained from constant rate of loading tests generally do not predict the material behaviour satisfactorily

at a constant rate of unloading. This is due to a more complicated form of the mechanical hysteresis curves under cyclic loading. More will be discussed about this in the next section. Recently, constant rate of loading tests have been used successfully by Fung and his collaborators in modelling the mechanical behaviour of soft biological tissues [34] - [39].

4. DISSIPATION OF ENERGY IN VISCOELASTIC MATERIALS

Consider an element of a tension bar with cross-section A. If forces σA are applied to this element of length dx at its ends, the strain will increase by $\dot{\epsilon}\,dt$ in the infinitesimal time interval dt. The work done by the external forces may be expressed as

$$dW \cdot A \cdot dx = \sigma A \cdot dx\, \dot{\epsilon}\, dt$$

where dW is the work done per unit volume of the bar. If the deformation continues for a finite time, the work done on a unit volume (specific work) for a sufficiently small value of deformation is given by

$$W = \int dW = \int \sigma\, \dot{\epsilon}\, dt \,. \qquad (2.12)$$

In an elastic bar this work is converted into elastic energy, which will be recovered completely upon unloading. In a viscoelastic bar, however, all or part of this work is transformed into heat and therefore dissipated. To quantitize the dissipated energy, we choose a harmonical strain history

$$\epsilon = \epsilon_0 e^{j\omega t}$$

The steady-state response will be given by

$$\sigma = \sigma_0 e^{j(\omega t + \phi)}$$

Substituting these expressions in eq. (2.12), integration over one period $T = 2\pi/\omega$ and discarting the imaginary part we get

$$W = \pi \epsilon_0^2 E''(\omega) \tag{2.13}$$

During every period of oscillation, an amount of energy given by eq. (2.13) is dissipated. In view of the second law of thermodynamics, the dissipated energy must be non-negative, whence

$$E'' > 0$$

for the given irreversible process. Also, dividing (2.13) by $T = 2\pi/\omega$, we obtain an expression for the average energy dissipated per unit time

$$W_d = \frac{1}{2} \epsilon_0^2 \omega E''(\omega) . \tag{2.14}$$

Owing to the energy dissipation in a viscoelastic material subjected to cyclic loading, the loading and unloading curves exhibit a hysteresis loop as shown in the stress-strain diagram below (Fig. 2.5).

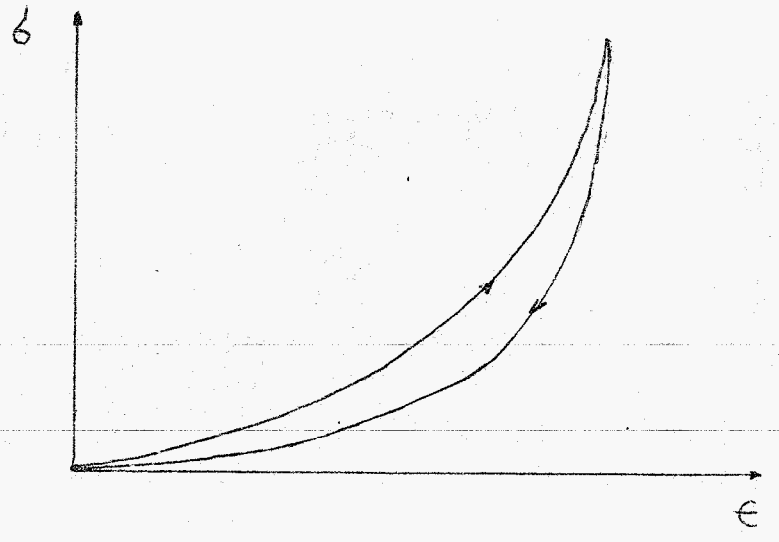


Fig. 2.5 The cyclic response of a typical biological material. The arrows show the loading and the unloading parts of the curve.

The hysteresis loop forms a closed curve when the mechanical properties are time-invariant, corresponding to a material with convolution-type kernel. Experimental investigations show that for certain biological tissues the hysteresis curve stabilizes only after a number of cycles (usually 6-10 cycles). This is called 'preconditioning' the given material. The tests are performed only after proper pre-conditioning under the testing conditions (strain rate, temperature, etc.) is achieved. If the testing conditions alter, then the specimen should be preconditioned once more prior to performing new tests [34], [35]. Data indicate that the hysteresis loop gets larger as the strain rate decreases. This result is in agreement with our intuition that for lower strain rates the material has more time to develop its hereditary action. It was seen that the material behaves almost elastically at very high strain rates and therefore exhibits negligible hysteresis.

5. STRESS-STRAIN CURVES

The most common means of characterizing the mechanical behaviour of traditional materials is through a stress-strain curve. A simple stress-strain curve is useful to obtain information about the various deformation regimes and, in particular, to define the region of essentially linear behaviour. For many engineering materials this is possible because the form of the stress-strain curve is not strongly sensitive to parameters such as the rate of loading, except possibly under extreme conditions. In contrast, the mechanical behaviour of viscoelastic materials is strongly time-dependent and the concept of an almost unique stress-strain curve does not exist. Yet, the stress-strain curves can be useful in defining the region of linearity and this will be pursued here a little further. Fig. 2.6 illustrates a number of possible

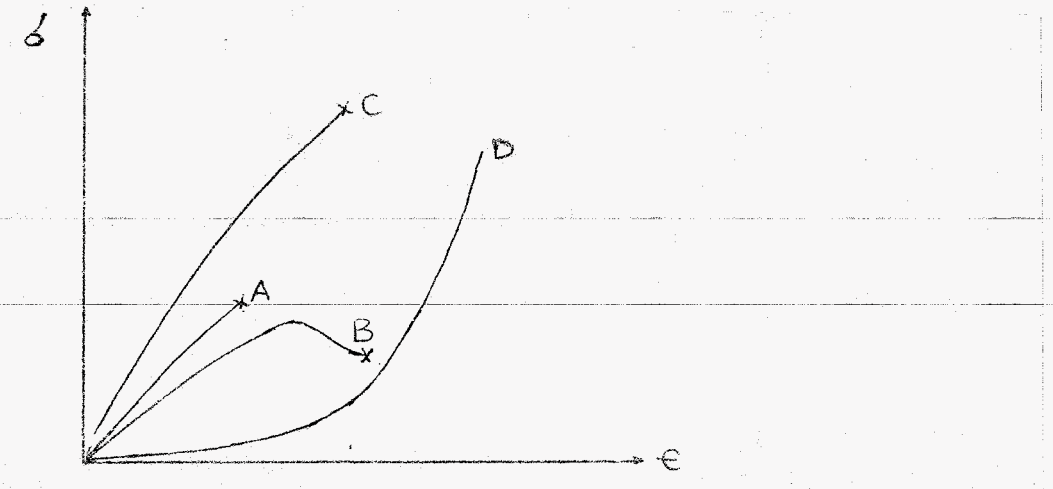


Fig. 2.6 Schematic stress-strain curves for various viscoelastic materials (x denotes failure)

stress-strain curves for various viscoelastic materials. Curves A and B are typical for respectively brittle and ductile materials. Many polymers assume a path like that of A or B on the stress-strain plane, depending on the rate of loading or other relevant parameters. Thus a material which is ductile at low rates of loading at a given temperature, having a curve like B in Fig. 2.6, may act in a brittle manner at high rates, that is like A. Curve C represents the behaviour of glass-reinforced plastics; it has a higher stiffness than that of the unreinforced materials due to the less ductile glass-fibers. Finally, curve D is typical of polymer foams under compression or of biological tissues under simple elongation. For the polymer foams the lower modulus at moderate strains is due to the collapse of the cell walls. As for the biological tissues, the load is mainly carried by elastine fibers at low strains, while the contribution of collagen fibers gradually increase with the increasing stress. The rate dependance of a typical biological material is illustrated in Fig. 2.7 below. Because of the strong rate-dependance, a whole set of curves is needed for the mechanical characterization in the entire deformation range. Yet, in practical applications, the strain is often restricted to a few percent, and if the assumption of linearity can be maintained within this low strain region the mechanical behaviour can be modelled adequately by the aid of the linear viscoelastic law.

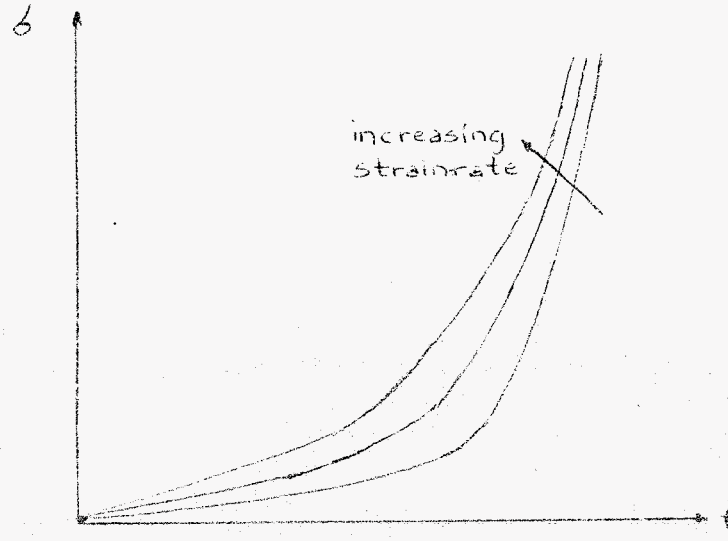


Fig. 2.7 Simple elongation of a typical biological material at various strain rates (percent/min)

When this is the case, it will be sufficient to determine the various material functions and the level of strain and the range of strain-rate for which the assumption of linearity can be maintained. The most useful strain-stress curves for viscoelastic materials are those that are obtained from 'isochronous' creep or relaxation data (Fig. 2.8). These cur-

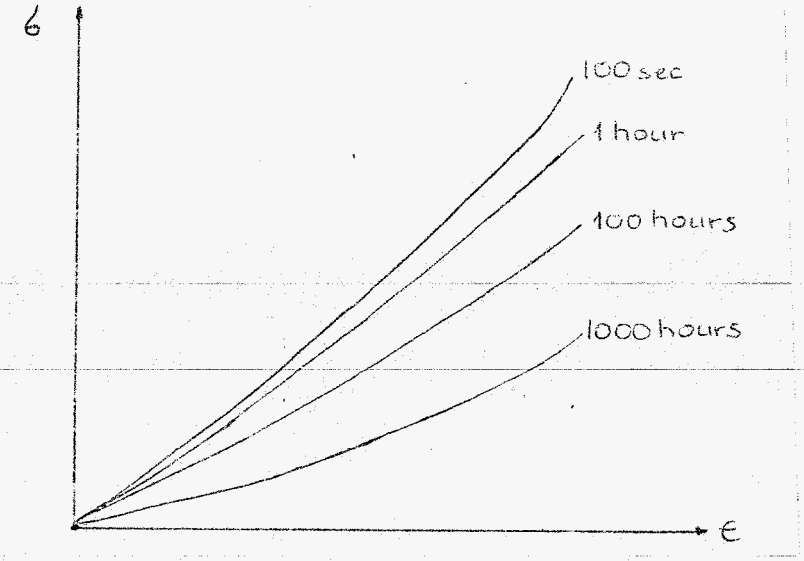


Fig. 2.8 Isochronous creep curves for a viscoelastic material

ves are obtained from the relaxation and creep tests for various values of respectively strain and stress. Each curve corresponds to the relaxation or creep values measured at a given time for a variety of applied strain or stress. As it will be shown presently, the linearity of a given material can only be deduced from the isochronous stress-strain curves.

