Literature study on the controlled synchronization problem

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Literature Study on the Controlled Synchronization Problem

Alexei Pavlov
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Professor :   Prof.dr. H. Nijmeijer
Coach   :   Dr.ir. N. van de Wouw

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Eindhoven University of Technology
Department of Mechanical Engineering
Section Dynamics and Control Technology

THE CONTROLLED SYNCHRONIZATION PROBLEM
Report on literature study

Alexei Pavlov

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Abstract

This report is devoted to the problem of synchronization of dynamic systems. General definitions of synchronization, asymptotic, approximate and controlled synchronization are given. The notions of generalized, partial and phase synchronization are discussed. Observer-based viewpoints on synchronization are reviewed. Observer designs for linear systems, linearizable error dynamics, systems with parametric uncertainties and high gain observers are considered. Similar designs for discrete-time systems including designs for nonlinear systems transformable to the extended Lur'e form are reviewed. Results on synchronization of two or more systems based on the passivity methodology are studied. Examples of synchronization in different engineering problems are discussed. Application of synchronization of chaotic systems in the field of communication is considered in detail.

1   Introduction

Starting with the work of Huygens [1], synchronization phenomena attracted attention of many researchers. Motivated by the study of chaotic systems, recent years have exhibited an increase in the interest in synchronization. In particular, quite recently the interest in synchronization was revived by Pecora and Carol [2], who studied master-slave synchronization of dynamical systems with chaotic behavior.

Synchronization is often desirable and sometimes necessary in many engineering systems. For example, AC power generators, being unified in a network, have to be synchronized in order to maintain the nominal frequency and phase of the current in the network. In communication, a receiver has to synchronize with the carrier signal generated by a transmitter in order to obtain the information signal. In television, synchronization is essential in the sense that the electron beams of receiver picture tubes should be at exactly the same spot on the screen at each instant as the beam in the television camera tube at the transmitter. In modern production lines a lot of mechanical actions carried out by robots must be synchronized in order to have fast production speed and to avoid buffers between operations. Even when riding a bicycle, the legs have to move synchronously – otherwise one may get in trouble.

A simple mechanical engineering problem, where the need in synchronization occurs, is controlling the H-drive as in Fig. 1A. The bridge is actuated by two motors and it can move forward and backward. If these motors do not act synchronously, then the bridge may stuck as in Fig. 1B. So, one has to control the motors in such way, that both sides of the bridge move synchronously.

Synchronization in its most general interpretation means correlated or corresponding in-time behavior of two or more processes. For example, the behavior of two dynamical systems, with states $x_1(t)$ and $x_2(t)$ respectively, may be called synchronized if the states are equal: $|x_1(t) - x_2(t)| \equiv 0$, for $t \geq t_0$. Or, if the difference between the states is not identically zero, but only tends to zero asymptotically, $|x_1(t) - x_2(t)| \to 0$, as $t \to \infty$, we can call these solutions asymptotically synchronized. Such type of behavior can occur, for example, if there is some kind of coupling between the systems. If two uncoupled systems do not exhibit (asymptotic) synchronization and we can affect one or both of them by control,
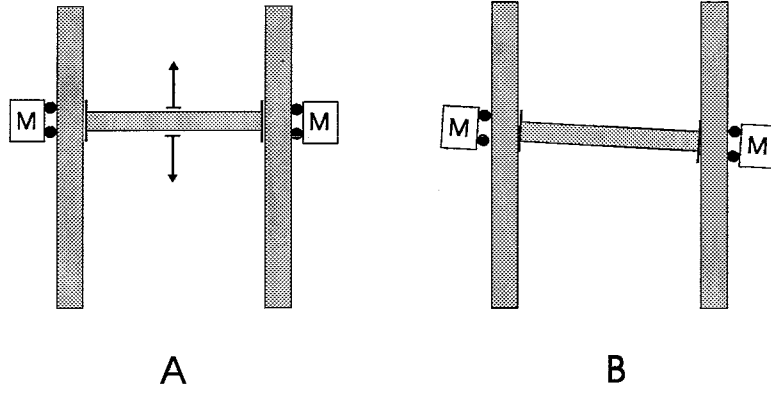
Figure 1: H-drive. A – operational, B – stuck due to asynchronous action of the motors.

we may try to make them asymptotically synchronize by establishing some coupling by means of control. The problem of designing such controllers can be referred to as the *controlled synchronization problem.*

Asymptotic synchronization is a kind of stability (attractivity) property. A remarkable fact is that synchronization can be observed in chaotic systems, which are inherently unstable. As an example of this fact, following [3], consider the Lorenz system:

$$
\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1) \\
\dot{y}_1 &= rx_1 - y_1 - x_1 z_1 \\
\dot{z}_1 &= -bz_1 + x_1 y_1.
\end{aligned}
\tag{1}
$$

System (1) is known to exhibit complex or chaotic motions for certain parameters σ, r, $b > 0$. With system (1) viewed as the transmitter or master system, we introduce the drive signal

$$
y = x_1,
\tag{2}
$$

which can be used at the receiver or slave system, to achieve asymptotic synchronization. We take as receiver dynamics

$$
\begin{aligned}
\dot{x}_2 &= \sigma(y_2 - x_2) \\
\dot{y}_2 &= rx_1 - y_2 - x_1 z_2 \\
\dot{z}_2 &= -bz_2 + x_1 y_2.
\end{aligned}
\tag{3}
$$

Notice that (3) consists of a copy of (1) with the state (x_2, y_2, z_2) and where in the (y_2, z_2)-dynamics the known signal x_1, see (2), is substituted for x_2. Introducing the error variables $e_1 = x_1 - x_2$, $e_2 = y_1 - y_2$, $e_3 = z_1 - z_2$, we obtain the error dynamics

$$
\begin{aligned}
\dot{e}_1 &= \sigma(e_2 - e_1) \\
\dot{e}_2 &= -e_2 - x_1 e_3 \\
\dot{e}_3 &= -be_3 + x_1 e_2,
\end{aligned}
\tag{4}
$$

which are linear and time-varying due to the fact that the error dynamics is excited by $x_1(t)$. The stability of $(e_1, e_2, e_3) = (0, 0, 0)$ is straightforwardly checked using the Lyapunov function

$$
V(e_1, e_2, e_3) = \frac{1}{\sigma}e_1^2 + e_2^2 + e_3^2
$$

with time derivative along solutions of (4)

$$
\dot{V}(e_1, e_2, e_3) = -2(e_1 - \frac{1}{2}e_2)^2 - \frac{3}{2}e_2^2 - 2be_3^2
$$

showing that (e_1, e_2, e_3) asymptotically converges to $(0, 0, 0)$. In other words, the receiver dynamics (3) asymptotically synchronize with the chaotic transmitter (1) no matter how (1) and (3) are initialized.

2

This example demonstrates two things. First, chaotic (inherently unstable) systems may exhibit some type of stability properties. In the state space $\mathbb{R}^6 = \{x_1, y_1, z_1, x_2, y_2, z_2\}$ of the combined system (1), (3) there is an asymptotically stable set $\mathcal{A}$ defined by the relations $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$. That means that for a given neighborhood of the set $\mathcal{A}$ any solution of the combined system (1), (3) starting close enough to $\mathcal{A}$ will remain in that neighborhood, and will converge to $\mathcal{A}$. Here the distance between a point X and the set $\mathcal{A}$ is defined as $\rho(X, \mathcal{A}) = \inf_{Y \in \mathcal{A}} |X - Y|$. Second, these stability properties (stability of sets) can be established (at least in some cases) by conventional stability analysis tools, for example using Lyapunov functions.

Stability properties make systems predictable in some sense and this is what we need to use a system in practice. For example, synchronization, being a kind of stability property, allows us to exploit the complex properties of chaotic systems. One of such applications is secure communication. Roughly speaking, the idea can be explained using the previous example: take the transmitter system (1) and transmit two signals: $y(t) = x_1(t)$ and $\bar{y}(t) = z_1(t) + d(t)$, where $d(t)$ is some information signal. The receiver asymptotically reconstructs $z_1(t)$ from $y(t)$, as it has been shown above. Subtracting this reconstructed signal from $\bar{y}(t)$, we asymptotically obtain the information signal $d(t)$. Due to the broadbanded spectrum of solutions of the Lorenz system, both $y(t)$ and $\bar{y}(t)$ look like noise if $d(t)$ is small compared to $z_1(t)$. This makes the transmission of information more secure, since the information is masked by a noise-like signal. Although this example is still far too naive for secure communication, it suggests the idea of using synchronization in secure communication. Other more complicated schemes of using synchronization in communication will be discussed in the sequel.

The examples presented above correspond only to one particular type of synchronization, namely synchronization of the states of two systems. Certainly, there are other types of synchronization. One can think of phase and frequency synchronization in case of systems having periodic orbits. Or, instead of the states one may want to synchronize certain outputs of the systems. All these different types of synchronization motivate to give a general definition of synchronization, which will be discussed later.

In this report, we will consider the synchronization and controlled synchronization problem as well as definitions, approaches, solutions, examples and applications concerning these two problems.

The report is organized as follows. General definitions of synchronization are given in Section 2, which is based on [4]. The observer-based approach to synchronization is reviewed in Section 3. Here, following [5], we give the observer problem statement and discuss different classes of dynamical systems, for which the observer problem can be (relatively) easily solved: linear systems, systems with linearizable error dynamics, systems which admit high gain observers (Sections 3.1-3.4). In case of systems with parametric uncertainties, one can try to find an adaptive observer capable to cope with the uncertainties. Such results from [6] are discussed in Section 3.5. Similar to continuous–time systems, observers can be designed for systems in discrete time. Some results on discrete-time observers from [7] are considered in Section 3.6. A fruitful approach to studying synchronization of identical systems is based on passivity. It allows to study synchronization not only of two systems, but also of multiple systems unified by coupling into a network. This approach is described in Section 4. The results in this section are taken from [8], [9]. Specific types of synchronization are discussed in Section 5. Possible applications of synchronization to communication are given in Section 6 and conclusions finish the report in Section 7.

2   General definitions of synchronization

As it was already mentioned in the introduction, one can think of different types of synchronization. To encompass most of the known definitions and applications, a general formalism for synchronization was proposed in [4]. In this section, we review some of the definitions given in [4].

