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# Only few graphs have bounded treewidth

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**Abstract**

We look at the treewidth of random graphs. Let $\delta > 1$ and $\epsilon < (\delta - 1)/(\delta + 1)$. Then almost all graphs with $n$ vertices and with at least $\delta n$ edges have treewidth $\geq n^{\epsilon}$. We also show that almost all graphs with $n$ vertices and $\delta n$ edges have treewidth $> b_{\delta} n$, where $b_{\delta}$ is strictly positive if $\delta > 1.18$. We show that for every $0 < b < 1$, there exists a $\delta_b$, such that almost all graphs with at least $\delta_b n$ edges have treewidth $> bn$. Our methods, together with recent results on minor theory, also show the following. Let $\delta \geq 1.18$. There exists a positive constant $c_{\delta}$, such that a.e. graph $G_{n,m}$ with $m \geq \delta n$ has a clique minor $K_s$ with $s \geq \lfloor c_{\delta} n^{1/3} \rfloor$. This extends earlier results of Bollobás et al. [5]. Finally, we show the following. Let $\mathcal{G}$ be a minor closed class of graphs (e.g. the class of planar graphs). For all $\delta \geq 1.18$, if $m \geq \delta n$ then a.e. graph with $m$ edges is *not* in $\mathcal{G}$.

## 1 Introduction

In this paper we look at the treewidth of random graphs. To be more precise, for random graphs with $n$ vertices, and $m \geq \delta n$ edges, we obtain asymptotic lower bounds for the treewidth. If $\delta \geq 1.18$, then there exists a positive constant $b_{\delta}$ such that a.e. graph with $m \geq \delta n$ edges has treewidth $\geq b_{\delta} n$. We proof that our results also show the following. Let $\mathcal{G}$ be a class of graphs which is closed under taking of minors. Let $\delta \geq 1.18$. Then almost every graph with $n$ vertices and $m$ edges, where $m \geq \delta n$, is *not* an element of $\mathcal{G}$. Part of our results, concerning clique minors of graphs, extends earlier results of Bollobás, Catlin and Erdös [5] and is related to results of Kostochka [11] and Thomason [21].

In this section we like to mention some earlier related results. The most important method we use in this paper is similar to that used in [22], for the *bandwidth* of

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random graphs. In this paper de la Véga proves that almost all graphs on $n$ vertices with $cn$ edges have bandwidth $\geq b_c n$ where $b_c$ is strictly positive for $c > 1$. We show that if $\delta \geq 1.18$, almost all graphs with $n$ vertices and $\delta n$ edges have treewidth $\geq b_\delta n$, where $b_\delta$ is strictly positive.

Bollobás et al. [5] obtain the following result. Let $0 < p < 1$ be fixed. For a.e. graph $G_{n,p}$, the maximum value $s$ such that $G_{n,p}$ has a minor $K_s$ is $(1 + o(1))\frac{n}{\sqrt{\log_d n}}$, where $d = \frac{1}{q}$. Let $c(s) = \inf\{c \mid e(G) \geq c|G| \Rightarrow G > K_s\}$, with $e(G)$ the number of edges of a graph $G$, $|G|$ the number of vertices and $G > K_s$ denotes that $G$ has a clique with $s$ vertices as a minor. The result of [5] shows that $c(s) \geq 0.265s\sqrt{\log_2 s}$ for large values of $s$ (see [21]). Subsequently, Kostochka [11] showed that $s\sqrt{\log s}$ is the correct order for $c(s)$. Thomason [21] shows the best upper bound as far as we know; $c(s) \leq 2.68s\sqrt{\log_2 s}\,(1 + o(1))$, for large $s$. For the treewidth problem this bound is of no interest; although a graph with $K_s$ as a minor has treewidth at least $s - 1$, a graph with treewidth at most $s$, can have at most $ns - \frac{1}{2}s(s + 1)$ edges. We extend the result of [5] as follows. Let $\delta > 1.18$. Then there exists a positive constant $c_\delta$ such that a.e. graph $G_{n,m}$ with $m \geq \delta n$ has a minor $K_s$ with $s \geq \lfloor c_\delta^{2/3} n^{1/3} \rfloor$.

