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# ON THE RELEVANCY OF NORMAL MODES TO NONLINEAR MECHANICAL SYSTEMS IN WHICH DAMPING AND PERIODIC FORCING ARE INCLUDED

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It is shown that when studying the resonance behaviour of externally forced and slightly damped nonlinear mechanical systems with many degrees of freedom, knowledge of the associated undamped freely vibrating systems is relevant. This knowledge is characterized by the so-called normal modes [1] which are the generalization of the eigenvibrations of linear systems to nonlinear systems. Starting from the natural vibrations of the undamped and unforced system, the concept of so-called natural forcing is used to show the relevancy of the normal modes to the resonance behaviour of the forced and slightly damped system.

## DEFINITION OF THE SYSTEMS TO BE CONSIDERED

We consider mechanical systems with equations of motion given by

$$M(\underline{x})\,\ddot{\underline{x}} + \varepsilon D(\underline{x},\dot{\underline{x}})\,\dot{\underline{x}} + \underline{F}(\underline{x}) = \underline{f}(t) \tag{1}$$

with

$$\underline{F}(\underline{x}) = \partial V(\underline{x})/\partial \underline{x} \tag{2}$$

In the above equations (1) the n degrees of freedom $x_1, x_2, \ldots, x_n$ of the system are brought together in the n*1 column $\underline{x}$.
The external force referred to the generalized coordinate $x_i$ is denoted by $f_i(t)$ and is an explicit function of the time t. The external forces $f_1(t), f_2(t), \ldots, f_n(t)$ are brought together in the n*1 column $\underline{f}(t)$.
The n*n mass matrix $M(\underline{x})$ is real, symmetric and positive definite.

The function $V(\underline{x})$ is the potential energy of the system.
The system is assumed to be dissipative and consequently the damping matrix $D(\underline{x},\dot{\underline{x}})$ is real, symmetric and positive semi-definite.
The parameter $\varepsilon$ in (1) is small and is introduced to express that we deal with a slightly damped system.

## NATURAL FORCING

Let us first consider the undamped forced system.

$$M(\underline{x})\,\ddot{\underline{x}} + \underline{F}(\underline{x}) = \underline{f}(t) \qquad (3)$$

By definition, the function $\underline{f}(t)$ is said to be a strictly natural forcing function if:

- the solution to (3) is periodic;
- $\underline{f}(t)$ = constant * $\underline{F}(\underline{x}(t))$ ($\underline{f}(t)$ and $\underline{F}(\underline{x}(t))$ similar).

The concept of natural forcing is taken from Harvey [2].
Without proof we remark that the concept can be used to deal with harmonically forced linear systems, to show the existence of subharmonic vibration of Duffing's equation and to demonstrate the stabilizing effect of periodic forcing on the inverted pendulum or Duffing's equation with a softening spring.
Now let $\underline{\xi}(t)$ be some periodic solution (period $T_0$) of the unforced system ((3) with $\underline{f}(t) = \underline{0}$). Defining the strictly natural forcing function $\underline{f}_n(t)$ by

$$\underline{f}_n(t) := (1-\alpha^2)\ \underline{F}(\underline{\xi}(\alpha t)) \qquad (4)$$

it follows that the function

$$\underline{x}(t) = \underline{\varphi}(t) := \underline{\xi}(\alpha t) \qquad (5)$$

solves the system (3) if $\underline{f}(t) = \underline{f}_n(t)$. The function $\underline{x}(t)$ is periodic, with period $T = T_0/\alpha$. From (5) and (4) we conclude that if $\alpha \to 1$ (i.e. $T \to T_0$), then the amplitude of $\underline{f}_n$ becomes infinitesimally small, whereas the amplitude of $\underline{\varphi}$ does not change at all. Hence the forced system is at resonance.

## NONLINEAR MULTI-DEGREES-OF-FREEDOM SYSTEMS WITH DAMPING AND PERIODIC FORCING

Consider the system defined by (1). Again, let $\underline{\xi}(t)$ be any periodic solution of the undamped freely vibrating system. Entirely in the spirit of the method of natural forcing we now take as forcing function

$$\underline{f}(t) = \underline{f}_p = (1-\alpha^2)\underline{F}(\underline{\xi}(\alpha t)) + \varepsilon\alpha D(\underline{\xi}(\alpha t),\underline{\xi}'(\alpha t))\ \underline{\xi}'(\alpha t) \qquad (6)$$

in which $(') = d/d(\alpha t)$. Note that, for sufficiently small $\varepsilon$, $\underline{f}_p$ can be considered as a perturbed strictly natural forcing function. Again we see that

$$\underline{x}(t) = \underline{\varphi}(t) := \underline{\xi}(\alpha t) \qquad (7)$$