Consider k dynamical systems

$$\Sigma_i = \{T, U_i, X_i, Y_i, \phi_i, h_i\}, \quad i = 1, \ldots, k,$$

where T is the common set of time instances, U_i, X_i, Y_i, are sets of inputs, states and outputs, respectively;

$\phi_i : T \times X_i \times U_i \to X_i$ are transition maps, $h_i : T \times X_i \times U_i \to Y_i$ are output maps (here one of the standard definitions of dynamical systems is used). In the sequel, we take as time set T either $T = \mathbb{R} \geq 0$ (continuous time) or $T = \mathbb{Z} \geq 0$ (discrete time).

First, consider the case when all U_i are just empty sets, i.e. inputs are not present and may be omitted in the formulation. Suppose l functionals $g_j : \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_k \times T \to \mathbb{R}^1$, $j = 1, \ldots, l$, are given. Here, $\mathcal{Y}_i$ are the sets of all functions from T into Y_i, i.e. $\mathcal{Y}_i = \{y : T \to Y_i\}$. For any $\tau \in T$ we then define σ_τ as the *shift operator*, i.e. $\sigma_\tau : \mathcal{Y}_i \to \mathcal{Y}_i$ is given as $(\sigma_\tau y)(t) = y(t + \tau)$ for all $y \in \mathcal{Y}_i$ and all $t \in T$.

Definition 1 *We call the solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ synchronized with respect to the functionals $g_1, \ldots, g_l$ if*

$$g_j(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t) \equiv 0, \quad j = 1, \ldots l, \tag{5}$$

is valid for all $t \in T$ and some fixed $\tau_1, \ldots, \tau_k \in T$, where $y_i(\cdot)$ denotes the output function of the system $\Sigma_i : y_i(t) = h(x_i(t), t)$, $t \in T$, $i = 1, \ldots, k$.
We say that solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ are approximately synchronized with respect to the functionals $g_1, \ldots, g_l$, if there are an $\varepsilon > 0$ and $\tau_1, \ldots, \tau_k \in T$ such that

$$|g_j(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t)| \leq \varepsilon, \quad j = 1, \ldots l, \tag{6}$$

for all $t \in T$.
The solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ are asymptotically synchronized with respect to the functionals $g_1, \ldots, g_l$, if for some $\tau_1, \ldots, \tau_k \in T$

$$\lim_{t \to \infty} g_j(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t) = 0, \quad j = 1, \ldots, l. \tag{7}$$

In many cases, the sets U_i, X_i, Y_i are finite-dimensional vector spaces and the systems Σ_i can be described by ordinary differential equations. First, consider the simplest case of disconnected systems without inputs

$$\Sigma_i : \dot{x}_i = F_i(x_i, t), \tag{8}$$

where F_i, $i = 1, \ldots k$, are time-dependent vector fields. Sometimes, synchronization may occur in disconnected systems (8) (e.g. all precise clocks are synchronized in the frequency sense). This case will be referred to as *natural synchronization*. A more interesting and important case, however, seems synchronization of interconnected systems. In this case, the system models are augmented with interconnections and can be described by the following differential equations:

$$\begin{aligned} \dot{x}_i &= F_i(x_i, t) + \tilde{F}_i(x_0, x_1, \ldots, x_k, t), \quad i = 1, \ldots k, \\ \dot{x}_0 &= F_0(x_0, x_1, \ldots, x_k, t), \end{aligned} \tag{9}$$

where the vector field F_0 describes the dynamics of the interconnection system, $\tilde{F}_i$ are vector fields describing the interconnections. The model (9) can formally not be considered within the given definition. To include the case of synchronization of interconnected systems we should introduce a dynamical system which describes the interconnections between the systems. To describe the possible interconnections, we now suppose that the input of each system Σ_i, $i = 1, \ldots, k$, can be composed from the output of the interconnection system $\Sigma_0 = \{T, U_0, X_0, Y_0, \phi_0, h_0\}$, where the transition and output maps are given by $\phi_0 : T \times X_0 \times U_0 \to X_0$ and $h_0 : T \times X_0 \times U_0 \to Y_0$ with $U_0 = Y_1 \times Y_2 \times \cdots \times Y_k$ and $Y_0 = U_1 \times U_2 \times \cdots \times U_k$. Now, it is possible to define synchronization of interconnected systems.

Definition 2 *We call the solutions $x_0(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ and interconnection system Σ_0 with initial conditions $x_0(0), \ldots, x_k(0)$ synchronized with respect to the functionals $g_1, \ldots, g_l$ if*

$$g_j(\sigma_{\tau_0} y_0(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t) \equiv 0, \quad j = 1, \ldots l, \tag{10}$$

is valid for all $t \in T$ and some $\tau_0, \ldots, \tau_k \in T$, where $y_i(\cdot)$ denotes the output function of the system $\Sigma_i : y_i(t) = h(x_i(t), t)$, $t \in T$, $i = 0, \ldots, k$.

We say that solutions $x_0(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ and interconnection system Σ_0 with initial conditions $x_0(0), \ldots, x_k(0)$ are approximately synchronized *with respect to the functionals $g_1, \ldots, g_l$, if there are an $\varepsilon > 0$ and $\tau_0, \ldots, \tau_k \in T$ such that*

$$|g_j(\sigma_{\tau_0} y_0(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t)| \le \varepsilon, \quad j = 1, \ldots, l, \tag{11}$$

for all $t \in T$.

The solutions $x_0(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ and interconnection system Σ_0 with initial conditions $x_0(0), \ldots, x_k(0)$ are asymptotically synchronized *with respect to the functionals $g_1, \ldots, g_l$, if for some $\tau_0, \ldots, \tau_k \in T$*

$$\lim_{t \to \infty} g_j(\sigma_{\tau_0} y_0(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t) = 0, \quad j = 1, \ldots, l. \tag{12}$$

A remarkable and widely used observation is that synchronization may exist, i.e. identity (10) (or (12)) may be valid in the interconnected system without any artificially introduced external action, i.e. when the interconnection system Σ_0 is given. In this case, the system (9) can be called *self-synchronized with respect to the functionals $g_1, \ldots, g_l$*. Similar definitions can be introduced for approximate and asymptotic self-synchronization. The above definitions do not yet include the possibility of controlling the system. Assume for simplicity that all Σ_i, $i = 0, \ldots, k$, are smooth finite-dimensional systems, described by differential equations with a finite-dimensional input, i.e.

$$\begin{aligned} \dot{x}_i &= F_i(x_i, t) + \tilde{F}_i(x_0, x_1, \ldots, x_k, u, t), \quad i = 1, \ldots, k, \\ \dot{x}_0 &= F_0(x_0, x_1, \ldots, x_k, u, t), \end{aligned} \tag{13}$$

where $u = u(t) \in \mathbb{R}^m$ is the input or control variable.

Definition 3 *The problem of* controlled synchronization with respect to the functionals g_j, $j = 1, \ldots l$, *(*controlled approximate *and* controlled asymptotic synchronization *with respect to the functionals g_j, $j = 1, \ldots, l$) is to find a control u as a feedback function of the states $x_0, x_1, \ldots, x_k$ and time t providing that (10) ((11) and (12), respectively) holds for the closed-loop system.*

Sometimes, the goal can be ensured without measuring any variables of the systems, for instance, by a time-periodic forcing. In this case, the control function u does not depend on system states and the problem of finding such a control is called an *open-loop-controlled (asymptotic) synchronization problem*. However, a more powerful approach assumes the possibility of measuring the states or some function of the system variables. Finding a control function in this case is called a *closed-loop* or *(asymptotic) feedback synchronization problem*.

In a variety of problems, complete information about the states of the systems $\Sigma_0, \Sigma_1, \ldots, \Sigma_k$ is not available and only some *output variables* $\tilde{y}_s$, $s = 1, \ldots, r$, with $\tilde{h}_s$ output functions of the interconnected system, so $\tilde{y}_s = \tilde{h}_s(x_0, x_1, \ldots, x_k, t)$, are available for use in the control law. The problem of *output feedback synchronization* can be posed as follows: find controller equations in the form of static output feedback

$$u(t) = \mathcal{U}(\tilde{y}_1, \ldots, \tilde{y}_r, t) \tag{14}$$

or in the form of dynamic output feedback

$$\begin{aligned} \dot{w} &= W(\tilde{y}_1, \ldots, \tilde{y}_r, w, t), \tag{15} \\ u(t) &= \mathcal{U}(\tilde{y}_1, \ldots, \tilde{y}_r, w, t) \tag{16} \end{aligned}$$

with $w \in \mathbb{R}^\nu$, $\tilde{y}_s \in \mathbb{R}^{p_s}$, $W : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_r} \times \mathbb{R}^\nu \times T \to \mathbb{R}^\nu$ and $\mathcal{U} : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_r} \times \mathbb{R}^\nu \times T \to \mathbb{R}^m$, are smooth parameterized vector fields (functions), such that the goal (12) in system (13), (14) (or (13), (15), (16)) is achieved.

3 An observer looks at synchronization

In the definitions from the previous section, the controlled synchronization problem is considered with respect to *given* systems. In practice, however, the problem is often the following: given a particular

dynamical system, the master, design a slave system, driven by the outputs of the master system, which asymptotically synchronizes with the master system. If synchronization is considered with respect to the functional $g(x_m(\cdot), x_s(\cdot), t) = |x_m(t) - x_s(t)|$, where x_m and x_s are the states of the master and slave system respectively, then the problem can be reformulated in the following way: given a dynamical system, the master, design a slave system driven by the outputs of the master system, such that it asymptotically reconstructs the state of the master system. The problem just described is the same as the observer problem from control theory. In this section, we review the observer design methodology and some related results from control theory, which can be applied to controlled synchronization problems. This section is based on [5], [6] and [7].

3.1 Problem statement

We consider two particular problems in the area of observer design, which are closely linked to synchronization. First, we introduce the full observer and, next, the reduced observer problem.

Consider dynamics governed by

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad t \geq t_0. \tag{17}$$

We assume that the vector field is smooth and that the system (17) has a unique solution $x(t, x_0)$ passing through the initial state $x(t_0, x_0) = x_0$ defined on the interval $[t_0, +\infty)$. The state is not directly available; only an output is measured:

$$y(t) = h(x(t), t) \in \mathbb{R}^p, \quad p < n. \tag{18}$$

We assume that h is smooth.

A *full order* observer for the system (17), (18) is defined as

$$
\begin{aligned}
\dot{\hat{x}} &= \hat{f}(\hat{x}(t), y(t), t) \quad \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}^n, \quad t \geq t_0 \\
\hat{y}(t) &= h(\hat{x}(t), t) \in \mathbb{R}^p,
\end{aligned} \tag{19}
$$

where $\hat{x} \in \mathbb{R}^n$ and $\hat{f}$ is a smooth vector field, parameterized by y and t, such that the error $e(t) = x(t) - \hat{x}(t)$ asymptotically converges to zero as $t \to \infty$ for all initial conditions $\hat{x}_0$, x_0 and t_0 and, moreover, if $e(t_0) = 0$ then $e(t) \equiv 0$ for all $t \geq t_0$.

