Parking a car in the smallest possible way

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PARKING A CAR IN THE SMALLEST

POSSIBLE WAY

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Contents

1 Introduction

Driving a car is a common skill nowadays. It is possible to master this art within a month. However there are a few manoeuvres which produce problems for inexperienced drivers. Parking a car in a narrow space is one of them. In the first place, drivers should know that the best way of doing this is to reverse. Even so, it is necessary to decide where to start and what trajectory to follow.
This model has been developed to answer those questions. Given the parameters of a car, the program will construct the optimal trajectory to permit the parked vehicle to occupy the smallest possible parking place.This work can also be helpful in planning parking places and serve too as a visual tool in driving schools.

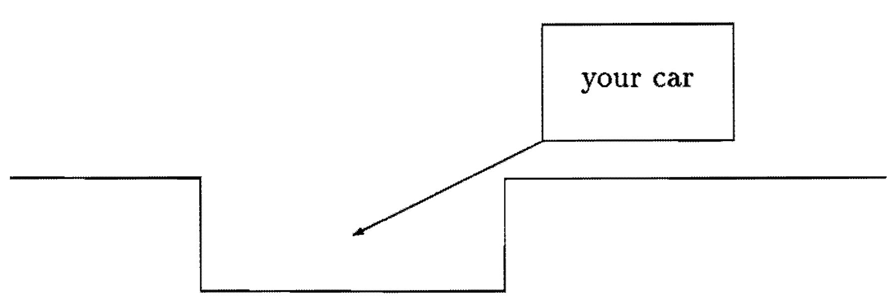

Figure 1: Ilustration of the problem.

2 Results

Given parameters of a car accompanied with ones of the parking place the program will construct a simple trajectory for you, following which you will find your car parked in the professional way. Below, there are some practical figures generated by the program.

models:	length	D_1	D_2	D_3
Volvo 440	4.310	1.523	0.093	0.306
Volvo 940	4.870	0.979	0.003	-0.380
Ford Escort	4.036	1.686	0.163	0.558
Ford Sierra	4.501	1.163	0.029	-0.046
Opel Astra	4.051	1.402	0.099	0.439
Opel Omega	4.740	1.465	0.043	0.045
VW Golf	4.020	1.661	0.149	0.045
VW Passat	4.569	1.296	0.031	0.003

Figure 2: Sample of results for modern cars.

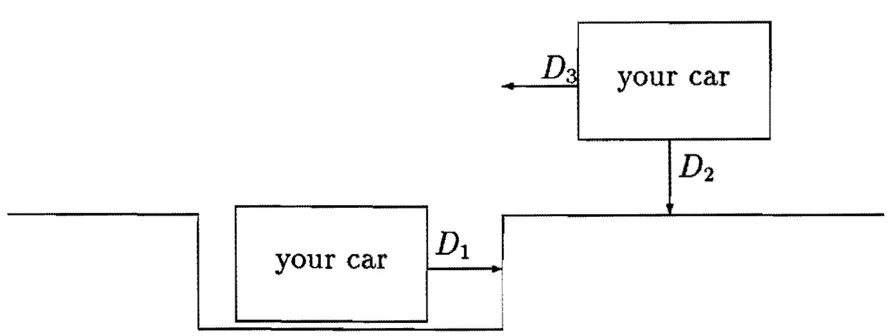

Figure 3: Crucial distances of the problem.

3 The model

3.1 Assumptions

- The position of the parked car is as close to the previously parked car and pavement as possible;
- The initial position of a car is parallel to pavement, with the wheels in the normal position;
- Velocities of rotating the wheels and motion of a car are constants (for simplicity of integration);

3.2 Parameters

R - the radius of the smallest circle the front of a car can make;

d - the width of a parking space (width of previous parked car);

c - the distance between the front of a car and its backwheels;

λ - angle velocity of turning wheels;

V - module of velocity of motion;

3.3 Variables

t - time;

f - complex coordinate vector of the right-front point of a car;

b - complex coordinate vector of the right-backwheel of a car;

γ - angle between $\mathbf{f} - \mathbf{b}$ and $\dot{\mathbf{f}}$;

3.4 Description of the model

We consider the opposite problem: suppose a car is parked in a parking-space, how can you get your car out? This problem is equivalent to the original one if the stating and ending conditions are martched. However, the latter problem has an advantage of deterministic initial conditions. When a car is standing in a small parking place the best way to leave it is to turn wheels maximum to the left [1],drive until the left-front point of a car passed the corner of the previous car and only then begin to think about any further manoeuvres.