Finally we like to mention two more related results. Cohen et al. [7] give the exact asymptotic probability that a graph is an interval graph and that a graph is a circular arc graph (this paper also contains numerous applications of interval graphs). In the common random graph model, interval graphs play only a minor role (if $m/n^{5/6} \to \infty$, where $m$ is the number of edges and $n$ is the number of vertices of a random graph, then the probability that such a graph is an interval graph goes to zero, if $n$ tends to infinity). For this reason, some work has been done to find separate models for random interval graphs [20]. This paper of Scheinerman, also contains results on maximum degree, Hamiltonicity, chromatic number etc., for these random interval graphs. In this paper we only look at the common random graph model, i.e. all graphs with a certain number of vertices and a certain number of edges are equiprobable.

In [8], results are obtained concerning the size of a chordal *subgraph*, given a graph with $n$ vertices and $m$ edges. For example, every graph with $n$ vertices and $m = \frac{n^2}{4} + 1$ edges, has a chordal subgraph with $\frac{3n}{2} - 1$ edges.

We can summarize the results of this paper as follows:

1. In section 3 we restate a result of Bollobás: almost all graphs with $< \frac{1}{2}n$ edges have treewidth $\leq 2$.
2. In section 4 we show that for all $\delta > 1$ and for all $0 < \epsilon < (\delta - 1)/(\delta + 1)$, a.e. graph $G_{n,m}$ with $m \geq \delta n$ has treewidth $\geq n^\epsilon$.
3. In section 5, we show that:

(a) For all $0 < b < 1$ there exists a constant $\delta$ such that if $m \geq \delta n$, then a.e. graph $G_{n,m}$ has treewidth $\geq bn$.

(b) For all $\delta \geq 1.18$ there exists a positive number $b$, such that if $m \geq \delta n$, then a.e. graph with $m$ edges has treewidth $\geq bn$.

4. In section 6 we prove the following related results:

   (a) Let $\delta \geq 1.18$. There exists a positive constant $c$, such that a.e. graph $G_{n,m}$ with $m \geq \delta n$ has a clique minor $K_s$ with $s \geq \lfloor cn^{1/3} \rfloor$.

   (b) Let $\mathcal{G}$ be a minor closed class of graphs. For all $\delta \geq 1.18$, a.e. graph $G_{n,m}$ with $m \geq \delta n$ is *not* in $\mathcal{G}$.

We do not know whether these results are optimal. Indeed, the smallest constant $c$, such that almost every graph with $cn$ edges has treewidth $bn$, for some $b > 0$, remains an open problem. Also, we do not have any results on random graphs of which the number of edges is in the range $(\frac{1}{2}n, n)$.

# 2 Preliminaries

For random graphs in general, the reader is referred to [6]. Let $P(Q)$ be the probability that a random graph with $n$ vertices and $m$ edges has a certain property $Q$. In most cases $m$ is a function of $n$. We say that almost every (a.e.) graph has property $Q$ if $P(Q) \to 1$ as $n \to \infty$. Throughout this paper we use $N = \binom{n}{2}$, where $n$ is the number of vertices of a graph ($N$ is the number of edges in the complete graph $K_n$ with $n$ vertices). We use $G_{n,p} \in \mathcal{G}(n,p)$ as a notation for a random graph with edge probability $p$, and $G_{n,m} \in \mathcal{G}(n,m)$ for a random graph with $n$ vertices and $m$ edges. In [6] it is shown that if $m$ is close to $pN = p\binom{n}{2}$, the two models $\mathcal{G}(n,m)$ and $\mathcal{G}(n,p)$ are practically interchangeable. In fact, since treewidth $\geq k$ is a *monotone increasing* property, it follows that, if $pqN \to \infty$ and $x$ is some fixed constant, almost every graph in $\mathcal{G}(n,p)$ has treewidth $\geq k$, if and only if a.e. graph in $\mathcal{G}(n,m)$ has treewidth $\geq k$, where $m = \lfloor pN + x(pqN)^{\frac{1}{2}} \rfloor$, ([6], page 35).

A $k$-tree is a graph defined recursively as follows (see e.g. [19]). A complete graph with $k$ vertices is a $k$-tree. A $k$-tree $T_{n+1}$ with $n+1$ vertices ($n \geq k$) can be constructed from a $k$-tree $T_n$ with $n$ vertices as follows. Take a new vertex $x$ and make this adjacent to all vertices of a $k$-clique in $T_n$ and to no other vertex of $T_n$. A *partial* $k$-tree is a subgraph of a $k$-tree. The *treewidth* of a graph $G$ is the minimum value $k$ for which $G$ is a partial $k$-tree. For computer science, classes of graphs with bounded treewidth are of interest, since many NP-complete problems are solvable in polynomial, and mostly even linear time for these graphs [3, 12]. In this paper we look at the treewidth of random graphs.