If we reconstruct the state via (19), we are reconstructing more information than necessary, since the output y already contains some information about the state. To discuss this point further let us specialize to the case where the output equation does not depend explicitly on time, $y(t) = h(x(t))$. Let us assume that there exists a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$
\begin{aligned}
\phi(x) &= \left(\begin{array}{c} h(x) \\ v(x) \end{array} \right) = \left(\begin{array}{c} y \\ z \end{array} \right), \\
x &= \phi^{-1}(y, z).
\end{aligned} \tag{20}
$$

Given the output y it suffices to reconstruct z in order to know x. Now, z is governed by the differential equation

$$\dot{z} = f_r(z(t), y(t), t), \quad z(t_0) = v(x_0), \quad t \geq t_0, \tag{21}$$

where the vector field f_r is uniquely determined by f and ϕ.

Let $\hat{z}$ be defined by:

$$\dot{\hat{z}} = f_r(\hat{z}(t), y(t), t), \quad \hat{z}(t_0) = \hat{z}_0, \quad t \geq t_0. \tag{22}$$

If the diffeomorphism ϕ can be chosen such that the error $e_r(t) = z(t) - \hat{z}(t)$ converges to zero asymptotically as $t \to \infty$ and if, moreover, $e_r(t_0) = 0$ implies that $e_r(t) \equiv 0$ for all $t \geq t_0$, then we call the system (22) a reduced order observer for the nonlinear system (17), (18). Once a reduced order observer is found, the full state is asymptotically recovered via $x(t) = \phi^{-1}(y(t), \hat{z}(t))$.

So far, we have considered observers in the form of a dynamical system having the same or lower order than the observed system. In some cases an observer having higher order than the original system can be constructed. In that case, the estimate of the state vector is given by some output of the observer. Such observer designs appear to be useful in case of parametric uncertainties in the observed system (see Section 3.5 for details).

For dynamical systems with inputs

$$\dot{x} = f(x, u, t), \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad t \geq t_0, \quad u(t)\text{-input}, \tag{23}$$
$$y = h(x, t) \in \mathbb{R}^p, \quad p < n,$$

the possibility of constructing an observer or even of distinguishing different states from the outputs in general depends on the inputs $u(t)$. It may happen that for certain inputs the system (23) may have two (or more) different solutions (due to different initial conditions) with the same outputs. That may require additional assumptions on the inputs or make an observer design even impossible (see [28] for details). For linear systems, however, if an observer can be designed for a system without inputs, than an observer can be also designed for this system with inputs added (assuming $u(t)$ is known).

A standard approach in solving the observer problem in control theory is to construct an observer as a copy of the original system (17) modified with a term depending on the difference between the output y and its prediction $\hat{y}$ derived from the observer system. Some approaches to solving the observer problem are discussed in the next sections.

3.2 Linear systems

In the case of linear time-invariant dynamics, the problem of constructing a full or reduced order observer is completely solved (see e.g. [11]). The relevant equations (17) and (18) now simplify to

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad A \in \mathbb{R}^{n \times n}$$
$$y(t) = Cx(t), \quad C \in \mathbb{R}^{p \times n}. \tag{24}$$

For any matrix A the solutions of (24) are defined on $(-\infty, +\infty)$, the observer problem is hence always well posed.

A. Full Observer
For the linear system (24), the observer system (19) takes the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K(\hat{y}(t) - y(t)), \quad \hat{x}(0) = \hat{x}_0$$
$$\hat{y}(t) = C\hat{x}(t). \tag{25}$$

Here $K \in \mathbb{R}^{n \times p}$ is known as an output injection matrix. The error $e(t) = x(t) - \hat{x}(t)$ is governed by

$$\dot{e}(t) = (A + KC)e(t). \tag{26}$$

This error dynamics represent a valid observer design if a gain matrix K can be found such that the matrix $A + KC$ has eigenvalues with negative real part. A sufficient condition for such matrix K to exist is observability of the pair of matrices (C, A). Observability means that the mapping $\mathcal{O}(x) = (Cx, CAx, \ldots, CA^{n-1}x)$ is invertible. Actually, if the pair (C, A) is observable, then the eigenvalues of the matrix $A + KC$ can be set arbitrary by means of appropriate choice of K. Existence of such K that $A + KC$ is a stable matrix is a requirement weaker than observability, called *detectability*. A necessary and sufficient condition for the pair (C, A) to be detectable is the following:

for all $\lambda \in \mathbb{C}$ such that $\text{rank} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < n$ it holds that $\text{Real}(\lambda) < 0$.

B. Reduced Order Observer
Clearly when measuring $y = Cx$, it appears that we only need to reconstruct $z = Hx$ where H is chosen

such that $[C^T\ H^T]^T$ has full column rank. Let us assume that C has full row rank p. This amounts to stating that there are no redundant measurements in the output y. We can then find $H \in \mathbb{R}^{(n-p) \times n}$ such that

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} C \\ H \end{bmatrix} x \quad and \quad x = [S\ T] \begin{pmatrix} y \\ z \end{pmatrix}.$$

The partial state z satisfies the differential equation

$$\dot{z}(t) = HATz(t) + HASy(t). \tag{27}$$

It can be shown that under the assumption that the matrix pair (A, C) be detectable, matrix H can be chosen such that HAT is asymptotically stable. A *reduced order* observer is then given by

$$\begin{aligned} \dot{\hat{z}}(t) &= HAT\hat{z}(t) + HASy(t) \\ \hat{x}(t) &= T\hat{z}(t) + Sy(t). \end{aligned} \tag{28}$$

Notice that if the matrix pair (A, C) is not observable, but detectable, it may be that a reduced order observer of lower dimension then $n - p$ exists, e.g. in the case that A is asymptotically stable, one could use as reduced order observer $\hat{x}(t) \equiv 0$.

3.3 Systems with linearizable error dynamics

From the previous examples, a straightforward extension toward nonlinear systems transpires. The idea is to consider systems that may give rise to linear error dynamics perhaps via an appropriate change of coordinates and/or rescaling of the output variables.

Consider the class of systems of *Lur'e type*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(Cx(t), t), \quad x(t_0) = x_0, \quad t \geq t_0 \\ y(t) &= Cx(t). \end{aligned} \tag{29}$$

Here A, C are constant matrices of appropriate dimensions. Suppose that the solutions of (29) are well defined on $[t_0, +\infty)$. Assuming that the matrix pair (A, C) is detectable, a full observer system takes the form:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + f(y(t), t) + K(\hat{y}(t) - y(t)) \\ \hat{x}(t_0) &= \hat{x}_0, \quad t \geq t_0 \\ \hat{y}(t) &= C\hat{x}(t). \end{aligned} \tag{30}$$

The error $e = x - \hat{x}$ is governed by

$$\dot{e} = (A + KC)e.$$

Thus, it suffices to choose K such that $A + KC$ is asymptotically stable. We can easily construct such observer due to the fact that the nonlinearity depends only on the measured output.

We can generalize such observer design in the following way. Starting from systems of the form (17) with $y(t) = h(x(t))$, find a coordinate transformation $\xi = \phi(x)$ and output transformation $\eta = \psi(y)$ such that in the new coordinates we have a system description of the form

$$\begin{aligned} \dot{\xi}(t) &= A\xi(t) + g(\eta(t), t), \quad \xi(t_0) = \xi_0, \quad t \geq t_0 \\ \eta(t) &= C\xi(t). \end{aligned} \tag{31}$$

Provided the matrix pair (A, C) is detectable we can then construct an observer which gives rise to linear error dynamics in the way explained earlier. Notice that for the observer problem to be well posed the solutions of (31) have to be well-defined on $[0, +\infty)$.

Conditions under which system (17) can be transformed into the form (31) by a state space transformation $\xi = \phi(x)$ and by an output transformation $\eta = \psi(y)$ are presented in [5], [12]. Here, we illustrate the

basic idea by the following example:

Example: Let us consider the Rössler system:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -x_2(t) - x_3(t) \\ x_1 + ax_2(t) \\ c + x_3(t)(x_1(t) - b) \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x, \tag{32}$$

where the coefficients $a, b, c > 0$. Assume also that $x_3(0) > 0$, then $x_3(t) = y(t) > 0$ for all $t \geq 0$ (because for $x_3 = 0$ its derivative is positive: $\dot{x}_3 = c > 0$ and, thus, $x_3(t)$ cannot come through zero). Keeping this restriction in mind we may use the comparison function $V = 1/2(x_1^2 + x_2^2) + x_3 > 0$. Taking the time-derivative of V along the solutions of (32) gives $\dot{V} = ax_2^2 + c - bx_3 \leq aV + c$ which implies that the solutions (with $x_3(0) > 0$) are well defined on $(0, +\infty)$ (integrating the inequality for $\dot{V}$ implies, that V may grow not faster than Ce^{at}, where $C > 0$ is some constant; thus, $V(x(t))$ is finite at each time instant $t \geq 0$ and, hence, $x(t)$ is finite and well-defined for all $t \geq 0$). Thus, the observer problem is well posed. Now, we introduce the following coordinates:

$$\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \ln(x_3) \end{pmatrix}, \quad \eta = \ln y. \tag{33}$$

In the new coordinates, the system equations are given by

$$\frac{d}{dt}\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} + \begin{pmatrix} -e^{\xi_3(t)} \\ 0 \\ -b + ce^{-\xi_3(t)} \end{pmatrix}$$
$$\eta(t) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \xi(t). \tag{34}$$

The linear part of (34) is again observable and the nonlinearity only incorporates η - the new measured output. Hence an observer with linear, asymptotically stable dynamics can be constructed as before.

The reasoning presented above is clearly relevant for the full observer problem. For reduced order observers the following statement holds [5]: if a full observer with linear error dynamics can be found, a reduced observer with linear error dynamics can also be constructed. The reverse may not be the case.