Data for stress-strain curves are commonly produced by one of the two test methods: constant strain rate tests and isochronous data from creep tests. Consider first the constant strain-rate tests. Let the strain history be given by

$$\epsilon(t) = \begin{cases} 0 & t<0 \\ kt & t \geq 0 \end{cases}$$

where k is a positive constant. Substituting this expression in eq. (2.9) yields

$$\sigma(t) = k \int_0^t G(t-\tau)\, d\tau = k\, T(t) \qquad (2.15)$$

(With $k = \dot{\epsilon}$ and $t = \epsilon/\dot{\epsilon}$ we write (2.15) in the form

$$\sigma(\epsilon, \dot{\epsilon}) = \dot{\epsilon} \cdot T(\epsilon/\dot{\epsilon})$$

In general, T is a non-linear function of its argument and accordingly, the stress-strain relation for a given $\dot{\epsilon}$ will generally be non-linear. Thus, a non-linear stress-strain curve obtained from

constant strain rate test does not in general rule out the linear mechanical behaviour, nor do linear curves imply linearity of the given material. A check for linearity is when the data plotted as $\sigma/\dot{\epsilon}$ versus $\epsilon/\dot{\epsilon}$ fall on the same curve for various values of $\dot{\epsilon}$, but such a test is not practically useful.

Consider now a creep test. We write completely analogous to eq. (2.9)

$$\epsilon(t) = \sigma(0)\, J(t) + \int_{0^+}^{t} J(t-\tau)\, \frac{d\sigma(\tau)}{d\tau}\, d\tau \qquad (2.16)$$

For a step-function of stress $\sigma = \sigma_0 U(t)$ the above equation yields

$$\epsilon(t) = \sigma_0\, J(t)$$

Hence, for isochronous data obtained at any given time, the linear mechanical behaviour is reflected in a linear strain-stress curve computed for different values of σ_0. As in Fig. 2.8 the mechanical behaviour corresponding to low strains is often only mildly non-linear. For many viscoelastic materials it is possible to define a low-strain region of approximately linear behaviour (up to 2-3 %). Also, it is possible to approximate the isochronous curves by two or more straight lines, depending on the required accuracy. If this is found insufficient to handle the given problem, then one should take refuge in a non-linear characterization. A difficulty however is that there exist a number of non-linear

theories and the experimental characterization programs appropriate to these theories are often rather involved. Several of these theories predict better results when compared to others, depending on the type of the stress or strain histories involved. Various nonlinear characterization methods in the present are discussed in [1], [11], [12], [34], [39]-[42]. We shall have more to say about the nonlinear characterization of viscoelastic materials in Chapter 4.

APPLICATIONS

III

CHARACTERIZATION OF LINEAR VISCOELASTIC MATERIALS THROUGH DIGITAL SPECTRAL ANALYSIS

1. SAMPLING THEOREM

In the present chapter we are concerned with the digital spectral analysis methods in so far as they are important for the direct determination of the quasi-static properties of linearly viscoelastic materials as discussed in section 2.2 above. No attempt will be made to cover the subject matter exclusively and attention will be confined mainly on the essential topics necessary for the application of the digital spectral analysis methods to the characterization of viscoelastic materials. At the end of this chapter we will develop several numerical procedures to obtain the material functions describing the short-time viscoelastic effects by the aid of digital spectral analysis. For a detailed discussion of the digital signal theory, the reader is referred to a vast number of textbooks written on the subject. For a mathematical fresh-up the discussion given in the appendices at the end of this paper will be useful.

Any absolutely integrable function can be sampled at regular intervals without loss of information. The following theorem is known [46],[47]: Let a function $\epsilon(t)$ be band limited, i.e. $\epsilon(t) \longleftrightarrow \epsilon(\omega)$ and $\epsilon(\omega) = 0$ for $|\omega| > \omega_c$ (the arrow denotes the Fourier transform). Then

$$\epsilon(t) = \sum_{n=-\infty}^{\infty} \epsilon(nT)\, \frac{\sin\{\omega(t-nT)\}}{\omega(t-nT)}$$

$$= \sum_{n=-\infty}^{\infty} \epsilon\left(\frac{n\pi}{\omega}\right) \frac{\sin(\omega t - n\pi)}{\omega t - n\pi} \qquad (3.1)$$

where $\omega \geq \omega_c$ and $T = \pi/\omega \leq \pi/\omega_c$. Thus a band-limited function may be represented by the sequence of its samples with a spacing of $T \leq \frac{\pi}{\omega_c}$. Equation (3.1) is known as the sampling theorem in the time domain. Similarly, given a function $\epsilon(t)$ such that $\epsilon(t) = 0$ for $|t| > T_c$, the sampling theorem in the frequency domain is given by

$$\epsilon(\omega) = \sum_{n=-\infty}^{\infty} \epsilon\left(\frac{n\pi}{T}\right) \frac{\sin(\omega T - n\pi)}{\omega T - n\pi} \qquad (3.2)$$

where $\omega \leq \pi/T_c$. Thus the frequency spectrum may equally be represented by the values $\epsilon\left(\frac{n\pi}{T}\right)$, spaced by a distance smaller than or equal to π/T_c. π/T_c is known as the Nyquist interval.

2. SAMPLING OF A CONTINUOUS TIME RECORD

Consider the periodic gate function (Fig. 3.1), which comprises of a train of rectangular pulses of height A

and gate width 2τ, repeating at intervals of T in the range $-\infty < t < \infty$.

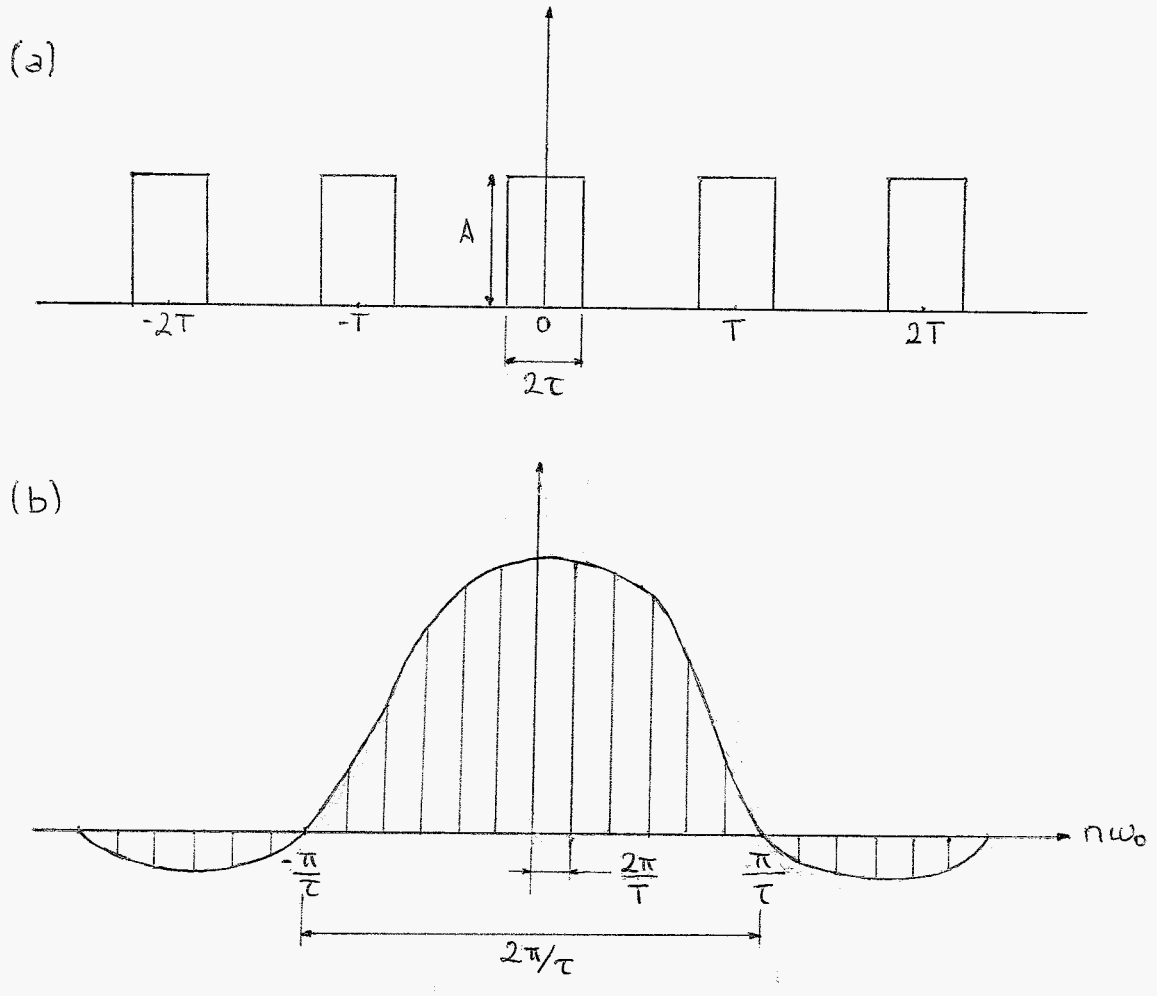


Fig. 3.1. (a) the periodic gate function
(b) the Fourier series representation (line spectrum) of the periodic gate function.

The Fourier series representation of the periodic gate function (Fig. 3.1.b) is given by

$$\epsilon_n = \frac{2A\tau}{T} \frac{\sin n\omega_0\tau}{n\omega_0\tau} \qquad (3.3)$$

This function has a sinc ($\sin x/x$) envelope which exists only for the integer values of n. Therefore, the spect

ral lines occur at harmonics of the repetition frequency $\omega_0 = 2\pi/T$. The distance between the zeros in the sinc envelope is given by $n\omega_0 = \pi/\tau$, such that the number of the spectral lines within the first lobe becomes $n = \pi/\omega_0\tau = T/2\tau$, truncated to the nearest integer. The amplitude at zero frequency, $\frac{2A\tau}{T}$, is called the d.c. amplitude. Now, if τ remains fixed while $T \to \infty$, then we obtain the Fourier transform of a single rectangular pulse of width 2τ. In this case a continuous spectrum is obtained represented by the sinc envelope in Fig. 3.1-b. On the other hand, if T remains finite but τ is reduced to zero in a limiting process such that each pulse has unit area ($A = 1/2\tau$), then the periodic gate function in Fig. 3.1-a becomes a train of equispaced unit impulses as shown in Fig. 3.2.

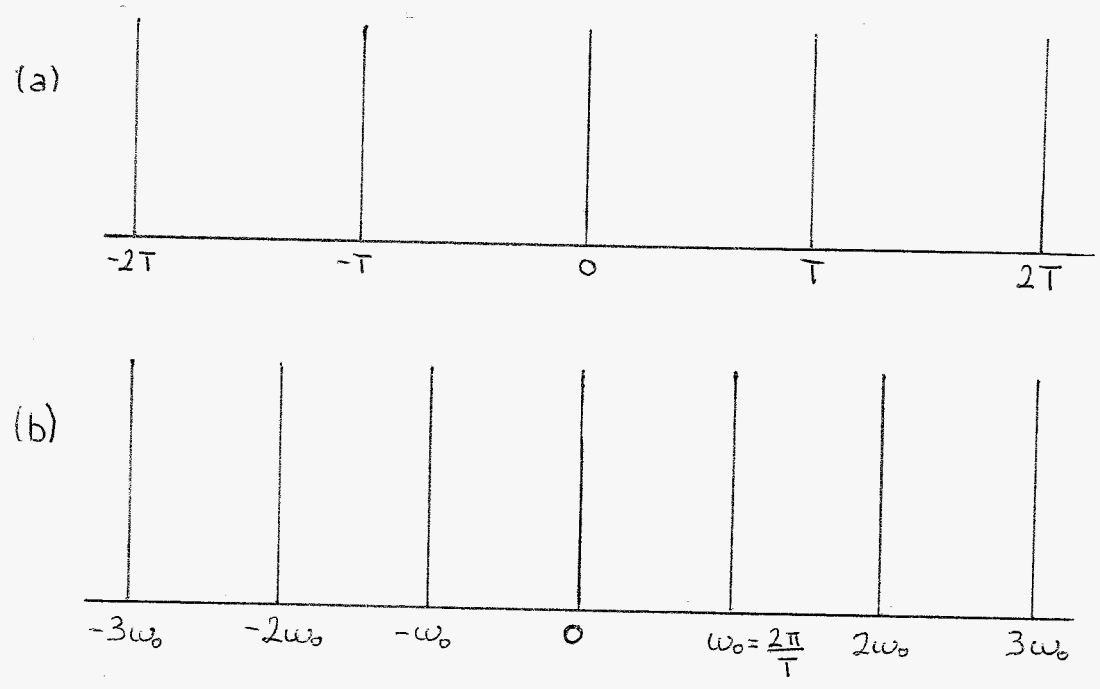


Fig. 3.2 (a) the sampling function with a sampling interval of $T = 2\pi/\omega_0$
(b) the spectrum of the sampling function

The unit impulse train of Fig. 3.2.a is also known as the 'sampling function'. It may be regarded as the limiting case of the periodic gate function as we have seen. The spectrum of the sampling function shown in Fig. 3.2.b is the limiting case of Fig. 3.1.b with the first zero occuring at infinite frequency. The interval between the spectral lines is the sampling frequency ω_s.
The sampling function has the sampling property that multiplication of an arbitrary continuous signal $\epsilon(t)$ by the impulse sequence of Fig. 3.2.a results in the sampled version of the signal in the time domain, as shown in Fig. 3.3 below.

