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ANALYTICITY SPACES, TRAJECTORY SPACES AND LINEAR MAPPINGS BETWEEN THEM

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Abstract.

Let A be a non-negative unbounded self-adjoint operator in a Hilbert space X. Introduce the analyticity space $S_{X,A}$ by

$$S_{X,A} = \{f \mid f \in X,\ \exists_{t>0}\exists_{h\in X}\ f = e^{-tA}h\} .$$

The trajectory space $T_{X,A}$ is defined to be the set of all mappings $F: (0,\infty) \to X$ which satisfy $\forall_{t>0}\forall_{\tau>0}\, e^{-tA}F(\tau) = F(t+\tau)$. Examples of such trajectories are $t \mapsto A^m e^{-tA}x$ with $x \in X$, $m \geq 0$.

Both $S_{X,A}$ and $T_{X,A}$ are linear spaces on which a topology and semi-norms are introduced. For the spaces $S_{X,A}$ and $T_{X,A}$ a pairing, properties and characterizations of morphisms and five Kernel Theorems are discussed. A list of examples is mentioned in which $S_{X,A}$ can be looked upon as a test function space and $T_{X,A}$ as a space of generalized functions. With this theory Dirac's formalism can be mathematically interpreted to a greater extend than with the usual rigged Hilbert space theory.

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In a very inspiring paper, De Bruijn [B] has introduced a theory of generalized functions, based on a specific one parameter semigroup of smoothing operators. De Bruijn's theory was generalized considerably by De Graaf [G]. Here we shall summarize the results in [G]. Proofs will mostly be omitted; they can be found in the cited paper [G]. Further, we shall give some examples of the theory and show its relation to unitary representations of Lie groups.

1. The space $S_{X,A}$

Let A be a non-negative, self-adjoint operator in a Hilbert space X. Then the semigroup $(e^{-tA})_{t\geq 0}$ consists of bounded linear operators on X. In order that this semigroup is smoothing, A is supposed to be unbounded. The test space $S_{X,A}$ is the dense linear subspace of X consisting of smooth elements $e^{-tA}h$, where $h \in X$ and $t > 0$. We have

$$S_{X,A} = \bigcup_{t>0} e^{-tA}(X) = \bigcup_{n\in\mathbb{N}} e^{\frac{1}{n}A}(X) .$$

Since each subspace $e^{-tA}(X)$ of X can be given its obvious Hilbert space structure, $S_{X,A}$ can be looked upon as a union of Hilbert spaces. We note that for each $f \in S_{X,A}$ there exist $\tau > 0$ such that $e^{\tau A}f$ makes sense as an element of X.

The strong topology in $S_{X,A}$ is the finest locally convex topology on $S_{X,A}$ for which the injections i_t: $e^{-tA}(X) \to S_{X,A}$, $t > 0$, are all continuous. In other words, we impose on $S_{X,A}$ the inductive limit topology with respect to the spaces $e^{-tA}(X)$, $t > 0$. We note that this inductive limit is not strict.

The function algebras $B(\mathbb{R})$ and $B_+(\mathbb{R})$ are defined as follows:

- $B(\mathbb{R})$ consists of all everywhere finite, real valued Borel functions ψ on $\mathbb{R}$ such that for all $t > 0$ the function $x \mapsto \psi(x)e^{-tx}$ is bounded on $[0,\infty)$.
- $B_+(\mathbb{R})$ consists of all $\psi \in B(\mathbb{R})$ with $\psi(x) \geq \varepsilon > 0$, $\varepsilon \in \mathbb{R}$.

By the spectral theorem for self-adjoint operators, the operators $\psi(A)$, $\psi \in B(\mathbb{R})$ are well defined, and the operators $\psi(A)e^{-tA}$, $t > 0$, are all bounded. Further for $f \in S_{X,A}$ and $\psi \in B(\mathbb{R})$

$$\psi(A)f = e^{-\tau A}(\psi(A)\, e^{-(t-\tau)A})e^{+tA} f \in S_{X,A}$$

if $t > 0$ sufficiently small and $0 < \tau < t$.

On $S_{X,A}$ the seminorms p_ψ are well-defined by

(1.1) $\quad p_\psi(f) = \|\psi(A)\, f\|$

where $\|\cdot\|$ denotes the usual norm in X. Then the following very fundamental theorem can be proved.