In the above examples, we were led to time-invariant linear error dynamics. This may not always be achievable, but it may be possible to attain linear time varying error dynamics. At the same time, in general it is extremely hard to establish stability properties for time-varying systems. For systems of the form $\dot{x}(t) = A(y(t))x(t) + B(y(t))$ with $y = Cx$, where A depends in a smooth way on y and such that the family of matrices $(A(y), C)$ is uniformly (in y) detectable, there is, however, a slightly more systematic design method for achieving a full order observer. It is based upon Lyapunov theory. The observer may be constructed as

$$\dot{\hat{x}}(t) = A(y(t))\hat{x}(t) + B(y(t)) + K(y(t))(\hat{y}(t) - y(t)),$$

where $K(y) = -R(y)^{-1}C^T$ and $R(y)$ is the unique symmetric positive definite solution of Riccati equation

$$0 = R(y)A(y) + A^T(y)R(y) + R(y)C^T CR(y) - C^T C.$$

This approach is non-trivial as it requires one to solve (analytically) a Riccati equation which depends on a parameter y.

3.4 High-gain Observer

If the dynamics is restricted to a compact set, then the following result is available [13]. Consider a time-invariant system of the form (17), (18). Let the output y be scalar. Assume that $\Omega \subset \mathbb{R}^n$ is a compact and a positively invariant set for the dynamics (17), (18). Assume that $\xi = \phi(x) := (h(x), L_f h(x), \cdots, L_f^{n-1}h(x))^T$ is a diffeomorphism on a open subset containing Ω. [1] The system equation

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[1]The iterative directional derivative $L_f^i h$ is defined in the following way: $L_f^1 h = \frac{\partial h}{\partial x}f$, $L_f^{k+1}h = L_f(L_f^k h)$.

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(17), (18) in the new coordinates ξ are represented by

$$\dot{\xi}(t) = F(\xi(t))$$
$$y(t) = \xi_1(t).$$
(35)

Consider also the system

$$\dot{\hat{\xi}}(t) = F(\hat{\xi}(t)) + K_\theta(\hat{y}(t) - y(t))$$
$$\hat{y}(t) = \hat{\xi}_1(t).$$
(36)

where the constant gain $K_\theta \in \mathbb{R}^n$ is defined via

$$K_\theta = -S_\theta^{-1} C^T,$$

where $S_\theta = S_\theta^T > 0$ solves

$$0 = \theta S_\theta + A^T S_\theta + S_\theta A - C^T C,$$

where

$$C = (1 \quad 0 \quad \ldots \quad 0) \in \mathbb{R}^{1 \times n}$$

and

$$A = \begin{pmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ 0 & 0_{1 \times (n-1)} \end{pmatrix}.$$

The system (36) is an observer for the system (35) for all sufficiently large $\theta > 0$, in that for all ξ_0 in $\phi(\Omega)$, the error $e(t) = \xi(t) - \hat{\xi}(t)$ decreases exponentially towards zero.

The equation $0 = \theta S_\theta^2 + A^T S_\theta + S_\theta A - C^T C$ to be solved for S_θ is known as an algebraic Riccati equation. The observability of (A, C) guarantees the existence of a positive definite solution S_θ with the property that $A + K_\theta C$ is an asymptotically stable matrix. The error dynamics are in general nonlinear, but due to the *large* gain K_θ the error dynamics on $\phi(\Omega)$ are essentially dominated by the stability of $A + K_\theta C$.

3.5 Adaptive observers

In the case of systems with parametric uncertainties, one can try to find an adaptive observer capable to cope with the uncertainties. In this section, we review some results from [6] concerning adaptive observer design.

Consider a single output nonlinear system with unknown constant parameters $\theta_i, i = 1, \ldots, N$.

$$\dot{x} = f(x) + \sum_{i=1}^{N} \theta_i q_i(x), \quad y = h(x),$$
(37)

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^1$, $f(0) = 0$, $h(0) = 0$, $\theta = (\theta_1, \ldots, \theta_N)^T$ is a vector of unknown parameters.

Definition 4 *A global adaptive observer for system (37) is a finite-dimensional system*

$$\begin{cases} \dot{w} = F_1(w, \hat{\theta}, y) \\ \dot{\hat{\theta}} = F_2(w, \hat{\theta}, y) \\ \hat{x} = H(w, \hat{\theta}, y), \end{cases}$$
(38)

where $w(t) \in \mathbb{R}^r, r \geq n$, $\hat{\theta}(t) \in \mathbb{R}^N$, $\hat{x}(t) \in \mathbb{R}^n$, if for every $x(0) \in \mathbb{R}^n$, $w(0) \in \mathbb{R}^r$, $\hat{\theta}(0) \in \mathbb{R}^N$ and any value of unknown parameters θ providing boundedness of the state vector of (37) $x(t)$, $0 \leq t < \infty$, the state vector of (38) $w(t)$, $\hat{\theta}(t)$ is also bounded and

$$\lim_{t \to \infty} \|x(t) - \hat{x}(t)\| = 0.$$
(39)

We first consider a special case, the so-called *adaptive observer form*, which is given as

$$\dot{z} = A_0 z + \phi_0(y) + B \sum_{i=1}^{N} \theta_i \psi_i(y), \quad y = C_0 z, \tag{40}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad C_0 = [0, \ldots, 0 \; 1],$$

$\psi_i(y)$ are smooth scalar functions, and $B = [b_1, b_2, \ldots, b_n]^\top \in \mathbb{R}^n$ is a constant vector such that $b_n > 0$ and the polynomial $b_n \lambda^{n-1} + \ldots + b_1$ is Hurwitz, i.e. all its roots have negative real parts. The conditions ensuring the possibility to transform (37) into the adaptive observer form (40) can be found in [15].

A solution to the adaptive observer design problem is provided by the following theorem [15].

Theorem 1 *The system*

$$\begin{cases} \dot{\widehat{z}} = A_0 \widehat{z} + \phi_0(y) + B \sum_{i=1}^{N} \widehat{\theta}_i \psi_i(y) + K(C_0 \widehat{z} - y) \\ \dot{\widehat{\theta}} = \Gamma \psi(y)(y - C_0 \widehat{z}) \end{cases} \tag{41}$$

is a global adaptive observer for system (40), where $\widehat{z}(t) \in \mathbb{R}^n$, $\widehat{\theta}(t) \in \mathbb{R}^N$, if $\Gamma = \Gamma^\top > 0$ is any $N \times N$ symmetric positive definite matrix and the vector $K \in \mathbb{R}^n$ is defined as

$$K = (A_0 B + \mu B)/b_n$$

with $\mu > 0$.

This result can be extended for systems of the form

$$\dot{z} = A_0 z + \phi_0(y) + \sum_{i=1}^{N} \theta_i \phi_i(y), \quad y = C_0 z, \tag{42}$$

where $\phi_i(y(t)) \in \mathbb{R}^n$, $i = 0, 1, \ldots, N$, are vectors of nonlinearities depending on the output $y = y(t)$. See [6] for details.

Furthermore, we review another solution to the adaptive observer design problem for systems of the form:

$$\dot{x} = Ax + \varphi_0(y) + B \sum_{i=1}^{m} \theta_i \varphi_i(y), \quad y = Cx, \tag{43}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^l$ is the vector of outputs, $\theta = (\theta_1, \ldots, \theta_m)^\top$ is the vector of uncertain parameters. It is assumed that the nonlinearities $\varphi_i(\cdot)$, $i = 0, 1, \ldots, m$, matrices A, C and vector B are known.

The proposed observer has the following form:

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + \varphi_0(y) + B \left[\sum_{i=1}^{m} \widehat{\theta}_i \varphi_i(y) + \widehat{\theta}_0 G(y - \widehat{y}) \right], \\ \widehat{y} &= C\widehat{x}, \end{aligned} \tag{44}$$

where $\widehat{x} \in \mathbb{R}^n$, $\widehat{\theta}_i \in \mathbb{R}$, $i = 0, \ldots, m$, and $G^T \in \mathbb{R}^l$ is the vector of weights. The adaptation algorithm is provided by standard adaptive control (speed-gradient) techniques as follows:

$$\dot{\theta}_i = -\gamma_i G(\hat{y} - y)\varphi_i(y), \quad i = 1, \ldots, m, \tag{45}$$

$$\dot{\theta}_0 = -\gamma_0 (G(\hat{y} - y))^2. \tag{46}$$

In order to formulate the conditions required for successful applicability of the proposed scheme we need some definitions.

Definition 5 *[16] The system $\dot{x} = \bar{A}x + \bar{B}u$, $y = \bar{C}x$ with transfer matrix $W(\lambda) = \bar{C}(\lambda I - \bar{A})^{-1}\bar{B}$, where u, $y \in \mathbb{R}^l$ and $\lambda \in \mathbb{C}$ is called hyper-minimum-phase if it is minimum-phase (i.e. the polynomial $\varphi(\lambda) = \det(\lambda I - \bar{A}) \det W(\lambda)$ is Hurwitz), and the matrix $\bar{C}\bar{B} = \lim_{\lambda \to \infty} \lambda W(\lambda)$ is symmetric and positive definite.*

Definition 6 *[17] A vector-function $f : [0, \infty) \to \mathbb{R}^m$ satisfies the persistency of excitation (PE) condition (or it is persistently exciting) on $[0, \infty)$, if it is piecewise continuous, bounded and if there exist positive constants $\alpha_1 > 0$, $\alpha_2 > 0$, $T > 0$ such that*

$$\alpha_1 I_m \leq \int_t^{t+T} f(s)f(s)^\top ds \leq \alpha_2 I_m$$

for all $t \geq 0$.

The following theorem gives sufficient conditions on applicability of the observer (44)-(46) as well as on its identification properties.

Theorem 2 *Assume that all trajectories of the system (43) are bounded and the linear system with the transfer function $W(\lambda) = GC(\lambda I - A)^{-1}B$ is hyper-minimum-phase. Then all trajectories of the observer (44), (45), (46) are bounded and $(\hat{x}(t) - x(t)) \to 0$, as $t \to \infty$. If, in addition, the vector-function $[\varphi_1(y(t)), \ldots, \varphi_m(y(t))]^T$ satisfies the PE condition, then also $(\hat{\theta}_i(t) - \theta_i) \to 0$, as $t \to \infty$, for $i = 1, \ldots, m$.*

Identification properties of the proposed observer scheme can be utilized for secure communication as it will be discussed in Section 6.

3.6 Discrete-time observers

Many continuous-time models are in the end – for instance, for the purpose of simulation and implementation – discretized or sampled. This motivates to look at the synchronization and observer problems for discrete-time systems. In this section, we review some results from [7] on discrete-time observer design.

Throughout this part of the report, we consider discrete-time nonlinear dynamics of the form

$$x(k+1) = f(x(k)), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{47}$$

where the state transition map f is a smooth mapping from $\mathbb{R}^n$ into itself. Let us assume that the solution $x(k, x_0)$ of (47) is not directly available, but only an *output* is measured, say

$$y(k) = h(x(k)), \tag{48}$$

where $y \in \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ is the smooth output map. For the sake of simplicity, we assume the output y to be scalar, i.e. $p = 1$.