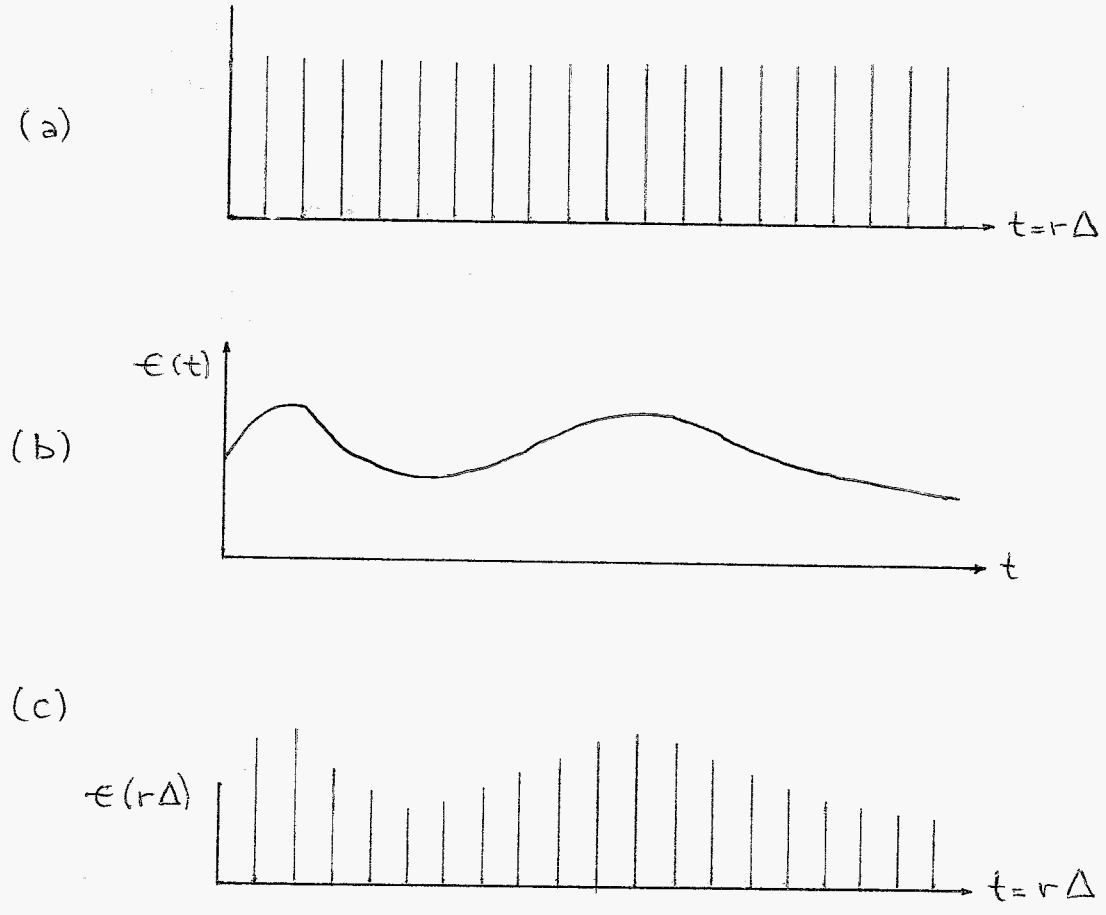


Fig. 3.3. The multiplication of an arbitrary function (b) with the impulse sequence (a) results into the sampled version of that function (c).

Hence the sampled signal may be interpreted as a sequence of impulses weighted by the instantaneous values of the continuous signal at the moments it was sampled. In the time domain the sampled signal may be expressed as

$$\epsilon(t) = \epsilon(t) \otimes \delta(t)$$

$$= \sum_{r=-\infty}^{\infty} \epsilon(r\Delta)\,\delta(t - r\Delta) \qquad (3.4)$$

where Δ is the period of the sampling function (the sampling interval).

The spectrum of the sampled signal can easily be derived from the fact that the convolution process in the time domain is a multiplication process in the frequency domain. Thus, the Fourier transform of (3.3) yields

$$\epsilon(n\omega_0) = \sum_{r=-\infty}^{\infty} \epsilon(r\Delta)\, e^{-j\omega r\Delta} \qquad (3.5)$$

where $\omega = n\omega_0$.

The discrete spectrum thus obtained is called the 'discrete Fourier transform', abbreviated as DFT. It is the Fourier transform of a signal which only exists at intervals of Δ.

Setting $z = e^{j\omega\Delta} = e^{s\Delta}$ in the above equation results in the 'z-transform'.

The discrete Fourier transform will be treated more extensively in the next section.

3. The DFT of a Periodic Signal

A continuous periodic signal $\epsilon(t)$ may be represented by the Fourier series

$$\epsilon(t) = \sum_{n=-\infty}^{\infty} \epsilon_n e^{jn\omega_0 t} \tag{3.6}$$

where the coefficients ϵ_n are given by

$$\epsilon_n = \frac{1}{T} \int_0^T \epsilon(t) e^{-jn\omega_0 t} dt \tag{3.7}$$

If the periodic signal is fed through an analogue-to-digital converter, it will be sampled at regularly spaced times as shown in Fig. 3.4.

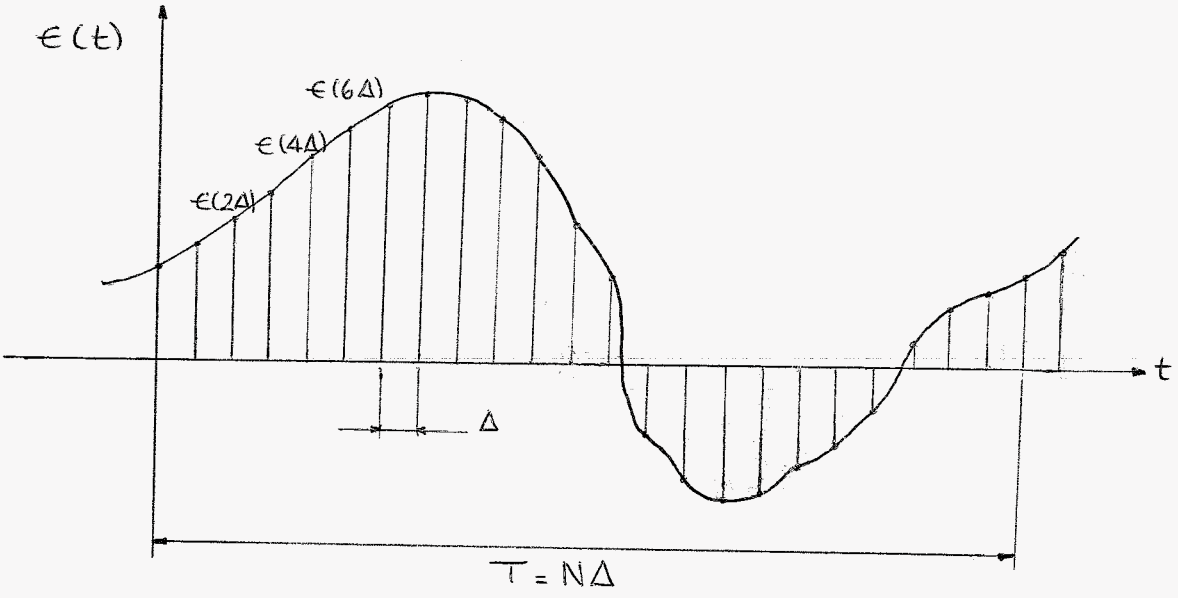


Fig. 3.4. Sampling of a periodic function at regular time intervals Δ.

If the sampling interval is Δ, then the discrete values of $\epsilon(t)$ at time $t = r\Delta$ can be denoted as $\epsilon(r\Delta)$ and the sequence $\{\epsilon(r\Delta)\}$, $r = 0, 1, 2, \ldots$, is called the 'discrete

time series' of $\epsilon(t)$. Since $\epsilon(t)$ is periodic with period $T = N\Delta$, it will suffice to obtain the discrete time series $\{\epsilon(r\Delta)\}$, where $r = 0, 1, 2, \ldots, (N-1)$. In this case the integral (3.7) can be approximated by the sum

$$\epsilon_n = \frac{1}{T} \sum_{r=0}^{N-1} \epsilon(r\Delta)\, e^{-j\left(\frac{2\pi n}{T}\right)(r\Delta)}\, \Delta$$

where $n = 0, 1, 2, \ldots, (N-1)$ and $\omega_0 = \frac{2\pi}{T}$.
Substituting $T = N\Delta$ into the above expression yields

$$\epsilon_n = \frac{1}{N} \sum_{r=0}^{N-1} \epsilon(r\Delta)\, e^{-j(2\pi nr/N)} \qquad (3.8)$$

where $n = 0, 1, 2, \ldots, (N-1)$.
Equation (3.8) is an approximate formula for calculating the coefficients of the Fourier series (3.6). It is the discrete Fourier transform of the discrete time series $\{\epsilon(r\Delta)\}$. Any typical value $\epsilon(r\Delta)$ of the sequence $\{\epsilon(r\Delta)\}$ may be regained exactly through the 'inverse discrete Fourier transform', IDFT, given by

$$\epsilon(r\Delta) = \sum_{n=0}^{N-1} \epsilon_n\, e^{j(2\pi nr/N)} \qquad (3.9)$$

where $r = 0, 1, 2, \ldots, (N-1)$.
The DFT of the discrete time series, $\{\epsilon_n\}$, obtained by the successive application of (3.8) for $n = 0, 1, 2, \ldots, (N-1)$ can be related to the Fourier transform of the underlying continuous function $\epsilon(\omega)$, provided that the underlying function $\epsilon(t)$ has no sudden discontinuities. The relation between the DFT and the Fourier transform is given by

$$\epsilon(\omega) = \epsilon(\omega) \otimes \delta(\omega)$$

$$= \sum_{n=-\infty}^{\infty} \epsilon_n \, \delta\left(\omega - \frac{2\pi n}{N\Delta}\right) \qquad (3.10)$$

When the sequence $\{\epsilon(r\Delta)\}$ is real, it may be shown that

$$\epsilon_{-n} = \epsilon_n^* = \epsilon_{N-n} \qquad \text{for all } n$$

$$\epsilon_{n+N} = \epsilon_n \qquad \text{for all } n \qquad (3.11)$$

$$\epsilon_{(N/2)+n} = \epsilon^*_{(N/2)-n} \qquad \text{for } n = 0, 1, 2, \ldots, (N/2)$$

A consequence of the sampling theorem is that the coefficients ϵ_n calculated by the DFT are the correct Fourier coefficients for frequencies

$$\omega_n = n\omega_0 = \frac{2\pi n}{N\Delta} \leq \frac{\pi}{\Delta} \qquad (3.12)$$

i.e. for n in the range $n = 0, 1, 2, \ldots (N/2)$.
Moreover, if there are frequencies above π/Δ present in the original signal, these introduce a distortion to the spectrum called 'aliasing'. The high frequency components fold back around the 'folding frequency' π/Δ and are superposed on the correct spectrum from 0 to $1/2\Delta$ Hz.
If ω_c is the maximum frequency component present in the signal, then aliasing can be avoided by ensuring that the sampling interval is small, such that $\omega_c < \pi/\Delta$ or, with $f_c = \omega_c/2\pi$

$$\Delta < \frac{1}{2f_c} \qquad (3.13)$$

In general, it is a good rule to select f_c one and a half or two times greater than the maximum anticipated frequency.

