EINDHOVEN UNIVERSITY OF TECHNOLOGY

Mathematical models based on free boundary problems

Citation for published version (APA):
Eijndhoven, van, S. J. L. (1990). *Mathematical models based on free boundary problems*. (Opleiding wiskunde voor de industrie Eindhoven : student report; Vol. 9001). Eindhoven University of Technology.

Document status and date:
Published: 01/01/1990

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Download date: 17. Nov. 2023

ARC 9001
02
IWD

Technische
Universiteit
Eindhoven

Opleiding Wiskunde voor de Industrie Eindhoven

REPORT 90-01

MATHEMATICAL MODELS BASED ON FREE BOUNDARY PROBLEMS

S.J.L. van Eijndhoven

February 1990

ECMI

Den Dolech 2
Postbus 513
5600 MB Eindhoven

MATHEMATICAL MODELS
BASED ON FREE BOUNDARY PROBLEMS

S.J.L. van Eijndhoven

This report is based on a course given by Prof. A. Fasano at the Eindhoven University of Technology from December 14 until December 22, 1989.

Foreword

A free boundary problem for a partial (or ordinary) differential equation is characterized by the fact that the boundary of the domain in which the differential equation is to be solved is at least partly unknown. So a free boundary problem consists of determining both the solution and the unknown boundary. In case of a fixed boundary problem, when the boundary is completely described, the problem is well-posed given a set of boundary data. If part of the boundary is unknown these boundary data are insufficient for the well-posedness of the problem and additional conditions must be specified. Conditions on the free boundary are naturally called free boundary conditions.
According to the above definition the term free is synonymous to unknown. Sometimes, however, a distinction is made between free and moving boundary problems referring to those cases in which the unknown boundary stays at rest or moves. In the terminology used here, there is no such difference and a moving boundary is a free boundary only if its motion is not prescribed.

In this report five examples of free boundary problems are discussed: the obstacle problem, the dam problem, the Stefan problem, the oxygen diffusion consumption problem and the flow problem of a Bingham fluid between two fixed plates.
In five appendices some mathematical prerequisites are gathered.

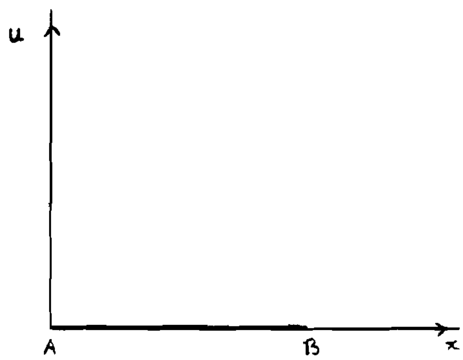
Contents

CHAPTER I

THE OBSTACLE PROBLEM

1. One-dimensional case

Take a rubber band and stretch it between two point A and B.



The equilibrium configuration will be the segment AB. The governing differential equation is given by

$$u''=0$$

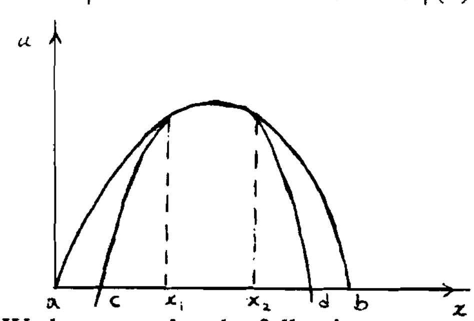
with boundary conditions

$$u(a)=u(b)=0.$$

Suppose we impose the constraint

$$u \geq \psi$$

where ψ is a C^1-function such that $\psi(a)=\psi(b)<0$ and $\psi>0$ on some interval (c,d).



The interval is divided into the set $\{u>\psi\}$ and in the *coincidence set* $\{u=\psi\}$. The boundary points x_1, x_2 of the coincidence set are unknown and they constitute the free boundary of the problem.

We have to solve the following

(I.1.1)

(i) $\quad u''=0 \;$ on $\; \{u>\psi\}$

(ii) $\quad u(0)=u(a)=0$

(iii) $\quad u=\psi$ over $\partial\{u=\psi\}$, i.e. $u(x_1)=\psi(x_1)$ and $u(x_2)=\psi(x_2)$.

The conditions (1.1.ii) and (1.1.iii) are not sufficient to determine u, x_1 and x_2. In addition we have to add

(iv) $\quad u'(x_1)=\psi'(x_1)$ and $u'(x_2)=\psi'(x_2)$.

Conditions (I.1.1.iii and iv) constitute the free boundary conditions.

Remark. Although the differential equation is linear and the conditions at $x=a$ and $x=b$ are linear, the problem itself is nonlinear. The nonlinearity is hidden in the free boundary conditions which involve the solution in some implicit way.

Now suppose for a moment that ψ is a C^2-function. Then we see that $\psi'' \leq 0$, and hence on (a,b) we have the following inequalities

$$(I.1.2) \qquad -u'' \geq 0 \ , \ u - \psi \geq 0 \ , \ u''(u-\psi) = 0 .$$

The problem is put in its *complementarity form*. In order to generalize this form to C^1-functions ψ we need a weak interpretation of (I.1.2) and seek for solutions u in a wider class. To this end we use the Sobolev space $W_0^{2,1}([a,b])$ consisting of all absolutely continuous functions w such that $\int_a^b |w'(x)|^2 dx < \infty$ and $w(a) = w(b) = 0$. Then by u'' we mean the second distributional derivative of u in $D'((a,b))$. For a distribution $F \in D'((a,b))$ we write $F \geq 0$ if $F(\phi) \geq 0$ for all positive $\phi \in D((a,b))$ (cf. Appendix C).
We remark that for $u \in W_0^{2,1}([a,b])$, $u'' \geq 0$, is equivalent with

$$(I.1.3) \qquad \int_a^b u'(x)\, \phi'(x)\, dx \geq 0 \ , \ \forall \phi \in D((a,b)) \, , \ \phi \geq 0 .$$

Further, since $D((a,b))$ is dense in $W_0^{2,1}([a,b])$, (*) is equivalent with

$$(I.1.4) \qquad \int_a^b u'(x)\, v'(x)\, dx \geq 0 \ , \ \forall v \in W_0^{2,1}([a,b]) \, , \ v \geq 0 .$$

In the new formulation the free boundary does not appear explicitly.
In its turn the complementarity problem (I.1.2) is equivalent to the following variational inequality. Define the convex and closed set $K := \{v \in W_0^{2,1}([a,b]) \mid v \geq \psi\}$ and look for $u \in K$ such that

$$(I.1.5) \qquad \int_a^b u'(x)\,(u'(x) - v'(x))\, dx \leq 0 \ , \ \forall_{v \in K} .$$

To show the equivalence we proceed as follows.

Suppose u satisfies (I.1.5). Let $\phi \in D((a,b))$ with $\phi \geq 0$. Then $u + \phi \in K$ and

$$\int_a^b u'(x)\,(u'(x) - (u'(x) + \phi'(x)))\, dx \leq 0$$

whence $u'' \geq 0$ in weak sense. Further, let C be a compact subset of $\{u > \psi\}$. Then, u and ψ being continuous, there exists $\delta > 0$ such that $u(x) - \psi(x) \geq \delta$ for all $x \in C$. So for all C^∞-functions ϕ with support in C there exists $t_\phi > 0$ such that

$$\forall_{t \in [-t_*, t_*]} \quad , \quad u + t\phi \in K$$

whence

$$\forall_{t \in [-t_*, t_*]} \; : \; t \int_a^b u'(x)\, \phi'(x)\, dx \geq 0 \, .$$

It follows that $\int_a^b u'(x)\, \phi'(x)\, dx = 0$ for all $\phi \in D((a,b))$ with support in $\{u > \psi\}$. Thus we conclude that $u'' = 0$ (weakly) on $\{u > \psi\}$.

Conversely, suppose u satisfies (I.1.2) in its weak interpretation. Then for all $v \in K$, $u - v \leq u - \psi = 0$, on $\{u = \psi\}$ and hence by (I.1.4)

$$\int_{\{u = \psi\}} u'(x)\, (u'(x) - v'(x))\, dx \leq 0 \, .$$

Moreover, on $\{u > \psi\}$ we have $u'' = 0$ weakly, i.e.

$$\forall_{\phi \in D((a,b))} \; : \; \int_{\{u > \psi\}} u'(x)\, \phi'(x)\, dx = 0$$

which yields, again because $D((a,b))$ is dense in $W_0^{2,1}([a,b])$,

$$\forall_{v \in K} \; : \; \int_{\{u > \psi\}} u'(x)\, (u'(x) - v'(x))\, dx = 0 \, .$$

Uniqueness of u can be established straightforwardly from (I.1.5). Indeed suppose u_1 and u_2 satisfy (I.1.5). Then

$$\int_a^b u_1'(x)\, (u_1'(x) - u_2'(x))\, dx \leq 0$$

and

$$\int_a^b u_2'(x)\, (u_2'(x) - u_1'(x))\, dx \leq 0$$

so that

$$\int_a^b (u_1'(x) - u_2'(x))^2\, dx \leq 0 \, .$$

Hence $u_1' = u_2'$. Since $u_j(a) = u_j(b) = 0$, $j = 1,2$, the result follows.

Finally, the variational problem (I.1.5) is equivalent with the problem of determining $u \in K$ for which the quadratic form

$$J(w) = \int_a^b |w'(x)|^2 \, dx \quad , \quad w \in K \tag{I.1.6}$$

is minimal. To see this, observe first that

$$J(\tilde{w}) - J(w) = \int_a^b \tilde{w}'(x) \, (\tilde{w}'(x) - w'(x)) \, dx \tag{*}$$

$$+ \int_a^b w'(x) \, (\tilde{w}'(x) - w'(x)) \, dx \, .$$

Let $u \in K$ satisfy (I.1.5).
Then for $\tilde{w}, w \in K$ with $\tilde{w} \neq u$

$$\int_a^b \tilde{w}'(x) \, (\tilde{w}'(x) - w'(x)) \, dx \geq 0 \, .$$

Hence both summands on the right hand side of (*) are positive if we take $\tilde{w} = v \in K$, $v \neq u$, and $w = u$.
It follows that $J(v) \geq J(u)$ for all $v \in K$.
Conversely, if u minimizes J over the convex set K, then

$$\frac{d}{dt} J(u + t(v-u)) \Big|_{t=0} \geq 0$$

from which (I.1.5) results.
Now $J(w)^{1/2}$ is the norm of w in $W_0^{2,1}([a,b])$ corresponding to the inner product

$$(w_1, w_2) = \int_a^b w_1'(x) \, w_2'(x) \, dx \, .$$

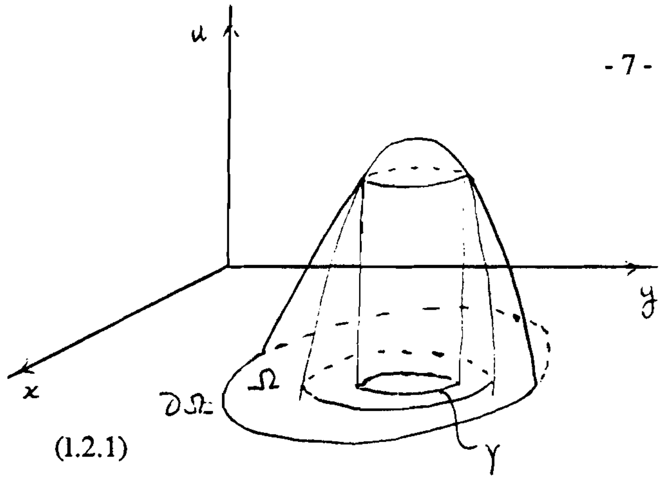
Since the set K is closed and convex in $W_0^{2,1}([a,b])$ a classical result from Hilbert space theory says that there exist a unique $u \in K$ such that

$$J(u) = \min \, \{J(w) \mid w \in K\} = \mathrm{dist}^2(0,K) \, .$$

Thus both existence and uniqueness of u is established.

2. Two-dimensional case

As we have seen the obstacle problem in one dimension has a simple solution. In two dimensions the problem becomes less trivial. In this case we consider a membrane stretched over a profile, e.g. the boundary of a given domain Ω at which $u = 0$, and with the same constraint that $u \geq \psi$ where now $\psi < 0$ on $\partial\Omega$. The classical formulation of this problem is rather complicated.



Look for a function u on $\Omega_0 = \Omega \setminus I$ such that

$$u \in C^2(\Omega_0) \cap C^1(\overline{\Omega}_0)$$

and such that u satisfies the following relations

(1.2.1)

(i) $\Delta u = 0$ in Ω_0

(ii) $u = 0$ on $\partial\Omega$

(iii) $u = \psi$ on γ

(iv) $\dfrac{\partial u}{\partial n} = \dfrac{\partial \psi}{\partial n}$ on γ.

Here I denotes the coincidence set $\{u = \psi\}$ and γ its C^1-boundary. Further, $\dfrac{\partial}{\partial n}$ denotes the normal derivative at γ, and the relation

$$\frac{\partial u}{\partial n} = \frac{\partial \psi}{\partial n}$$

expresses that the membrane is tangent to the obstacle. Under the assumption that $\psi \in C^2(\Omega)$ we can prove again that the problem (I.2.1) is equivalent to the complementarity problem valid in the whole domain Ω

(I.2.2) $\quad -\Delta u \geq 0 \;,\; u - \psi \geq 0 \;,\; -\Delta u(u - \psi) = 0 .$

In order to interpret problem (I.2.2) for a more general class of functions ψ we take $\psi \in W^{2,1}(\Omega)$ and seek for solutions $u \in W_0^{2,1}(\Omega)$, i.e. the closure of $D(\Omega)$ in $W^{2,1}(\Omega)$. Thus (I.2.2) is given the following weak interpretation.

(I.2.3) $\quad -\Delta u \geq 0$ means $\forall_{\phi \in D(\Omega), \phi \geq 0} : \int_\Omega (\nabla u \cdot \nabla \phi)\, dx \geq 0$

$\Delta u(u - \psi) = 0$ means $\int_\Omega \nabla u \cdot \nabla \phi = 0$ for $\phi \in D(\Omega)$ with support contained in $\{u > \psi\}$.

(Observe that we have applied Green's first identity.)