The observer problem for (47), (48) deals with the question how to reconstruct the state trajectory $x(k, x_0)$ on the basis of the measurements $y(k)$. A *full order observer* (or briefly *observer*) for the system (47), (48) is a dynamical system of the form

$$\hat{x}(k+1) = \hat{f}(\hat{x}(k), y(k)), \quad \hat{x}(0) = \hat{x}_0 \in \mathbb{R}^n, \tag{49}$$

where $\hat{x} \in \mathbb{R}^n$, and $\hat{f}$ is a smooth mapping on $\mathbb{R}^n$ parameterized by y, such that the error $e(k) := x(k) - \hat{x}(k)$ asymptotically converges to zero as $k \to \infty$ for all initial conditions x_0 and $\hat{x}_0$. Moreover, we require that if $e(k_0) = 0$ for some k_0, then $e(k) = 0$ for all $k \geq k_0$.

Analogous to the case of continuous-time systems, we first consider the case of linear systems and then its extensions to nonlinear systems.

For linear systems the dynamics (47), (48) take the form

$$x(k+1) = Ax(k), \quad y(k) = Cx(k). \tag{50}$$

The observer proposed for this dynamics is

$$\hat{x}(k+1) = A\hat{x}(k) + L(C\hat{x}(k) - y(k)), \tag{51}$$

where L is an output injection matrix. Obviously, the error $e(k) = \hat{x}(k) - x(k)$ satisfies the following equation

$$e(k+1) = (A + LC)e(k).$$

The error $e(k)$ tends to zero if and only if all the matrix $A + LC$ is asymptotically stable (in discrete time sense), i.e. all the eigenvalues of the matrix $A + LC$ lie in the open unit disk (the set $\{z \in \mathbb{C} \mid |z| < 1\}$). The question now is under what conditions there exists such L that $A + LC$ is an asymptotically stable matrix? Observability of the pair (C, A) is, like in the continuous–time case, a sufficient condition for that. But necessary and sufficient conditions are somewhat different from the case of continuous–time systems:

$$\text{for all } \lambda \in \mathbb{C} \text{ such that rank} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < n \quad \text{it holds that } |\lambda| < 1.$$

Next, consider a class of nonlinear systems that is slightly more general than linear systems, namely systems in Lur'e form. Assume that the master dynamics are governed by the following system of difference equations

$$x(k+1) = Ax(k) + \varphi(y(k)), \quad y(k) = Cx(k), \tag{52}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^1$ is the scalar output, $\varphi : \mathbb{R}^1 \to \mathbb{R}^n$ is a smooth function and A, C are constant matrices of appropriate dimensions. Similar to the case of continuous-time systems the proposed observer has the form

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + \varphi(y(k)) + L(\hat{y}(k) - y(k)) \\ \hat{y}(k) = C\hat{x}(k) \end{cases} \tag{53}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the estimate of $x(k)$ and L is a $n \times 1$ matrix.

Subtracting (52) from (53), one can easily see that the error vector $e(k) := \hat{x}(k) - x(k)$ obeys the following linear difference equation

$$e(k+1) = (A + LC)e(k). \tag{54}$$

Therefore, if all eigenvalues of $A + LC$ lie in the open unit disc (i.e., the set $\{z \in \mathbb{C} \mid |z| < 1\}$), then (53) is an observer for (52). Thus, the observer design problem reduces to finding such L that $A + LC$ is asymptotically stable. As it was mentioned before, a sufficient condition for the existence of such L is observability of the pair (C, A).

The proposed observer design can be applied for systems in Lur'e form. This poses the following question: what can we do if the transmitter dynamics are not in the form (52)? A possible solution is to transform the system by means of change of coordinates into Lur'e form.

Let a discrete–time system (47), (48) with scalar output be given, and assume that $f(0) = 0$, $h(0) = 0$. The problem is to find conditions ensuring existence of an invertible coordinate change $z = T(x)$ such that the system (47) is locally (or globally) equivalent to the following Lur'e system

$$z(k+1) = Az(k) + \varphi(y(k)), \quad y(k) = Cz(k) \tag{55}$$

where the pair (C, A) is observable.

As one can see from the problem statement, the coordinate change $z = T(x)$ can be either locally or globally defined (i.e., the inverse mapping T^{-1} can exist on a neighborhood of the origin or everywhere). In the first case, the systems (47), (48) and (55) are equivalent if for all k one has that $\|x(k)\|$ is sufficiently small. In the second case, there are no restrictions of such kind. The following result gives a (local) solution to the problem.

Theorem 3 *A discrete–time system (47), (48) with scalar output is locally equivalent to a system in Lur'e form (55) with observable pair (C, A) via a coordinate change $z = T(x)$ if and only if*

(i) *the pair $(\partial h(0)/\partial x, \partial f(0)/\partial x)$ is observable,*

(ii) *the Hessian matrix of the function $h \circ f^n \circ \mathcal{O}^{-1}(s)$ is diagonal, where $x = \mathcal{O}^{-1}(s)$ is the inverse map of*

$$\mathcal{O}(x) = \left[h(x), h \circ f(x), \ldots, h \circ f^{n-1}(x) \right]^{\top}, \tag{56}$$

with $h \circ f(x) := h(f(x)), f^1 := f, f^j := f \circ f^{j-1}$.

The condition (ii) of Theorem 3 is quite restrictive. Therefore, the question arises whether, and in what way, it may be relaxed. To answer this question, we will assume that at time k besides $y(k)$ also past output measurements $y(k-1), \cdots, y(k-N)$ for some $N > 0$ are available, and first consider nonlinear dynamics of the following form:

$$\begin{cases} x(k+1) &= Ax(k) + \varphi(y(k), y(k-1), \cdots, y(k-N)) \\ y(k) &= Cx(k) \end{cases} \tag{57}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^1$, $\varphi : \mathbb{R}^{N+1} \to \mathbb{R}^n$ is a smooth mapping, and A, C are matrices of appropriate dimensions. Note that the dynamics (57) for $N = 0$ are just the dynamics (52). Therefore, we refer to dynamics of the form (57) as dynamics in *extended Lur'e form with buffer* N. Assume that the pair (C, A) is observable. As it has been shown, there exists a matrix L such that all eigenvalues of $A + LC$ lie in the open unit disc. Then, it may be shown that the following dynamics represent an observer for (57):

$$\begin{cases} \widehat{x}(k+1) &= A\widehat{x}(k) + \varphi(y(k), \cdots, y(k-N)) + L(\widehat{y}(k) - y(k)) \\ \widehat{y}(k) &= C\widehat{x}(k) \end{cases} \tag{58}$$

Now we ask ourselves the question under what conditions the discrete-time system (47), (48) may be transformed into an extended Lur'e form for some $N \geq 0$. The transformations, which we are going to use here, are more general than the transformation used for systems in Lur'e form, in the sense that we also allow them to depend on the past output measurements $y(k-1), \cdots, y(k-N)$. More specifically, we will be looking at parameterized transformations $z = T(x, \xi_1, \cdots, \xi_N)$, where $z \in \mathbb{R}^n$, with the property that (locally or globally) there exists a mapping $T^{-1}(\cdot, \xi_1, \cdots, \xi_N) : \mathbb{R}^n \to \mathbb{R}^n$ parameterized by $(\xi_1, \cdots, \xi_N)$, such that for all $(\xi_1, \cdots, \xi_N)$ we have

$$T(T^{-1}(z, \xi_1, \cdots, \xi_N), \xi_1, \cdots, \xi_N) = z.$$

A mapping having this property will be referred to as an *extended coordinate transformation*. We will then say that the system (47), (48) may be transformed into an extended Lur'e form with buffer N if there exists an extended coordinate transformation $T(\cdot, \xi_1, \cdots, \xi_N) : \mathbb{R}^n \to \mathbb{R}^n$ parameterized by $(\xi_1, \cdots, \xi_N)$ such that the variable

$$z(k) := T(x(k), y(k-1), \cdots, y(k-N)) \tag{59}$$

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satisfies (57), where the pair (C, A) is observable. As pointed out above, one may then build an observer (58) for $z(k)$ in (59). From this observer, one then obtains estimates $\widehat{x}(k)$ for $x(k)$ by inverting the extended coordinate change T:

$$\widehat{x}(k) := T^{-1}(\widehat{z}(k), y(k-1), \cdots, y(k-N)) \tag{60}$$

Necessary and sufficient conditions under which a system (47), (48) may be transformed into an extended Lur'e form with buffer N can be found in [7]. Here, we mention a simple yet important corollary to this result: a system (47), (48) for which the mapping $\mathcal{O}$ in (56) is a local (global) diffeomorphism may always be locally (globally) transformed into an extended Lur'e form with buffer $n-1$.

Observers for perturbed linear systems
So far the design procedure for observers has been based on the assumption that for the discrete–time system under consideration the mapping $\mathcal{O}$ in (56) is a (local or global) diffeomorphism. In the sequel, we consider a particular class of systems for which this might not be the case. Namely, we consider systems of the form

$$\begin{cases} x(k+1) &= Ax(k) + Bf(x(k)) \\ y(k) &= Cx(k) \end{cases} \tag{61}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^1$ is the scalar output, the function $f : \mathbb{R}^n \to \mathbb{R}^1$ is smooth, A, B, C are matrices of appropriate dimensions, and the pair (C, A) is observable. Clearly, depending on the specific structure of f and B, the system (61) may have a mapping $\mathcal{O}$ that is not a diffeomorphism. Nevertheless, we may derive conditions on (61) that guarantee the existence of an observer.

Define the rational function $G(s)$ by $G(s) := C(sI - A)^{-1}B$. Then $G(s)$ has the form $G(s) = \frac{q(s)}{p(s)}$, where q and p are polynomials in s, with $\deg(p) > \deg(q)$. Assume that $\deg(p) - \deg(q) = 1$, which is equivalent to the fact that $CB \neq 0$. To obtain an observer for (61), we first define new coordinates in the following way. Since $CB \neq 0$, there exists an $(n-1) \times n$ matrix N such that $NB = 0$ and the matrix $S := \begin{bmatrix} C^T & N^T \end{bmatrix}^T$ is invertible. Thus, $(\xi, z) := (Cx, Nx)$ forms a new set of coordinates for (61). It is straightforwardly checked that in these new coordinates the system (61) takes the form

$$\begin{cases} \xi(k+1) &= \bar{f}(\xi(k), z(k)) \\ z(k+1) &= A_1\xi(k) + A_2z(k) \\ y(k) &= \xi(k) \end{cases} \tag{62}$$

We now assume the following:

A1 The mapping $\bar{f}$ in (62) is globally Lipschitz with respect to z, i.e., there exists an $L > 0$ such that

$$|\bar{f}(\xi, \bar{z}) - \bar{f}(\xi, z)| < L\|\bar{z} - z\| \quad \forall \xi \in \mathbb{R}, \quad \forall z, \bar{z} \in \mathbb{R}^{n-1}.$$

A2 All zeros of the polynomial $q(s)$ are located in the open unit disc.