Another way of avoiding aliasing is to filter the data prior to sampling so that information above a maximum frequency of interest is removed. This method is preferable above the first, since the cut-off frequency may be chosen equal to the maximum frequency of interest and a better computing economy may be achieved.

4. THE DFT OF A FINITE NONDETERMINISTIC SIGNAL

The Fourier series representation of a 'nondeterministic' or 'random' signal does not exist. Furthermore, since stationary random processes, which are convenient to determine the quasi-static properties of a linear system, theoretically continue forever, the condition $\int_{-\infty}^{\infty} |e(t)|dt < \infty$ is not satisfied. This renders the application of the Fourier transform impossible.
The DFT applies to signals which repeat indefinitely with period T, through which the finite length of record to be transformed is specified, i.e. $N = T/\Delta$.
Now, since practically all data are finite, the length of a finite random record can always be considered as the period of the random signal and the DFT can still be applied. Hence the non-deterministic signal can be assumed to repeat with a period equal to the record length for analytical convenience. In order to be more explicit on this matter, it may be said that the Fourier series and Fourier transforms of

data differ in their theoretical properties but not, for most practical purposes, in their digital computational details. This is because only a finite range Fourier series or transform can actually be computed with digitized data, and this finite range can always be considered as the period of an associated Fourier series. However, this assumption inevitably introduces an error called 'signal leakage', which we shall now discuss.

A finite signal $\epsilon(t)$ recorded in the time domain such that $\epsilon(t)=0$ for $|t| > T/2$ can analytically be represented by

$$\epsilon(t)\, B_{T/2}(t) = \begin{cases} \epsilon(t) & \text{for } |t| < T/2 \\ 0 & \text{for } |t| > T/2 \end{cases} \qquad (3.14)$$

where $B_{T/2}$ is the Boxcar function (Fig. 3.5).

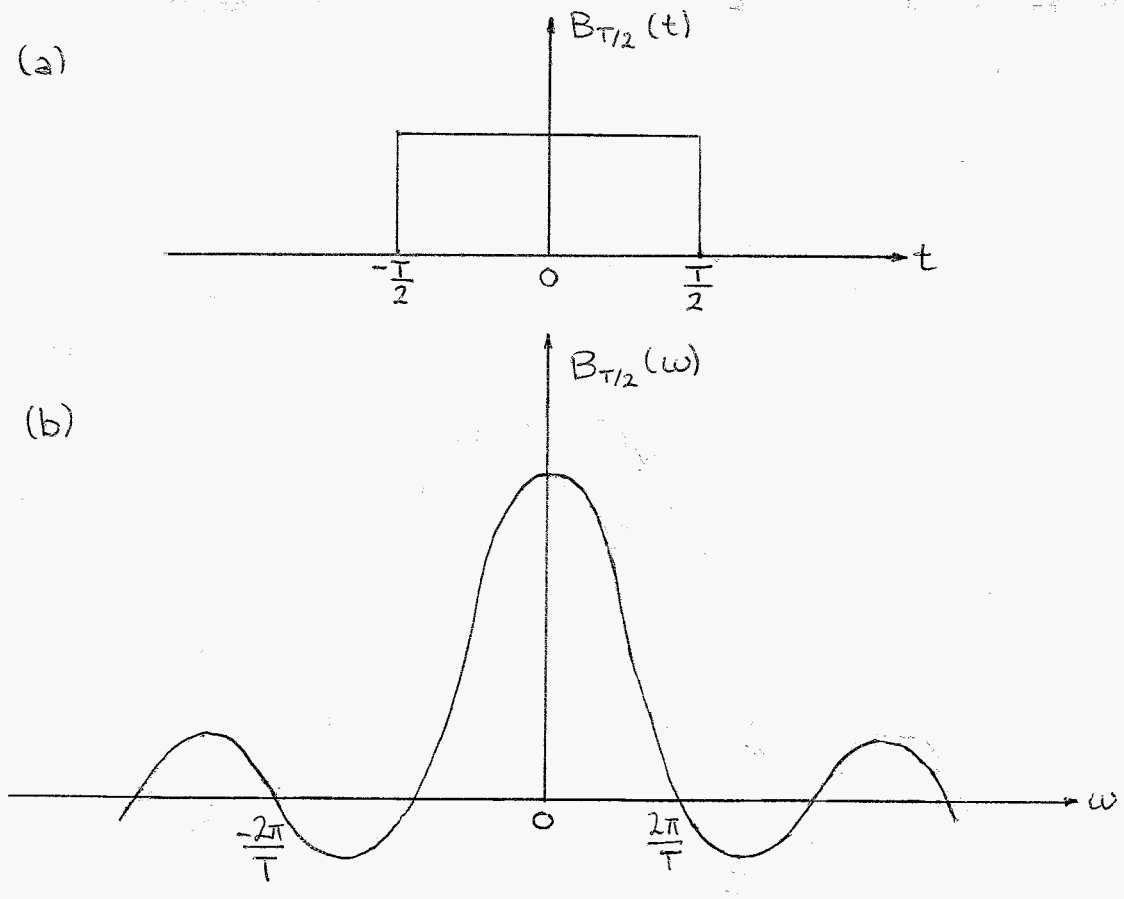


Fig. 3.5. (a) the Boxcar function
(b) the spectrum of the Boxcar function

This process of cutting a signal off in the time domain corresponds to 'weighting' the spectrum of the signal by the spectrum of the boxcar function given in Fig. 3.5.b. The latter spectrum is sometimes called the 'rectangular window' spectrum.
The Fourier transform of (3.14) yields

$$\epsilon(t)\, B_{T/2}(t) \longleftarrow \frac{1}{2\pi}\, \epsilon(\omega) \otimes T\left(\frac{\sin(\omega T/2)}{\omega T/2}\right)$$

$$\longleftarrow \int_{-\infty}^{\infty} \epsilon(u)\, \frac{\sin\{T/2\,(\omega-u)\}}{\pi(\omega-u)}\, du \qquad (3.15)$$

By 'weighting' we mean the convolution of the spectrum of a finite signal with the basic lag window (see also Fig. 3.1.b) in the frequency domain. Theoretically, the correct result would be obtained through

$$\epsilon(t)\cdot 1 \longleftarrow \epsilon(\omega) \otimes \delta(\omega)$$

which is the convolution with the delta function $\delta(\omega)$. The boxcar weighting causes 'signal leakage' because a sharp spectral line is smeared out over a wide range of frequencies (theoretically an infinite number of side lobes in Fig. 3.5.b, the half of which are negative). Particularly the first two side lobes contribute to an erroneous result since they have a height of approximately one-fifth of that of the main lobe. To improve the accuracy of weighting, it is necessary to modify the boxcar weighting in the time domain such that the main lobe in the spectrum is broadened and the side lobes are suppressed.
This may be achieved by 'smoothing' the spectrum through the application of a different window

function. For this purpose, the 'cosine taper data window' shown in Fig. 3.6 is often used.

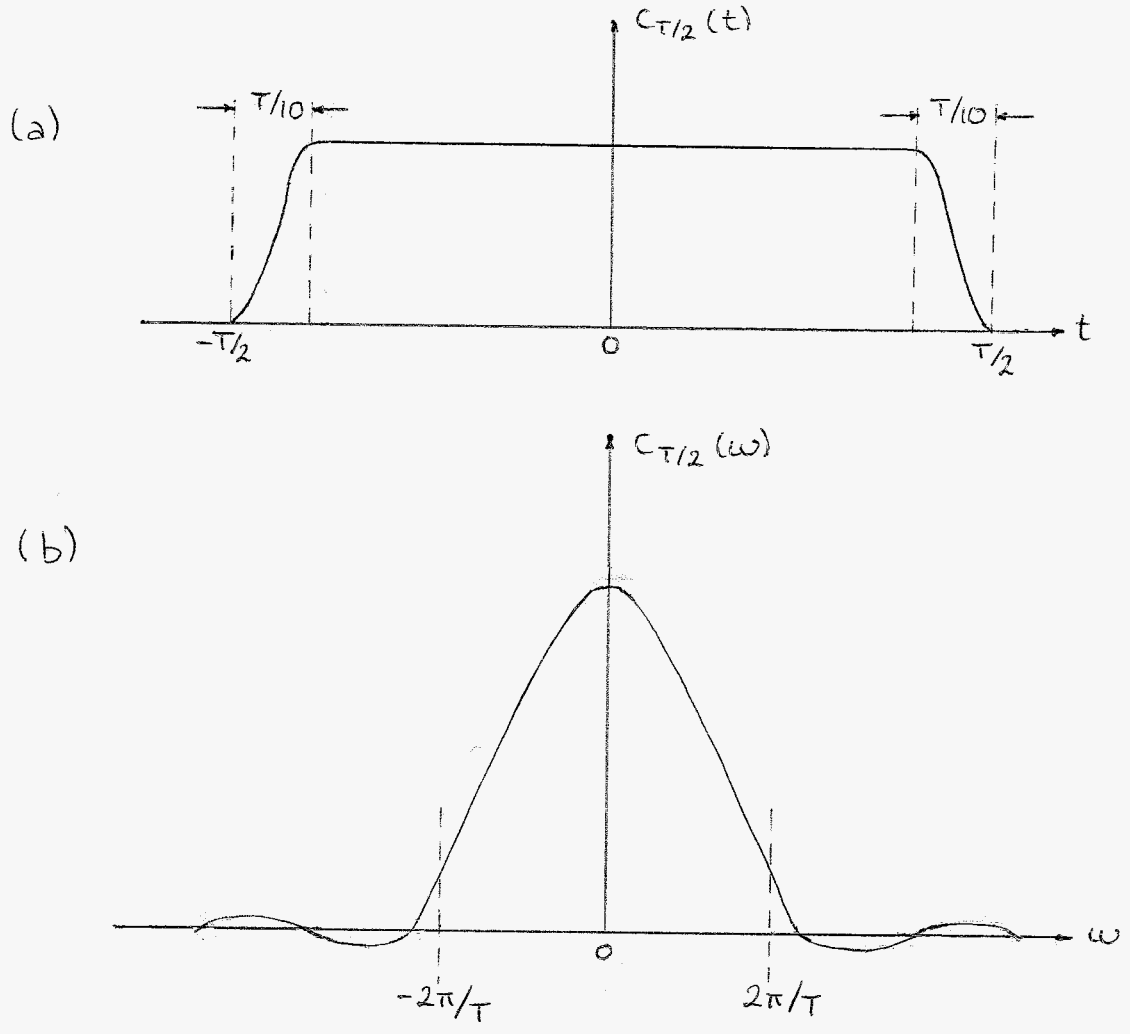


Fig. 3.6 (a) the cosine taper data window
(b) the cosine taper spectral window.

By tapering the original time record at each end, the middle portion of the record is weighted more heavily than the two ends. Since we regard the original time record as a periodic function with period T, by tapering we reach the effect of smoothing the discontinuities at the 'joints' of the periodic function.
A comparison of Fig. 3.5.b and 3.6.b shows that the smoothed spectrum has a larger main lobe and

that the side lobe leakage is suppressed over that of the rectangular window spectrum.

An effect of tapering is to reduce the variance of the tapered data relative to the original data. This variance change given by the ratio of the areas under $B_{T/2}(t)$ and $C_{T/2}(t)$ is $1/0.875$. The smooth spectrum estimates should be multiplied by this factor in order to obtain the correct scale when the cosine taper data window is used.