From the weakly interpreted complementarity form we arrive at the following variational inequality: Introduce the convex and closed set $K \subset W_0^{2,1}(\Omega)$ by

$$K = \{v \in W_0^{2,1}(\Omega) \mid v \geq \psi\} .$$

Then (I.2.2) is equivalent with the problem of searching $u \in K$ for which

(I.2.4) $$\int_\Omega \nabla u \cdot (\nabla u - \nabla v)\, dx \le 0 \quad , \quad \forall_{v \in K} \, .$$

It can be proved that the norm in $W_0^{2,1}(\Omega)$ is equivalent with

$$\|u\|^0_{2,1} = (\int_\Omega |\nabla u|^2 dx)^{1/2} \, .$$

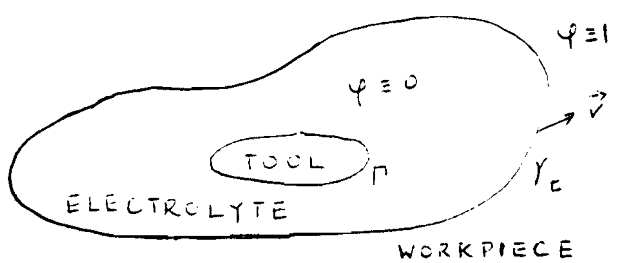
Now (I.2.4) is, in its turn, equivalent with searching $u \in K$ that minimizes the functional

$$J(w) = \int_\Omega |\nabla w|^2 dx = (\|w\|^0_{2,1})^2 \, , \quad w \in K \, .$$

Since K is closed and convex such a unique $u \in K$ exists.

3. A comparable problem

In comparison with the obstacle problem we present the problem of refining metal surfaces using electrolytic processes. The workpiece and the tool are respectively the anode and the cathode in an electrolytic circuit. The current flows in the electrolytic solution under the action of a potential difference between the electrodes and causes dissolution of the anode surface which allows for micrometric machining of the work-piece.



Clearly the free boundary in this problem is the anode surface. Keeping the potential difference between the electrodes constant we can use a non dimensional potential ϕ with value $\phi = 0$ on the cathode surface Γ and $\phi = 1$ on the *moving* anode surface γ_t. The function ϕ is harmonic in the region Ω_t occupied by the electrolyte. So we have the following formulation of the problem.
Given the cathode surface Γ and the initial configuration γ_0 of the anode surface, find the pair (γ_t, ϕ) such that

(I.3.1)

(i) $\Delta\phi = 0$ in Ω_t , $t > 0$

(ii) $\phi \big|_\Gamma = 0$, $t > 0$

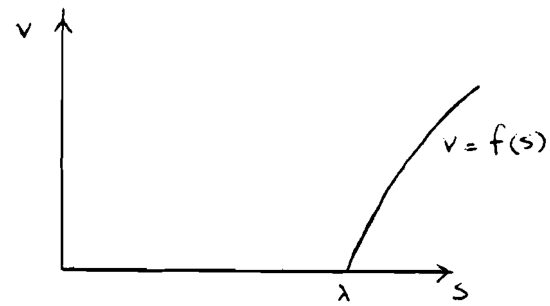
(iii) $\phi \big|_{\gamma_t} = 1$, $t > 0$

(iv) $\gamma_{t=0} = \gamma_0$.

Of course, we need an additional free boundary condition relating the local dissolution rate, i.e. the normal component v_n of the velocity of the anode surface, to the value of the electric field. We take

(v) $$v_n = f\left[\frac{\partial\phi}{\partial n}\right] \quad \text{on} \quad \gamma_t .$$

A realistic form of f exhibits the presence of a threshold current below which no or very little machining occurs.



The free boundary will be steady if

$$\frac{\partial\phi}{\partial n} \le \lambda .$$

Therefore the threshold current model has a limiting steady state (ϕ, γ) satisfying the equations

(I.3.2)

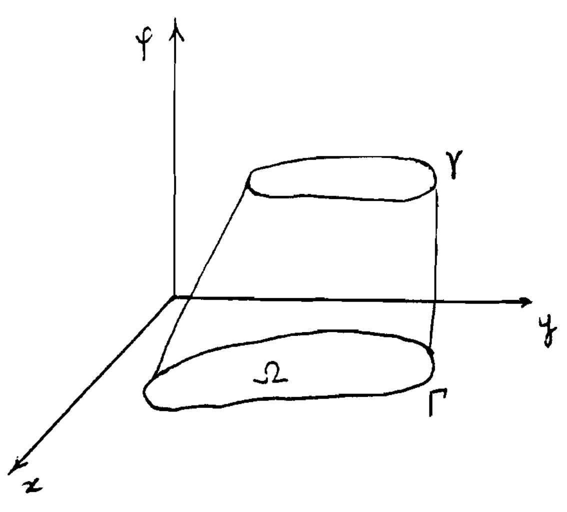
(i) $\Delta\phi = 0 \;\text{ in }\; \Omega$

(ii) $\phi \,\Big|_{\Gamma} = 0$

(iii) $\phi \,\Big|_{\gamma} = 1$

(iv) $\dfrac{\partial\phi}{\partial n} \,\Big|_{\gamma} = \lambda .$

Problem (I.3.2) can be interpreted as a membrane equilibrium problem: Given the profile Γ on $\phi = 0$ look for a profile γ on $\phi = 1$ such that the slope of the membrane complies with the condition (iv).

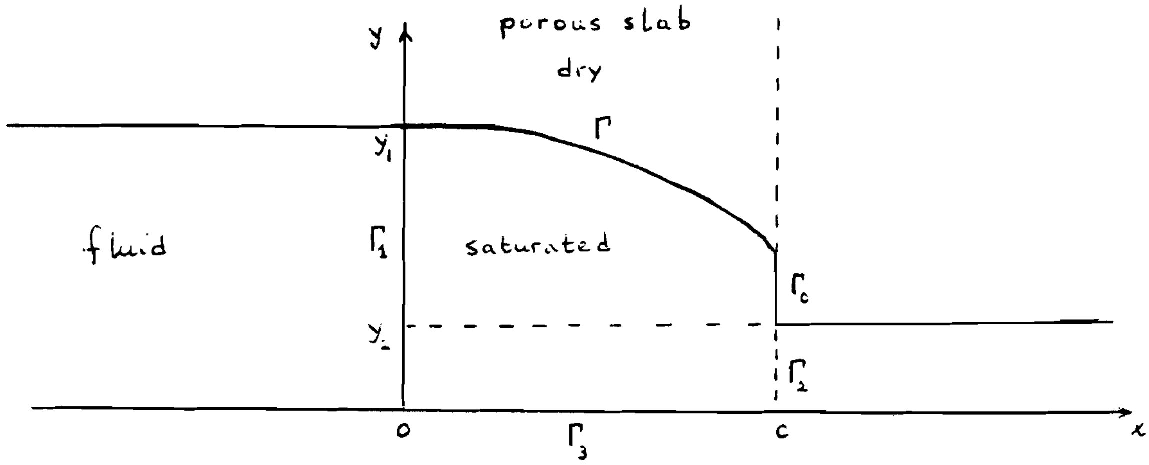


This problem is much harder to solve than the previous one since it cannot be made variational.

CHAPTER II

THE DAM PROBLEM

In this chapter we consider the steady filtration of a fluid through a porous dam. The problem is sketched in the following figure.



We assume complete saturation, there is no capillarity.

Recall that a fluid flow in a (saturated) porous medium is governed by Darcy's law (cf. Appendix B)

$$\vec{v} = -k\,\nabla(p + \rho\, g\, y)$$

where $\vec{v}$ is the volumetric velocity,
k the hydraulic conductivity,
ρ the fluid density,
g gravity, and
y upwardly directed vertical coordinate.

Incompressibility of the fluid implies that div $\vec{v} = 0$ and so that $\Delta p = 0$ on the region $\{p > 0\}$.

We have to find p in the saturated region and the free boundary Γ which we describe by $y = \phi(x)$, $0 \le x \le c$. First we derive the conditions on the fixed boundaries $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3. We normalize such that $k = \rho\, g = 1$.

$$\begin{aligned}
&\text{on } \Gamma_1 : \quad p(0,y) = y_1 - y \quad, \quad 0 \le y \le y_1\,,\\
&\text{on } \Gamma_2 : \quad p(c,y) = y_2 - y \quad, \quad 0 \le y \le y_2\,,\\
&\text{on } \Gamma_0 : \quad p(c,y) = 0 \qquad\quad, \quad y_2 \le y \le \phi(c)\,.
\end{aligned}$$

For the boundary condition on Γ_3 we apply Darcy's law

$$v_x = -\frac{\partial p}{\partial x} \quad , \quad v_y = -\left[\frac{\partial p}{\partial y} + 1\right] = 0$$

and so

$$\text{on } \Gamma_3 \ : \ \frac{\partial p}{\partial y}(x,0) = -1 \ , \quad 0 \le x \le c \ .$$

On the free boundary Γ we have

$$p(x, \phi(x)) = 0 \quad , \quad 0 \le x \le c$$

and the additional free boundary condition

$$\frac{\partial (p+y)}{\partial n} \Big|_{\Gamma} = 0 \, .$$

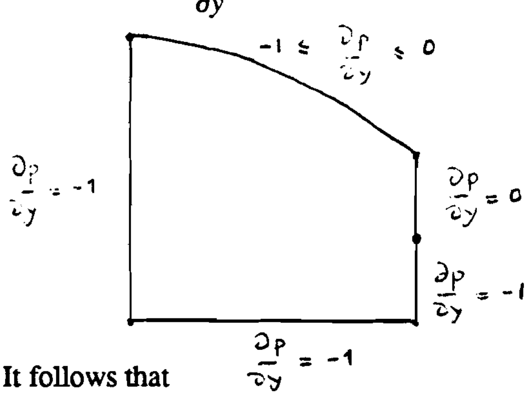
Explanation: At Γ the fluid flows tangent to the curve Γ. It follows that $\vec{v} \cdot \vec{n} = 0$ and hence by Darcy's law

$$\nabla (p+y) \cdot \vec{n} = 0 \quad \text{on} \quad \Gamma \, .$$

We have the following observations.

(II.1) Because of the maximum principle p cannot be negative in Ω since otherwise p would be strictly negative in a point of Γ_3 which yields a contradiction. In fact even the stronger assertion that $p > 0$ in Ω is valid.

(II.2) We have $\dfrac{\partial p}{\partial y} > -1$ in Ω.



Indeed, the curve Γ is a level line of p since

$$p(x, \phi(x)) = 0 \, , \quad 0 \le x \le c \, .$$

So $\nabla p \, \Big|_{\Gamma}$ is normal to Γ and hence

$$\vec{v} \cdot \nabla p = 0 \quad \text{on} \quad \Gamma \, .$$

It follows that

$$\nabla (p+y) \cdot \nabla p = 0 \quad \text{on} \quad \Gamma \, .$$

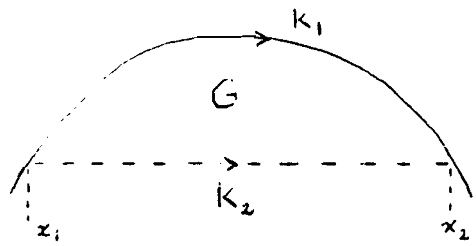
This means that

$$\left[\frac{\partial p}{\partial x}\right]^2 + \left[\frac{\partial p}{\partial y} + \frac{1}{2}\right]^2 = \frac{1}{4} \quad \text{on} \quad \Gamma \, .$$

Since $\Delta \left[\dfrac{\partial p}{\partial y}\right] = 0$ the result follows from the maximum principle.

(II.3) $\phi(x)$ is decreasing.

* Assume



Applying Gauss' divergence theorem to G yields

$$0 = \int_{\partial G} (\nabla p \cdot \vec{n})\, ds = \int_{K_1} \nabla p \cdot \vec{n}\, ds - \int_{K_2} \nabla p \cdot \vec{n}\, ds\ .$$

Now

$$\nabla p \cdot \vec{n} = -\,|\nabla p| \quad \text{on} \quad K_1 \ : \ (t, \phi(t)),\ x_1 \le t \le x_2$$

and

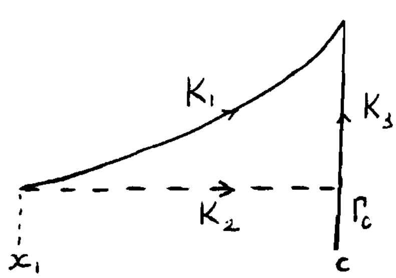
$$\nabla p \cdot \vec{n} = -\frac{\partial p}{\partial y} \quad \text{on} \quad K_2 \ : \ (t, \phi(x_1)),\ x_1 \le t \le x_2\ .$$

It follows that

$$\int_{K_1} |\nabla p|\, ds = \int_{K_2} \frac{\partial p}{\partial y}\, ds < 0$$

which yields a contradiction.

* Assume

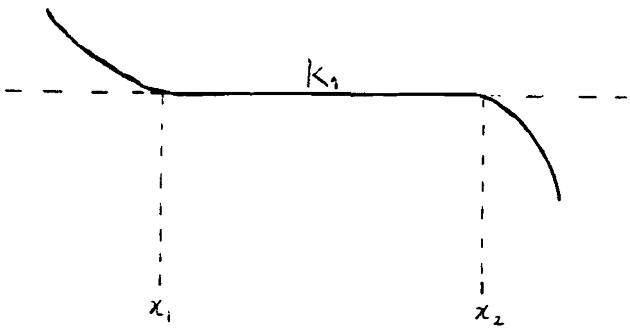


Then with Gauss' divergence theorem

$$\int_{K_1} |\nabla p|\, ds = \int_{K_2} \frac{\partial p}{\partial y}\, ds + \int_{K_3} \frac{\partial p}{\partial x}\, ds < 0$$

(observe that $v_x > 0$ on K_3)
and again we arrive at a contradiction.

* Assume ϕ is constant on (x_1, x_2) with $\phi(x_1) = y'$.