As an observer candidate we take the following system:

$$\begin{cases} \widehat{\xi}(k+1) &= \bar{f}(y(k), \widehat{z}(k)) \\ \widehat{z}(k+1) &= A_1y(k) + A_2\widehat{z}(k) \end{cases} \tag{63}$$

We then have the following result.

Theorem 4 *Assume that for (61) we have that the pair (C, A) is observable, that $CB \neq 0$, and that assumptions **A1** and **A2** hold. Then (63) is an observer for (62).*

This theorem allows us to construct observers for discrete-time systems, which can not be transformed into an (extended) Lur'e form.

4 Passivity-based design of synchronizing systems

A fruitful approach to studying synchronization is based on utilizing passivity properties of synchronizing systems. In this section, we discuss some results obtained within this approach in [8] and [9].

Let us first introduce some notations and definitions. A function $V : X \to \mathbb{R}^+$ defined on a subset X of $\mathbb{R}^n$, $0 \in X$ is *positive definite* if $V(x) > 0$ for all $x \in X \setminus \{0\}$ and $V(0) = 0$. It is *radially unbounded* if $V(x) \to \infty$ as $|x| \to \infty$. In the sequel, we will use the following notations: let $V : \mathbb{R}^n \to \mathbb{R}^1$ be a continuously differentiable vector function, then $\nabla V(x)$ stands for the vector (in $\mathbb{R}^n$) of its first partial derivatives calculated at the point x: $(\nabla V(x))^\top = \partial V(x)/\partial x$.

Consider the nonlinear time-invariant affine control system:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{64}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input which is assumed to be a continuous and bounded function of time, $y(t) \in \mathbb{R}^l$ is the output; $f : \mathbb{R}^n \to \mathbb{R}^n$, the columns of the matrix $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are smooth vector fields, $f(0) = 0$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ is a smooth mapping.

Definition 7 *The system (64) is called C^r-semipassive if there exist a C^r-smooth, $r \geq 0$ nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$ and a function $H : \mathbb{R}^n \to \mathbb{R}^1$ such that for any initial conditions $x(0)$ and any admissible input $u \in C^0 \cap \mathcal{L}_\infty$ the following dissipation inequality*

$$V(x(t)) - V(x(0)) \leq \int_0^t (y(s)^\top u(s) - H(x(s)))ds \tag{65}$$

holds for all $0 \leq t < T_{u,x_0}$, where the function H is nonnegative outside some ball:

$$\exists \rho > 0 \ |x| \geq \rho \implies H(x) \geq 0.$$

If the function H is positive outside some ball, i.e.,

$$\exists \rho > 0 \ |x| \geq \rho \implies H(x) \geq \varrho(|x|)$$

for some continuous positive function ϱ defined for $|x| \geq \rho$, then system (64) is said to be strictly semipassive. The function $V(x)$ is called a storage function.

The most useful property of semipassive systems is that being linearly interconnected they possess bounded solutions (see [8] for details).

Now we can apply the concept of semipassivity to the synchronization of two systems. Consider the following system:

$$\begin{cases} \dot{x}_1 = f(x_1) + g_1(x_1)u_1 \\ \dot{x}_2 = f(x_2) + g_2(x_2)u_2 \end{cases} \tag{66}$$

where $x_1(t) \in \mathbb{R}^n$ and $x_2(t) \in \mathbb{R}^n$ are the states variables, $u_1(t) \in \mathbb{R}^m$, $u_2(t) \in \mathbb{R}^m$ are the control inputs and $y_1(t) \in \mathbb{R}^m$ and $y_2(t) \in \mathbb{R}^m$ are the outputs of the first and second subsystems given by

$$y_1 = h(x_1), \quad y_2 = h(x_2). \tag{67}$$

It is assumed that f, g_1, g_2 and h are continuously differentiable to ensure existence and uniqueness of the solutions of (66) with continuous inputs $u_1(t), u_2(t)$ at least on some time interval. The problem of synchronization is to find an appropriate feedback law for both u_1 and u_2 such that $|x_1(t) - x_2(t)| \to 0$ as $t \to \infty$ for all initial conditions $x_1(0), x_2(0)$.

We assume that there exists a globally defined coordinate transformation such that in the new coordinates system (66) is represented in the following normal form (see [10] for the conditions sufficient for the

possibility of such transformation):

$$\begin{cases} \dot{z}_1 = q(z_1, y_1) \\ \dot{y}_1 = a(z_1, y_1) + b_1(z_1, y_1)u_1 \\ \\ \dot{z}_2 = q(z_2, y_2) \\ \dot{y}_2 = a(z_2, y_2) + b_2(z_2, y_2)u_2. \end{cases} \tag{68}$$

The problem is to find an appropriate control algorithm as a static output feedback to ensure the goal of synchronization. We will seek for the conditions which ensure synchronization by the simplest controller:

$$u_1 = -\gamma(y_1 - y_2), \quad u_2 = -\gamma(y_2 - y_1), \tag{69}$$

where $\gamma \in \mathbb{R}^1$ is the synchronization gain, also referred in the literature to as a *coupling constant*. As we will see, under certain conditions, if γ exceeds some threshold value then synchronization occurs for all initial conditions from a given compact set.

This result is formulated in the following theorem, which establishes sufficient conditions of the semiglobal synchronization:

Theorem 5 *Assume that*

A1. *The functions q, a, b_1, b_2 are continuous and locally Lipschitz.*

A2. *The system*

$$\begin{cases} \dot{z}_i = q(z_i, y_i) \\ \dot{y}_i = a(z_i, y_i) + b_i(z_i, y_i)u_i, \end{cases} \tag{70}$$

$i = 1, 2$, *is C^0-semipassive with respect to the input u_i and output y_i with the radially unbounded storage function $V_i : \mathbb{R}^n \to \mathbb{R}^+$.*

A3. *There exist a C^2-smooth positive definite function $V_0 : \mathbb{R}^{n-m} \to \mathbb{R}^+$ and positive number α such that the following inequality is satisfied*

$$(\nabla V_0(z_1 - z_2))^\top (q(z_1, y_1) - q(z_2, y_1)) \le -\alpha|z_1 - z_2|^2.$$

for all $z_1, z_2 \in \mathbb{R}^{n-m}, y_1 \in \mathbb{R}^m$.

A4. *The matrix $b_1(z_1, y_1) + b_2(z_2, y_2)$ is positive definite:*

$$b_1(z_1, y_1) + b_2(z_2, y_2) > 2\beta I_m, \beta > 0.$$

Then, for any initial conditions $z_1(0), z_2(0), y_1(0), y_2(0)$ there exists $\bar{\gamma}$ such that for all $\gamma > \bar{\gamma}$ all solutions $z_1(t), z_2(t), y_1(t), y_2(t)$ are bounded for all $t \ge 0$ and the goal of synchronization is achieved.

Condition **A1** is a regularity condition, which guarantees at least local existence and uniqueness of solutions of the systems (68). Semipassivity of (68) is required in **A2**. Condition **A3** is a kind of stability property of the zero dynamics of the system (68). It guarantees that if the outputs are identical, $y_1(t) \equiv y_2(t)$ then the difference between $z_1(t)$ and $z_2(t)$ asymptotically converges to zero.

Remark. Condition **A3** resembles what is sometimes called *the active-passive decomposition* (APD) (see [19]). APD can be briefly described as follows. Given a dynamical system

$$\dot{x} = F(x) \tag{71}$$

decompose its state into x^1 and x^2. Then, the dynamics of x^1 and x^2 are given by $\dot{x}^i = F_i(x^1, x^2)$, $i = 1, 2$. Now consider a separate system

$$\dot{y} = F_2(x^1(t), y) \tag{72}$$

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driven by the solution $x^1(t)$. If for every initial condition of the systems (71) and (72), the response $y(t)$ asymptotically converges to $x^2(t)$ then the decomposition $x \to (x^1, x^2)$ is called an *active-passive decomposition*. The nice thing about the APD is that we can asymptotically reconstruct the *passive* component $x^2(t)$ from the *active* component $x^1(t)$. This, in turn, resembles what is called *a reduced order observer* in control theory.

Theorem 5 establishes sufficient conditions under which two identical systems synchronize. From a practical point of view this result is not always satisfactory. Indeed, to find $\bar{\gamma}$ the model of the system and perhaps the initial conditions must be known. Thus, it is interesting to find an adaptation algorithm which tunes γ until synchronization occurs. Such an algorithm can be easily found. The following result is valid:

Theorem 6 (Gain adaptation). *Assume that the hypothesis of Theorem 5 holds and the synchronization gain is updated by*

$$
\begin{aligned}
\gamma(t) \quad &= \quad \gamma_0 + \lambda_1 (y_1(t) - y_2(t))^\top (y_1(t) - y_2(t)) \\
&+ \quad \lambda_2 \int_0^t (y_1(s) - y_2(s))^\top (y_1(s) - y_2(s)) ds,
\end{aligned}
$$

where $\lambda_1 \geq 0, \lambda_2 > 0$ are some numbers.

Then, all solutions of the whole system are bounded and the goal of synchronization is achieved for arbitrary initial conditions $z_i(0), y_i(0), \gamma_0$.

The design methods of synchronizing systems discussed in this section appear to be also effective in case of multiple interconnected systems. In order to proceed in that direction, let us first give a definition of *diffusively coupled systems*.

Definition 8 *Given the smooth systems*

$$
\begin{cases} \dot{x}_j = f(x_j) + B u_j \\ y_j = C x_j \end{cases} ,
\tag{73}
$$

where $j = 1, \ldots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the j-th system, $u_j(t) \in \mathbb{R}^m$ is the input, $y_j(t) \in \mathbb{R}^m$ is the output of the j-th system, $f(0) = 0$, and B, C are constant matrices of appropriate dimension. We say that systems (73) are diffusively coupled if the matrix CB is similar to a positive definite matrix and systems (73) are interconnected by the following feedback

$$
u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \ldots - \gamma_{jk}(y_j - y_k)
\tag{74}
$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ are constants such that $\sum_{j \neq i}^k \gamma_{ji} > 0$ for all $i = 1, \ldots, k$.