It is seen that by using a larger main lobe the variance of the measurements is improved. But, this occurs at the cost of frequency resolution, since the frequencies which are close to one another can not easily be distinguished.

Above it is discussed under which conditions the DFT operation may be applied to nondeterministic finite signals. It should also be noted that the sampling process should be performed in accordance with the Nyquist criterion (3.13) such that aliasing will not occur.

For a more detailed treatment of the subject matter in this section, the reader is referred to [47]-[49].

5. ESTIMATION OF THE COMPLEX MODULUS AND COMPLIANCE FOR RANDOM SIGNALS

The complex modulus and compliance of linear viscoelastic materials are defined under steady state conditions. A finite random process due to the random

excitation of the material over a finite length of time is per definition nonstationary and it is not possible to obtain direct information about the frequency composition of the quasi-static properties of the material. This difficulty can be overcome by Fourier analysing, not the sample series of the signals themselves, but their autocorrelation and cross-correlation functions. How this can be done is the subject of the present section.

The mean value, or the statistical expectation of a function $\epsilon(t)$ is defined by

$$m_\epsilon = E[\epsilon] = \int_{-\infty}^{\infty} \epsilon\, p(\epsilon)\, d\epsilon$$

where $p(\epsilon)$ is the probability density function of ϵ. An exact knowledge of the probability density function is in general not available. A possible method of estimating the mean value based upon N independent observations is

$$\overline{m}_\epsilon = \frac{1}{N} \sum_{i=1}^{N} \epsilon_i$$

where $\overline{m}_\epsilon$ is known as the sample mean.

Now the autocorrelation function $R_{\epsilon\epsilon}(\tau)$ of a random function $\epsilon(t)$ is defined as the mean value of the product $\epsilon(t)\,\epsilon(t+\tau)$, i.e.

$$R_{\epsilon\epsilon}(\tau) = E[\epsilon(t)\,\epsilon(t+\tau)] \qquad (3.16)$$

At very large time intervals, i.e. $\tau \to \infty$, the random function will be uncorrelated so that

$$R_{\epsilon\epsilon}(\tau \to \infty) \longrightarrow m_\epsilon^2 \qquad (3.17)$$

In the same way, the cross-correlation function between the random signals $\epsilon(t)$ and $\delta(t)$ is defined as

$$R_{\epsilon\sigma}(\tau) = E[\epsilon(t)\sigma(t+\tau)] \tag{3.18}$$

If the random signals ϵ and σ are uncorrelated for a large time separation τ, then

$$R_{\epsilon\sigma}(\tau \longrightarrow \infty) \longrightarrow m_\epsilon m_\sigma \tag{3.19}$$

When the random excitation $\epsilon(t)$ and its response $\sigma(t)$ have no periodic components, then their mean values $m_\epsilon = E[\epsilon]$ and $m_\sigma = E[\sigma]$ may be normalized such that

$$R_{\epsilon\epsilon}(\tau \longrightarrow \infty) = R_{\epsilon\sigma}(\tau \longrightarrow \infty) = 0 \tag{3.20}$$

Under these conditions we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} |R_{\epsilon\epsilon}(\tau)|\, d\tau < \infty \\ &\int_{-\infty}^{\infty} |R_{\epsilon\sigma}(\tau)|\, d\tau < \infty \end{aligned} \tag{3.21}$$

When the above statistical precautions are taken then the autocorrelation and the cross-correlation functions may be Fourier transformed to give

$$S_{\epsilon\epsilon}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{\epsilon\epsilon}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{3.22}$$

$$S_{\epsilon\sigma}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{\epsilon\sigma}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{3.23}$$

where $S_{\epsilon\epsilon}$ is the auto-spectral density function of strain and $S_{\epsilon\sigma}$ is the cross-spectral density function between the strain and stress signals.
It may be shown that the auto-spectral density function and the cross-spectral density function are related through

$$S_{\epsilon\sigma}(\omega) = H(\omega)\, S_{\epsilon\epsilon}(\omega) \tag{3.24}$$

and

$$S_{\sigma\varepsilon}(\omega) = H^*(\omega)\, S_{\varepsilon\varepsilon}(\omega) \tag{3.25}$$

where $H(\omega)$ is the complex compliance and $H^*(\omega)$ is its complex conjugate. Similar relations hold for the complex modulus and its complex conjugate

$$S_{\varepsilon\varepsilon}(\omega) = E(\omega)\, S_{\varepsilon\sigma}(\omega) \tag{3.26}$$

and

$$S_{\varepsilon\varepsilon}(\omega) = E^*(\omega)\, S_{\sigma\varepsilon}(\omega) \tag{3.27}$$

We may therefore write

$$E(\omega) = \frac{1}{H(\omega)} = \frac{S_{\varepsilon\varepsilon}(\omega)}{S_{\varepsilon\sigma}(\omega)} . \tag{3.28}$$

Now, if the random signals $\varepsilon(t)$ and $\sigma(t)$ are properly sampled, then the DFT coefficients $\{\varepsilon_n\}$ and $\{\sigma_n\}$ can be obtained through the successive application of (3.8). In that case the auto-spectral density and the cross-spectral density functions can easily be computed from

$$S_{\varepsilon\varepsilon_n} = \varepsilon_n^* \varepsilon_n \tag{3.29}$$

and

$$S_{\varepsilon\sigma_n} = \varepsilon_n^* \sigma_n \tag{3.30}$$

after which the complex modulus and compliance can be estimated as

$$E(n\omega) = \frac{S_{\varepsilon\varepsilon_n}}{S_{\varepsilon\sigma_n}} = \frac{1}{H(n\omega)} \tag{3.31}$$

The above calculations can be performed quickly through the application of fast Fourier transform (FFT) procedures. The fast Fourier transform is a computer algorithm which economizes the computation of the DFT to a great extend. For common FFT algorithms it is required that the num-

ber of samples N is a power of two, i.e. $N = 2^x$ where x is a positive integer. This may easily be achieved by extending the record length through adding a sufficient number of zero's.

The spectral window which is 'inherent' to the most common FFT procedures is a triangular window given by

$$W_{T/2}(t) = \begin{cases} 1 - \frac{2|t|}{T} & \text{for } |t| \leq T/2 \\ 0 & \text{elsewhere} \end{cases} \tag{3.32}$$

The non-uniform weighting may be compensated by frequency smoothing, but the resulting error can only be reduced at the expense of frequency resolution as before. The FFT procedures are discussed at length in [48], [49].

It is noted finally that the digital composition of the complex modulus and the complex compliance can be expressed in accordance with eq. (3.10) in the continuous form as

$$E(\omega) = \sum_{n=-\infty}^{\infty} E(n\omega)\, \delta\left(\omega - \frac{2\pi n}{N\Delta}\right) \tag{3.33}$$

and

$$H(\omega) = \sum_{n=-\infty}^{\infty} H(n\omega)\, \delta\left(\omega - \frac{2\pi n}{N\Delta}\right) \tag{3.34}$$

Equivalently, equation (3.2) may apply. Thus, we may write for instance

$$E(\omega) = \sum_{n=-\infty}^{\infty} E(n\omega)\, \frac{\sin(\omega N\Delta - n\pi)}{\omega N\Delta - n\pi} \tag{3.35}$$

6. DETERMINATION OF THE RELAXATION AND CREEP FUNCTIONS FROM QUASI-STATIC MEASUREMENTS

In the previous sections we have discussed the applications of the digital spectral analysis methods to the characterization of the quasi-static material functions. The determination of the complex modulus and the complex compliance functions from periodical histories is preferred above their estimation from random histories, since the former method leads to more accurate results by the aid of a less involved procedure. Specifically, no special care must be taken for the signal leakage, provided that the signals are sampled over a whole period and in accordance with the Nyquist criterion. In both cases, the FFT algorithms may be efficiently employed to economize the computation procedure.

Once the complex modulus and the complex compliance are known, the corresponding kernels can be computed by the aid of the inverse discrete Fourier transform. Thus, if, for instance, the original time series $\{\epsilon(r\Delta)\}$ is periodic with $T = N\Delta$, then in view of eq. (3.9) equation (1.13) can be written in the form

$$g(r\Delta) = \sum_{n=0}^{N-1} \frac{b_n}{\epsilon_n} e^{j(2\pi nr/N)} + A \cdot \delta(r\Delta) \qquad (3.36)$$

where $r = 0, 1, \ldots, (N-1)$. Eq. (3.36) enables one to calculate the relaxation kernel from the digital data on the composition of the complex modulus.

The relaxation function can be obtained from the relaxation kernel through

$$G(t) = \int_0^t \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} (A - E(j\omega)) e^{j\omega\tau} d\omega \right] d\tau$$

which may be written in discrete form as

$$G(r\Delta) = -\sum_{r=0}^{N-1} \sum_{n=0}^{N-1} \frac{b_n}{\epsilon_n} e^{j(2\pi nr/N)} \Delta + A \cdot \Delta \sum_{r=0}^{N-1} \delta(r\Delta)$$

$$= -\sum_{r=0}^{N-1} \sum_{n=0}^{N-1} \frac{b_n}{\epsilon_n} e^{j(2\pi nr/N)} \cdot \Delta + A\Delta \cdot U(r\Delta) \qquad (3.37)$$

The relaxation function can also be obtained directly from the complex modulus. Putting

$$\psi_n = \frac{b_n/\epsilon_n}{j\omega_n} = \frac{b_n/\epsilon_n}{j\frac{2\pi n}{N\Delta}} \qquad (3.38)$$

eq. (1.22) can be written as

$$G(r\Delta) = \sum_{r=0}^{N-1} \psi_n e^{j(2\pi nr/N)} \qquad (3.39)$$

To obtain a continuous expression for the relaxation function one can write in accordance with the sampling theorem (4.1)

$$G(t) = \sum_{r=0}^{N-1} G(r\Delta) \frac{\sin\{\omega(t-r\Delta)\}}{\omega(t-r\Delta)} \qquad (3.40)$$

where $\omega = 2\pi/N\Delta$.

Similar results are obtained for the creep measures by interchanging the input and the output signals. The creep kernel can be calculated from the quasi-static data on the composition of the complex compliance function by the formula

$$h(r\Delta) = \sum_{n=0}^{N-1} \frac{\epsilon_n}{\sigma_n} e^{j(2\pi nr/N)} - B \cdot \delta(r\Delta) \qquad (3.41)$$

where $r = 0, 1, \ldots\ldots, (N-1)$, Δ is the sampling interval, N is the record-length and $T = N\Delta$.
The creep function can be obtained from the creep kernel through

$$J(r\Delta) = \sum_{r=0}^{N-1} \sum_{n=0}^{N-1} \frac{\epsilon_n}{\sigma_n} e^{j(2\pi nr/N)} \cdot \Delta - B \cdot \Delta \cdot U(r\Delta) \qquad (3.42)$$

Or alternatively, it may be obtained directly from the complex compliance by virtue of the inverse discrete Fourier transform

$$J(r\Delta) = \sum_{r=0}^{N-1} \theta_n e^{j(2\pi nr/N)} \qquad (3.43)$$

where

$$\theta_n = \frac{\epsilon_n/\sigma_n}{j\frac{2\pi n}{N\Delta}} . \qquad (3.44)$$

To obtain a continuous expression for the creep function we write in this case

$$J(t) = \sum_{r=0}^{N-1} J(r\Delta) \frac{\sin\{\omega(t-r\Delta)\}}{\omega(t-r\Delta)} \qquad (3.45)$$

where again $\omega = 2\pi/N\Delta$.

The above calculations can be economized by the aid of FFT algorithms. This should be preceded by the determination of the impact modulus A and the equilibrium moduli G_∞ and J_∞.

7. DETERMINATION OF THE VISCOELASTIC SPECTRA FROM QUASI-STATIC MEASUREMENTS

It was already discussed in section 1.6 under which conditions the viscoelastic spectra can be obtained from their corresponding kernels through the application of the Fourier transforms. Here we will simply reproduce the relations through which the spectra - if they exist - can be obtained from the quasi-static data.