(1.2) Theorem.

The seminorms p_ψ of (1.1) are continuous on $S_{X,A}$ and they generate the strong topology on $S_{X,A}$.

Although the inductive limit is not strict, because of Theorem (1.2) most results for strict inductive limits are also valid in our $S_{X,A}$ space. In [G] the following results have been proved with ad hoc arguments.

(1.3) Theorem.

A subset $B \subset S_{X,A}$ is bounded iff there is $t > 0$ such, that B is a bounded subset of $e^{-tA}(X)$.

(1.4) Theorem.

A subset $K \subset S_{X,A}$ is compact iff there is $t > 0$ such, that K is a compact subset of $e^{-tA}(X)$.

(1.5) Theorem.

A sequence (f_n) in $S_{X,A}$ is Cauchy iff (f_n) is a Cauchy sequence in some $e^{-tA}(X)$.

Hence $S_{X,A}$ is sequentially complete, because each $e^{-tA}(X)$ is complete. The elements of $S_{X,A}$ can be characterized as follows.

(1.6) Lemma.

Let $f \in X$, and suppose $f \in D(\psi(A))$ for all $\psi \in B_+(\mathbb{R})$. Then $f \in S_{X,A}$. Employing the standard terminology of topological vector spaces, the properties of $S_{X,A}$ are the following.

(1.7) Theorem.

I $S_{X,A}$ is complete.

II $S_{X,A}$ is bornological.

III $S_{X,A}$ is barreled.

IV $S_{X,A}$ is Montel, iff for every $t > 0$ the operator e^{-tA} is compact on X.

V $S_{X,A}$ is nuclear iff for every $t > 0$ the operator e^{-tA} is Hilbert-Schmidt on X.

2. The space $\mathcal{T}_{X,A}$

In X consider the evolution equation

(2.1) $$\frac{dF}{dt} = -AF\,.$$

A solution F of (2.1) is called a trajectory if F satisfies

(2.2.i) $\forall_{t>0}\forall_{\tau>0}:\ e^{-\tau A}F(t) = F(t+\tau)$

(2.2.ii) $\forall_{t>0}:\ F(t) \in X$.

We emphasize that $\lim_{t\downarrow 0} F(t)$ does not necessarily exist in X-sense. The complex vector space of all trajectories is denoted by $\mathcal{T}_{X,A}$. For $F \in \mathcal{T}_{X,A}$ we have $F(t) \in \mathcal{S}_{X,A}$, $t > 0$. The Hilbert space X can be embedded in $\mathcal{T}_{X,A}$. To this end, define emb: $X \to \mathcal{T}_{X,A}$ by

(2.3) $$\text{emb}(x)\ (t) = e^{-tA}x \quad , \quad x \in X.$$

Thus X can be considered as a subspace of $\mathcal{T}_{X,A}$, and we have

$$\mathcal{S}_{X,A} \subset X \subset \mathcal{T}_{X,A}\ .$$

The characterization of the elements of $\mathcal{T}_{X,A}$ is as follows.

(2.4) <u>Theorem.</u>

Let $F \in \mathcal{T}_{X,A}$. Then there exists $w \in X$ and $\psi \in \mathcal{B}_+(\mathbb{R})$ such that $F(t) = \psi(A)e^{-tA}w$, $t > 0$.

The strong topology in $\mathcal{T}_{X,A}$ is the locally convex topology induced by the seminorms

(2.5) $\rho_n(F) = \|F(\frac{1}{n})\|$, $n \in \mathbb{N}$.

With this topology $T_{X,A}$ becomes a Frêchet space, i.e. a metrizable and complete space.

It is not hard to see that $S_{X,A}$ is dense in $T_{X,A}$. For $F \in T_{X,A}$ just take the sequence $(F(\frac{1}{n})) \subset S_{X,A}$. This sequence converges to F in the strong topology of $T_{X,A}$. Further in [G], ch. II, the following results have been proved:

(2.6) Theorem.

A set $B \subset T_{X,A}$ is bounded iff each of the sets $\{F(t) \mid F \in B\}$, $t > 0$, is bounded in X.