Since ∇p is normal to the curve $(t, \phi(t))$ we have

$$\frac{\partial p}{\partial x} = 0 \quad \text{on} \quad K_1$$

$$\frac{\partial p}{\partial y} = -1 \quad \text{on} \quad K_1$$

and of course also

$$p = 0 \quad \text{on} \quad K_1 \ .$$

The solution of the problem

$$\Delta \tilde{p} = 0 \quad \text{in} \quad (x_1, x_2) \times (0, y')$$

$$\frac{\partial \tilde{p}}{\partial x} = 0 \ , \quad \frac{\partial \tilde{p}}{\partial y} = -1 \quad \text{and} \quad \tilde{p} = 0 \quad \text{on} \quad K_1$$

$$\frac{\partial \tilde{p}}{\partial y} = -1 \quad \text{on} \quad (x_1, x_2)$$

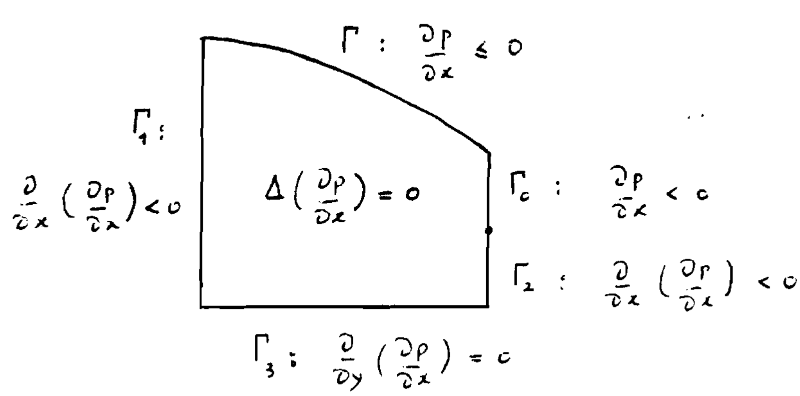
is given by

$$\tilde{p}(x,y) = y' - y \ .$$

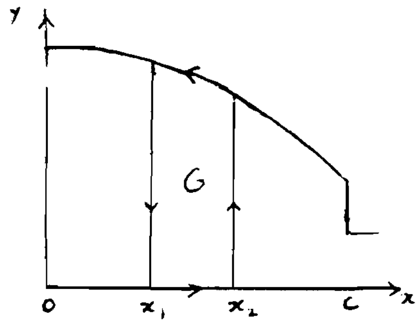
This solution can be extended in a unique way to a harmonic function in the rectangle $(0,c) \times (0,y')$. Since the looked for solution p agrees with $\tilde{p}$ on $(x_1,x_2) \times (0,y')$ and is harmonic in Ω we must have $p = \tilde{p}$ on $\Omega \cap (0,c) \times (0,y')$. Thus we get a contradiction with the boundary conditions on Γ_1 or on Γ_2.

(II.4) $\quad \dfrac{\partial p}{\partial x} < 0 \quad \text{in} \quad \Omega .$

Indeed on Γ_1 and Γ_2 we have $\dfrac{\partial p}{\partial y} = -1$ and so $\dfrac{\partial}{\partial x}\left[\dfrac{\partial p}{\partial x}\right] = 0$ on Γ_1 and Γ_2. The fluid flows tangent to Γ_3, so $\dfrac{\partial}{\partial y}\left[\dfrac{\partial p}{\partial x}\right] = 0$ on Γ_3. Moreover $\dfrac{\partial p}{\partial x} < 0$ on Γ_0, because $v_x > 0$ on Γ_0.
It is clear that $\dfrac{\partial p}{\partial x} \le 0$ on Γ.
Now apply the maximum principle:



(II.5)



Consider the region G as indicated in the figure.

Since div $\vec{v} = 0$, Gauss' divergence theorem yields

$$\int_{\partial G} (\vec{v} \cdot \vec{n})\, ds = 0 .$$

Now $(\vec{v} \cdot \vec{n}) = 0$ on Γ and Γ_3, what yields

$$\int_0^{\phi(x_1)} v_x(x_1,y)\, dy = \int_0^{\phi(x_2)} v_x(x_2,y)\, dy = q \quad \text{(same discharges)} .$$

So $q\, c = \int_0^c \Big(\int_0^{\phi(x)} \big(-\frac{\partial p}{\partial x}(\xi,\eta)\big)\, d\eta \Big)\, d\xi = \frac{1}{2}(y_1^2 - y_2^2)$.

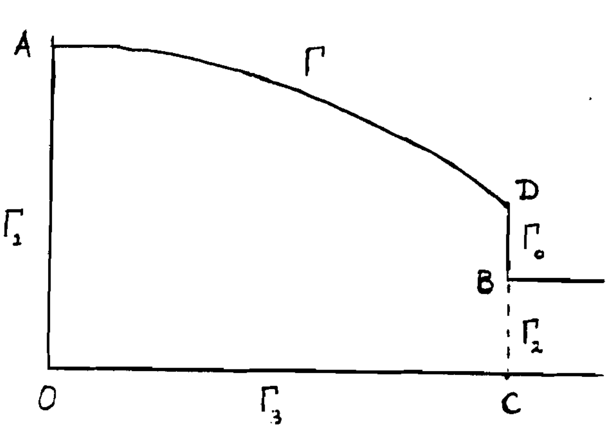
(II.6) The curve Γ_0 really exists.

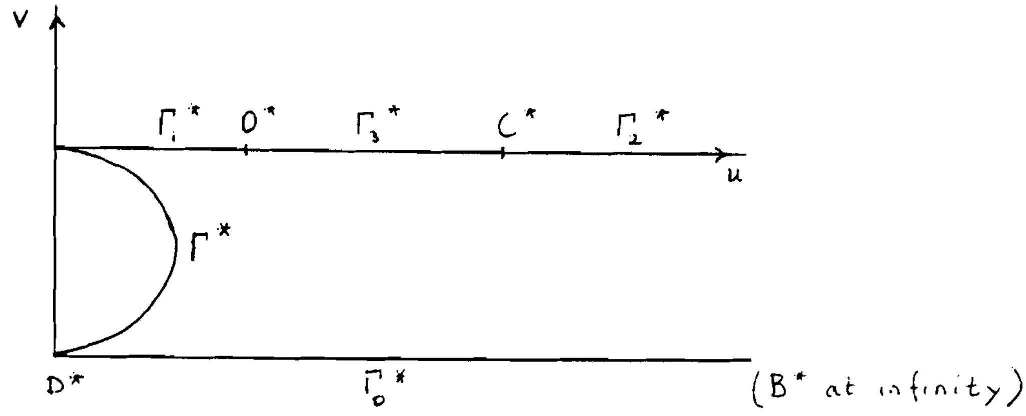
We have $\vec{v} = (u,v)$, $\vec{v} = -\nabla(p+y)$, $\operatorname{div}(\vec{v}) = 0$.
So $u = -\frac{\partial p}{\partial x}$ and $v = -\frac{\partial p}{\partial y} - 1$.

From div $\vec{v} = 0$ we obtain

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y} \end{aligned}\right\} \text{ Cauchy–Riemann equations .}$$

Put $z = x + iy$ and $f(z) = u(x,y) + i\, v(x,y)$. Then f is holomorphic in $\Omega \subset \mathbb{C}$ and f maps the region Ω in the (x,y)-plane onto the region Ω^* in the (u,v)-plane with boundary segments $\Gamma_0^*, \Gamma_1^*, \Gamma_2^*, \Gamma_3^*$ and Γ_4^*.
In the latter plane we have





Explanation:

As we have seen on Γ we have $\left[\frac{\partial p}{\partial x}\right]^2 + \left[\frac{\partial p}{\partial y} + \frac{1}{2}\right]^2 = \frac{1}{4}$ implying that

$$u^2 + (v + \tfrac{1}{2})^2 = \tfrac{1}{4} .$$

So $\Gamma^* = \{(u,v) \mid u^2 + (v+\frac{1}{2})^2 = \frac{1}{4} , u \geq 0\}$.

Further on Γ_1, Γ_2 and Γ_3 we have $\frac{\partial p}{\partial y} = -1$, i.e. $v = 0$.

Now if D were equal to B, then D^* should be equal to B^*, which cannot be the case.

On Γ_0 we have $\frac{\partial p}{\partial y} = 0$, i.e. $v = -1$. It follows that the intersection point B of Γ_0 and Γ_2 is send to infinity by the mapping f. Hence u has a singular point in B.

Intuitively, water is coming down from D to B and at Γ_2 the y-component of the velocity is zero. So there would be water accumulation if the x-component of the velocity at B were not zero.

Having done some qualitative analysis for the problem, the next question is how to prove existence of a solution. For convenience we state the problem again

(II.7)

(i) $\quad \Delta p = 0 \;\text{ in }\; \Omega : \{(x,y) \mid 0 < y < \phi(x), 0 < x < c\}$

(ii) $\quad p(0,y) = y_1 - y, \;\; 0 \leq y \leq y_1 ,$

(iii) $\quad p(c,y) = y_2 - y, \;\; 0 \leq y \leq y_2 ,$

(iv) $\quad p(c,y) = 0 \;\; , \;\; y_2 \leq y \leq \phi(c) ,$

(v) $\quad \frac{\partial p}{\partial y}(x,0) = -1 , \;\; 0 \leq x \leq c$

(vi) $\quad p \big|_{\Gamma} = 0$

(vii) $\quad \frac{\partial}{\partial n}(p+y)\Big|_{\Gamma}=0\,, \quad \Gamma : \{(x,\phi(x)) \mid 0\le x\le c\}\,.$

Enclose the region Ω in a sufficiently large rectangle $D=[0,c]\times[0,\bar{y}]$ $\bar{y}>y_1$. The pressure p is zero in $D\setminus\Omega$. For $\psi\in C_c^\infty(D)$ consider the expression

$$\psi\,\Delta(p+y)\,.$$

Then by Green's first identity

$$\int_\Omega \psi\,\Delta(p+y)\,dx = -\int_\Omega (\nabla\psi\cdot\nabla(p+y))\,dx + \int_{\partial\Omega} \psi\,\frac{\partial(p+y)}{\partial n}\,ds\,.$$

Due to the boundary conditions the integral $\int_{\partial\Omega}\cdots ds$ is zero. So we end up with the following weak formulation of the problem

$$0=\int_\Omega(\nabla\psi\cdot\nabla(p+y))\,dx =$$

$$=\int_\Omega \frac{\partial\psi}{\partial y}\,dx + \int_\Omega(\nabla\psi\cdot\nabla p)\,dx$$

$$=\int_D \chi_\Omega\,\frac{\partial\psi}{\partial y}\,dx + \int_\Omega(\nabla\psi\cdot\nabla p)\,dx\,.$$

Using distributional derivatives we get the distributional equation

(II.8) $\quad -\Delta p=\frac{\partial\chi_\Omega}{\partial y}\;$ in $\;D\,.$

Next we apply the so called *Baiocchi transform*

(II.9) $\quad w(x,y)=\int_y^{\bar{y}} p(x,\eta)\,d\eta\,, \quad \frac{\partial w}{\partial y}=-p\,.$

It follows that

$$-\Delta p=\frac{\partial}{\partial y}(\Delta w)\,.$$

Since $w(x,\bar{y})\equiv 0$ and $\chi_\Omega(\bar{y})=0$ it follows that

(II.10) $\quad \Delta w=\chi_\Omega\;$ in $\;D\,.$

We look for a solution $w\in W^{2,1}(D)$ with $w\ge 0$ and values on the boundary ∂D given by

$$w(x,\bar{y})=0\,,\quad 0\le x\le c$$

$$w(0,y) = \int_y^{\bar y} (y_1 - \eta)^+ \, d\eta \quad , \quad 0 < y < \bar y$$

$$w(c,y) = \int_y^{\bar y} (y_2 - \eta)^+ \, d\eta \quad , \quad 0 < y < \bar y \, .$$

For the boundary condition at $y = 0$ we have

$$q = \int_0^{\bar y} v_x(x,\eta)\, d\eta = -\frac{d}{dx} \int_0^{\bar y} p(x,\eta)\, d\eta$$

so that

$$\int_0^{\bar y} p(x,\eta)\, d\eta = c - qx$$

with

$$c = \int_0^{\bar y} p(0,\eta)\, d\eta = \int_0^{y_1} (y_1 - \eta)\, d\eta = \frac{1}{2}\, y_1^2 \; .$$

It follows that

$$w(x,0) = \frac{1}{2}\, y_1^2 - \frac{1}{2c}\, (y_1^2 - y_2^2)\, x \quad , \quad 0 \le x \le c$$

where we used (II.5).
So for all $u \in W^{2,1}(D)$ such that $u \,\big|_{\partial D} = w \,\big|_{\partial D}$ and $u \ge 0$

$$\int_D \nabla w \cdot \nabla (u - w)\, dx = - \int_D \chi_\Omega (u - w)\, dx$$

yielding the integral inequality

$$\int_D \nabla w \; \nabla (u - w)\, dx \ge - \int_D (u - w)\, dx \; .$$

We can conclude from this that there exists exactly one solution. In fact we have to minimize the functional

$$I(v) = \int_D |\nabla v|^2 \, dx + 2 \int_D v \, dx$$

over the closed convex set

$$K = \{ v \in W^{2,1}(D) \mid v \ge 0 \, , \;\; v \,\big|_{\partial D} = w \,\big|_{\partial D} \} \, .$$

Next we discuss a generalization of the dam problem.