Applying IDFT to eq. (3.36) one obtains for the relaxation spectrum

$$F(\alpha_n) = \sum_{r=0}^{N-1} \left\{ -\sum_{n=0}^{N-1} \frac{b_n}{\epsilon_n} e^{j(2\pi nr/N)} \right\} e^{j(2\pi nr/N)}$$

$$+ A \sum_{r=0}^{N-1} \delta(r\Delta) e^{j(2\pi nr/N)}$$

$$= -\sum_{r=0}^{N-1} \sum_{n=0}^{N-1} \frac{b_n}{\epsilon_n} e^{j(4\pi nr/N)} + A \qquad (3.46)$$

where $n = 0, 1, \ldots\ldots, (N-1)$.

Similarly one has from (3.41) for the creep spectrum

$$S(\beta_n) = \sum_{r=0}^{N-1} \sum_{n=0}^{N-1} \frac{\epsilon_n}{b_n} e^{j(4\pi nr/N)} + B \qquad (3.47)$$

These calculations can be performed economically by the aid of the FFT. In a similar fashion, the viscoelastic spectra can also be obtained from the relaxation and creep functions. The derivation of these relations is left as an exercise to the reader.

IV

APPLICATIONS TO NONLINEAR CHARACTERIZATION

1. NONLINEAR THEORIES

There exist numerous nonlinear theories of viscoelasticity in the literature and there is no general agreement as to which theory is suitable to model the nonlinear mechanical behaviour of viscoelastic materials. Many attempts have been made to model experimental data by the aid of the existing theories and to compare them with one another, although this was not sufficient to evaluate conclusively any theory. It seems that the success or failure of a given theory depends on the specific application.

It is beyond the scope of the present paper to discuss the mathematical details of all the existing nonlinear theories of viscoelasticity. Basing ourselves on the summarizing works in the literature [11], [40]-[42] we will consider here the possible applications of the quasi-static methods to the nonlinear characterization of viscoelastic materials. A detailed treatment of the major part of the discussion given here, as well as the determination of the material parameters occuring in these nonlinear theories can be found in the above-mentioned works.

The most general nonlinear theory of viscoelasticity is the Green-Rivlin theory, in which the homogeneous functional relating strain to stress is expressed as a polynomial expansion of linear functionals of the strain history. The one-dimensional

form of this theory for non-aging materials is given by the expression

$$\sigma(t) = \int_{-\infty}^{t} G_1(t-\tau)\,\dot{\epsilon}(\tau)\,d\tau + \int_{-\infty}^{t}\int G_2(t-\tau_1, t-\tau_2)\,\dot{\epsilon}(\tau_1)\,\dot{\epsilon}(\tau_2)\cdot$$

$$d\tau_1 d\tau_2 + \int_{-\infty}^{t}\int\int G_3(t-\tau_1, t-\tau_2, t-\tau_3)\,\dot{\epsilon}(\tau_1)\,\dot{\epsilon}(\tau_2)\,\dot{\epsilon}(\tau_3)\,d\tau_1 d\tau_2 d\tau_3$$

$$+ \cdots \qquad (4.1)$$

where the integration functions G_n can be assumed, without loss of generality, to be symmetric with respect to their arguments.

Eq. (4.1) is not suitable for practical applications and various approximate methods of characterization have been proposed [42]. One important approximation method is the so-called modified superposition principle, wherefore the higher order kernel functions G_n are restricted to a function of the smallest time argument, or of the most recent occurence. In this way, it is possible to reduce the multiple integral polynomial into a single integral representation so that eq. (4.1) becomes

$$\sigma(t) = \int_{-\infty}^{t} G_1(t-\tau)\frac{d\epsilon}{d\tau}\,d\tau + \int_{-\infty}^{t} G_2(t-\tau)\frac{d}{d\tau}\left[\epsilon^2(\tau)\right]d\tau +$$

$$+ \int_{-\infty}^{t} G_3(t-\tau)\frac{d}{d\tau}\left[\epsilon^3(\tau)\right]d\tau + \cdots \qquad (4.2)$$

where

$$G_2(t-\tau) = G_2(t-\tau, t-\tau), \text{ etc.}$$

For practical applications, the integral polynomials of infinite order are truncated after terms of

some order, often the third. Even then the resulting one-dimensional theory is too complicated for routine practical application.

Another special theory is obtained through the generalization of the linear viscoelastic law such that

$$F[\sigma] = \int_{-\infty}^{t} G(t-\tau) \frac{d}{d\tau} H[\epsilon] d\tau \qquad (4.3)$$

Special forms of eq. (4.3) have been used by Leaderman, Rabotnow and Fung. It is shown in [40] and [50] that the eq. (4.3) can be obtained as a specialization of the Green-Rivlin theory. It is evident that eq. (4.3) is less general than the Green-Rivlin formulation, since the former involves only a single linear functional. In addition to the above-mentioned theories, there are several other important theories one may come across in the literature. The important methods of non-linear characterization can be summarized as

1) Green-Rivlin theory;
2) Approximations of the Green-Rivlin theory through special assumptions on the form of the kernel functions;
3) Leaderman's theory
4) Rabotnow's theory
5) Semi-empirical theory of Fung
6) Lianis's theory of finite linear viscoelasticity
7) BKZ-theories (Bernstein, Kearsley and Zapas)
8) Schapery's theory of stress-time superposition
9) Empirical function fitting.

In addition to these, the general theory of viscoelasticity due to Volterra is discussed in [1].

It is noted that the tests performed so far to verify these theories are relatively simple in their time dependence; the modelling of the material behaviour has been obtained in most cases for single and multi-step loading. There are but few experiments reported in the literature in which the material characterization is obtained through other types of histories.

2. Complex Modulus Formulation

The complex modulus formulation corresponding to the Green-Rivlin theory is given in [1]. To reproduce it we write eq. (4.1) in the form

$$\sigma(t) = \sum_{n=1}^{N} \sigma_n(t) \tag{4.4}$$

where

$$\sigma_n(t) = \int_0^\infty \cdots \int_0^\infty G_n(\tau_1, \ldots, \tau_n)\, \dot{\epsilon}(t-\tau_1) \cdots \dot{\epsilon}(t-\tau_n)\, d\tau_1 \cdots d\tau_n \tag{4.5}$$

Eq. (4.4) can be Fourier transformed as

$$\sigma(\omega) = \sum_{n=1}^{N} \sigma_n(\omega) \tag{4.6}$$

where the Fourier transform of the relaxation functions G_n are defined by

$$G_n(\omega_1,\omega_2,\ldots,\omega_n)=\int_0^\infty\cdots\int_0^\infty G_n(\tau_1,\tau_2,\ldots,\tau_n)\cdot e^{-j(\omega_1\tau_1+\omega_2\tau_2+\cdots+\omega_n\tau_n)}\cdot d\tau_1 d\tau_2\cdots d\tau_n \qquad (4.7)$$

Analogous to eq. (1.21) and with $G_n(\infty)=0$, we define a set of complex moduli $\bar{E}_n$ by

$$\bar{E}_n(\omega_1,\ldots,\omega_n)=j^n\omega_1\omega_2\cdots\omega_n G_n(\omega_1,\ldots,\omega_n) \qquad (4.8)$$

Using (4.7) and (4.8), the Fourier transform of (4.5) can be expressed as

$$\sigma_n(\omega)=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\bar{E}_n(\omega_1,\ldots,\omega_n)\,\epsilon(\omega_1)\cdots\epsilon(\omega_n)\cdot d\omega_1\cdots d\omega_n \qquad (4.9)$$

Equations (4.6) and (4.9) represent the complex modulus formulation of the Green-Rivlin theory. For $n=1$ they reduce to

$$\sigma(\omega)=\bar{E}_1(\omega)\,\epsilon(\omega)$$

which is equivalent to the complex modulus formulation used in the linear theory. The experimental determination of the quasi-static material functions defined above is rather involved. A large number of quasi-static tests are required and the efficient digital spectrum analysis techniques discussed in the previous chapter can no longer be employed. In the remaining part of this section we will briefly

discuss the determination of the stress components for an alternating strain history. Consider a strain history with a finite number of frequency components such that

$$\epsilon(t) = \sum_{m=1}^{M} \epsilon_m \sin \omega_m t$$

Substituting this expression in (4.5) yields with $n=1$

$$b_1(t) = \sum_{m=1}^{M} \epsilon_m \omega_m \int_0^{\infty} G_1(\tau) \cos\{\omega_m (t-\tau)\} d\tau$$

$$= \sum_{m=1}^{M} \epsilon_m \left\{ \bar{E}_1'(\omega_m) \cos\omega_m t + \bar{E}_1''(\omega_m) \sin\omega_m t \right\} \quad (4.10)$$

where $\bar{E}_1'$ and $\bar{E}_2''$ denote the real and the imaginary parts of the first-order complex modulus $\bar{E}_1$ respectively. Similarly, for $n=2$ one gets

$$b_2(t) = \sum_{m=1}^{M} \sum_{k=1}^{M} \epsilon_m \epsilon_k \omega_m \omega_k \int_0^{\infty}\int_0^{\infty} G_2(\tau_1, \tau_2) \cos\{\omega_m (t-\tau_1)\}$$

$$\cdot \cos\{\omega_k (t-\tau_2)\} d\tau_1 d\tau_2 .$$

Using (4.8) and trigonometric identities, this can be written as

$$b_2(t) = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{M} \epsilon_m \epsilon_k \left\{ -\bar{E}_2'(\omega_m, \omega_k) \cos[(\omega_m + \omega_k)t] + \right.$$

$$\left. + \bar{E}_2''(\omega_m, \omega_k) \sin[(\omega_m + \omega_k)t] + \bar{E}_2'(\omega_m, \omega_k) \cos[(\omega_m - \omega_k)t] \right.$$

$$- \bar{E}_2''(\omega_m, -\omega_k) \sin[(\omega_m - \omega_k)t] \Big\} \qquad (4.11)$$

The higher stress components can be calculated in a similar way. We will now illustrate the above theory for the special case of $n=2$ and $M=1$. A discussion of the second-order theory with two frequency components ($M=2$) can be found in [40]. For a strain input $\epsilon(t) = \epsilon_1 \sin\omega_1 t$, eq.'s (4.4), (4.10) and (4.11) yield

$$\sigma(t) = X(\omega_1, \epsilon_1) + \sum_{n=1}^{2} \Big\{ Y_n(\omega_1, \epsilon_1) \cos n\omega_1 t + Z_n(\omega_1, \epsilon_1) \sin n\omega_1 t \Big\} \qquad (4.12)$$

where

$$\begin{aligned} X(\omega_1, \epsilon_1) &= \tfrac{1}{2} \epsilon_1^2 \bar{E}_2'(\omega_1, -\omega_1) \\ Y_1(\omega_1, \epsilon_1) &= \epsilon_1 \bar{E}_1''(\omega_1) \\ Y_2(\omega_1, \epsilon_1) &= -\tfrac{1}{2} \epsilon_1^2 \bar{E}_2'(\omega_1, \omega_1) \\ Z_1(\omega_1, \epsilon_1) &= \epsilon_1 \bar{E}_1'(\omega_1) \\ Z_2(\omega_1, \epsilon_1) &= \tfrac{1}{2} \epsilon_1^2 \bar{E}_2''(\omega_1, \omega_1) \end{aligned} \qquad (4.13)$$