(2.7) Theorem.

A set $K \subset T_{X,A}$ is compact iff each of the sets $\{F(t) \mid F \in K\}$, $t > 0$, is compact in X.

With the aid of the standard terminology of topological vector spaces $T_{X,A}$ can be described as follows.

(2.8) Theorem.

I $T_{X,A}$ is bornological.
II $T_{X,A}$ is barreled.
III $T_{X,A}$ is Montel iff the operators e^{-tA} are compact on X for all $t > 0$.
IV $T_{X,A}$ is nuclear iff the operators e^{-tA} are Hilbert-Schmidt on X for all $t > 0$.

3. The pairing of $\mathcal{S}_{X,A}$ and $\mathcal{T}_{X,A}$

On $\mathcal{S}_{X,A} \times \mathcal{T}_{X,A}$ the sesquilinear form $\langle \cdot\, , \cdot \rangle$ is defined by

$$(3.1) \qquad \langle g\, , F\rangle := (e^{tA} g\, , F(t))\ ,$$

where as usual $(\cdot\, , \cdot)$ denotes the inner product of X. We note that this definition makes sense for $t > 0$ sufficiently small, and does not depend on the choice of $t > 0$ because of the trajectory property (2.2.ii) satisfied by F.

The spaces $\mathcal{S}_{X,A}$ and $\mathcal{T}_{X,A}$ can be considered as the strong topological dual spaces of each other by this pairing. So we have

(3.2) Theorem.

I Let ℓ be a linear functional on $\mathcal{S}_{X,A}$. Then ℓ is continuous iff there exists $F \in \mathcal{T}_{X,A}$ such, that $\ell(h) = \langle h\, , F\rangle$, $h \in \mathcal{S}_{X,A}$.

II Let m be a linear functional on $\mathcal{T}_{X,A}$. Then m is continuous iff there exists $f \in \mathcal{S}_{X,A}$ such, that $m(G) = \overline{\langle f\, , G\rangle}$, $G \in \mathcal{T}_{X,A}$.

As usual, the linear functionals of $\mathcal{S}_{X,A}$ resp. $\mathcal{T}_{X,A}$ induce the weak topology on $\mathcal{T}_{X,A}$ resp. $\mathcal{S}_{X,A}$ in the following way:

(3.3.i) The weak topology on $\mathcal{S}_{X,A}$ is the topology induced by the seminorms, $p_F(h) = |\langle h\, , F\rangle|$, $F \in \mathcal{T}_{X,A}$.

(3.3.ii) The weak topology on $\mathcal{T}_{X,A}$ is the topology induced by the seminorms $\rho_f(G) = |\langle f\, , G\rangle|$, $f \in \mathcal{S}_{X,A}$.

A simple argument [CH], II. §22, shows, that $\mathcal{S}_{X,A}$ and $\mathcal{T}_{X,A}$ are reflexive both in the strong and the weak topology.

(3.4) Theorem. (Banach-Steinhaus)

Weakly bounded sets in $S_{X,A}$ resp. $T_{X,A}$ are strongly bounded.

In the next two theorems weak convergence of sequences in $S_{X,A}$ as well as in $T_{X,A}$ are characterized.

(3.5) Theorem.

$f_n \to 0$ in the weak topology of $S_{X,A}$ iff

$$\exists_{t>0}: (f_n) \subset e^{-tA}(X) \text{ and } f_n \to 0, \text{ weakly, in } e^{-tA}(X) .$$

As a corollary it immediately follows that strong convergence of a seqence in $S_{X,A}$, implies its weak convergence. Further, any bounded sequence in $S_{X,A}$ has a weakly convergent subsequence.

(3.6) Theorem.

$F_n \to 0$ weakly in $T_{X,A}$ iff $\forall_{t>0}: F_n(t) \to 0$ weakly in X.

So again it follows that strongly converging sequences in $T_{X,A}$ are weakly convergent. By a diagonal argument it can be proved that any bounded sequence in $T_{X,A}$ has a weakly converging subsequence.

When are weakly convergent sequences always strongly convergent? The next theorem deals with this question.

(3.7) Theorem.

The following three statements are equivalent:

I For each $t > 0$, the operator e^{-tA} is compact on X.