So, the first part of the $park^{-1}$ trajectory is well defined :- it is an arc of a circle with radius **R**. For the rest we consider a profile for γ discribed by Figure 4.

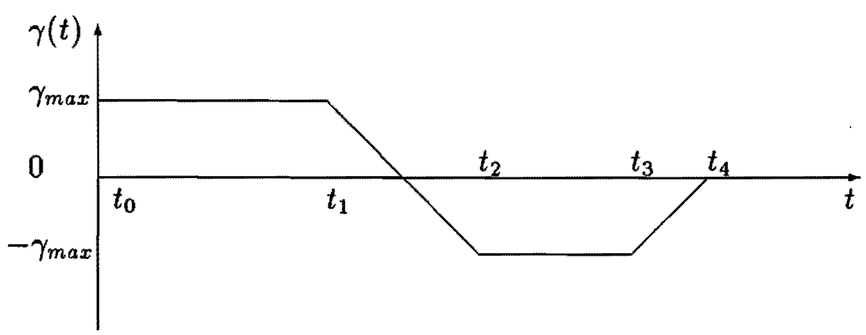

Figure 4: Profile for γ.

You can interprete this profile in terms of trajectory of front right wheel as follows:

$[t_0, t_1]$ - arc of circle with radius **R**.

$[t_1, t_2]$ - arc of clothoid with parameter $\mathbf{V}/\pi$ (see reference [1]).

$[t_2, t_3]$ - arc of circle with radius **R**.

[1]This description is valid for the right-side trafic. In order to apply it to the left-side trafic substitute "right" instead of "left" and vice versa.

$[t_3, t_4]$ - arc of clothoid with parameter $\mathbf{V}/\pi$.

At the end of the trajectory car stands on the road aside of previously parked car with its wheels in normal position. We looked for the trajectory to minimize D_3 keeping D_2 small positive (see figure 1), such that does not touch parked car anywhere.
We introduce rectangular complex coordinates in such a way : right-back wheel has a coordinates (0,0) and the correspondent front wheel (c,0) in the initial position.

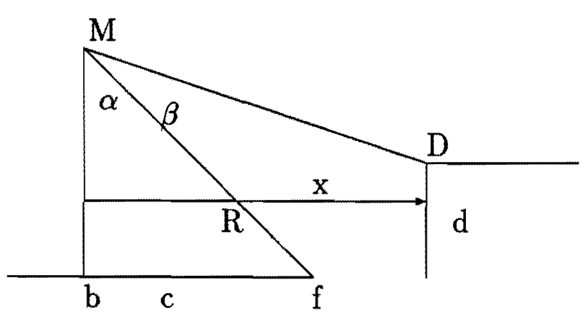

Figure 5: Initial position. **M** is a center of a turning circle; **D** is a left-back point of privious car ; α is $\angle bMf$; β is $\angle bMD$; **x** is a minimal distance to a privious car.

Consider a unit vector in the direction of a car : $\frac{(f-b)}{c}$, then increment of **b** is in the direction of this vector, meawhile **f** is shifting in the direction of front wheels with speed **v**. Hence, differential equations follow :

$$\dot{f} = e^{i\gamma(t)} \cdot \frac{(f-b)}{c} v, \qquad \dot{b} = K(t) \frac{(f-b)}{c} v, \tag{1}$$

where **K**(t) is a real valued function. We are intrested in the motion of the car as the whole, therefore we introduce new variable **z** in the manner:

$$z = f - b \qquad\qquad and \qquad\qquad \dot{z} = \dot{f} - \dot{b}. \tag{2}$$

Initial length of vector **z** is

$$z(t_0) = c. \tag{3}$$

Furthermore, the length should be preserved : $z \cdot \bar{z} = c^2$, for all **t**; and so

$$\dot{z} \cdot \bar{z} + z \cdot \dot{\bar{z}} = 0. \tag{4}$$

Using this fact, **K**(t) can be determined :

$$K(t) = \cos(t). \tag{5}$$

So, the equations of the problem are :

$$f(t) = \frac{v}{c} \int_{t_0}^{t} e^{i\gamma(x)} \cdot z(x)dx + f(t_0), \tag{6}$$

$$b(t) = \frac{v}{c} \int_{t_0}^{t} \cos(\gamma(x)) \cdot z(x)dx + b(t_0), \tag{7}$$

where **z** is given by :

$$z(t) = z(t_0)e^{\frac{v}{c} \cdot i \int_{t_0}^{t} \sin(\gamma(x))dx}. \tag{8}$$

Though, we can not deal with them analitically, because all of them contain the integral of the function :

$e^{\sqrt{1-x^2}}$,
which does not have a solution in the elementary functions according to [2], we can apply a numerical procedure to obtain solution of (7) and (8).