If $G$ is a graph and $(u,v)$ is an edge of $G$, the graph obtained by *contracting* the edge $(u,v)$ is the graph obtained from $G[V \setminus \{u,v\}]$, by adding a new vertex $z$ and adding edges $(z, w)$ for all $w \in (Adj(u) \cup Adj(v)) \setminus \{u, v\}$. A graph $H$ obtained from $G$ by a series of edge deletions and contractions is called a *minor* of $G$. We state some of the most important results in graph minor theory.

1. Kuratowski showed that a graph is planar if and only if it does not contain $K_5$ or $K_{3,3}$ as a minor.

2. Hadwiger's conjecture states that if a graph does not have $K_s$ as a minor, then the chromatic number is less than $s$. Wagner showed that in the case of $s = 5$, this is implied by the 4-Colour-Conjecture [23].

3. Wagner's conjecture, stating that every minor closed class of graphs has a finite obstruction set, was proved by Robertson and Seymour [16].

4. If $X$ is a graph, and $\mathcal{G}[HX]$ the class of graphs with no minor isomorphic to $X$, then there is a constant $c$ such that all graphs of $\mathcal{G}[HX]$ have treewidth $\leq c$ if and only if $X$ is planar. This was proved by Robertson and Seymour [18].

5. For every graph $H$, there is an $O(n^3)$ algorithm to test if $H$ is a minor of a given graph $G$ with $n$ vertices. It follows that every minor closed class of graphs is recognizable in $O(n^3)$ time [16].

Notice that for each constant $k$, the class of partial $k$-trees is minor closed. For $k = 2, 3$ the obstruction sets have been determined [2].

# 3 Sparse graphs

As a first result on treewidth we restate a result of Bollobás, ([6], page 99). A connected unicyclic graph with $t$ vertices, is a connected graph with $t$ edges.

**Lemma 3.1** *Suppose* $p = \frac{c}{n}$, $0 < c < 1$. *Then a.e.* $G_{n,p}$ *is such that every connected component is a tree or a unicyclic graph.*

Notice that a unicyclic graph has treewidth at most 2.

**Corollary 3.1** *If* $m < \frac{1}{2}n$, *then a.e. graph* $G_{n,m}$ *has treewidth at most two.*

In the next section we show that almost all graphs with $\delta n$ edges, have treewidth $\geq n^\epsilon$, for all fixed $\epsilon < \frac{\delta - 1}{\delta + 1}$.

# 4 Subgraphs of $k$-trees

Recall that the number of $k$-trees is given by the following formula, which was shown in different manners in a number of papers [4, 9, 14, 15].

$$T_k(n) = \binom{n}{k} (1 + k(n-k))^{n-k-2}$$

**Lemma 4.1** *Let $0 < \epsilon < 1$, and let $k \leq n^\epsilon$. Then $T_k(n) = o\left(n^{(1+\epsilon)(n-2)}\right)$.*

*Proof.*

$$\begin{aligned} T_k(n) &= \binom{n}{k} (1 + k(n-k))^{n-k-2} \\ &\leq n^k (nk)^{n-k-2} \\ &\leq n^{n-2} n^{\epsilon(n-k-2)} \\ &= n^{(1+\epsilon)(n-2)} n^{-\epsilon k} = o\left(n^{(1+\epsilon)(n-2)}\right) \ (n \to \infty) \end{aligned}$$

This proves the lemma. □

**Lemma 4.2** *Let $k$ be some integer. The number of partial $k$-trees with $m$ edges is at most $T_k(n)\binom{nk}{m}$.*

*Proof.* The number of edges in a $k$-tree is $nk - \frac{1}{2}k(k+1)$ (see e.g. [4]). It follows that the number of partial $k$-trees with $m$ edges is at most

$$T_k(n) \binom{nk - \frac{1}{2}k(k+1)}{m} \leq T_k(n) \binom{nk}{m}$$

□

**Theorem 4.1**

$$\forall_{\delta>1} \forall_{0<\epsilon<(\delta-1)/(\delta+1)} \forall_{m \geq \delta n} [\text{ a.e. } G_{n,m} \text{ has treewidth } \geq n^\epsilon ]$$