The saturation σ of a porous medium is defined as follows

(II.11) $$\sigma = \frac{\text{volume occupied by the fluid}}{\text{total available volume}} \quad , \quad 0 \le \sigma \le 1 \, .$$

Until now we considered the situation $\sigma = 0$ (dry) or $\sigma = 1$ (complete saturation). It is clear that a complete saturation requires a certain pressure p_s:

$$p \ge p_s \;\Rightarrow\; \sigma = 1 \, .$$

In this more general situation time has to be taken into account and instead of the incompressibility condition div $\vec{v} = 0$ we have

(II.12) $$\frac{\partial \sigma}{\partial t} + \operatorname{div} \vec{v} = 0$$

$$\vec{v} = -k(\sigma)\, \nabla(p + \rho g y) \, .$$

A combination of the above relations gives

$$\frac{\partial \sigma}{\partial t} - \nabla \cdot (k(\sigma)\, \nabla(p + \rho g y)) = 0 \, .$$

If gravity can be neglected, then $p \sim \sigma$, $0 < \sigma < 1$, and we end up with a heat equation of type

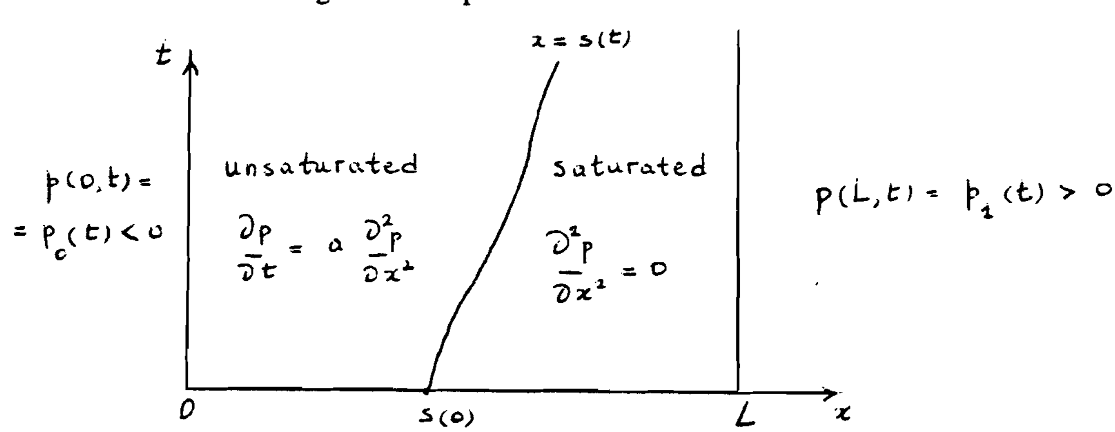
$$\frac{\partial p}{\partial t} - \nabla \cdot (a(p) \nabla p) = 0 \, .$$

We consider the problem with $a(p)$ constant and for only one space variable. So we look at the differential equation

$$\frac{\partial p}{\partial t} - a \frac{\partial^2 p}{\partial x^2} = 0$$

valid in the unsaturated region $0 \le x \le s(t)$, $s(t)$ denoting the free boundary point dependent on t. Also, we rescale such that $p_s = 0$.

Consider the following schematic plot



Statement of the problem

(II.13)

(i) $$\frac{\partial^2 p}{\partial x^2} = 0 \quad , \quad s(t) < x < L$$

(ii) $$\frac{\partial p}{\partial t} = a \, \frac{\partial^2 p}{\partial x^2} \quad , \quad 0 < x < s(t)$$

(iii) $$p(0,t) = p_0(t) \quad , \quad t \geq 0$$

(iv) $$p(L,t) = p_1(t) \quad , \quad t \geq 0$$

(v) $p(s(t), t) = 0$ (first condition on the free boundary) .

So in the saturated region we have

$$p(x,t) = p_1(t) \, \frac{(x - s(t))}{(L - s(t))} \quad , \quad s(t) \leq x \leq L \, .$$

Consequently

(vi) $\dfrac{\partial p}{\partial x} (s(t), t) = \dfrac{p_1(t)}{L - s(t)}$ (second condition on the free boundary) .

CHAPTER III

THE STEFAN PROBLEM

We consider a heat conducting medium occupying a given domain Ω in $\mathbb{R}^n$ in which phase change from solid to liquid or from liquid to solid is taken place. By Ω_s we denote the solid part of the domain and by Ω_l the liquid part. The crucial point of the mathematical scheme is the description of what happens at the interface Γ. The temperature in each phase obeys the heat equation. So

$$\text{(III.1)} \qquad \rho_s c_s \frac{\partial \theta_s}{\partial t} - k_s \Delta \theta_s = 0 \quad \text{in} \quad \Omega_s$$

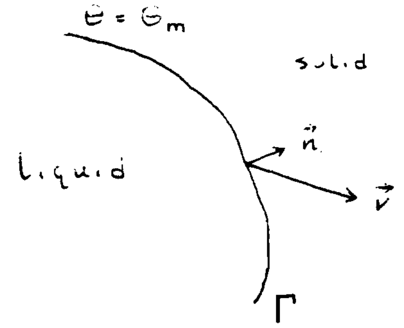
and

$$\text{(III.2)} \qquad \rho_l c_l \frac{\partial \theta_l}{\partial t} - k_l \Delta \theta_l = 0 \quad \text{in} \quad \Omega_l \,.$$

Here ρ_s, c_s and k_s are respectively the density, the specific heat and the thermal conductivity of the solid phase. Similarly, ρ_l, c_l and k_l are defined.
At the interface Γ the temperature must be equal to the phase change temperature θ_m which is normally a constant. So we derive the condition

$$\text{(III.3)} \qquad \theta_s(x,t) = \theta_l(x,t) = \theta_m \,, \quad x \in \Gamma(t) \,, \quad t \geq 0 \,.$$

This is not the only condition for θ_s and θ_l on Γ. An additional condition is derived from the heat balance at the interface.
Let $\vec{n}$ be the unit normal vector at Γ pointing towards the solid phase and let $\vec{v}$ be the velocity of a point of Γ. The the normal component of $\vec{v}$, i.e. $\vec{v} \cdot \vec{n}$, represents the local rate of melting or of solidification if positive or negative, respectively. If L denotes the heat absorbed (or released) for melting (or solidifying) a unit volume of the material, then $L\,\vec{v} \cdot \vec{n}$ is the local rate of heat absorption (or heat release) in the process. Further, the heat coming to the interface from the liquid phase equals $k_l \dfrac{\partial \theta_l}{\partial n}$ and the heat flux leaving it through the solid phase equals $k_s \dfrac{\partial \theta_s}{\partial n}$. There is balance of heat whenever



$$\text{(III.4)} \qquad L\,\vec{v} \cdot \vec{n} = -k_l \frac{\partial \theta_l}{\partial n} + k_s \frac{\partial \theta_s}{\partial n} \quad \text{on} \quad \Gamma \,.$$

Let the interface Γ be described by the equation

(III.5) $\quad S(x,t) = 0$

where S is a continuously differentiable function. Then we have

(III.6) $\quad \vec{n}(x,t) = \pm \nabla_x S(x,t) \,/\, |\nabla_x S(x,t)|$.

From (III.5) taking the total time derivative we obtain

(III.7) $$0 = \frac{d}{dt} S(x(t),t) = \vec{v} \cdot \nabla_x S(x,t) + \frac{\partial S}{\partial t}(x,t) .$$

Thus we arrive at the *Stefan condition*

(III.8) $$-L \frac{\partial S}{\partial t} = [-k_l \nabla_x \theta_l + k_s \nabla_x \theta_s] \cdot \nabla_x S .$$

The problem of determining (S, θ_l, θ_s) is called a Stefan problem. In its classical formulation S is required to be C^1 for t in some interval $(0,T)$, θ_l and θ_s must be $C^{2,1}$ in $\Omega_l \times (0,T)$ and $\Omega_s \times (0,T)$, respectively. Moreover, the temperature θ composed of θ_s and θ_s, must be continuous in $\overline{\Omega} \times [0,T]$.

1. One-dimensional case

In the one-dimensional problem x is a scalar variable and the free boundary can be expressed by the equation

(III.1.1) $\quad S(x,t) = x - s(t) = 0$.

The Stefan condition (III.8) takes the form

(III.1.2) $$L\,\dot{s}(t) = -k_l \frac{\partial \theta_l}{\partial x} + k_s \frac{\partial \theta_s}{\partial x} .$$

which can be written as

$$L\,\dot{s}(t) = [\![-k \frac{\partial \theta}{\partial x}]\!]_s^l$$

where $[\![f]\!]_s^l$ denotes the jump of f from the liquid side to the solid side at the interface. Whenever either θ_s or θ_l is identically equal to θ_m the Stefan condition simplifies to

(III.1.3)

(i) $$L\,\dot{s}(t) = -k_l \frac{\partial \theta_l}{\partial x} \quad \text{(melting, liquid phase problem)} .$$

(ii) $$L\,\dot{s}(t) = k_s \frac{\partial \theta_s}{\partial x} \quad \text{(solidifying, solid phase problem)} .$$

We have a so called one phase problem.

Let us consider the liquid phase problem, which becomes after rescaling.

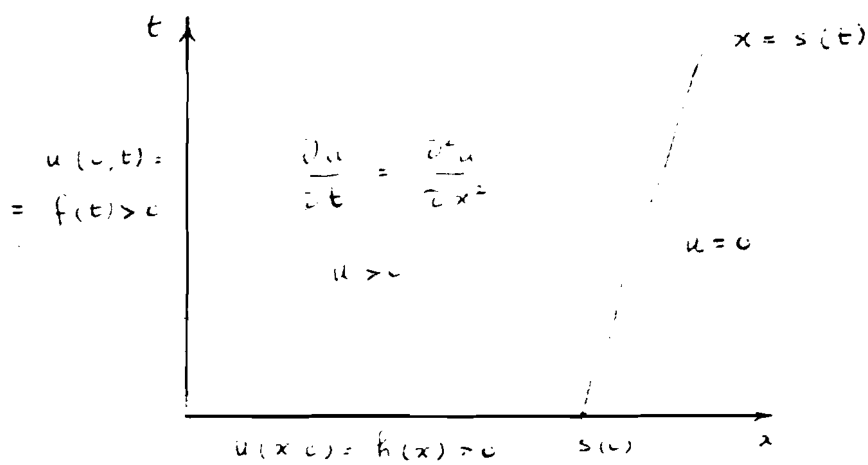
(III.1.4)

(i) $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

(ii) $$u(x,t) = 0 \quad , \quad x \geq s(t) \quad , \quad t > 0$$

(iii) $$\dot{s}(t) = -\frac{\partial u}{\partial x}\,(s(t), t) \quad , \quad t > 0 .$$

Observe that u denotes the rescaled temperature such that $u = 0$ is the melting temperature. The situation is sketched in the following plot



First we construct explicit solutions using self-similar solutions of the heat equation. Therefore, take $u(x,t) = f(a(t)x)$. Then we have

$$\frac{\partial^2 u}{\partial x^2}\,(x,t) = (a(t))^2\, f''(a(t)x)$$

$$\frac{\partial u}{\partial t}\,(x,t) = \dot{a}(t)\, f'(a(t)x) .$$

Taking $\eta = a(t)x$ it follows that

$$\eta\, \dot{a}(t)\, f'(\eta) = (a(t))^3\, f''(\eta) .$$

Separation of variables yields

$$\frac{\dot{a}}{a^3} = \lambda \quad \text{and} \quad f'' = \lambda \eta f'$$

what for $\lambda = -2$ leads to the solution

$$a(t) = \frac{1}{2\sqrt{t}} \quad \text{and} \quad f'(\eta) = A\, e^{-\eta^2} .$$

Thus the following self-similar solutions have been constructed

(III.1.5) $\quad u(x,t) = A\,[\mathrm{erf}(\tfrac{1}{2}\,\alpha) - \mathrm{erf}\left[\frac{x}{2\sqrt{t}}\right]]$

where

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{y} e^{-\eta^2}\, d\eta .$$

Introducing III.1.5 into the free boundary condition

$$u(s(t),\, t) = 0$$

yields

(III.1.6) $\quad s(t) = \alpha\sqrt{t} \quad , \quad t > 0$

whence

$$\dot{s}(t) = \frac{\alpha}{2\sqrt{t}} \quad , \quad t > 0 .$$

Moreover, $\dfrac{\partial u}{\partial x} = \dfrac{-2A}{\sqrt{\pi}}\, e^{-x^2/4t}\, \dfrac{1}{2\sqrt{t}}$.

So the second free boundary condition

$$\dot{s}(t) = -\frac{\partial u}{\partial x}\,(s(t),\, t)$$

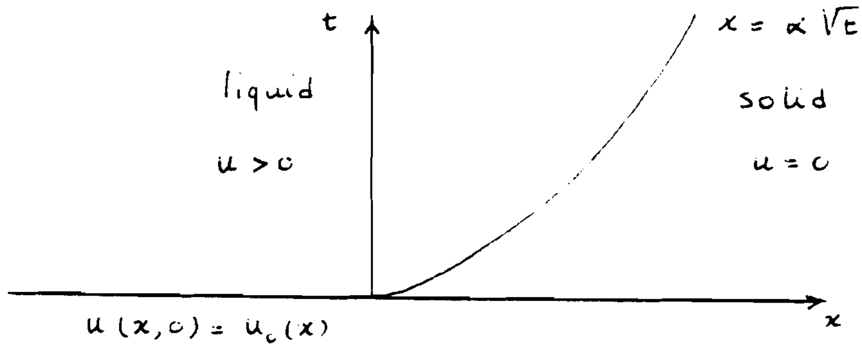
is fulfilled whenever

(III.1.7) $\quad A = \frac{1}{2}\sqrt{\pi}\,\alpha\, e^{\alpha^2/4} .$

It follows that the free boundary can be any parabola. The result is the so called Neumann solution

$$u(x,t) = \alpha\, e^{\alpha^2/4} \int_{x/2\sqrt{t}}^{\alpha/2} \exp(-\eta^2)\, d\eta \quad , \quad x < \alpha\sqrt{t} .$$

We have found a one-parameter family (α) of solutions.



At $t = 0$ the half space $x < 0$ is occupied by the liquid and the half space $x > 0$ is occupied by the solid at zero temperature. We can find the initial value of u, $u_0(x)$, for $x < 0$ by letting t tend to zero. This yields

$$\text{(III.1.8)} \qquad u_0(x) = \alpha\, e^{\alpha^2/4} \int_{-\infty}^{\alpha/2} e^{-\eta^2}\, d\eta \quad , \quad x < 0\,.$$

We also see that the temperature at $x = 0$ remains constant

$$u(0,t) = \alpha\, e^{\alpha^2/4} \int_{0}^{\alpha/2} e^{-\eta^2}\, d\eta \quad , \quad t > 0\,.$$

This way we can solve problems with constant initial data u_0 or constant boundary data u_1. Therefore we have to find α such that

$$u_0 = \alpha\, e^{\alpha^2/4} \int_{-\infty}^{\alpha/2} e^{-\eta^2}\, d\eta$$

or

$$u_1 = \alpha\, e^{\alpha^2/4} \int_{0}^{\alpha/2} e^{-\eta^2}\, d\eta\,.$$

It can be shown that for each $u_0 > -1$ there exists a unique α. Moreover the corresponding solution u has the same sign as u_0. So for $-1 < u_0 < 0$ we are dealing with a supercooled fluid

$$-1 < u(x,t) < 0 \quad , \quad x < s(t)\,, \ t > 0\,.$$

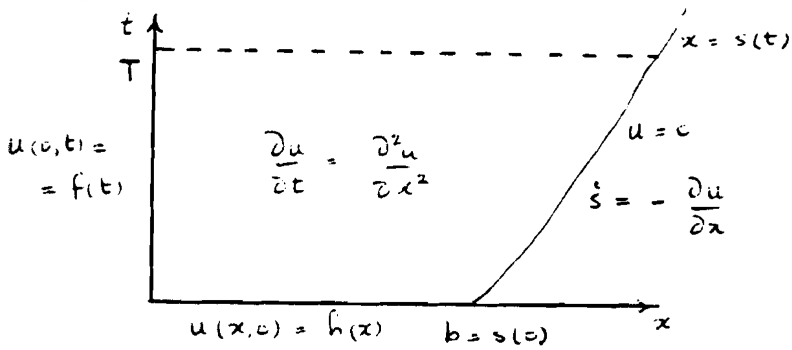
There appears a discontinuity at $(x,t) = (0,0)$ since the temperature jumps from u_0 to 0.