II Each weakly convergent sequence in $S_{X,A}$ converges strongly in $S_{X,A}$.

III Each weakly convergent sequence in $T_{X,A}$ converges strongly in $T_{X,A}$.

4. Characterization of continuous linear mappings between the spaces $S_{X,A}$, $T_{X,A}$, $S_{Y,B}$ and $T_{Y,B}$

Let B be a non-negative self-adjoint operator in the separable Hilbert space Y. In this section we give conditions implying continuity of linear mappings $S_{X,A} \to S_{Y,B}$, $S_{X,A} \to T_{Y,B}$, $T_{X,A} \to T_{Y,B}$ and $T_{X,A} \to S_{Y,B}$. Further, there are given conditions on a linear operator in X such that it can be extended to a continuous linear mapping on $T_{X,A}$. The next theorem is an immediate consequence of the fact that $S_{X,A}$ is bornological.

(4.1) Theorem.

Let R be an arbitrary locally convex topological vector space. A linear mapping $\mathcal{L}: S_{X,A} \to R$ is continuous iff

I for each $t > 0$ the mapping $\mathcal{L}\, e^{-tA}: X \to R$ is continuous.

II for each null sequence $(u_n) \subset S_{X,A}$, the sequence $(\mathcal{L} u_n)$ is a null sequence in R.

In [G], De Graaf gives several equivalent conditions on linear mappings of one of the mentioned types to be continuous. Each of these conditions is useful in its own context. The next theorem deals with continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.

(4.2) Theorem.

Suppose P: $S_{X,A} \to S_{Y,B}$ is a linear mapping. Then P is continuous iff one of the following conditions is satisfied

I $f_n \to 0$ strongly in $S_{X,A}$ implies $Pu_n \to 0$ strongly in $S_{Y,B}$.

II For each $t > 0$ the operator Pe^{-tA} is continuous from X into Y.

III For each $t > 0$ there exists $s > 0$ such that $Pe^{-tA}(X) \subset e^{-sB}(Y)$ and $e^{sB}Pe^{-tA}$ is a bounded linear operator from X into Y.

IV There exists a dense linear subspace $\Xi \subset Y$ such that for each fixed $y \in \Xi$ the linear functional $\ell_{P,y}(f) = (Pf, y)_Y$ is continuous on $S_{X,A}$.

V For each $t > 0$ the adjoint $(Pe^{-tA})^*$ of Pe^{-tA} is continuous from X into Y.

The next corollary is important for applications.

(4.3) Corollary.

Let Q be a densely defined closable operator: $X \to Y$. If $D(Q) \supset S_{X,A}$ and $Q(S_{X,A}) \subset S_{Y,B}$, then Q maps $S_{X,A}$ continuously into $S_{Y,B}$.

(4.4) Theorem.

Let K: $S_{X,A} \to T_{Y,B}$ be a linear mapping. Then K is continuous iff

I For each $t > 0$, $s > 0$ the operator $e^{-sB}Ke^{-tA}$ is continuous from X into Y.

II For each $s > 0$ the mapping $e^{-sB}K$ is continuous from $S_{X,A}$ into $S_{Y,B}$.

(4.5) Theorem.

Let V: $T_{X,A} \to S_{Y,B}$ be a linear mapping, and let V_r: X → Y denote its restriction to X. Then V is continuous iff one of the following conditions is satisfied

I $V_r^*(Y) \subset S_{X,A}$.

II There exists t > 0 such that $V_r^*(Y) \subset e^{-tA}(X)$ and $e^{tA} V_r^*$ is bounded as an operator from X into Y.

III There exists t > 0 such that $V_r e^{tA}$ with domain $e^{-tA}(X) \subset X$ is bounded as an operator from X into Y.

IV There exists t > 0 and a continuous linear mapping Q: $S_{X,A} \to S_{Y,B}$ such, that $V = Q e^{-tA}$.

(4.6) Theorem.

Let Φ: $T_{X,A} \to T_{Y,B}$ be a linear mapping. Let Φ_r: X → $T_{Y,B}$ denote the restriction of Φ to X. Then Φ is continuous iff one of the following conditions is satisfied.

I For each g ∈ $S_{Y,B}$ the linear functional F → $\overline{\langle y , \Phi F \rangle}$ is continuous on $T_{X,A}$.