*Proof.* Let $k \leq n^\epsilon$. The total number of graphs with $m$ edges is $\binom{N}{m}$, where $N = \binom{n}{2}$. Let $F$ be the fraction of all graphs with $m$ edges, that have treewidth $\leq k$. We show that $F \to 0$. If $m \geq nk$, then clearly $F = 0$. Assume henceforth that $m < nk$. We have, if $n$ is large enough:

$$F \leq T_k(n) \frac{\binom{nk}{m}}{\binom{N}{m}} \leq T_k(n) \left(\frac{nk}{N-m}\right)^m \leq T_k(n) \left(3\frac{n^{1+\epsilon}}{n^2}\right)^{\delta n}$$

since $m < nk \leq n^{1+\epsilon}$, $N - m \geq \frac{1}{3}n^2$ if $n$ is large enough. We also used $m \geq \delta n$, and $k \leq n^\epsilon$. It follows that for large enough $n$ (using lemma 4.1):

$$F \leq T_k(n) \left(3n^{\epsilon-1}\right)^{\delta n} \leq 3^{\delta n} n^{n(\epsilon(\delta+1)-(\delta-1))} \to 0$$

since $\epsilon(\delta+1) - (\delta-1) < 0$. □

In the next section we show that if $\delta$ is somewhat larger, we already have a lower bound for the treewidth which is linear in $n$.

# 5 The separator method

In this section we show the following. Let $\delta \geq 1.18$. There exists a *positive* number $b_\delta$, such that for $m \geq \delta n$, a.e. graph $G_{n,m}$ does not have a balanced separator of size $\leq b_\delta n$. It is well known that a partial $k$-tree has a balanced separator with at most $k+1$ vertices. We show that a random graph with at least $\delta n$ edges does not have such a separator for small $k$.

In this section a balanced separator is a set $C$ with $k+1$ vertices such that every connected component has at most $\frac{1}{2}(n-k)$ vertices. The following lemma shows there exist such separators in partial $k$-trees. For a slightly more general result see also [17].

**Lemma 5.1** *Let $G = (V, E)$ be a $k$-tree with at least $k+1$ vertices. There is a clique $C$ with $k+1$ vertices such that every connected component of $G[V - C]$ has at most $\frac{1}{2}(n-k)$ vertices.*

*Proof.* Consider the following algorithm. Start with any $k+1$-clique $S_0$. Assume There is a connected component $C$ in $G[V-S_0]$ which has more than $\frac{1}{2}(n-k)$ vertices. Notice that the other components together have less than $\frac{1}{2}(n-k)-1$ vertices. There exists a vertex $x$ in $C$ which has $k$ neighbors in $S_0$. Let $y \in S_0 \setminus Adj(x)$. Define $S_1 = \{x\} \cup (Adj(x) \cap S_0)$. Notice that $S_1$ also has $k+1$ neighbors. The algorithm continues with $S_1$.

We show that this algorithm terminates. In order to prove this we show that in each step of the algorithm the number of vertices in the largest component decreases. Notice that $G[V-S_1]$ has two types of components. One type consists only of vertices of $C \setminus \{x\}$. If the largest component of $G[V-S_1]$ is among these, the number of vertices has clearly decreased. The other type of components consists only of vertices of $\{y\} \cup V \setminus (C \cup S_0)$. By the remark above, the total number of vertices in this set is less than $\frac{1}{2}(n-k)$. Since the largest component of $G[V-S_0]$ has more than this number of vertices, this shows that the number of vertices in the largest component of $G[V-S_1]$ is at least one less than the number of vertices in the largest component of $G[V-S_0]$. This proves that the algorithm terminates, and this proves the lemma. □

**Corollary 5.1** *Let $G = (V, E)$ be a graph with at least $k+1$ vertices and with treewidth $\leq k$. Then there exists a set $S$ of $k+1$ vertices such that every component of $G[V-S]$ has at most $\frac{1}{2}(n-k)$ vertices.*

To ease the forthcoming computations somewhat, we partition the vertices in three sets.