If $\bar{E}_2 = 0$, then eq.'s (4.12) - (4.13) reduce to the linear theory. When a spectral analysis of the stress response to the applied strain is made, one will obtain values for $\bar{E}_1(\omega_1)$, $\bar{E}_1''(\omega_1)$, $\bar{E}_2'(\omega_1, \omega_1)$, $\bar{E}_2'(\omega_1, -\omega_1)$ and $\bar{E}_2''(\omega_1, \omega_1)$ at the particular frequency ω_1. To express these moduli as functions of frequency, this procedure must be repeated for a sufficient number of frequencies ω_m in the desired range. It is noted finally that the following identities hold:

$$\bar{E}_1'(-\omega_1) = \bar{E}_1'(\omega_1) \qquad \bar{E}_1''(-\omega_1) = -\bar{E}_1''(\omega_1)$$

$$\bar{E}_2'(-\omega_1,-\omega_1) = \bar{E}_2'(\omega_1,\omega_1) \qquad \bar{E}_2''(-\omega_1,-\omega_1) = -\bar{E}_2''(\omega_1,\omega_1)$$

$$\bar{E}_2''(\omega_1,-\omega_1) = 0.$$

3. RELAXATION KERNEL FORMULATION

An alternative theory which is much easier to use is given by eq. (4.3) above. The corresponding complex modulus is obtained analogous to the linear theory. Consider for this purpose the equation

$$F[\sigma] = A\,H[\epsilon] - \int_0^t g(t-\tau)\,H[\epsilon(\tau)]\,d\tau \qquad (4.14)$$

where F and H are linear functionals of respectively σ and ϵ. The Fourier transform of (4.14) yields

$$F(\omega) = \bar{E}(j\omega)\,H(\omega) \qquad (4.15)$$

where the complex modulus $\bar{E}(j\omega)$ is of the form (1.11). In order to express (4.14) in terms of the relaxation function we substitute $g(t-\tau) = \frac{dG(t-\tau)}{d\tau}$ so that

$$F[\sigma] = A\,H[\epsilon(t)] - \int_0^t \frac{dG(t-\tau)}{d\tau}\,H[\epsilon(\tau)]\,d\tau$$

or, upon partial integration with respect to τ

$$F[\sigma] = G(t)\,H[\epsilon(0)] + \int_0^t G(t-\tau)\,\frac{d}{d\tau}\,H[\epsilon(\tau)] \qquad (4.16)$$

where we put $A = G(0)$. An equation of the form (4.16) with $F[\sigma] \equiv \sigma$ has been used by Fung [35] and his associates in the description of various soft biological tissues. The theory of Leaderman is of the form (4.3), i.e. $H[\epsilon(0)] = 0$ in (4.16) and further $F[\sigma] \equiv \sigma$. The theory used by Rabotnow is similar to that proposed by Leaderman in being of the form

(4.3) with $F[\sigma]\equiv\sigma$, but it differs from it in that different forms of the functional H are used for increasing and decreasing regions of the strain history. In [34] a semi-empirical method is described to determine the relation between $F[\sigma]\equiv\sigma$ and $H[\dot{\varepsilon}]$ through a sufficiently high rate of loading tests. This is especially useful to predict the short-time behaviour of the given material. For the description of the viscoelastic effects at longer times, the relation between F and H can be determined from the isochronous data over the relevant time range. Equations of the form (4.14) or (4.16) are linear even when nonlinear stress and strain measures F and H are used. This enables one to apply the characterization procedures described in the previous chapters also in the nonlinear range since the integral transforms involved are valid regardless the forms of the functionals F and H.

A more general theory is proposed in [1], where the Volterra equation (4.14) has a uniformly continuous kernel which need not necessarily be a function of the difference of its arguments. Equations of this form are still linear and can be solved by the method of successive approximations. Such a theory is especially useful when the mechanical properties of the given material alters in time as in the case of various plastic resins, concrete and biological materials that age. It is also possible to treat the short-time and the long-time viscoelastic effects separately, since a uniformly continuous

kernel $k(t,\tau)$ can be decomposed into two parts such that

$$k(t,\tau) = g(t,\tau) + p(t,\tau) .$$

The kernel $g(t,\tau)$ may be chosen appropriate to describe the short-time viscoelastic effects. Since the material properties can be assumed to be time-invariant for sufficiently short times, there is nothing against setting $g(t,\tau) \equiv g(t-\tau)$. This enables one to determine the short-time behaviour from quasi-static methods discussed above. The kernel $p(t,\tau)$ is a degenerate kernel which may be written in the form

$$p(t,\tau) = \sum_{k=1}^{n} a_k(t)\, b_k(\tau)$$

The functions a_k and b_k can be selected to approximate the relaxation and creep curves to any desired degree of accuracy. A possible choice is of course

$$p(t,\tau) = \sum_{k=1}^{n} F_k\, e^{\alpha_k t}\, e^{-\alpha_k \tau}$$

where the coefficients F_k, α_k may be interpreted as a viscoelastic spectrum. In this way, it is possible to combine the integral and phenomenological theories of viscoelasticity within a special nonlinear theory accounting both for the short-term and the long term viscoelastic effects. It is noted that power functions are more likely to be successful in predicting the behaviour of various polymeric materials [12].

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APPENDIX A : Fourier Series and Transform

A.1. Fourier Series

The basis of the Fourier series representation is that a given periodical function can be represented by a sum of a set of orthogonal functions and that the representation improves as the number of the functions in the orthogonal set increases.
Let the function $\epsilon(t)$ be piecewise smooth and periodic with period T, i.e. $\epsilon(t+T) = \epsilon(t)$ for all t; then the Fourier series of $\epsilon(t)$ is convergent with the sum

$$\frac{1}{2}\left\{\epsilon(t^+) + \epsilon(t^-)\right\} = \sum_{n=-\infty}^{\infty} \epsilon_n e^{jn\omega_0 t} \qquad (A.1)$$

where $\omega_0 = 2\pi/T$ and the coefficients ϵ_n are given by

$$\epsilon_n = \frac{1}{T}\int_0^T \epsilon(t) e^{-jn\omega_0 t}\, dt \qquad (A.2)$$

Further, if $\epsilon(t)$ is a continuous function of t, then $\epsilon(t^+) = \epsilon(t^-) = \epsilon(t)$ and the sum of the Fourier series equals $\epsilon(t)$. A periodic signal $\epsilon(t)$ is therefore represented in the frequency domain by the coefficients ϵ_n. ω_0 is known as the 'fundamental frequency' or the 'repetition frequency'. The complex coefficients ϵ_n form a discrete frequency spectrum as shown in Fig. 2.4 above.

A.2 Fourier Transform

When the function $\epsilon(t)$ is not periodic, it may still be conceived as a periodic function with period $T \to \infty$ and its frequency domain representation will in this case be given by the Fourier transform. Let $\epsilon(t)$ be an absolutely integrable piecewise smooth function in the interval $[-\frac{1}{2}T, \frac{1}{2}T]$. Then $\epsilon(t)$ can be represented by the integral sum of harmonical components $\epsilon(\omega)e^{j\omega t}$ as $T \to \infty$ such that

$$\epsilon(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon(\omega) e^{j\omega t} d\omega \qquad (A.3)$$

For an infinite period $\Delta\omega \to 0$ in Fig. 2.4 and a continuous frequency spectrum is obtained. The function $\epsilon(\omega)$ is called the 'Fourier transform' or the 'spectrum' of $\epsilon(t)$. It can be conceived as the limiting case of the Fourier series representation of a periodical function whose period tends to infinity. It is obtained from the original function by the formula

$$\epsilon(\omega) = \int_{-\infty}^{\infty} \epsilon(t)\, e^{-j\omega t} dt \qquad (A.4)$$

The functions $\epsilon(t)$ and $\epsilon(\omega)$ form a 'Fourier transform pair'. Symbolically we write

$$\epsilon(t) \longleftrightarrow \epsilon(\omega)$$

Eq. (A.3) is called the 'inverse Fourier transform. If the function $\epsilon(t)$ is real – as is the case for a signal – then it can be shown that $\epsilon(\omega)$ is a hermi-

tic function, i.e. $\epsilon(-\omega) = \epsilon^*(\omega)$ where $\epsilon^*(\omega)$ is the complex conjugate of $\epsilon(\omega)$. Also the reverse is true.

APPENDIX B: Definitions of Some Important Signal Functions

1) Heaviside Unit function:

$$U(t) = \begin{cases} 0 & \text{for} \quad t<0 \\ 1 & \text{for} \quad t>0 \end{cases} \tag{B.1}$$

2) Boxcar function:

$$B_T(t) = \begin{cases} 1 & \text{for} \quad |t| < \frac{1}{2}T \\ 0 & \text{for} \quad |t| > \frac{1}{2}T \end{cases} \qquad T>0 \tag{B.2}$$

3) Triangular pulse:

$$q_T(t) = \begin{cases} 1 - \frac{|t|}{T} & \text{for} \quad |t| \le T \\ 0 & \text{for} \quad |t| > T \end{cases} \qquad T>0 \tag{B.3}$$

$$= \frac{1}{T} B_T(t) \otimes B_T(t)$$

4) Sign function:

$$\text{sgn}(t) = \begin{cases} -t & \text{for} \quad t<0 \\ 0 & \text{for} \quad t=0 \\ t & \text{for} \quad t>0 \end{cases} \tag{B.4}$$

5) Dirac delta function:

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{B.5}$$

Note: The symbol $\otimes$ denotes the convolution process (Appendix D).

APPENDIX C: Generalized Functions and Generalized Fourier Transform

C.1. Generalized Functions

The definitions of the Dirac delta function given by eq.'s (B.5) are contradictory, because if a function equals zero except at one point, then the integral of that function also equals zero. A remedy to this difficulty was given by Schwartz who introduced the concept of generalized function or distribution, which is a generalization of the classical concept of function. Consider the set of functions $P_\alpha(t)$

$$P_\alpha(t) = \begin{cases} 1/\alpha & \text{for } |t| < \frac{1}{2}\alpha \\ 0 & \text{for } |t| > \frac{1}{2}\alpha \end{cases} \qquad (C.1)$$

where $\alpha > 0$. $P_\alpha(t)$ describes a rectangular pulse with amplitude $1/\alpha$ and duration α. If, further a continuous function $f(t)$ is given, then we have the following rule:

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = \lim_{\alpha \to 0} \int_{-\infty}^{\infty} P_\alpha(t) f(t) dt = f(0) \qquad (C.2)$$

This leads to the alternative definition of the delta function

$$\delta(t) = \text{'}\lim_{\alpha \to 0}\text{'} \; P_\alpha(t). \qquad (C.3)$$

According to this definition, the delta function is the 'limit' of a train of pulse with increasing height and decreasing duration so that the area of each of the

pulses always equals unity. For this reason, $\delta(t)$ is also known as the unit-impulse function. The delta function is the integral of the Heaviside unit function since

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(\tau)\,u(t-\tau)\,d\tau = \delta(t) \otimes u(t) = u(t) . \tag{C.4}$$

One may therefore write

$$\delta(t) = u'(t)$$

where the derivative is taken in the 'generalized sense'. Note that $u(t)$ is not differentiable at $t=0$ in the usual sense. In the following, we list the properties of the delta function.