Remark. In general (so not only for similarity solutions) the following can be said

- The temperature jump at the origin cannot exceed -1.
- If there is some neighbourhood of $x = 0$ in which $u_0(x) > -1$ then the supercooled problem has one unique solution.

– If $u_0(x) \leq -1$ in some neighbourhood of $x = 0$ then no solution exists.

We return to the one-phase problem in which we attach boundary data at $x = 0$ and initial data for $t = 0$.



We assume the following: h and f are continuous, $h(b) = 0$, $0 \leq h(x) \leq H(b-x)$ and f is bounded with $0 \leq f(t) \leq Hb$.

(III.1.9) Formulation of the problem.

(i) $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad , \quad 0 < x < s(t), \; t > 0$$

(ii) $$u(x,0) = h(x) \quad , \quad 0 < x < b$$

(iii) $$u(0,t) = f(t) \quad , \quad t > 0$$

(iv) $$s(0) = b > 0$$

(v) $$u(s(t),t) = 0 \quad , \quad t > 0$$

(vi) $$\frac{\partial u}{\partial x}(s(t),t) = -\dot{s}(t), \; t > 0 .$$

(III.1.10) **Theorem.**

There exists precisely one solution u (global in time) of (III.1.9) with $0 \leq \dot{s} \leq H$.

Proof.
We use Schauder's fixed point theorem to prove existence.

Consider the following family of curves

$$S(T,A) = \{ s \in C((0,T)) \mid s(0) = b \wedge \forall_{0 < t_1, t_2 < T} : \\ 0 \leq \frac{s(t_1) - s(t_2)}{t_1 - t_2} \leq A \} .$$

Take a fixed $s \in S(T,A)$ and consider the problem

$$\text{(III.1.11)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0, & 0 < x < s(t), \ 0 < t < T \\ u(s(t), t) = 0, & 0 < t < T \\ u(x, 0) = h(x), & 0 < x < b \\ u(0,t) = f(t), & 0 < t < T. \end{cases}$$

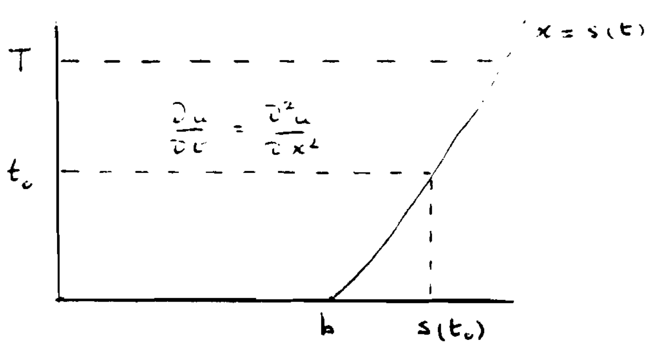
Since the curve s is Lipschitz continuous, $\dfrac{\partial u}{\partial x}$ is continuous up to the curve s. (This is an old theorem of Gevrey, 1913.) Next solve the equation

$$\begin{cases} \dot{\sigma}(t) = -\dfrac{\partial u}{\partial x}(s(t), t) \\ \sigma(0) = b \end{cases}$$

where u denotes the solution of (III.1.11). This generates a new curve σ. The described procedure leads to an operator t mapping a curve s on a curve σ. If there is a constant A such that t maps $S(T,A)$ into ifself, then a fixed point of t yields a solution of the free boundary Stefan problem. To find such a constant A we first observe that we have assumed that

* $h(x) < H(b-x)$
* $f(t) < Hb$
* s is increasing.

Take a fixed $t_0, 0 < t_0 < T$.



Let v be defined by

$$v(x) = H(s(t_0) - x).$$

Then $v(x) \geq h(x)$,

$v(0) \geq f(t)$,

$v(s(t)) > 0, \ 0 < t < t_0$,

and

v is a solution of the heat equation.

Now let $w = v - u$. Then we have

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, \quad 0 < x < s(t), \ 0 < t < t_0$$

$$w(0,t) > 0, \quad 0 < t < t_0$$

$$w(x, 0) > 0, \quad 0 < x < b$$

$$w(s(t),t) > 0 \quad , \quad 0 < t < t_0$$

$$w(s(t_0),t_0) = 0 \, .$$

So the maximum principle says that $w \geq 0$ for $0 \leq x \leq s(t)$ and $0 \leq t \leq t_0$, which means

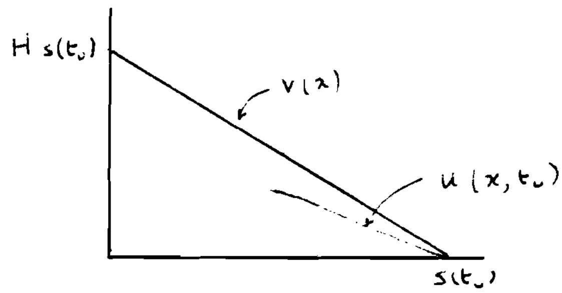
$$v(x) \geq u(x,t) \, , \quad 0 \leq x \leq s(t) \, , \quad 0 \leq t \leq t_0$$

$$v(s(t_0)) = u(s(t_0),t_0) = 0 \, .$$

It follows that

$$\frac{\partial u}{\partial x}(s(t_0),t_0) \geq -H$$

(see picture).



As a simple consequence of the maximum principle, yielding $u > 0$, we also must have

$$\frac{\partial u}{\partial x}(s(t_0),t_0) \leq 0 \, .$$

So t_0 being arbitrary, we obtain

$$\forall_{t>0} \; : \; -H \leq \frac{\partial u}{\partial x}(s(t),t) \leq 0 \, .$$

Thus we find $0 \leq \dot{\sigma} \leq H$ and we can therefore take $A = H$, i.e. t maps $S(T,H)$ into $S(T,H)$.

Now we are in the following position

- t is a mapping from $S(T,H)$ into $S(T,H)$.
- $S(T,H)$ is a closed convex bounded subset of the Banach space $C([0,T])$ with $\|f\| = \max\limits_{0 \leq t \leq T} |f(t)|, f \in C([0,T])$.

In order to be able to apply Schauder's fixed point theorem we prove that

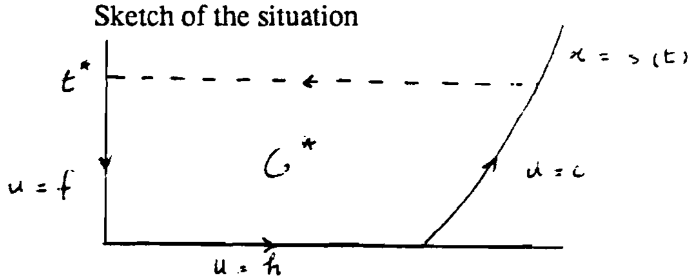
$$\text{I} \; : \; S(T,H) \text{ is compact in } C([0,T])$$

$$\text{II} : \; \text{t} : S(T,H) \to S(T,H) \text{ is continuous} \, .$$

I. By definition the set $S(T,H)$ is equicontinuous. So by Ascoli's theorem the set $S(T,H)$ is compact in $C([0,T])$.

II. Proving continuity of t requires a more lengthy proof.

Sketch of the situation



$0 \leq h(x) \leq H(b-x)$,
$0 \leq f(t) \leq Hb$,
s is increasing with $s(0 = b$
Lipschitz constant $\leq H$
$\Rightarrow \; s(t) \leq Ht.$

Consider the vector field

$$\vec{v}(x,t) = (xu \, , \, x \frac{\partial u}{\partial x} - u) \, .$$

Then by Green's integral identity

$$\int_{G^*} \left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial t} \right] d\sigma = \int_{\partial G^*} (\vec{v}, \vec{T}) \, ds \, .$$

It follows that

$$(*) \qquad 0 = \int_0^b x \, h(x) \, dx - \int_0^{s(t^*)} x \, u(x,t^*) \, dx + \int_0^{t^*} f(\tau) \, d\tau$$

$$+ \int_0^{t^*} s(\tau) \frac{\partial u}{\partial x} (s(\tau), \tau) \, d\tau$$

(observe that $u(s(\tau), \tau) = 0$).

We have defined $\dot{\sigma}(t) = - \frac{\partial u}{\partial x} (s(t), t)$, $\sigma(0) = b$.

Take $s_1, s_2 \in S(T,H)$ and use (*) to get

$$0 = - \int_0^{s_1} x \, u_1(x,t^*) \, dx + \int_0^{s_2} x \, u_2(x,t^*) \, dx$$

$$- \int_0^{t^*} s_1(\tau) \, \dot{\sigma}_1(\tau) \, d\tau + \int_0^{t^*} s_2(\tau) \, \dot{\sigma}_2(\tau) \, d\tau \, .$$

This yields

$$(**) \qquad \int_0^{t^*} s_1(\tau) \, (\dot{\sigma}_1(\tau) - \dot{\sigma}_2(\tau)) \, d\tau = - \int_0^{t^*} \dot{\sigma}_2(\tau) \, (s_1(\tau) - s_2(\tau)) \, d\tau$$

$$- \int_0^{\min(s_1(t^*), s_2(t^*))} x(u_1(x,t^*) - u_2(x,t^*)) + (-1)^j \int_{\min(s_1(t^*), s_2(t^*))}^{\max(s_1(t^*), s_2(t^*))} x \, u_j(x,t^*) \, dx \, .$$

Integrating the left hand side by parts in a slightly generalized sense we get

$$(***) \qquad s_1(t^*)\,(\sigma_1(t^*)-\sigma_2(t^*)) = \int_0^{t^*} \dot{s}_1(\tau)\,[\sigma_1(\tau)-\sigma_2(\tau)]\,d\tau + \cdots$$

Put $\|z\|_{t^*} = \max\{\,|z(x,\tau)|\ \mid 0 \le x \le s(\tau),\ 0<\tau<t^*\}$.
Then (**) and (***) gives

$$(****) \qquad b\ |\sigma_1(t^*)-\sigma_2(t^*)| \le s_1(t^*)\ |\sigma_1(t^*)-\sigma_2(t^*)| \le$$

$$\le H\,t^*\ \|\sigma_1-\sigma_2\|_{t^*} + H\,t^*\ \|s_1-s_2\|_{t^*} +$$

$$+\frac{1}{2}(b+H\,t^*)^2\ \|u_1-u_2\|_{t^*} + \frac{1}{6}H\ \|s_1-s_2\|_{t^*}^3$$

where we used that $u_j(x,t^*) \le H(s_j(t^*)-x)$, $0 \le x \le s_j(t^*)$.

By the maximum principle

$$\|u_1-u_2\|_{t^*} \le \sup_{0<\tau<t^*} |u_1-u_2|_{x=\min(s_1(\tau),s_2(\tau))} \le H\ \|s_1-s_2\|_{t^*}\ .$$

Now take $T^* = \frac{1}{2}\ b/H$ and conclude that

$$(*****) \qquad \frac{1}{2}\ b\ \|\sigma_1-\sigma_2\|_{T^*} \le C\ \|s_1-s_2\|_{T^*}$$

with C a constant depending on b and H.
Hence t is a continuous mapping from $S(T^*,H)$ into $S(T^*,H)$.

Remark. We have obtained a solution up to $t=T^*$. However we know that $0 \le u(x,T^*) \le H(s(T^*)-x)$. So taking

$$h^*(x) = u(x,T^*)$$

we can solve the problem

$$\frac{\partial u^*}{\partial t} = \frac{\partial^2 u^*}{\partial x^2}\quad ,\quad 0<x<s(t),\ T^*<t<T^{**}$$

$$u^*(0,t) = f(t)\quad ,\quad T^*<t<T^{**}\ ,$$

$$u^*(x,0) = h^*(x)\quad ,\quad 0<x<s(T^*)\ ,$$

$$u^*(s(t),t) = 0\quad ,\quad T^*<t<T^{**}\ ,$$

$$\dot{s}(t) = \frac{-\partial u}{\partial x}(s(t),t)\quad ,\quad T^*<t<T^{**}\ .$$

Thus the solution can be extended up to $t = T^{**} = \frac{1}{2}\ s(T^*)/H$.
In other words the solution exists for all $T>0$.

Our next aim is to prove uniqueness.

(III.1.12) **Lemma.**

For any solution (s,u) we have the following identity

$$\frac{1}{2}\,(s(t))^2 = \frac{1}{2}\,b^2 + \int_0^b x\,h(x)\,dx - \int_0^{s(t)} x\,u(x,t)\,dx + \int_0^t f(\tau)\,d\tau\,.$$

Proof.

As we have seen in the above proof

$$0 = \int_0^b x\,h(x)\,dx - \int_0^{s(t)} x\,u(x,t)\,dx + \int_0^b f(\tau)\,d\tau +$$

$$+ \int_0^t s(\tau)\,\frac{\partial u}{\partial x}\,(s(\tau),\tau)\,d\tau\,.$$

Since $-\dot{s}(\tau) = \dfrac{\partial u}{\partial x}\,(s(\tau),\tau)$ the result follows. □

The following result is on the monotone dependence of the free boundary on the data.

(III.1.13) **Theorem.**

Let (s_1,u_1) and (s_2,u_2) be solutions corresponding to the respective data (b_1,h_1,f_1) and (b_2,h_2,f_2) satisfying the requirements

$$0 \le h_j(x) \le H_j(b_j - x)\,,$$

$$0 \le f_j(t) \le H_j b_j \quad , \quad j = 1,2\,.$$

If $b_1 \ge b_2$, $h_1 \ge h_2$ and $f_1 \ge f_2$ then $s_1 \ge s_2$.