**Definition 5.1** *Let $G = (V, E)$ be a graphwith $n$ vertices. A partition $(S, A, B)$ of the vertices is a* balanced $k$-partition *if the following three conditions are satisfied:*

1. $|S| = k+1$

2. $\frac{1}{3}(n-k-1) \leq |A|, |B| \leq \frac{2}{3}(n-k-1)$

3. *$S$ separates $A$ and $B$, i.e. there are no edges between vertices of $A$ and vertices of $B$.*

**Lemma 5.2** *Let $G = (V, E)$ be a partial $k$-tree with $n$ vertices such that $n \geq k+4$. Then $G$ has a balanced $k$-partition.*

*Proof.* Let $S$ be a balanced separator of which the existence is guaranteed by corollary 5.1. Let $C_1, \ldots, C_t$ be the connected components of $G[V-S]$. Hence, $|C_i| \leq \frac{1}{2}(n-k)$. We consider two cases:

**Case 1** There exists a component $C_i$ such that $|C_i| \geq \frac{1}{3}(n-k-1)$. In this case let $A = C_i$ and $B$ the union of the other components. Then clearly, since $n-k \geq 4$, and $|C_i| \leq \frac{1}{2}(n-k)$ it follows that $|A| \leq \frac{2}{3}(n-k-1)$.

**Case 2** All components have less than $\frac{1}{3}(n-k-1)$ vertices. In this case, choose $s$ such that:

$$|C_1| + \ldots + |C_s| \leq \frac{2}{3}(n-k-1) < |C_1| + \ldots + |C_{s+1}|$$

It follows that:

$$|C_{s+2}| + \ldots + |C_t| < \frac{1}{3}(n-k-1)$$

Let $A = C_{s+1} \cup \ldots \cup C_t$ and $B = C_1 \cup \ldots \cup C_s$. Then:

$$|A| < |C_{s+1}| + \frac{1}{3}(n-k-1) \leq \frac{2}{3}(n-k-1)$$

Clearly, also:

$$|A| = n-k-1-|B| \geq \frac{1}{3}(n-k-1)$$

□

We show that the fraction of all graphs that have a balanced $k$-partition is negligible if the number of edges is not too small.

**Lemma 5.3** *Let $L_k(n,m)$ be the number of graphs with $m$ edges, that have a balanced $k$-partition. Then the following upper bound holds:*

$$L_k(n,m) \leq \frac{1}{2} \sum_{\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1)} \binom{n}{k+1} \binom{n-k-1}{a} \binom{N - a(n-k-1-a)}{m}$$

*where $N = \binom{n}{2}$.*

*Proof.* First we choose a separator $S$ with $k+1$ vertices, and a set $A$ with $a$ vertices, where $\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1)$. Finally, we choose $m$ edges. Since no edges between $A$ and $B$ are allowed, these must be chosen from a set of $N-a(n-k-1-a)$ available edges. Since $A$ and $B$ are interchangeable, we divide by 2 to find an upper bound for the number of graphs which have a balanced $k$-partition. □

**Lemma 5.4**

$$L_k(n,m) \leq \binom{n}{k+1} \cdot 2^{n-k-2} \cdot \binom{N - \frac{2}{9}(n-k-1)^2}{m}$$

*Proof.* Notice that:

$$\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1) \Rightarrow N - a(n-k-1-a) \leq N - \frac{2}{9}(n-k-1)^2$$

And hence

$$\binom{N - a(n-k-1-a)}{m} \leq \binom{N - \frac{2}{9}(n-k-1)^2}{m}$$

Also notice that

$$\sum_{\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1)} \binom{n-k-1}{a} \leq 2^{n-k-1}$$

Using these upperbounds, the lemma follows from lemma 5.3. □

**Definition 5.2** *Let $F_k(n,m)$ be the fraction of all graphs with $n$ vertices and $m$ edges that have a balanced $k$-partition, i.e.*

$$F_k(n,m) = \frac{L_k(n,m)}{\binom{N}{m}}$$

*where $N = \binom{n}{2}$.*

**Lemma 5.5**

$$F_k(n,m) \leq 2^{n-k-2} \cdot \binom{n}{k+1} \cdot \left(1 - \frac{\frac{4}{9}(n-k-1)^2}{n^2}\right)^m$$

*Proof.* Let $t = \frac{2}{9}(n-k-1)^2$. Then

$$\begin{aligned}
\frac{\binom{N-t}{m}}{\binom{N}{m}} &= \frac{N-t}{N} \cdot \frac{N-t-1}{N-1} \cdot \ldots \cdot \frac{N-t-m+1}{N-m+1} \\
&= \left(1 - \frac{t}{N}\right)\left(1 - \frac{t}{N-1}\right) \ldots \left(1 - \frac{t}{N-m+1}\right) \\
&\leq \left(1 - \frac{t}{N}\right)^m \leq \left(1 - \frac{2t}{n^2}\right)^m
\end{aligned}$$