The coefficients γ_{ji} in (74) can be unified in one symmetric $k \times k$ matrix Γ, which is defined in the following way:

$$
\Gamma = \begin{pmatrix}
\sum_{i=2}^k \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\
-\gamma_{21} & \sum_{i=1, i \neq 2}^k \gamma_{2i} & \cdots & -\gamma_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki}
\end{pmatrix},
\tag{75}
$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ and all row sums are zero. The matrix Γ is symmetric and, therefore, all its eigenvalues are real. It can be shown (see, e.g. [8]) that all eigenvalues of Γ are nonnegative, that is, the matrix Γ is positive semidefinite. Let the eigenvalues of the matrix Γ be ordered as: $0 = \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_k$. The matrix Γ completely defines the topology of linear interconnections in a network of diffusively coupled identical systems.

It is shown in [14] that, using the nonsingularity of CB, the systems (73) may, via a linear change of coordinates, be transformed into the form

$$\begin{cases} \dot{z}_j = q(z_j, y_j) \\ \dot{y}_j = a(z_j, y_j) + CBu_j \end{cases} \tag{76}$$

where $z_j(t) \in \mathbb{R}^{n-m}$, $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$, $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$.

The following theorem gives sufficient conditions for asymptotic synchronization of solutions (z_j, y_j), $j = 1, \ldots, k$, of the systems (76).

Theorem 7 *Consider k smooth diffusively coupled systems (73), (74), which because of the nonsingularity of CB are rewritten as (76), (74). Assume that*

A1. *The system*

$$\begin{cases} \dot{z} = q(z, y) \\ \dot{y} = a(z, y) + CBu \end{cases} \tag{77}$$

is strictly semipassive with respect to the input u and output y with a radially unbounded storage function $V : \mathbb{R}^n \to \mathbb{R}_+$.

A2. *There exist a C^2-smooth positive definite function $V_0 : \mathbb{R}^{n-m} \to \mathbb{R}_+$ and a positive number α such that the following inequality is satisfied*

$$(\nabla V_0(z_1 - z_2))^\top (q(z_1, y_1) - q(z_2, y_1)) \le -\alpha|z_1 - z_2|^2.$$

for all $z_1, z_2 \in \mathbb{R}^{n-m}, y_1 \in \mathbb{R}^m$.

Then, for all positive semidefinite matrices Γ as in (75) all solutions of the closed loop system (76), (74) are ultimately bounded and there exists a positive $\bar{\gamma}$ such that for all positive semidefinite matrices Γ with eigenvalues $0 = \gamma_1 \le \gamma_2 \le \ldots \le \gamma_k$ for which $\gamma_2 > \bar{\gamma}$ there exists a globally asymptotically stable compact subset of the diagonal set $\mathcal{A} = \{y_j \in \mathbb{R}^m, z_j \in \mathbb{R}^{(n-m)} : y_i = y_j, \ z_i = z_j, \ i,j = 1, \ldots, k\}.$

Let us explain the result of Theorem 7. It claims that under the conditions imposed the diagonal set $\mathcal{A} = \{x_j \in \mathbb{R}^n : x_1 = x_2 = \ldots = x_k\}$ contains a bounded closed invariant globally attractive set $\mathcal{A}_1 \subset \mathcal{A}$, that is the distance between any solution $x(t)$ and this set vanishes with time. Additionally, it claims that this set is Lyapunov stable: the maximum of the distance between $x(t)$ and $\mathcal{A}_1$ depends continuously on the initial distance between $x(0)$ and $\mathcal{A}$.

5   Types of synchronization

In Section 2, we considered definitions of synchronization with respect to some *given* functional. In the physical literature, however, systems are sometimes called synchronized if (asymptotically) *there exists some* functional relation between solutions of the systems. In this section we review some of these definitions of synchronization.

A. Generalized synchronization
We start this section with a motivating example. Consider again the Lorenz system (1) and its "copy" (3) driven by the output of the system (1). As it was shown in Section 1, these coupled systems exhibit asymptotic synchronization, i.e. $|X_1(t) - X_2(t)| \to 0$ as $t \to \infty$, where $X_1 = (x_1, y_1, z_1)^T$, $X_2 = (x_2, y_2, z_2)^T$ are the states of the systems. This is so-called *identical synchronization*. Now consider system (1) after a coordinate transformation $X_1 = \phi(Z_1)$, where $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism. Since we have not changed the systems, but only changed the coordinate representation of one of them, the difference between X_1 and X_2 still tends to zero as t tends to infinity. This in turn implies the following relation between the state of the system (3) and the state of the system (1) written in the new coordinates:

$$|\phi(Z_1(t)) - X_2(t)| \to 0 \quad \text{as} \quad t \to \infty. \tag{78}$$

Certainly this relation alone indicates that there is a kind of synchronization between the systems with the states Z_1 and X_2. But this is not an identical synchronization any more. In literature, such type of synchronization is referred to as *generalized synchronization*, the term coined in [18]. Several attempts were made to give a definition of the generalized synchronization, see [18] and [19]. In [19], the following definition is given:

Definition 9 *Generalized synchronization of the uni-directionally coupled systems*

$$\begin{aligned} \text{drive} \quad \dot{x} &= f(x), \quad x \in \mathbb{R}^n \\ \text{response} \quad \dot{y} &= g(y,x), \quad y \in \mathbb{R}^m \end{aligned} \tag{79}$$

occurs for the attractor $A_x \subset \mathbb{R}^n$ of the drive system if an attracting synchronization set

$$M = \{(x,y) \in A_x \times \mathbb{R}^m : y = H(x)\}$$

exists that is given by some function $H : A_x \to \mathbb{R}^m$ and that possesses an open basin $B \supset M$ such that

$$\lim_{t\to\infty} |y(t) - H(x(t))| = 0 \quad \forall (x(0), y(0)) \in B. \tag{80}$$

This definition is taken "as is" from [19]. Definitely, in order for the expression (80) to make sense, one has to demand H to be defined not only on the attractor A_x, but in some open neighborhood of A_x.

In [20], an attempt was made to give necessary and sufficient conditions for the generalized synchronization to occur. The central idea is that for the drive-response setup (79), if the response is stable (for each $x(t)$ the solution $y(t)$ is asymptotically unique), generalized synchronization in (79) occurs. As it is argued in [21] this is in general not true, although the stable drive-response scenario is interesting and deserves more study.

B. Invariant manifolds and synchronization

Another approach to deal with generalized synchronization is via the notion of invariant manifolds, as in [22]. The definition of synchronization given in [22] is as follows.

Definition 10 *Consider two coupled dynamical systems*

$$\dot{x} = f(x,y) \quad \dot{y} = g(y,x), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \tag{81}$$

where $f, g : \mathbb{R}^{2n} \to \mathbb{R}^n$ are smooth functions. The systems x and y synchronize if there exists a smooth, compact manifold M being the graph $y = \phi(x)$ for some diffeomorphism $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, with boundary, such that M is invariant under the flow, inflowing and locally attractive. M will be referred as the synchronization manifold.

Let us explain the definition. Let M be the manifold from the definition. Since M is invariant under the flow generated by (81), then all solutions of (81) starting in M will stay in M at least for some time. An "inflowing" manifold is a compact manifold with boundary, such that on the boundary the vector field is directed into the interior of the manifold. Thus the solutions starting on M will not leave M. The manifold M is locally attractive, thus any solution of (81) such that $(\phi(x(0)) - y(0))$ is small enough (or to say $(x(0), y(0))$ is close enough to M) will tend to M. Since M is the graph of a mapping ϕ, this means that for such solutions $|y(t) - \phi(x(t))| \to 0$ as $t \to \infty$.

In order for such synchronization to be of physical interest, it should persist under small perturbations of the two subsystems. The dynamics on the synchronization manifold can be quite complicated. In order to ensure that synchronization persists under perturbation of the system, it is required that the rate at which trajectories are attracted toward the manifold is greater than the rates of contraction or expansion within the manifold. If the rate of attraction is k times greater than the expansion or contraction rates within the manifold, the synchronization manifold is called *normally k-hyperbolic*. For a precise formulation of normal k-hyperbolicity see [23]. The crucial fact is that normally k-hyperbolic invariant manifolds persist as smooth manifolds under small perturbations of the underlying dynamical system. This leads to the following definition.

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Definition 11 *The synchronization of x and y is called stable if the synchronization manifold M is normally k-hyperbolic for some k ≥ 1.*

This definition suggests a mathematical framework within which the problem of structurally stable (generalized) synchronization can be discussed and analyzed.

C. Partial synchronization
Sometimes, it can be noticed that in a network of coupled systems one or more clusters of systems exist, within which systems (identically) synchronize with each other, but do not synchronize with the systems outside of the cluster. Or, in the case of two coupled systems sometimes one can notice that some of the state variables do synchronize with each other (the difference between them tends to zero), while the other do not. In these two examples the so-called *partial synchronization* occurs (see, e.g., [31]).

For uncoupled systems with inputs one can define a control problem to introduce by means of control such coupling between the systems that for the coupled systems partial synchronization occurs. Such *controlled partial synchronization problem* is very close to what is known in control theory as *the output regulation problem* (see [26]).

D. Phase synchronization
Initially, the term synchronization corresponded only to systems with periodic trajectories, and it meant what is now called *phase synchronization*. We will give an idea of phase synchronization [29], [30]. Readers interested in rigorous definitions of this phenomenon are referred to [29].

For a system with periodic trajectories, the dynamics of a point on a periodic trajectory can be described by

$$\frac{d\phi}{dt} = \omega_0, \tag{82}$$

where $\phi(t)$ is the phase, $\omega_0 = 2\pi/T_0$ is the frequency and T_0 is the period of the periodic solution. Consider two systems of the form (82) coupled by a sinusoidal term:

$$\dot{\phi}_1 = \omega_1 + K\sin(\phi_2 - \phi_1) \tag{83}$$

$$\dot{\phi}_2 = \omega_2 + K\sin(\phi_1 - \phi_2), \quad \omega_1 \neq \omega_2, \quad K > 0 \tag{84}$$

It can be noticed that for large values of the parameter K the difference between the phases tends to some constant value C: $(\phi_2(t) - \phi_1(t)) \to C$, as $t \to \infty$. Thus, despite the frequencies ω_1 and ω_2 of the uncoupled systems are different, the coupled systems after a transient generate phases with the same frequency $\bar{\omega} = \omega_1 + K\sin C = \omega_2 - K\sin C$. This phenomenon is called *phase locking* or *phase synchronization*. In our case, it can be described by the relation $(\dot{\phi}_2(t) - \dot{\phi}_1(t)) \to 0$, as $t \to \infty$. In the case of other couplings between (83) and (84) one can sometimes observe a more general type of phase synchronization, which can be described by the relation $(n\dot{\phi}_2(t) - m\dot{\phi}_1(t)) \to 0$, as $t \to \infty$, where n, m are integers. In this case, the frequencies of the oscillations of the coupled systems are related through the integer numbers n and m.