II For each s > 0 the linear mapping $e^{-sB}\Phi$ is continuous from $T_{X,A}$ into $S_{Y,B}$.

III For each s > 0 $(e^{-sB}\Phi_r)^* \subset S_{X,A}$.

IV For each s > 0 there exists t > 0 such that $e^{-sB}\Phi_r e^{tA} = e^{-sB}\Phi e^{tA}$ on the domain $e^{-tA}(X)$ is bounded as an operator form X into Y.

An interesting class of densely defined linear operators is established by those operators in X which can be extended to continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$. This class is characterized as follows.

(4.7) Theorem.

Let E be a densely defined linear operator from X into Y. E can be extended to a continuous linear mapping $\bar{E}: T_{X,A} \to T_{Y,B}$ iff E has a densely defined adjoint $E^*: D(Q^*) \supset S_{Y,B} \to X$ with $E^*(S_{Y,B}) \subset S_{X,A}$.

As a corollary of this theorem it follows that a continuous linear mapping $Q: S_{X,A} \to S_{Y,B}$ can be extended to a continuous mapping $\bar{Q}: T_{X,A} \to T_{Y,B}$ iff its adjoint Q^* satisfies $D(Q^*) \supset S_{Y,B}$ and $Q^*(S_{Y,B}) \subset S_{X,A}$.

5. Topological tensor products and Kernel theorems

Let $X \otimes Y$ denote the set of Hilbert-Schmidt operators from X into Y. $X \otimes Y$ is a Hilbert space, which can be regarded as a complete topological tensor product of the Hilbert spaces X and Y. Further, in $X \otimes Y$ the operator $A \boxplus B$ is defined to be the unique self-adjoint extension of the operator $A \otimes I + I \otimes B$ which is well defined on the algebraic tensor product $D(A) \otimes_a D(B)$. We have $e^{-t(A \boxplus B)} = e^{-tA} \otimes e^{-tB}$, $t > 0$. So $(e^{-t(A \boxplus B)})_{t>0}$ is a semigroup of smoothing operators on $X \otimes Y$.

Now, according to section 1 and 2, we introduce the spaces $S_{X\otimes Y, A\boxplus B}$ and $T_{X\otimes Y, A\boxplus B}$. They can be regarded as topological completions of the algebraic tensor products $S_{X,A} \otimes_a S_{Y,B}$ c.q. $T_{X,A} \otimes_a T_{Y,B}$.

An element $J \in S_{X\otimes Y, A\boxplus B}$ can be considered as a linear operator $J: S_{X,A} \to S_{Y,B}$ in the following way: Let $F \in T_{X,A}$. Define JF by

$$JF = e^{-\varepsilon B}(e^{\varepsilon B} J e^{\varepsilon A})\, F(\varepsilon)\ .$$

For $\varepsilon > 0$ and sufficiently small this definition makes sense and does not depend on the choice of ε.

(5.1) Kernel theorem.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then $S_{X\otimes Y,A\boxplus B}$ comprises all continuous linear mappings from $T_{X,A}$ into $S_{Y,B}$.

An element $K \in T_{X\otimes Y,A\boxplus B}$ can be considered as a linear operator K: $S_{X,A} \to T_{Y,B}$ in the following way: Let $f \in S_{X,A}$. Define $Kf \in T_{Y,B}$ by

$$(Kf)(t) := e^{-(t-\varepsilon)B} K(\varepsilon)\, e^{\varepsilon A} f, \quad t > 0.$$

For any $f \in S_{X,A}$ and $t > 0$ this definition makes sense for $\varepsilon > 0$ sufficiently small. Moreover $(Kf)(t)$ does not depend on the choice of ε.

(5.2) Kernel theorem.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then $T_{X\otimes Y,A\boxplus B}$ comprises all continuous linear mappings from $S_{X,A}$ into $T_{Y,B}$.

Next, in order to describe continuous linear mappings P: $S_{X,A} \to S_{Y,B}$ and Φ: $T_{X,A} \to T_{Y,B}$ De Graaf introduces two more topological tensor products:

The subspace Σ_A' of $T_{X\otimes Y,A\otimes I}$ defined by

$$\Sigma_A' := \{P \mid P \in T_{X\otimes Y,A\otimes I}\,,\ \forall_{t>0}: P(t) \in S_{X\otimes Y,A\boxplus B}\}\,.$$

This is a topological completion of $T_{X,A} \otimes_a S_{Y,B}$.