Proof.

Suppose first that $b_1 > b_2$.

We shall show that $s_1(t) > s_2(t)$ for all $t \in [0,T]$.

Assume this were not true. Then there exists $t_0 \in (0,T]1$ such that $s_1(t_0) = s_2(t_0)$ while $s_1(t) > s_2(t)$ for $0 \le t < t_0$.

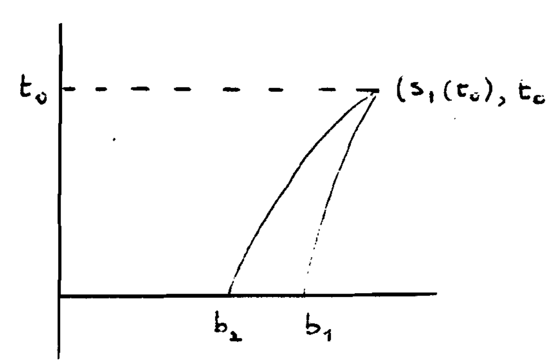
The function $u_1 - u_2$ satisfies

$$(u_1 - u_2)\,(0,t) = f_1(t) - f_2(t) \ge 0$$

$$(u_1 - u_2)\,(x,0) = h_1(x) - h_2(x) \ge 0\,,$$

$$0 \le x \le b_2$$

$$(u_1 - u_2(s_2(t),t) = u_1(s_2(t),t) \ge 0\,,$$



$$0 \le t \le t_0 .$$

So by the maximum principle $u_1 \ge u_2$ on $0 \le x \le s_2(t)$, $0 \le t \le t_0$.
Since $u_1(s_1(t_0), t_0) = u_2(s(t_0), t_0) = 0$ it also follows that

$$\frac{\partial u_1}{\partial x}(s_1(t_0), t_0) < \frac{\partial u_2}{\partial x}(s_2(t_0), t_0)$$

because of the so called boundary point principle.
Hence $\dot{s}_1(t_0) > \dot{s}_2(t_0)$ which yields a contradiction.

Next consider the case $b_1 = b_2 = b$.
In this case take a family of solutions (s^δ, u^δ) to the problem with data

$$s^\delta(0) = b + \delta$$

$$u^\delta(x,0) = h_1(x) \quad , \quad 0 \le x < b$$

$$u^\delta(x,0) = 0 \quad , \quad b \le x \le b + \delta$$

$$u^\delta(0,t) = f_1(t) .$$

From the above arguments we see that $s_1 < s^\delta$ and $s_2 < s^\delta$ for all t. For the difference $s^\delta - s_1$ we have by the previous lemma

$$\frac{1}{2}\,(s^\delta(t) - s_1(t))\,(s^\delta(t) + s_1(t)) =$$

$$= \frac{1}{2}\,\delta(2b+\delta) - \int_0^{s_1(t)} x(u^\delta(x,t) - u_1(x,t))\,dx$$

$$- \int_{s_1(t)}^{s^\delta(t)} x\,u^\delta(x,t)\,dx .$$

Since $u^\delta \ge 0$ and $u^\delta \ge u_1$ we have

$$\frac{1}{2}\,(s^\delta(t) - s_1(t))\,(s^\delta(t) + s_1(t)) \le \frac{1}{2}\,\delta(b+\delta) .$$

Moreover, since $s^\delta(t) + s_1(t) \ge 2b + \delta$, we have

$$s^\delta(t) - s_1(t) \le \delta$$

and using $s_2(t) < s^\delta$,

$$s_2(t) < s_1(t) + \delta .$$

The result follows by letting δ tend to zero. $\square$

(III.1.14) **Corollary.**
Problem (III.1.9) has exactly one solution (s,u) with s in the class $S(T,H)$.

Proof.

If (s_1,u_1) and (s_2,u_2) are solutions corresponding to the same set of data, then by the previous theorem both $s_1 \geq s_2$ and $s_2 \leq s_1$. Hence $s_1 = s_2$ and a posteriori $u_1 = u_2$. □

2. More dimensional Stefan problem

Throughout we assume that no supercooling or superheating is allowed. So each phase is characterized by the sign of the temperature, taking zero as the melting point. The thermal energy (enthalpy) stored in a unit volume of the solid can be taken equal to $\rho_s c_s \theta$ while in the liquid the thermal energy can be taken equal to $\rho_l c_l \theta + L$, L denoting the latent heat. So we can define

(II.2.1) $$E(\theta) = \rho_s c_s \theta \quad \text{for} \quad \theta < 0$$

$$E(\theta) = \rho_l c_l \theta + L \quad \text{for} \quad \theta > 0 ,$$

leaving aside for the moment the question of defining E for $\theta = 0$.

The heat equation can be written in the following form

(II.2.2) $$\frac{\partial E}{\partial t} = k_s \Delta \theta \quad , \quad \theta < 0 ,$$

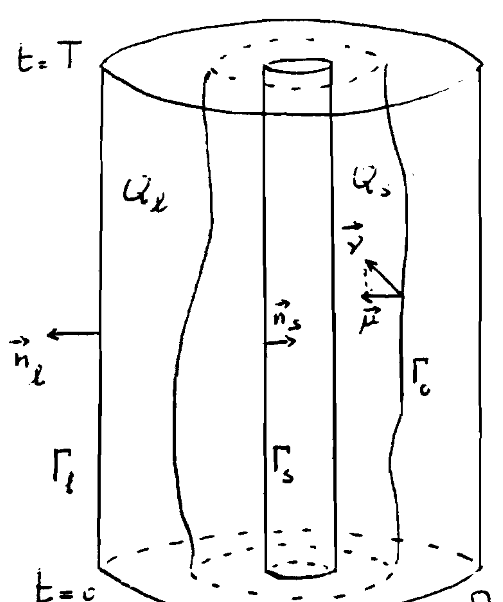
$$\frac{\partial E}{\partial t} = k_l \Delta \theta \quad , \quad \theta > 0 ,$$

where k_s and k_l are supposed to be constant.

Consider the following space of test functions

(II.2.3) $$V = \{\phi \in C^\infty(\overline{\Omega} \times [0,T]) \mid \phi = 0 \ \text{on} \ \partial\Omega \times (0,T) \ \wedge \ \phi(x,T) = 0\} .$$

With this test function space a weak formulation of the Stefan problem will be derived.



$\Omega_0 = \Omega \times \{0\}$

Γ_l : outer boundary with external normal $\vec{n}_l$

$k_l \theta \big|_{\Gamma_l} = f(x,t) > 0$

Γ_s : inner boundary with external normal $\vec{n}_s$

$k_s \theta \big|_{\Gamma_s} = g(x,t) < 0$

initial condition

$E(x,0) = E_0(x)$ in Ω.

$\vec{\nu}$ normal to the free boundary Γ_0 in $\mathbb{R}^{n+1}$ (directed towards the solid)
$\vec{\mu}$ projection of $\vec{\nu}$ on $\mathbb{R}^n$
$\vec{n}$ normal to $\Gamma_0 \cap \Omega \times \{t\}$ in $\mathbb{R}^n$
$Q = \Omega \times (0,T)$, $Q_l(Q_s)$ subset of Q occupied by the liquid (solid).

Observe that $\vec{\nu}(x,t) = (\nu_1(x,t), \ldots, \nu_n(x,t), \nu_0(x,t))$.
Since $\frac{\partial}{\partial t}(\phi E) = \phi \frac{\partial E}{\partial t} + E \frac{\partial \phi}{\partial t}$, $\phi \in V$, we find by Gauss' divergence theorem applied to the region Q_l

(i) $$\int_{Q_l} \phi \frac{\partial E}{\partial t}\, dx\, dt = -\int_{Q_l} E \frac{\partial \phi}{\partial t}\, dx\, dt + \int_{\Omega_{0l}} \phi E_0\, d\sigma + \int_{\Gamma_0} L \cdot \phi \cdot \nu_0\, d\sigma .$$

Similarly for the region Q_s

(ii) $$\int_{Q_s} \phi \frac{\partial E}{\partial t}\, dx\, dt = -\int_{Q_s} E \frac{\partial \phi}{\partial t}\, dx\, dt + \int_{\Omega_{0s}} \phi E_0\, d\sigma .$$

Further, applying Gauss' integral identity again,

$$\int_{Q_l} \nabla_x \cdot (\phi \nabla_x \theta - \theta \nabla_x \phi)\, dx\, dt =$$

$$= \int_{\Gamma_l} (\phi \frac{\partial \theta}{\partial n_l} - \theta \frac{\partial \phi}{\partial n_l})\, d\sigma + \int_{\Gamma_0} (\phi \nabla_x \theta \,\Big|_l - \theta \nabla_x \phi \,\Big|_l) \cdot \vec{\mu}\, d\sigma .$$

(Observe that $\nabla_x \phi$, $\nabla_x \theta \in \mathbb{R}^n \times \{0\}$.)

It yields

(iii) $$\int_{Q_l} k_l \phi \Delta \theta\, dx\, dt = -\int_{\Gamma_l} f \frac{\partial \phi}{\partial n_l}\, d\sigma + \int_{\Gamma_0} \phi k_l \nabla_x \theta \,\Big|_l \cdot \vec{\mu}\, d\sigma$$
$$+ \int_{Q_l} k_l \theta \Delta \phi\, dx\, dt .$$

Similarly,

(iv) $$\int_{Q_s} k_s \phi \Delta \theta\, dx\, dt = -\int_{\Gamma_s} g \frac{\partial \phi}{\partial n_s}\, d\sigma - \int_{\Gamma_0} \phi k_s \nabla_x \theta \,\Big|_s \cdot \vec{\mu}\, d\sigma$$
$$+ \int_{Q_s} k_s \theta \Delta \phi\, dx\, dt .$$

Combining the relations (i)-(iv) we get

$$\int_Q (E \frac{\partial \phi}{\partial t} + k(\theta)\, \theta \Delta \phi)\, dx\, dt =$$

$$= \int_{\Omega_0} \phi E_0 \, dx + \int_{\Gamma_l} f \frac{\partial \phi}{\partial n_l} \, d\sigma + \int_{\Gamma_s} g \frac{\partial \phi}{\partial n_s} \, d\sigma +$$

$$+ \int_{\Gamma_0} \phi \, \{ L \nu_0 - k_l \nabla_x \theta \Big|_l \cdot \vec{\mu} - k_s \nabla_x \theta \Big|_s \cdot \vec{\mu} \} \, d\sigma$$

where

$$k(\theta) = \begin{cases} k_s & , \ \theta < 0 \\ k_l & , \ \theta > 0 \, . \end{cases}$$

Describing the free boundary Γ_0 by $S(x,t) = 0$ we have

$$\nu_0(x,t) = (\partial S / \partial t) \, / \, (|\nabla_x S|^2 + (\partial S / \partial t)^2)^{1/2}$$

and

$$\vec{\mu}(x,t) = \nabla_x S \, / \, (|\nabla_x S|^2 + (\partial S / \partial t)^2)^{1/2} \, .$$

So because of the Stefan condition

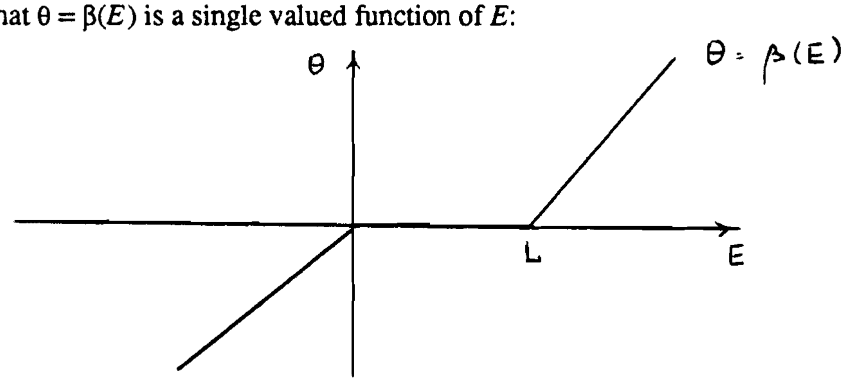
$$L \, \nu_0 = k_l \nabla_x \theta \Big|_l \cdot \vec{\mu} + k_s \nabla_x \theta \Big|_s \cdot \vec{\mu} \, .$$

Thus we arrive at the following weak formulation of the Stefan problem.

(II.2.4) $$\int_Q (E \frac{\partial \phi}{\partial t} + k(\theta) \, \theta \, \Delta \phi) \, dx \, dt =$$

$$= \int_{\Omega_0} \phi \, E_0 \, dx + \int_{\Gamma_l} f \frac{\partial \phi}{\partial n_l} \, d\sigma + \int_{\Gamma_s} g \frac{\partial \phi}{\partial n_s} \, d\sigma \, .$$

We note that $\theta = \beta(E)$ is a single valued function of E:



We define a *weak solution* of the Stefan problem as a measurable bounded function E, satisfying the below integral relation for all test function ϕ in V.

(II.2.5) $$\int_Q \left(E \frac{\partial \phi}{\partial t} + k(\beta(E))\,\beta(E)\,\Delta\phi\right) dx\,dt =$$

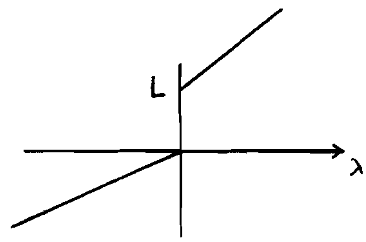
$$= \int_{\Omega_0} \phi\, E_0\, dx + \int_{\Gamma_l} f \frac{\partial \phi}{\partial n_l}\, d\sigma + \int_{\Gamma_s} g \frac{\partial \phi}{\partial n_s}\, d\sigma\,.$$

Remark. We have $\{(x,t) \in Q \mid \theta(x,t) = 0\} = \{(x,t) \in Q \mid 0 < E(x,t) < L\}$.

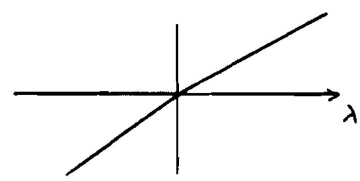
It may happen that this set has nonzero measure. In this case we have a *mushy region*, i.e. an intermediate phase between pure states.