solves the system of equations (1) with forcing (6). Now, let $t^*$ be any instant of time for which $\overset{\circ}{\underline{\varphi}}$ vanishes ($\overset{\circ}{(\ )} = d/dt$), so that

$$\overset{\circ}{\underline{\varphi}}(t^*) = \underline{0} \qquad (8)$$

Furthermore, let

$$\underline{a} := \underline{\varphi}(t^*) \qquad (9)$$

At $t^*$ all components of $\underline{\varphi}$ reach extremal values. Hence $||\underline{\varphi}(t)||^2 := \underline{\varphi}^T\underline{\varphi}$ reaches a local maximum $(\underline{a}^T\underline{a})$. If no damping were present ($\varepsilon$=0), the same would hold for $||\underline{f}_p||^2 = \underline{f}_p{}^T\underline{f}_p$. However, the damping slightly perturbes not only the shape of the forcing function, but also causes a slight phase shift which is generally different for each component $f_i$ of $\underline{f}_p$. As a result, the components of $\underline{f}_p$ reach their respective extrema at different times. For this reason it makes sense to search for extremal values of

$$h(t;\varepsilon,\alpha) := \underline{f}_p{}^T\underline{f}_p \qquad (10)$$

A necessary condition which must then apply is

$$h_t(t;\varepsilon,\alpha) := \frac{\partial h}{\partial t} = 0 \qquad (11)$$

From (6) and (10) we have

$$h(t;\varepsilon,\alpha) = (1-\alpha^2)^2 \underline{F}^T\underline{F} + \varepsilon^2\alpha^2\underline{\xi}'^T D^T D\underline{\xi}' + 2\varepsilon\alpha(1-\alpha^2)\underline{F}^T D\underline{\xi}' \tag{12}$$

with $\underline{\xi} = \underline{\xi}(\alpha t)$, $\underline{F} = \underline{F}(\underline{\xi})$ and $D = D(\underline{\xi},\alpha\underline{\xi}')$.
Putting $\varepsilon = 0$ in (12), i.e. considering the undamped system, and differentating with respect to time yields

$$h_t(t;0,\alpha) = 2\alpha(1-\alpha^2)\underline{F}^T J\underline{\xi}' \tag{13}$$

in which the matrix $J = J(\underline{\xi})$ is the Jacobian matrix of $\underline{F}$, so that the elements of J are $J_{ij} = \partial F_i/\partial x_j$. Because $\underline{\xi}'(\alpha t^*) = \underline{0}$ we have $h_t(t^*;0,\alpha) = 0$, as should be expected from the foregoing.
Differentiating (13) with respect to t, and using the fact that $\underline{\xi}'(\alpha t^*) = \underline{0}$ and that $\underline{\xi}(t)$ satisfies the equations of motion for the undamped, freely vibrating system, yields (with $M = M(\underline{x})$)

$$h_{tt}(t^*;0,\alpha) = -2\alpha^2(1-\alpha^2)^2\underline{F}^T(\underline{a})J(\underline{a})M^{-1}\underline{F}(\underline{a}) \tag{14}$$

From (14) we conclude that $h_{tt}(t^*;\ 0,\alpha) \neq 0$ if $\alpha \neq 0, \alpha^2 \neq 1$, $\underline{F}^T(\underline{a})J(\underline{a})M^{-1}\underline{F}(\underline{a}) \neq 0$. Assuming that these conditions apply it then follows from the implicit function theorem that a function $\delta(\varepsilon)$ and an interval $(-\varepsilon_0,\varepsilon_0)$ exist, so that

$$h_t(t^* + \delta(\varepsilon);\ \varepsilon,\alpha) = 0 \quad \forall\ \varepsilon \in (-\varepsilon_0,\ \varepsilon_0) \tag{15}$$

It is found that

$$\delta(\varepsilon) = -\left\{\frac{C_1(\underline{a})}{(1-\alpha^2)C_2(\underline{a})}\right\}\varepsilon + B\varepsilon^2 + O(\varepsilon^3) \tag{16}$$

in which

$$C_1(\underline{a}) := \underline{F}^T(\underline{a})D(\underline{a},\underline{0})M^{-1}\underline{F}(\underline{a})\ ;\ C_2(\underline{a}) := \underline{F}^T(\underline{a})J(\underline{a})M^{-1}\underline{F}(\underline{a}) \tag{17}$$