Using this the lemma follows from lemma 5.4. □

**Definition 5.3** *For $0 < b < 1$ and $\delta > 0$, define:*

$$\varphi(b,\delta) = 2^{1-b} \cdot \frac{\left(1 - \frac{4}{9}(1-b)^2\right)^{\delta}}{b^b(1-b)^{1-b}}$$

**Theorem 5.1** *Let $0 < b < 1$ and $\delta$ be fixed. Let $m \geq \delta n$ and let $k+1 = \lceil bn \rceil$. Then*

$$F_k(n,m) = o\left(\varphi(b,\delta)^n\right)$$

*Proof.* Since $k+1 \to \infty$ and $n-k-1 \to \infty$, we find with the aid of Stirling's formula:

$$\binom{n}{k+1} \sim \left(\frac{n}{k+1}\right)^{k+1} \cdot \left(\frac{n}{n-k-1}\right)^{n-k-1} \cdot \frac{1}{\sqrt{2\pi(k+1)(1-\frac{k+1}{n})}}$$

Since $bn \leq k+1 \leq bn+1$ it follows that $\left(\frac{n}{k+1}\right)^{k+1} \leq \left(\frac{1}{b}\right)^{bn+1}$. And also

$$\begin{aligned} \left(\frac{n}{n-k-1}\right)^{n-k-1} &\leq \left(\frac{n}{n-bn-1}\right)^{n-bn} \\ &= \left(\frac{1}{1-b}\right)^{(1-b)n} \cdot \frac{1}{\left(1-\frac{1}{n(1-b)}\right)^{n(1-b)}} \sim e \cdot \left(\frac{1}{1-b}\right)^{(1-b)n} \end{aligned}$$

We have, since $m \geq \delta n$

$$\begin{aligned} \left(1 - \frac{\frac{4}{9}(n-k-1)^2}{n^2}\right)^m &\leq \left(1 - \frac{\frac{4}{9}(n-bn-1)^2}{n^2}\right)^{\delta n} \\ &\leq \left(1 - \frac{4}{9}(1-b)^2 + \frac{8}{9n}(1-b)\right)^{\delta n} \\ &\leq \left(1 - \frac{4}{9}(1-b)^2\right)^{\delta n} \cdot \left(1 + \frac{8(1-b)}{9n(1-\frac{4}{9}(1-b)^2)}\right)^{\delta n} \\ &\sim \left(1 - \frac{4}{9}(1-b)^2\right)^{\delta n} \cdot \exp\left(\frac{8\delta(1-b)}{9-4(1-b)^2}\right) \end{aligned}$$

The result now follows from the fact that $\sqrt{2\pi(k+1)(1-\frac{k+1}{n})} \to \infty$ and lemma 5.5. □

**Theorem 5.2** *For all $0 < b < 1$ there exists a $\delta_b$ such that if $m \geq \delta_b n$ a.e. graph $G_{n,m}$ has treewidth $\geq bn$.*

*Proof.* Notice that

$$\lim_{\delta \to \infty} \varphi(b,\delta) = 0$$

The result follows from theorem 5.1. □

**Lemma 5.6** *For $\delta \geq 1.18$, there exists a* positive number $b_\delta$ *such that* $\varphi(b_\delta, \delta) < 1$.

*Proof.* Notice that

$$\lim_{b \downarrow 0} \left(\frac{1}{b}\right)^b \cdot \left(\frac{2}{1-b}\right)^{1-b} = 2$$

Hence $\varphi(0, \delta) = 2(\frac{5}{9})^\delta < 1$ if $\delta \geq 1.18$. The result now follows from the fact that $\varphi$ is a continuous function for $b$ in $(0, 1)$. □

**Theorem 5.3** *Let $\delta \geq 1.18$. There exists a positive contstant $b_\delta$ such that if $m \geq \delta n$ $F_k(n, m) \to 0$.*

*Proof.* This follows immediately from lemma 5.6 and theorem 5.1. □

**Corollary 5.2** *Let $\delta \geq 1.18$. Then a.e. graph $G_{n,m}$ with $m \geq \delta n$ has treewidth $\Theta(n)$.*

# 6 Related results

In this section, we show some results in graph minor theory. Apart from theorem 5.3, the following theorem is the main ingredient. In [1] Alon et al. proved the following theorem.