We want to prove the existence of a weak solution.
First we define the function E and K as

$$E(\lambda) = \begin{cases} \rho_s c_s \lambda & , \ \lambda < 0\,, \\ \rho_l c_l \lambda + L & , \ \lambda > 0 \end{cases}$$



$$K(\lambda) = \begin{cases} \lambda k_s & , \ \lambda < 0\,, \\ \lambda k_l & , \ \lambda > 0\,. \end{cases}$$



We introduce two sequences of smooth functions: a sequence E_m with $E_m' > 0$ and $E_m(\lambda) = E(\lambda)$, $|\lambda| \geq \frac{1}{m}$, a sequence K_m with $K_m' > 0$ and $K_m(\lambda) = K(\lambda)$, $|\lambda| \geq \frac{1}{m}$, and $K_m \leq K$.
Define for each $m \in \mathbb{N}$ $\tilde{E}_m$ by

$$\tilde{E}_m(K_m(\lambda)) = E_m(\lambda)\,,$$

i.e. $\tilde{E}_m = E_m \circ K_m^{\leftarrow}$. Then $\tilde{E}_m$ is monotoneously increasing and smooth.

Let $\tilde{\theta}_m$ denote the solution of the non-linear initial value problem

$$\frac{\partial \tilde{E}_m(\tilde{\theta}_m)}{\partial t} - \Delta\,\tilde{\theta}_m = 0 \quad \text{in} \quad Q$$

$$\tilde{\theta}_m \Big|_{\Gamma_l} = f(x,t)\ , \quad \tilde{Q}_m \Big|_{\Gamma_s} = g(x,t)$$

$$\tilde{\theta}_m(x,0) = \tilde{\theta}_{m0}(x) = K_m(\theta_0(x))\,.$$

Remark. In fact we solve the problem

$$\frac{\partial E_m(\theta_m)}{\partial t} - \Delta K_m(\theta_m) = 0$$

$$k_l\,\theta_m \Big|_{\Gamma_l} = f\ , \quad k_s\,\theta_m \Big|_{\Gamma_s} = g$$

$$\theta_m(x,0) = \theta_0(x) \quad \text{with} \quad E_0(x) = E(\theta_0)\,.$$

It can be proved on the basis of a (generalized) maximum principle that

(II.2.6)

- $\tilde{\theta}_m$ is uniformly bounded in Q, i.e.

$$\sup_m \sup_{(x,t)\in Q} \tilde{\theta}_m(x,t) < \infty$$

- $\dfrac{\partial\tilde{\theta}_m}{\partial n_s}, \dfrac{\partial\tilde{\theta}_m}{\partial n_l}$ are uniformly bounded on Γ_s and Γ_l, respectively.

Since there exists $c > 0$ such that $\tilde{E}_m{}' \geq c > 0$ we have the estimation

$$c \int_{Q_{t^*}} \left| \frac{\partial\tilde{\theta}_m}{\partial t} \right|^2 dxdt \leq \int_{Q_{t^*}} \tilde{E}_m{}'(\tilde{\theta}_m) \left| \frac{\partial\tilde{\theta}_m}{\partial t} \right|^2 dxdt =$$

$$= \int_{Q_{t^*}} \frac{\partial\tilde{\theta}_m}{\partial t} \Delta\tilde{\theta}_m \, dx\, dt\,.$$

Now

$$\frac{\partial\tilde{\theta}_m}{\partial t} \Delta\tilde{\theta}_m = \nabla_x \cdot \left[\frac{\partial\tilde{\theta}_m}{\partial t} \nabla_x \tilde{\theta}_m \right] - \frac{1}{2} \frac{\partial}{\partial t} |\nabla_x \theta_m|^2$$

so that the right hand side is equal to

$$\int_{Q_{t^*}} \nabla_x \cdot \left[\frac{\partial\tilde{\theta}_m}{\partial t} \nabla_x \tilde{\theta}_m \right] dx\, dt - \frac{1}{2} \int_0^{t^*} \Big(\frac{d}{dt} \int_{\Omega_t} |\nabla_x \theta_m|^2 \, dx \Big) dt\,.$$

(Observe that $Q_{t^*} = \Omega \times [0,t^*]$ and $\Omega_t = \Omega \times \{t\}$.)

By Gauss

$$\int_{Q_{t^*}} \nabla_x \cdot \left[\frac{\partial\tilde{\theta}_m}{\partial t} \nabla_x \tilde{\theta}_m \right] dx\, dt = \int_{\Gamma_{l,t^*}} \frac{\partial\tilde{\theta}_m}{\partial n_l} \cdot \frac{\partial f}{\partial t} d\sigma + \int_{\Gamma_{s,t^*}} \frac{\partial\tilde{\theta}_m}{\partial n_s} \frac{\partial g}{\partial t} d\sigma\,.$$

So consequently

$$c \int_{Q_{t^*}} \left| \frac{\partial\tilde{\theta}_m}{\partial t} \right|^2 dx\, dt + \frac{1}{2} \int_{\Omega(t^*)} |\nabla_x \tilde{\theta}_m|^2 \, dx \leq$$

$$\leq \int_{\Gamma_l} \frac{\partial\tilde{\theta}_m}{\partial n_l} \frac{\partial f}{\partial t} d\sigma + \int_{\Gamma_s} \frac{\partial\tilde{\theta}_m}{\partial t} d\sigma + \int_{\Omega_0} |\nabla_x \tilde{\theta}_m|^2 \, dx\,.$$

Since the above inequality is valid for all $t^* \in (0,T]$ it follows that

$$\int_Q \left(\left| \frac{\partial \bar{\theta}_m}{\partial t} \right|^2 + |\nabla \bar{\theta}_m|^2 \right) dx\, dt$$

$$\leq \frac{T}{\min(cT, \frac{1}{2})} \left\{ \int_{\Gamma_l} \frac{\partial \bar{\theta}_m}{\partial n_l} \frac{\partial f}{\partial t}\, d\sigma + \int_{\Gamma_s} \frac{\partial \bar{\theta}_m}{\partial n_s} \frac{\partial g}{\partial t}\, d\sigma + \int_{\Omega_0} |\nabla \bar{\theta}_m|^2\, dx \right\} .$$

Now

$$\int_{\Omega_0} |\nabla \bar{\theta}_m|^2\, dx = \int_\Omega |K_m'(\theta_0)|^2\, |\nabla \theta_0|^2\, dx \leq$$

$$\leq \max(k_s^2, k_l^2) \int_\Omega |\nabla \theta_0|^2\, dx .$$

So applying (II.2.6) there exists a constant C_1 independent of m such that

(II.2.7.i) $\quad \forall_m : \int_Q \left(\left| \frac{\partial \bar{\theta}_m}{\partial t} \right|^2 + |\nabla \bar{\theta}_m|^2 \right) dx\, dt \leq C_1 .$

Further since the sequence $(\bar{\theta}_m)$ is uniformly bounded in Q there exist constants $C_2 > 0$ and $C_3 > 0$ such that

(II.2.7.ii) $\quad \forall_m : \int_Q |\bar{\theta}_m|^2\, dx\, dt < C_2$

(II.2.7.iii) $\quad \forall_m : \int_Q |\bar{E}_m(\bar{\theta}_m)|^2\, dx\, dt < C_3 .$

Now use the following well-known theorem.

> Let (ψ_m) be a bounded sequence in a Hilbert space H with inner product $(\cdot\,,\cdot)_H$. Then there exists a subsequence (ψ_{m_k}) and $\psi \in H$ such that for all $\phi \in H$
>
> $$\lim_{k \to \infty} (\psi_{m_k}, \phi)_H = (\psi, \phi)_H \quad \text{(weak convergence !)} .$$

We conclude from (II.2.7.i) and (II.2.7.ii) that there exists a subsequence $(\bar{\theta}_{m_k})$ which tends weakly to some $\bar{\theta}$ in the Hilbert space $W^{2,1}(Q)$ and for which also by (II.2.7.iii) the sequence $(\bar{E}_{m_k}(\bar{\theta}_{m_k}))$ tends to some $\hat{E}$ weakly in the Hilbert space $L_2(Q)$. Thus we derive the following

$$\forall_{\phi \in V} : \lim_{k \to \infty} \int_a \left(\bar{E}_{m_k}(\bar{\theta}_{m_k}) \frac{\partial \phi}{\partial t} + \bar{\theta}_{m_k} \Delta \phi \right) dx\, dt =$$

$$= \int_Q \left(\hat{E} \frac{\partial \phi}{\partial t} + \bar{\theta} \Delta \phi \right) dx\, dt .$$

On the other hand it can be proved (cf. page 33) that

$$\int_Q (\bar{E}_m(\bar{\theta}_{m_k}) \frac{\partial\phi}{\partial t} + \bar{\theta}_{m_k} \Delta\phi)\, dx\, dt =$$

$$= \int_{\Omega_0} \phi\, \bar{E}_{m_k}(\bar{\theta}_{m_k 0})\, dx + \int_{\Gamma_l} f \frac{\partial\phi}{\partial n_l}\, d\sigma + \int_{\Gamma_s} g \frac{\partial\phi}{\partial n_s}\, d\sigma\,.$$

Now $\bar{E}_{m_k}(\bar{\theta}_{m_k 0}) = \bar{E}_{m_k}(K_{m_k}(\theta_0)) = E_{m_k}(\theta_0) \to E_0 (k \to \infty)$.

It follows that $\hat{E}$ is a weak solution if we can prove that $\hat{E} = E(\theta)$ for $\theta = K^{\leftarrow}(\bar{\theta})$.
To show this we observe that the canonical injection from $W^{2,1}(Q)$ into $L_2(Q)$ is compact. So the sequence $(\bar{\theta}_{m_k})$ tends to $\bar{\theta}$ in the norm of $L_2(Q)$. Consequently, taking a subsequence if necessary, $\bar{\theta}_{m_k} \to \bar{\theta}$ almost uniform in Q. Hence

$$\bar{E}_{m_k}(\bar{\theta}_{m_k}) \to \bar{E}(\bar{\theta}) = E(\theta)\,.$$

Also uniqueness of the weak solution can be proved. In short, we sum some arguments for this.

Let E_1 and E_2 be weak solutions and suppose that the set $\Psi = \{(x,t) \in Q \mid E_1(x,t) \neq E_2(x,t)\}$ has non-zero measure. It follows that

$$\forall_{\phi \in V} : \int_\Psi [(E_1 - E_2) \frac{\partial\phi}{\partial t} + (K(\beta(E_1)) - K(\beta(E_2))\Delta\phi]\, dx\, dt = 0\,.$$

Put

$$\sigma = \frac{K(\beta(E_1)) - K(\beta(E_2))}{E_1 - E_2} \quad , \quad (x,t) \in \Psi\,.$$

Then $0 \leq \sigma \leq \sigma_0$ for some constant σ_0.
We extend σ to the whole of Q as a bounded measurable function, whence

$$\forall_{\phi \in V} : \int_Q (E_1 - E_2) (\frac{\partial\phi}{\partial t} + \sigma\Delta\phi)\, dx\, dt = 0\,.$$

Next consider a sequence of smooth functions in Q such that $\sigma_n \geq \frac{1}{n}$ and $\sigma_n \to \sigma$. Then it can be proved that for each $\psi \in C^\infty(Q)$ there exists $\phi_n \in V$ such that

$$\frac{\partial\phi_n}{\partial t} + \sigma_n \Delta\phi_n = \psi\,.$$

It follows that

$$\int_Q [(E_1 - E_2)\psi + (E_1 - E_2)(\sigma - \sigma_n)\Delta\phi_n]\, dx\, dt = 0\,.$$

Taking the limit $n \to \infty$,

$$\int_Q (E_1 - E_2)\, \psi \, dx\, dt = 0 .$$

Being valid for all $\psi \in C_c^\infty(Q)$ we must have $E_1 = E_2$.

CHAPTER IV

REACTION-DIFFUSION PROBLEMS

A substance diffuses through a medium and at the same time undergoes a chemical reaction which may involve heat absorption or heat release. The concentration of the diffusing substance is denoted by c and the absolute temperature by T. Then c and T are usually described by the following system of (nonlinear) partial differential equations

$$\text{(IV.1)} \qquad \frac{\partial c}{\partial t} - d\,\Delta c = -A\,c^m \exp(-E/RT)$$

in the set $\{c > 0\}$ and

$$\text{(IV.2)} \qquad C\,\frac{\partial T}{\partial t} - k\,\Delta T = Q\,A\,c^m \exp(-E/RT)$$

throughout the medium.

The right hand side of equation (IV.1) describes the rate at which the chemical is used in the reaction. Here A is a constant, the so called pre-exponential factor, $m \geq 0$ is the order of the reaction, $E \geq 0$ the activation energy and R the universal gas constant. The right hand side of the second equation is the rate at which heat is released ($Q > 0$) or absorbed ($Q < 0$). If $Q = 0$ the reaction is isothermal, i.e. the temperature remains constant.

Let Ω be the domain occupied by the medium. A fundamental problem is whether c can vanish identically over a subset D of Ω, D is called a dead core. In this case, besides c and T one should determine the evolution of the free boundary ∂D.

We investigate the case of a one-dimensional stationary isothermal reaction-diffusion in a slab $0 \leq x \leq a$ with prescribed boundary conditions

$$c(0,t) = c(a,t) = c_0 > 0\,.$$

Using the non-dimensional variable $u = c/c_0$, the differential equation reduces to

$$\text{(IV.3)} \qquad \frac{\partial^2 u}{\partial x^2} = \lambda\,u_+^m \quad , \quad (\lambda > 0)$$

where $u(0) = u(a) = 1$.

As a preliminary we consider the problem in the half space $x > 0$ imposing the conditions

$$u(0) = 1 \quad \text{and} \quad \lim_{x\to\infty} u(x) = 0\,.$$

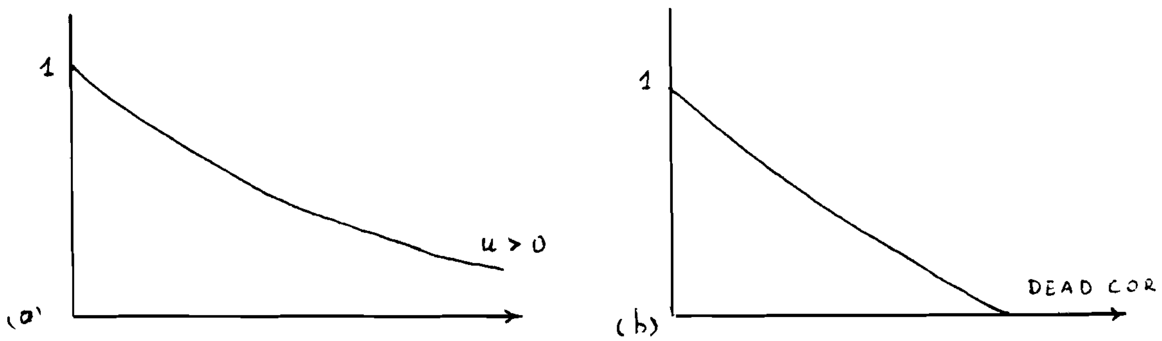
From (IV.3) we see that $\dfrac{d^2u}{dx^2}$ is nonnegative whence $\dfrac{du}{dx}$ is nondecreasing.