C.2 Properties of the Delta Function

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{C.5}$$

$$\int_{-\infty}^{\infty} \delta(t)\,f(t)\,dt = f(0) \tag{C.6}$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,f(t)\,dt = f(t_0) \tag{C.7}$$

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t) \quad ; \quad \delta(t) = u'(t) \tag{C.8}$$

$$f(t) \otimes \delta(t) = f(t) \tag{C.9}$$

$$t\,\delta(t) = 0 \tag{C.10}$$

$$f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0) \tag{C.11}$$

$$\delta(at) = \frac{1}{|a|}\,\delta(t) \quad ; \quad \delta(-t) = \delta(t) \tag{C.12}$$

C.3 Generalized Fourier Transform

The Fourier transform of a function $\epsilon(t)$ exists under the condition $\int_{-\infty}^{\infty} |\epsilon(t)|dt < \infty$. There are however some important functions such as constant functions or step-functions which are not absolutely integrable. To overcome this difficulty, the classical Fourier transform can be generalized in the following way:

$$\epsilon(\omega) = \lim_{\alpha \downarrow 0} \int_{-\infty}^{\infty} e^{-\alpha|t|}\, \epsilon(t)\, e^{-j\omega t} dt \qquad (C.13)$$

It can be shown that if $\epsilon(t)$ is absolutely integrable, then eq. (C.13) reduces to the definition of the classical Fourier transform (A.4). The inverse generalized Fourier transform is defined as

$$\epsilon(t) = \lim_{\alpha \downarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha|\omega|}\, \epsilon(\omega)\, e^{j\omega t} d\omega \qquad (C.14)$$

The generalized Fourier transform of the constant function $\epsilon(t) = 1$ can be obtained as

$$1 \longrightarrow \lim_{\alpha \downarrow 0} \int_{-\infty}^{\infty} e^{-\alpha|t|}\, e^{-j\omega t} dt = \lim_{\alpha \downarrow 0} \frac{2\alpha}{\alpha^2+\omega^2}$$

$$= 2\pi\, \delta(\omega) \qquad (C.15)$$

As for the Heaviside unit function one may write

$$u(t) \longrightarrow \lim_{\alpha \downarrow 0} \int_{0}^{\infty} e^{-\alpha t}\, e^{-j\omega t} dt = \lim_{\alpha \downarrow 0} \frac{1}{\alpha + j\omega}$$

$$= \lim_{\alpha \downarrow 0} \left\{ \frac{\alpha}{\alpha^2+\omega^2} - \frac{j\omega}{\alpha^2+\omega^2} \right\} = \pi\, \delta(\omega) + \frac{1}{j\omega} \qquad (C.16)$$

C.4 Some Useful Fourier Transform Pairs

$$\delta(t) \longleftrightarrow 1$$

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

$$u(t) \longleftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\int_{-\infty}^{t} f(\tau)\,d\tau \longleftrightarrow \frac{f(\omega)}{j\omega} + \pi f(0)\delta(\omega)$$

$$f'(t) \longleftrightarrow j\omega f(\omega) \quad \text{if} \quad \lim_{t\to\pm\infty} f(t) = 0$$

$$f^{(n)}(t) \longleftrightarrow (j\omega)^n f(\omega) \quad \text{if} \quad \lim_{t\to\pm\infty} f^{(k)}(t) = 0 \; ; \; k = 0, 1, \cdots, (n-1)$$

$$\delta(t - t_0) \longleftrightarrow e^{-j\omega t_0}$$

$$e^{j\omega t_0} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$\cos\omega_0 t \longleftrightarrow \pi\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

$$\operatorname{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

$$B_T(t) \longleftrightarrow \frac{2\sin(\frac{1}{2}\omega T)}{\omega}$$

$$\frac{\sin Tt}{\pi t} \longleftrightarrow B_{2T}(\omega)$$

$$q_T(t) \longleftrightarrow \frac{4\sin^2\left(\frac{\omega T}{2}\right)}{T\omega^2}$$

$$\frac{\sin^2(Tt)}{\pi T t} \longleftrightarrow q_{2T}(\omega)$$

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2 + \omega^2} \qquad \operatorname{Re}\{a\} > 0$$

$$e^{-a|t|}u(t) \longleftrightarrow \frac{1}{a + j\omega} \qquad \operatorname{Re}\{a\} > 0$$

$$e^{at}u(-t) \longleftrightarrow \frac{1}{a - j\omega} \qquad \operatorname{Re}\{a\} > 0$$

$$e^{-at^2} \longleftrightarrow \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \qquad a>0$$

$$f^*(t) \longleftrightarrow f^*(-\omega)$$

$$f(at) \longleftrightarrow \frac{1}{a} f\left(\frac{\omega}{a}\right)$$

$$f(t-t_0) \longleftrightarrow f(\omega) e^{-j\omega t_0}$$

$$f(t) e^{j\omega_0 t} \longleftrightarrow f(\omega-\omega_0)$$

$$(-jt)^n f(t) \longleftrightarrow f^{(n)}(\omega)$$

$$e^{-at}\cos bt\, u(t) \longleftrightarrow \frac{a+j\omega}{(a+j\omega)^2+b^2} \qquad \mathrm{Re}\{a\}>0$$

$$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^n} \qquad \mathrm{Re}\{a\}>0\ ;\ n=1,2,\ldots$$

APPENDIX D: Convolution Process

The convolution of two functions $f(t)$ and $g(t)$ is defined by

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \qquad (D.1)$$

The sufficient condition for the existence of $f(t) \otimes g(t)$ is that either $f(t)$ is bounded and $g(t)$ is absolutely integrable or $g(t)$ is bounded and $f(t)$ is absolutely integrable. It can easily be shown that the convolution is a commutative process. Substituting $\xi = t-\tau$ in (D.1) one gets

$$\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} g(\xi)\, f(t-\xi)\, d\xi$$

It follows that $f(t) \otimes g(t) = g(t) \otimes f(t)$.

It is readily seen that if $f(t)$ and $g(t)$ are both causal functions, then the convolution (D.1) is also causal since

$$f(t) \otimes g(t) = \int_0^t f(\tau) g(t-\tau) d\tau .$$

The convolution theorem in the time domain is stated as follows. Let $f(t)$ and $g(t)$ be two absolutely integrable functions so that their Fourier transform exist. Then the Fourier transform of the convolution $f(t) \otimes g(t)$ is obtained as

$$f(t) \otimes g(t) \longleftrightarrow \int_{-\infty}^{\infty} e^{-j\omega t} dt \int_{-\infty}^{\infty} f(t) g(t-\tau) d\tau =$$

$$= \int_{-\infty}^{\infty} f(\tau) d\tau \int_{-\infty}^{\infty} g(t-\tau) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(\tau) g(\omega) e^{-j\omega\tau} d\tau$$

$$= f(\omega) g(\omega) \qquad (D.2)$$

Thus the convolution process in the time domain is a multiplication process in the frequency domain. Conversely, the convolution theorem in the frequency domain is stated as (verify):

$$f(t) g(t) \longleftrightarrow \frac{1}{2\pi} f(\omega) \otimes g(\omega) . \qquad (D.3)$$

It can be shown that the convolution theorems remain valid when one of the given functions is a

constant function, sign function or unit function. For instance, the (generalized) Fourier transform of $g(t) \otimes U(t)$ is obtained as follows. Using (C.16) we write

$$g(t) \otimes U(t) \longleftrightarrow g(\omega)\left[\pi\,\delta(\omega) + \frac{1}{j\omega}\right].$$

In view of the rule (C.6) this becomes

$$g(t) \otimes U(t) \longleftrightarrow \pi\, g(0)\,\delta(\omega) + \frac{g(\omega)}{j\omega} \qquad (D.4)$$

where $g(0) = \int_{-\infty}^{\infty} g(\tau)\,d\tau$

See also the Fourier transform of the integral $\int_{-\infty}^{t} f(\tau)d\tau$ in Appendix C.4.

APPENDIX E: Hilbert Transform

The Fourier transform of a function $\epsilon(t)$ can be written in the form $\epsilon(\omega) = a(\omega) + j\,b(\omega)$. If $\epsilon(t)$ is a causal function, then the real and the imaginary components of the Fourier transform are not independent from eachother. Denoting the causal function $\epsilon(t)$ by $\epsilon(t)U(t)$, we obtain its Fourier transform from the convolution theorem

$$\epsilon(t)U(t) \longleftrightarrow \frac{1}{2\pi}\,\epsilon(\omega) \otimes \left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right]$$

$$\longleftrightarrow \frac{1}{2\pi}\,\epsilon(\omega) \otimes \frac{1}{j\omega} + \frac{1}{2}\epsilon(\omega) = \frac{1}{\pi}\,\epsilon(\omega) \otimes \frac{1}{j\omega}$$

Decomposing the above result into real and ima-

ginary parts we may write

$$a(\omega) + j\, b(\omega) = \frac{1}{\pi j} \int_{-\infty}^{\infty} \frac{a(u) + j\, b(u)}{\omega - u}\, du \qquad (E.1)$$

or

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{b(u)}{\omega - u}\, du$$

$$b(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a(u)}{\omega - u}\, du \qquad (E.2)$$

Eq.'s (E.2) are called the Hilbert transform. Symbolically we write

$$a(\omega) = -\mathcal{H}\{b(\omega)\} \;; \quad b(\omega) = \mathcal{H}\{a(\omega)\}.$$

If the function $\epsilon(t)$ contains an impulse component $A\delta(t)$, we define a new function $\epsilon(t) - A\delta(t)$ with the Fourier transform $\epsilon(\omega) - A = a(\omega) - A + j\, b(\omega)$. Since

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A}{\omega - u}\, du = \lim_{\varepsilon \to 0,\, \gamma \to \infty} \frac{1}{\pi} \left(\int_{-\gamma}^{\omega - \varepsilon} + \int_{\omega + \varepsilon}^{\gamma} \right) \frac{A}{\omega - u}\, du$$

$$= \lim_{\varepsilon \to 0,\, \gamma \to \infty} \frac{E}{\pi} \left\{ -\ln\varepsilon + \ln|\omega + \gamma| - \ln|\omega - \gamma| + \ln\varepsilon \right\}$$

$$= 0$$

eq.'s (E.2) are generalized as

$$a(\omega) = A + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{b(u)}{\omega - u}\, du$$

$$b(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a(u)}{\omega - u}\, du \qquad (E.3)$$

Equations (E.2) often result into integrals which are difficult to evaluate. A more tractable method to proceed further is then given by the Fourier transforms. The relations (E.2) can be written in the form

$$a(\omega) = \frac{1}{2\pi}\, j\, b(\omega) \otimes \frac{2}{j\omega} \;;\; j\, b(\omega) = \frac{1}{2\pi}\, a(\omega) \otimes \frac{2}{j\omega} \qquad (E.4)$$

Let $a(t) \longleftrightarrow a(\omega)$ and $b(t) \longleftrightarrow b(\omega)$. Then the inverse Fourier transform of (E.4) is obtained as

$$a(t) = b(t)\, \mathrm{sgn}(t) \;;\; b(t) = a(t)\, \mathrm{sgn}(t). \qquad (E.5)$$

Accordingly, the inverse Fourier transform of $\epsilon(\omega) = a(\omega) + j\, b(\omega)$, i.e. $\epsilon(t)$ is given by

$$\epsilon(t) = a(t) + b(t) = a(t)\,[1 + \mathrm{sgn}(t)] \qquad (E.6)$$

From the above equation it is readily seen that $\epsilon(t) = 0$ for $t < 0$, or in other words, $\epsilon(t)$ is a causal function. Now, let $a(\omega)$ be known and we require an expression for $b(\omega)$. One may proceed as follows. Determine first $a(t) \longleftrightarrow a(\omega)$. Then from (E.5) $b(t) = a(t)\, \mathrm{sgn}(t)$ and one has from (E.4) $a(t)\,\mathrm{sgn}(t) \longleftrightarrow j\, b(\omega)$. The required Fourier transforms can eventually be obtained from a table of Fourier transform pairs.

APPENDIX F: Laplace Transform

The Laplace transform can be conceived as a generalization of the Fourier transform, where a convergence factor $e^{-s_0 t}$ is introduced inside the integral. Let a function $\epsilon(t)$ be defined in $[-\varepsilon, \infty)$ for $\varepsilon > 0$ and integrable for each finite sub-interval. Then the uni-lateral Laplace transform (or shortly the Laplace transform) of $\epsilon(t)$ is defined as

$$\epsilon(s) = \mathcal{L}\{\epsilon(t)\} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} \epsilon(t)\, e^{-s_0 t}\, e^{-j\omega t}\, dt$$

$$= \int_{0^-}^{\infty} \epsilon(t)\, e^{-st}\, dt \qquad (F.1)$$

with $s = s_0 + j\omega$.

If the convergence factor s_0 equals zero, the Laplace transform reduces to the Fourier transform. A sufficient condition for this is that the given function $\epsilon(t)$ is absolutely integrable. Analogous to the Fourier transform, the inverse (uni-lateral) Laplace transform is defined as

$$\epsilon(t) = \mathcal{L}^{-1}\{\epsilon(s)\} = \frac{1}{2\pi} \int_{0^-}^{\infty} \epsilon(s)\, e^{-st}\, d\omega \qquad (F.2)$$

The Laplace transform of the delta function is unity. In the above equations the lower integration limit is taken as 0^-, appropriate to include the possibility that the given function $\epsilon(t)$ contains an impulse term.