Now, we are able to predict position of a car after a period of time δt if we know function $\gamma(t)$ for the same period of time. As soon as all parameters of the problem are given, the trajectory of a car is uniquely defined by the values of t_0, t_1, t_2, t_3 and t_4 (see figure 4).
So, what can we say about t_2 and $t4$? Definitely, that

$$t_2 = t_1 + \frac{2 \cdot \gamma_{max}}{\lambda} \qquad and \qquad t_4 = t_3 + \frac{\gamma_{max}}{\lambda}. \tag{9}$$

In addition, a car has to end up in a horizontal position:

$$z(t_4) = z(t_0), \tag{10}$$

which means that :

$$\int_{t_0}^{t_4} \sin(\gamma(x))dx = 0, \tag{11}$$

and so

$$t_3 = \frac{(\cos(\gamma_{max}) - 1)}{\lambda \cdot \sin(\gamma_{max})} + 2 \cdot (t_1 + \frac{\gamma_{max}}{\lambda}). \tag{12}$$

Thus, there is still one degree of freedom left in t_1, so we are going to use it in optimization process. There are two resonable constraints on t_1. First, a car is not supposed to make a full circle between t_0 and t_1, or even half of it.

Second, $Im(f(t_1)) \geq$d, in order to avoid privious car. To obtain latter recall figure 5. Notice that if Im(M)≤d, then x = R, else :

$$(Im(M) - d)^2 + x^2 = R^2, \tag{13}$$

so

$$x^2 = c^2 + 2 \cdot Im(M) \cdot d - d^2. \tag{14}$$

Therefore :

$$t_{min} = t_0 + \frac{R(\beta - \alpha)}{v}, \tag{15}$$

where

$$\sin(\alpha) = \frac{c}{R} \qquad \textit{and} \qquad \sin(\beta) = \frac{x}{R}. \tag{16}$$

4 The computer program

The object of the computer program is to determine numerically the only remaining unknown t-value : t_1.

Two bounds for t_1 are given up front :

$$t_1 \geq arccos(\frac{\sqrt{R^2 - c^2}}{R} - \lambda)\frac{R}{V} = t_{1,min} \tag{17}$$

and

$$t_1 \leq arccos(\frac{\sqrt{R^2 - c^2} - d}{\sqrt{R^2 - c^2}})\frac{R}{V} = t_{1,max} \tag{18}$$

The first one means that we consider turning the front wheel after the front of the car has passed the corner of the previously parked car (point **D** on the figure 5) and the second one means that after the back wheel is at heigt d, no further investigations is needed, since the car then definitely passed point **D** and the back wheel position only increases in the imaginary direction.

The last observation is that the value D_3 increases when t_1 increases.

These three observation gives us the following:

Find the minimal value of t_1 such that the car does not enter the region occupied by the previous parked car.

This gives rise to the following function :

$$g(t_1) = \begin{cases} 0 & \text{if the car enters the place occupied by the already parked car} \\ 1 & \text{if not} \end{cases}$$

Whether the forbidden domain is violated by the linesegment between f and b can be checked by a simple analysis and we will not go into that. In fact, since back wheel is not the end of a car, one can extend the linesegment between f and b a bit futher than b.

More interesting is how to determine f and b. We used the classical Runge-Kutta method with stepsize h and constructed a stepsize control mechanism for it with stepsize h/2.

The procedure to find the t_1 value is the simple bisection method.

A typical trajectory for the car is depicted in Appendix A.

5 References

[1] D.S.Meek R.S.D.Thomas A guided clothoid spline, Computer Aided Geometric Design 8, North-Holland, 1991.
[2] I.S.Gradshteyn I.M.Ryshik Table of Integrals, Series and Products Academic press,1980.