The subspace Σ'_B of $T_{X\otimes Y, I\otimes B}$ defined by

$$\Sigma'_B := \{\Phi \mid \Phi \in T_{X\otimes Y, I\otimes B}\ ,\ \forall_{t>0}: \Phi(t) \in S_{X\otimes Y, A\boxplus B}\}.$$

Σ'_B is a topological completion of $S_{X,A} \otimes_a T_{Y,B}$.

On the spaces Σ'_A and Σ'_B complete sets of seminorms are introduced. An element $P \in \Sigma'_A$ can be considered as a linear operator $P: S_{X,A} \to S_{Y,B}$ as follows: For $f \in S_{X,A}$ define $Pf \in S_{Y,B}$ by

$$Pf = P(\varepsilon)\ e^{\varepsilon A} f\ .$$

Then $Pf \in S_{Y,B}$, because $P(\varepsilon) \in S_{X\otimes Y, A\boxplus B}$. The definition makes sense for $\varepsilon > 0$ sufficiently small and does not depend on the choice of ε.

(5.3) <u>Kernel theorem</u>.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then Σ'_A comprises all continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.

Finally, an element $\Phi \in \Sigma'_B$ can be considered as a linear operator $\Phi: T_{X,A} \to T_{Y,B}$ in the following way: For $F \in T_{X,A}$ define $\Phi F \in T_{Y,B}$ by

$$(\Phi F)(t) := \Phi(t)\ e^{\varepsilon(t)A} F(\varepsilon(t))\ .$$

This definition makes sense for each $t > 0$ and $\varepsilon(t) > 0$ sufficiently small. The result does not depend on the specific choice of $\varepsilon(t)$.

(5.4) Kernel theorem.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then Σ_B' comprises all continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$.

For more details and proofs the reader is referred to [G], Ch. VI. In [E_2] the spaces Σ_A' and Σ_B' will be defined in a more elegant way and discussed in a wider context. Further investigations in this theory of generalized functions led to a fifth Kernel theorem for those continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$, which can be extended to a continuous linear mapping from $T_{X,A}$ into $T_{Y,B}$, the so called extendable linear mappings. See [E_2].

6. Examples of $S_{X,A}$-spaces

(1) The S_α^β-spaces of Gelfand-Shilov

De Bruijn's theory of generalized function is based on the test function space $S_{L_2(\mathbb{R}),H}$, where H is the Hamiltonian operator of the harmonic oscillator,

$$H = \frac{1}{2}\left(- \frac{d^2}{dx^2} + x^2 + 1 \right) .$$

The space $S_{L_2(\mathbb{R}),H}$ consists of entire analytic functions f satisfying

$$|f(x + iy)| \le C \exp(-\tfrac{1}{2}Ax^2 + \tfrac{1}{2}By^2) \quad , \quad x,y \in \mathbb{R},$$

where A, B en C are some positive constants only dependent on f. The space $S_{L_2(\mathbb{R}),H}$ equals the space $S_{\frac{1}{2}}^{\frac{1}{2}}$ introduced in the books of Gelfand-Shilov [GS_2].

Recently, it has been proved that the Gelfand-Shilov spaces $S_{1/k+1}^{k/k+1}$, $k \in \mathbb{N}$, are $S_{X,A}$-type spaces. (see [EGP]). To this end, put

$$B_k = \left(- \frac{d^2}{dx} + x^{2k} \right)^{k+1/2k} .$$

Then $S_{1/k+1}^{k/k+1} = S_{L_2(\mathbb{R}),B_k}$. By applying the Fourier transform it easily follows that

$$S_{k/k+1}^{1/k+1} = S_{L_2(\mathbb{R}),\tilde{B}_k}$$

where $\tilde{B}_k = \left(\left(- \frac{d^2}{dx^2} \right)^k + x^2 \right)^{k+1/2k}$.

We conjecture that a great number of Gelfand-Shilov spaces S_α^β are of type $S_{X,A}$.