Phase synchronization is important in electric power production. In this process, AC generators are the oscillators coupled by an electric network and affected by power consumers. The problem is to control the generators and/or the network in such way that the alternating currents produced by different generators have the same (or nearly the same) frequency and phase.

After dealing with the phase synchronization for periodic oscillators one can ask a question whether something similar can occur for systems with more complex dynamics, for example chaotic ones. Simulations of some chaotic systems driven by external harmonic excitation suggest that the answer is 'yes'. At the same time, the analysis of such problem encounters a lot of difficulties. The very first of them is how to define phase and frequency for complex dynamics. Here, we will not go into details and we refer the interested reader to [29].

6 Applications to communication

Synchronization and controlled synchronization of complex/chaotic systems is a topic that has become popular in particular because of its possible use in secure communication, see [2], [24], [6]. The basic idea is to transmit an information signal with a broadband chaotic carrier signal and to use synchronization to recover the information at the receiver. Further, in this section, we review different implementations of this general concept (following [19]) and pay more attention to the so-called parameter modulation method (following [24]).

One of the ways of using synchronization in secure communication has already been briefly explained in Section 1. It is called *chaotic masking*. The information is added to a chaotic carrier and the synchronization of the response in the receiver is used to recover the message. Another method is called *chaos shift keying*. Within this method binary information signals are encoded by switching between different drive systems. At the receiver, the message can be recovered by monitoring the synchronization of the corresponding response systems of the receiver. In the third method, *parameter modulation*, the information signal is used to modulate a parameter of the drive system and the receiver reproduces the messages using identification algorithms. The variations of the information signal have to be slow compared to the dynamics of the drive system.

Let us consider the last method in greater detail. Consider a transmitter (drive) system Σ_T of the form

$$
\begin{aligned}
\dot{x} &= f(x, \lambda), \quad x \in \mathbb{R}^n & (85) \\
y &= h(x), \quad y \in \mathbb{R}, & (86)
\end{aligned}
$$

where λ is a time-varying message satisfying $\lambda_{min} \leq \lambda \leq \lambda_{max}$ ($\forall t$) and $y \in \mathbb{R}$ is the transmitted signal (i.e., the coded message). It is assumed that the system Σ_T is chaotic (or at least sufficiently complex) for all constant λ satisfying $\lambda_{min} \leq \lambda \leq \lambda_{max}$. The task is now to build a receiver system Σ_R that reconstructs the message $\lambda(t)$ from the coded message $y(t)$.

If the system Σ_T is in the form

$$
\dot{x} = Ax + \varphi_0(y) + B\lambda\varphi(y), \quad y = Cx, \tag{87}
$$

one can use the result of Theorem 2 to design a system reconstructing the slowly varying parameter (information signal) λ from the transmitted signal y. The idea lying behind that algorithm is that synchronization between the transmitter and the designed receiver occurs only if the value of $\hat{\lambda}$, the estimate of the unknown λ, is equal to λ. Then, this estimate is tuned in such a way that the synchronization error is minimized. Under certain conditions, this strategy leads to successful identification of λ.

A slightly different idea is used in the following setup for communication using chaotic signals which was proposed in [25]. The transmitter is a three-dimensional system Σ_T of the form

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, x_3) + g(x_1, x_2, x_3)\lambda \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) & (88) \\
y &= x_1
\end{aligned}
$$

where λ is a message that is mainly slowly time varying (i.e., λ is slowly time varying for most of the time, but may exhibit occasional jumps) and satisfies $\lambda_{min} \leq \lambda \leq \lambda_{max}$ ($\forall t$). Furthermore, $y \in \mathbb{R}$ is the transmitted signal. Moreover, a second system is considered that has the form

$$
\begin{aligned}
\dot{\hat{x}}_2 &= f_2(y, \hat{x}_2, \hat{x}_3) & (89) \\
\dot{\hat{x}}_3 &= f_3(y, \hat{x}_2, \hat{x}_3).
\end{aligned}
$$

It is assumed that system (89), driven by the output of (88) y, asymptotically synchronizes with the (x_2, x_3) subsystem in (88) in the sense that for (88), together with the system (89), it holds for all initial conditions that

$$
\lim_{t \to +\infty} (x_i(t) - \hat{x}_i(t)) = 0, \quad (i = 2, 3). \tag{90}
$$

22

We now show that the problem of estimating λ may be viewed as a linear parameter identification problem. If one assumes that the systems (88) and (89) have synchronized, the dynamics of y in (88) is given by

$$\dot{y} = u_1(t) + \lambda u_2(t) \tag{91}$$

where

$$
\begin{aligned}
u_1(t) &:= f_1(y(t), \hat{x}_2(t), \hat{x}_3(t)) \\
u_2(t) &:= g(y(t), \hat{x}_2(t), \hat{x}_3(t)). \tag{92}
\end{aligned}
$$

We then see that (91) may be interpreted as a linear time-invariant system with output y and inputs u_1 and u_2. Our task is now to obtain a mechanism that estimates λ for the linear system (91), based on the measurements y, u_1, u_2. This problem may be interpreted as a linear parameter identification problem and it may be solved using standard identification techniques (see for example [17]). In [24], the following solution to the problem is given:

$$
\begin{aligned}
\dot{w}_0 &= = -kw_0 + y \\
\dot{w}_1 &= -kw_1 + u_1 \\
\dot{w}_2 &= -kw_2 + u_2 \\
\hat{y} &= kw_0 + w_1 + \hat{\lambda}w_2 \\
\dot{\hat{\lambda}} &= -\nu \frac{\text{sign}(w_2)}{1 + |w_2|}(\hat{y} - y), \quad \nu > 0.
\end{aligned}
\tag{93}
$$

System (93) generates estimates $\hat{\lambda}$ of the parameter λ. If the function $|w_2(t)|/(1 + |w_2(t)|)$ is persistently exciting, then $(\hat{\lambda}(t) - \lambda(t)) \to 0$ as $t \to \infty$. The condition of persistent excitation is not restrictive in case $y(t)$, $u_1(t)$ and $u_2(t)$ are generated by a chaotic system. It follows from the fact that chaotic systems produce signals with a broad continuous power spectrum, that indeed the PE condition may be expected to be satisfied.

The example presented above can be generalized to partially linearizable transmitters. A partially linear transmitter is a transmitter of the form

$$
\begin{aligned}
\dot{x}_1 &= A(\lambda)x_1 + B(\lambda)\chi(y, x_2) \\
\dot{x}_2 &= f_2(y, x_2) \\
y &= C(\lambda)x_1
\end{aligned}
\tag{94}
$$

where $x_1 \in \mathbb{R}^q$, $x_2 \in \mathbb{R}^{n-q}$, $\chi : \mathbb{R} \times \mathbb{R}^{n-q} \to \mathbb{R}^m$, $f_2 : \mathbb{R} \times \mathbb{R}^{n-q} \to \mathbb{R}^{n-q}$, and $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ are matrices of appropriate dimensions that linearly depend on λ. For system (94), the following is assumed:

(A) The x_2 subsystem synchronizes with a copy of itself, i.e., the dynamics

$$\dot{\hat{x}}_2 = f_2(y, \hat{x}_2) \tag{95}$$

satisfy

$$\lim_{t \to \infty} |\hat{x}_2(t) - x_2(t)| = 0$$

whatever the initial conditions of (94), (95) are.

(B) For any solution of (94) the signal $\chi(y(t), x_2(t))$ is persistently exciting.

If (A) and (B) are satisfied, a reconstruction mechanism for λ can be obtained by applying standard linear identification techniques to the system

$$
\begin{aligned}
\dot{z} &= A(\lambda)z + B(\lambda)u \\
y &= C(\lambda)z
\end{aligned}
\tag{96}
$$

where $u := \chi(y, \hat{x}_2)$.

23

7   Conclusions

Synchronization of dynamic systems attracts attention of researches from different fields, resulting in different approaches to investigation of this phenomenon. In this report we have tried to review these approaches and some results on synchronization. Taking into account the vast amount of literature on the topic, the report does not pretend to be complete. At the same time, we have tried to review the main ideas. We started with the general definitions of synchronization as well as approximate, asymptotic, self- and controlled synchronization. These definitions are based on a synchronization criterion given in the form of a functional with respect to which synchronization is considered. Different choices of the functional lead to different particular types of synchronization including identical, partial and phase synchronization. At the same time, if a functional relation between the states of the systems exists, sometimes it is not important of what particular kind it is. This results in the notion of generalized synchronization (GS). The field of GS is not properly elaborated yet and it deserves further study. In many situations design of synchronizing systems can be considered as a particular case of the observer design problem from control theory. Hence, we have reviewed some results regarding observer design. Observers for linear systems, systems with linearizable error dynamics, systems with uncertainties and systems which admit high-gain observer design were considered. In many situations, observers for discrete-time systems can be designed similar to the continuous-time case. Thus, we have also reviewed discrete observer design. A fruitful approach to studying synchronization of two or more coupled dynamic systems is based on passivity theory. The results on semiglobal, global and adaptive synchronization between two or more coupled systems were considered. The growing interest in synchronization (in particular in synchronization of chaotic systems) was probably caused by its potential application to secure communication. In the report, we have discussed possible methods of such applications of synchronization.

Most of the results considered in this report correspond to the so-called identical synchronization, i.e., asymptotic convergence to zero of the difference between the states of two (or more) dynamical systems. This problem (as well as partial and generalized synchronization) can be viewed as a particular case of the *output regulation problem* (see [26]) in which the control goal is to make some output depending on the states of two systems asymptotically tend to zero. The output regulation problem has been solved recently in the local setting and under the rather restrictive assumption of neutral stability of one of the systems (see [26] for details). This assumption is not satisfied in case of complex or chaotic dynamical systems. Thus, successful solutions of the controlled synchronization problem for chaotic systems can be viewed as examples of problems for which the assumption of neutral stability is not satisfied, but for which there still exists a solution. This observation, indicated in [27], sets further directions for research in the field of output regulation, with the application to the problem of controlled synchronization in mind.

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