B is a complicated expression. Its precise form is of no concern because

B contributes only to those terms of $h(t^*+\delta(\varepsilon);\varepsilon,\alpha)$ which are of third or higher order in $\varepsilon$. On setting $t=t^*+\delta(\varepsilon)$ in expression (12), expanding $h(t^*+\delta(\varepsilon);\varepsilon,\alpha)$ in a series with powers of $\varepsilon$, and retaining only those terms which are of second order in $\varepsilon$ we obtain

$$h(t^* + \delta(\varepsilon);\varepsilon,\alpha) = (1-\alpha^2)^2 \underline{F}^T(\underline{a})\underline{F}(\underline{a}) + \varepsilon^2\alpha^2 \frac{C_1^2(\underline{a})}{C_2(\underline{a})} + O(\varepsilon^3) \tag{18}$$

The function $h(t^* + \delta(\varepsilon);\varepsilon,\alpha)$ expresses how extremal values of $\underline{f}_p^T\underline{f}_p$ change in the neighbourhood of $t^*$ if the damping is varied. We now seek to determine a value of $\alpha$ in such a way that h is minimal (at least locally) if, for fixed values of $\varepsilon$ and $\underline{a}$, it is considered as a function of $\alpha$. This amounts to searching for a "frequency" $\omega$ (:= 1/T, T=period) of the forcing so that the "amplitude" $\sqrt{h}$ of $\underline{f}_p$ is minimal, because $\alpha$ is connected with $\omega$ by:

$$\alpha = \frac{T_0}{T} = \frac{\omega}{\omega_0} \tag{19}$$

$T_0$: period of the solution $\underline{F}(t)$ ($\underline{a}$: vector of amplitudes) of the freely vibrating undamped system.
Note that $(\omega_0, \underline{a}) \in$ backbone curve, which represents the amplitude-period relation of the undamped freely vibrating system.

T: period of the forcing of the damped system.

Differentiating (18) with respect to $\alpha$ gives that h assumes a local minimum if, up to the considered order

$$\alpha = \alpha_r = 1 - \frac{\varepsilon^2 \, C_1^2(\underline{a})}{4C_2(\underline{a})\underline{F}^T(\underline{a})\underline{F}(\underline{a})} \tag{20}$$

As the difference between $\omega_r = \alpha_r\,\omega_0$ and $\omega_0$ is of the order $\varepsilon^2$, the point $(\omega_r, \underline{a})$ lies in the vicinity of the backbone curve if the damping coefficient is sufficiently small. For this reason the backbone curve represents an approximation of the points $(\omega, \underline{a})$ for which the ratio $||\underline{a}||/\sqrt{h}$ is maximal. As a damped system is said to be at resonance, precisely if this ratio peaks (considered as function of the forcing frequency), we conclude that knowledge of the backbone curve is indeed relevant when studying the resonance behaviour of the forced (slightly) damped system.

Fig. 1 illustrates this conclusion for a classical example of a non-linear system with one degree of freedom, i.e. the Duffing system with linear viscous damping and harmonic forcing.

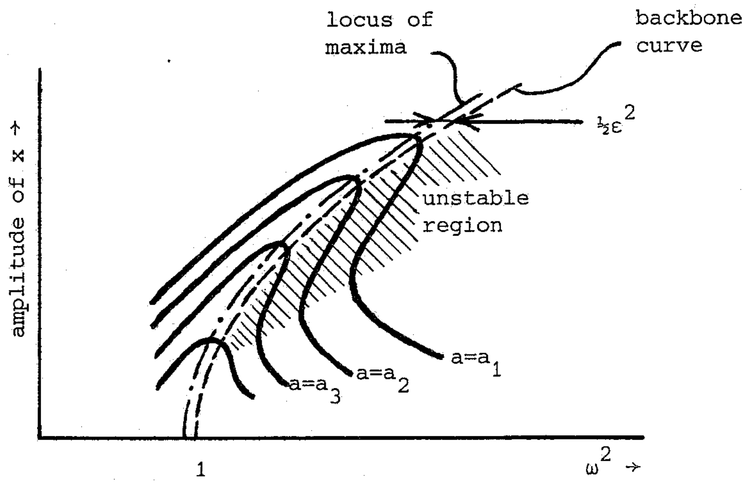


Fig. 1. Response curves and backbone curve for the damped Duffing system with harmonic forcing:

$$\ddot{x} + \varepsilon \dot{x} + x + \varepsilon\beta x^3 = \varepsilon a \cos \omega t; \; \beta>0, \; 0<\varepsilon \ll 1.$$

Different values of a, but ε and β fixed.

## REFERENCES


[1] Rosenberg, R.M., On nonlinear vibrations of systems with many degrees of freedom. Advances in applied mechanics 9 (1966), pp. 155-242.

[2] Harvey, T.J., Natural forcing functions in nonlinear systems. J. Appl. Mech. 25 (1958), pp. 352-356.