(2) <u>Hankel invariant distribution spaces</u>

For $\alpha > -1$, the Hankel transform $\mathbb{H}_\alpha$ is formally defined by

$$(\mathbb{H}_\alpha f)(x) = \int_0^\infty \mathcal{J}_\alpha(xy) \sqrt{xy}\, f(y)dy \quad , \quad x > 0,$$

where $\mathcal{J}_\alpha$ is the Bessel function of order α. The Hankel transform extends to a unitary operator on $Z = L_2(0,\infty)$. The generalized Laguerre functions $\mathcal{L}_n^{(\alpha)}$, $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{L}_n^{(\alpha)}(x) = \left(\frac{2\,\Gamma(n+1)}{\Gamma\,(n+\alpha+1)} \right)^{\frac{1}{2}} x^{\alpha+\frac{1}{2}}\, e^{-\frac{1}{2}x^2}\, L_n^{(\alpha)}(x^2) \quad , \quad x > 0,$$

where $L_n^{(\alpha)}$ is the n-th generalized Laguerre polynomial of type α, satisfy

$$\mathbb{H}_\alpha \mathcal{L}_n^{(\alpha)} = (-1)^n \mathcal{L}_n^{(\alpha)} .$$

They establish a complete orthonormal basis of eigenfunctions in Z for the positive self-adjoint operator A_α

$$A_\alpha : -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - \frac{1}{4}}{x^2} - 2\alpha \ .$$

Their respective eigenvalues are $4n+2$, $n \in \mathbb{N} \cup \{0\}$.

By routine methods it can be shown that the space S_{Z,A_α} is invariant under the unitary operator $\mathbb{H}_\alpha$. So $\mathbb{H}_\alpha$ extends to a continuous bijection on the distribution space T_{Z,A_α}. In [E_1], [EG] the elements of S_{Z,A_α} are characterized as follows

$f \in S_{Z,A_\alpha}$ iff

(i) $z \mapsto z^{-(\alpha+\frac{1}{2})} f(z)$ extends to an entire analytic and even function

and (ii) there are positive constants A, B and C such that

$$|z^{-(\alpha+\frac{1}{2})} f(z)| \le C \exp(-\tfrac{1}{2}Ax^2 + \tfrac{1}{2}By^2)$$

where $z = x + iy$.

(3) Nuclear $S_{X,A}$-spaces for given sets of operators in X

In [E_2], there will be given a matrix calculus for the continuous linear mappings from a nuclear $S_{X,A}$ space into itself. With the aid of this calculus we have been able to construct a nuclear $S_{X,A}$ space for a finite number of bounded linear operators on a Hilbert space X, and also for a finite number of commuting, self-adjoint operators in X. The existence of such nuclear $S_{X,A}$ space is very important for our theory of generalized eigenfunctions and our interpretation of Dirac's formalism (see [E_2]).

7. Analytic vectors

In [Ne_1], Nelson introduced the notion analytic vector. Let A be a self-adjoint operator in X. Then $\mathfrak{f} \in X$ is an analytic vector for A iff

$$\| A^n \mathfrak{f} \| \leq a b^n n! \quad , \quad n = 0,1,2,\ldots$$

for some fixed constants a, b only dependent on $\mathfrak{f}$. The space of analytic vectors for A is denoted by $C^{\omega}(A)$, and called the analyticity domain of A. Nelson showed that for a non-negative self-adjoint operator A the vector $\mathfrak{f} \in C^{\omega}(A)$ can be written as $\mathfrak{f} = e^{-tA} w$ where $t > 0$ and $w \in X$. Hence $C^{\omega}(A) = S_{X,A}$.

The notion analytic vector was also introduced for unitary representations of Lie groups (see [Ne_1], [Wa], [Go] and [Na]):

Let G be a finite dimensional Lie group. A unitary representation U of G is a mapping

$$g \mapsto U(g) \quad , \quad g \in G$$

from G into the unitary operators on some Hilbert space X.

A vector $\mathfrak{f} \in X$ is called an analytic vector for the representation U, if the mapping

$$g \mapsto U(g)\mathfrak{f}$$

is analytic on G. We shall denote the space of analytic vectors for U by $C^{\omega}(U)$.