**Theorem 6.1** *Let $h \geq 1$ be an integer and let $G = (V, E)$ be a graph with $n$ vertices and no $K_h$ minor. There exists a subset $X \subseteq V$, with $|X| \leq h^{3/2}n^{1/2}$ such that every connected component of $G[V - X]$ has at most $\frac{1}{2}n$ vertices.*

Together with our results this proves the following theorem.

**Theorem 6.2** *Let $\delta \geq 1.18$. There exists a positive constant $c_\delta$, such that a.e. graph $G_{n,m}$ with $m \geq \delta n$ edges has a clique $K_h$ with $h \geq \lfloor c_\delta n^{1/3} \rfloor$ as a minor.*

*Proof.* By theorem 5.3, there is a positive number $b_\delta$ such that a.e. graph $G_{n,m}$ with $m \geq \delta n$, does not have a balanced $k$-partition for $k \leq b_\delta n$. Let $c_\delta = (\frac{1}{3}b_\delta)^{2/3}$ and let $h = \lfloor c_\delta n^{1/3} \rfloor$. By theorem 6.1, if a graph does not have $K_h$ as a minor, then it has a separator $X$ with $|X| \leq c_\delta^{3/2} n \leq \frac{1}{3}b_\delta n$, such that every component has at most $\frac{1}{2}n$ elements. By a similar argument as in lemma 5.2 there is a partition of the vertices $(X, A, B)$ such that $|A|, |B| \leq \frac{2}{3}n$. Assume this is not a balanced $k$-partition. Then without loss of generality we may assume that $|A| > \frac{2}{3}(n - |X|)$. Let $t = 3|A| - 2(n - |X|)$. Take $t$ vertices out of $A$ and move them into $X$. Call the new sets $X'$ and $A'$. Clearly, $X'$ separates $A'$ and $B$, and $|A'| = \frac{2}{3}(n - |X'|)$. Since $|A| \leq \frac{2}{3}n$:

$$|X'| = |X| + t = |X| + 3|A| - 2(n - |X|) \leq 3|X|$$

It follows that $(X', A', B)$ is a balanced $k$-partition with $k + 1 = |X'| \leq b_\delta n$. □

Let $\mathcal{G}$ be a minor closed class of graphs (e.g. the class of planar graphs). Recently, Robertson and Seymour [16] proved Wagner's conjecture; there is a *finite* set of forbidden minors. From this result the following lemma easily follows.

**Lemma 6.1** *Let $\mathcal{G}$ be a minor closed class of graphs. there exists an integer $h$ such that $K_h$ is not a minor of any graph $G \in \mathcal{G}$.*

*Proof.* Take a finite set $S$ of forbidden minors. Let $h$ be the minimal number of vertices of any element of $S$. □

**Theorem 6.3** *Let $\mathcal{G}$ be a minor closed class of graphs. For all $\delta \geq 1.18$, a.e. graph $G_{n,m}$ with $m \geq \delta n$ is* not *in $\mathcal{G}$.*

*Proof.* Take $h$ such that no graph in $\mathcal{G}$ has $K_h$ as a minor. From theorem 6.2, there exists a positive number $c_\delta$, such that a.e. graph $G_{n,m}$ with $m \geq \delta n$ has a $K_t$ minor with $t \geq \lfloor c_\delta n^{1/3} \rfloor$. It immediately follows that a.e. graph $G_{n,m}$ is not in $\mathcal{G}$. □

# 7 Conclusions

In this paper we discussed the treewidth of random graphs. We showed that if the number of edges is more than $n$, a random graph with $n$ vertices does not have a small treewidth. The methods we used together with results in minor theory [1] also showed that random graphs with $n$ vertices and at least $1.18n$ edges, do have a large clique, with size $\Omega(n^{1/3})$, as a minor. Also any minor closed class of graphs contains almost no random graph with at least $1.18n$ edges.

We do not know whether the results presented in this paper are optimal. Indeed, finding the smallest constant $c$, such that almost every graph with $cn$ edges has treewidth at least $bn$, for some $b > 0$, remains an open problem. Furthermore, we do not have any results on the treewidth of graph of which the number of edges is in the range $(\frac{1}{2}n, n)$.

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