Let $A(G)$ denote the Lie algebra of the Lie group G, and let $\{p_1,\ldots,p_d\}$ be a basis for $A(G)$. Then for every $p \in A(G)$

$$s \mapsto U(\exp(sp))$$

is a one parameter group of unitary operators on X. By Stone's theorem its infinitesimal generator, denoted by $\partial U(p)$, is skew-adjoint. Thus the Lie algebra $A(G)$ is represented by skew-adjoint operators in X.

Put

$$\Delta = I - \sum_{k=1}^{d} (\partial U(p_k))^2.$$

Nelson, [Ne_1], has proved that the operator Δ can be uniquely extended to a positive, self-adjoint operator in X. Denote its extension by Δ, also. Then we have (see [Ne_1], [Go])

(7.1) Theorem.

The space of analytic vectors for the representation U, $C^{\omega}(U)$ equals the space $S_{X,\Delta^{\frac{1}{2}}}$.

The following result tells something about the action of $\partial U(p)$, $p \in A(G)$ on the space $S_{X,\Delta^{\frac{1}{2}}}$.

(7.2) Theorem.

The linear operators $\partial U(p)$, $p \in A(G)$, are continuous as linear mappings from $S_{X,\Delta^{\frac{1}{2}}}$ into itself.

Proof. Let $p \in A(G)$.

Following [Go], proposition 2.1, the operator $\partial U(p)$ maps $S_{X,\Delta^{\frac{1}{2}}}$ into itself. Since $\partial U(p)$ is skew-adjoint, continuity follows from section 4, Theorem 4.2. □

In several cases the space $S_{X,\Delta^{\frac{1}{2}}}$ is nuclear. Here we mention the following cases. Possibly, other cases can be found in the book of Warner,

[Wa]. For a proof we refer to [Na].

$S_{X,\Delta^{\frac{1}{2}}}$ is nuclear if U is an irreducible unitary representation of G on X and one of the following statements is satisfied:

(i) G is semi-simple with finite center.

(ii) G is the semi-direct product of $A \otimes K$ where A is an abelian invariant subgroup and K is a compact subgroup, e.g. the Euclidian groups.

(iii) G is nilpotent.

Again we note that nuclearity of $S_{X,\Delta^{\frac{1}{2}}}$ is very important for our theory of generalized functions and interpretation of Dirac's formalism.

References.

[B] Bruijn, N.G. De, A theory of generalized functions, with applications to Wigner distribution ans Weyl correspondence, Nieuw Archief voor Wiskunde (3), XXI, 1973, pp. 205 - 280.

[E_1] Eijndhoven, S.J.L. Van, On Hankel invariant distribution spaces, EUT-report-82-WSK-01, Eindhoven University of Technology, 1982.

[E_2] Eijndhoven, S.J.L. Van, Ph.D. thesis, to appear 1983.

[EG] Eijndhoven, S.J.L. Van and Graaf, J. De, Some results on Hankel invariant distribution spaces. To appear in Proc. Koninklijke Nederlandse Akademie van Wetenschappen.

[EGP] Eijndhoven, S.J.L. Van, Graaf, J. De, Pathak, R.S., A characterization of the spaces $S_{1/k+1}^{k/k+1}$ by means of holomorphic semigroups. To appear.

[G] Graaf, J. De, A theory of generalized functions based on holomorphic semigroups., TH-report-79-WSK-02, Eindhoven University of Technology, 1979.

[Go] Goodman, R., Analytic and entire vectors for representations of Lie groups, Trans. Am. Math. Soc., 143, (1969) 55.

[GS_2] Gelfand, I.M. and Shilov, G.E., Generalized functions, Vol. 2, Academic Press, New York (1968).

[Na] Nagel, B., Generalized eigenvectors, in A.O. Barut (Ed.), Proceedings NATO ASI on Math. Phys., Istanbul, 1970, Reidel, 1971.

[Ne] Nelson, E., Analytic vectors, Ann. Math. 70 (1959), pp. 572 - 615.

[Wa] Warner, G., Harmonic analysis on semi simple Lie groups, Springer, 1972.

[Ch] Choquet, G., Lectures on analysis, Vol. II, W.A. Benjamin inc., 1969.