Since $\lim\limits_{x\to\infty} u(x) = 0$, it follows that also $\lim\limits_{x\to\infty} u'(x) = 0$.

Multiplying by $\dfrac{du}{dx}$ and integrating over $(0,x)$ yields as long as $u(x) > 0$

(IV.4) $\frac{1}{2}((u'(x))^2 - (u'(0))^2) = \lambda((u(x))^{m+1} - 1)/(m+1)$.

We see that there are two possibilities



From (IV.4) we deduce by taking $x \to \infty$ (extending the validity of (IV.4) to the dead core in case (b))

$$u'(0) = -[2\lambda/m+1]^{1/2}$$

so that (IV.4) takes the form

$$u'(x) = -(2\lambda/m+1)^{1/2}(u_+(x))^{m+1/2}.$$

As long as $u > 0$ separation, of variables is permitted and yields

$$\lambda^{1/2} x = \left[\frac{m+1}{2}\right]^{1/2} \int_u^1 z^{-(m+1)/2}\, dz$$

$$= \left[\frac{m+1}{2}\right]^{1/2} \left[\begin{array}{ll} \log u & \text{as } m=1 \\ \frac{2}{m-1}(u^{-(m-1)/2}-1) & \text{as } m \neq 1. \end{array}\right.$$

Taking the limit $u \downarrow 0$ we see that $x \to \infty$ as $m \geq 1$ and $x \to \lambda^{-1/2}(2(m+1))^{1/2}/(1-m)$ as $m < 1$. It follows that for $m \geq 1$ the solution has to be positive everywhere, while for $m < 1$ a dead core appears at

(IV.5) $x_d = \lambda^{-1/2}(2(m+1))^{1/2}/(1-m)$.

Coming back to our original problem for a slab $0 \leq x \leq a$ we can now say that a dead core can be expected if and only if $a > 2x_d$.

In case $m = 0$ we deal with the so called *oxygen-diffusion consumption problem*. The term c_+^m has to be replaced by the Heaviside function $H(c)$ taking the value 1 if $c > 0$ and the value 0 otherwise.

The non-stationary problem is classically described as follows

(IV.6)

(i) $\qquad \dfrac{\partial c}{\partial t} - \dfrac{\partial^2 c}{\partial x^2} = -1 \quad , \quad 0 < x < s(t) \quad , \quad 0 < t < T$

(ii) $\qquad c(x,0) = u_0(x) \geq 0 \quad , \quad 0 < x < 1 \, ,$

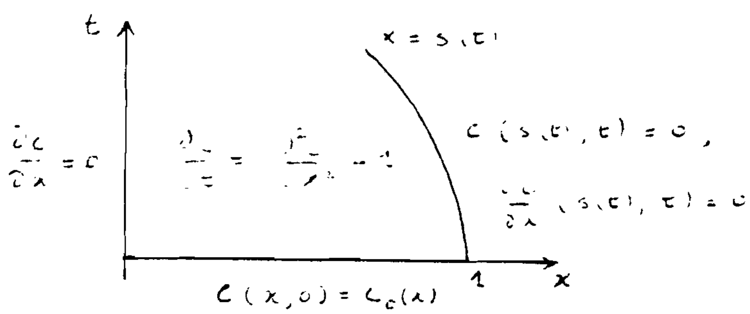
(iii) $\qquad \dfrac{\partial c}{\partial x}(0,t) = 0 \quad , \quad 0 < t < T \, ,$

(iv) $\qquad s(0) = 1 \, ,$

(v) $\qquad c(s(t),t) = 0 \quad , \quad 0 < t < T \, ,$

(vi) $\qquad \dfrac{\partial c}{\partial x}(s(t),t) = 0 \quad , \quad 0 < t < T \, .$

For the corresponding stationary problem we know that a dead core appears at $x_d = (2/\lambda)^{1/2} = \sqrt{2} \quad (\lambda = 1)$.



The weak formulation consists in writing the equation $\dfrac{\partial c}{\partial t} - \dfrac{\partial^2 c}{\partial x^2} = -H(c)$ in the whole domain, and it is equivalent to a variational inequality.

Put $u = \dfrac{\partial c}{\partial t}$. Then $\dfrac{\partial^2 c}{\partial x^2} = u + 1$ for $0 < x < s(t)$ and $0 < t < T$. It can be checked that u satisfies the following equations

(IV.7)

(i) $\qquad \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 \quad , \quad 0 < x < s(t) \, , \; 0 < t < T \, ,$

(ii) $\qquad u(x,0) = u_0(x) = c_0''(x) - 1 \, ,$

(iii) $\qquad \dfrac{\partial u}{\partial x}(0,t) = 0 \, ,$

(iv) $\qquad s(0) = 1$

(v) $\quad u(s(t), t) = 0$

(vi) $\quad \dfrac{\partial u}{\partial x}(s(t), t) = -\dot{s}(t)$.

Explanation.

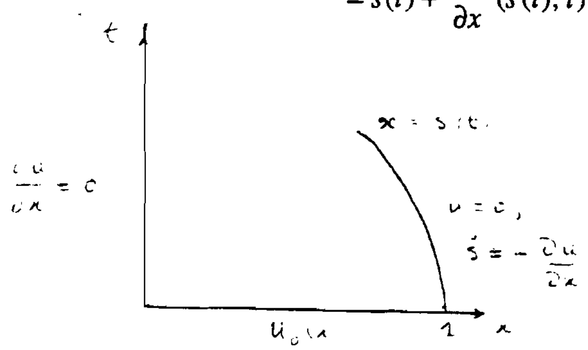
$$* \quad 0 = \frac{d}{dt}\left(c(s(t), t)\right) = \dot{s}(t)\, \frac{\partial c}{\partial x}(x,t) \Bigg|_{x=s(t)} + \frac{\partial c}{\partial t}(s(t), t)$$

$$= u(s(t), t)$$

$$* \quad 0 = \frac{d}{dt}\left(\frac{\partial c}{\partial x}(s(t), t)\right) = \dot{s}(t)\, \frac{\partial^2 c}{\partial x^2}(x,t) \Bigg|_{x=s(t)} + \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial t}(s(t), t)\right)$$

$$= \dot{s}(t)\,(u(x,t)+1) \Bigg|_{x=s(t)} + \frac{\partial u}{\partial x}(s(t), t) =$$

$$= \dot{s}(t) + \frac{\partial u}{\partial x}(s(t), t) .$$



If $c_0'' - 1 < 0$, then $u_0 < 0$ and it follows that $u < 0$. In this case the equations for u corresponds to the Stefan model for the solidification of a supercooled fluid.

We have

$$c(x,t) = \int_x^{s(t)} d\xi \int_\xi^{s(t)} \frac{\partial^2 c}{\partial x^2}(\eta,t)\, d\eta =$$

$$= \int_x^{s(t)} d\xi \int_\xi^{s(t)} [u(\eta,t) + 1]\, d\eta ,$$

whence

$$c_0(x) = \int_x^1 d\xi \int_\xi^1 [u_0(\eta) + 1]\, d\eta$$

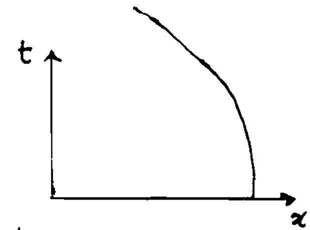
and

$$c_0'(0) = -\int_0^1 [u_0(\eta) + 1]\, d\eta = -Q .$$

We distinguish three cases

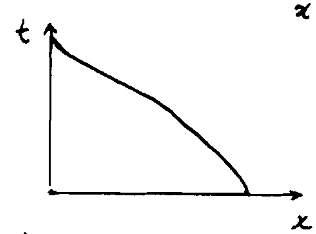
A.

There exists a global solution for $0 < x < s(t),\ t > 0$.



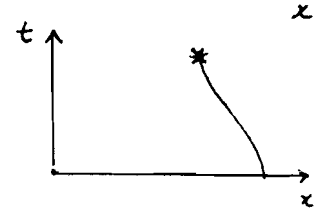
B.

There is extinction in a finite time: $\exists_T : s(T) = 0$.



C.

There is a blow up $\exists_T : \dot{s}(t) \to -\infty,\ t \to T$



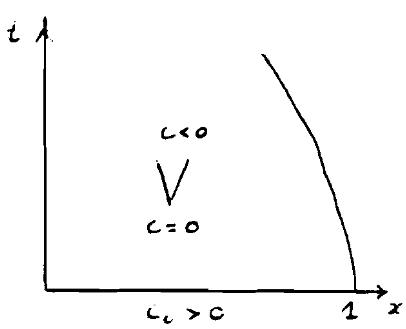
It can be proved that

$$A \Rightarrow Q > 0$$

$$B \Rightarrow Q = 0$$

$$C \Leftarrow Q < 0$$

Suppose a negativity set is formed. Of course then we are dealing with a wrong model because c has to be nonzero. But a mathematical meaning is there.



The following statements are valid.

(1) The negativity set expands.

(2) The negativity set is bound to meet the free boundary in finite time.

(3) The meeting point is a point of essential blow up.

(4) There are no other cases of essential blow up.

Physically, essential blow ups do not occur.

When c becomes zero in a point, one has to introduce a new dead core and solve a new free boundary problem.

CHAPTER V

BINGHAM FLUIDS

Consider a Newtonian fluid flow in some region of $\mathbb{R}^2$ with velocity field $\vec{v} = \vec{v}(x,t)$. Let $p = p(x,t)$ denote the pressure of the fluid. Then p and $\vec{v}$ satisfy the Navier-Stokes equations

$$\text{(V.1)} \qquad \rho \frac{\partial \vec{v}}{\partial t} + \tfrac{1}{2} (\nabla \vec{v})^2 + (\text{curl}\, \vec{v}) \times \vec{v} = -\nabla p + \eta \Delta \vec{v}$$

where ρ denotes the homogeneous mass density and η the viscosity of the fluid. If the fluid is incompressible we have div $\vec{v} = 0$.

Consider a flow given by $\vec{v} = (v(x,y,t), 0)$ and assume that the fluid is incompressible. Then the Navier-Stokes equations reduce to

$$\text{(V.2)} \qquad \rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0 .$$

It follows that $v(x,y,t) = v(y,t)$ and $p(x,y,t) = p_1(t)x + p_0(t)$.

For a Newtonian fluid the stress tensor is related to the gradient of the velocity field in the following way

$$\text{(V.3)} \qquad \mathbf{T} = \eta\, (\nabla \vec{v} + (\nabla \vec{v})^T) .$$

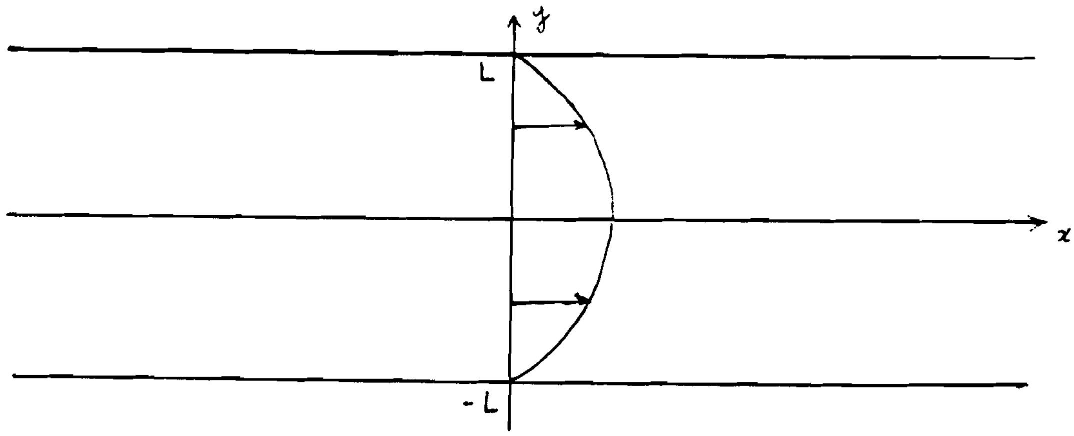
So in our particular situation

$$\tau_{xx} = 2\eta \frac{\partial v}{\partial x} = 0 \quad , \quad \tau_{yy} = 0$$

$$\tau_{xy} = \eta \frac{\partial v}{\partial y} \equiv \eta \sigma .$$

Here $\tau \equiv \tau_{xy}$ is the shear stress and $\sigma \equiv \dfrac{\partial v}{\partial y}$ is the strain rate.

Example.
Consider the stationary flow of an incompressible Newtonian fluid between two (infinitely long) plates under the assumption that there are no volume forces.



For the velocity field we take $\vec{v}(x,y) = (v(x,y), 0)$, i.e. a Poiseuille flow. Adding no-slip conditions at the boundary, i.e. $v(x,L) = v(x,-L) = 0$ we obtain the solution

$$p(x,y) = p_0 x + p_1$$

$$v(x,y) = \frac{p_0}{2\eta}(L^2 - y^2)\,.$$

Observe that in this case $\tau = \tau(y) = -p_0 y$.

A Bingham fluid is a non-Newtonian fluid characterized by the presence of a threshold value τ_0 for the shear stress, such that if the shear stress τ is less than τ_0 the fluid behaves like a rigid body, while for $\tau > \tau_0$ it behaves as a fluid where the relationship between shear stress and strain rate is linear, i.e.

(V.4) $$\tau = \tau_0 + \eta\sigma\,.$$

The dynamics of a Bingham fluid, described by a velocity field $\vec{v}$ obeys the Navier-Stokes equations in the region $\{\tau > \tau_0\}$, while on the boundary with the rigid core (the free boundary) we have $\tau = \tau_0$, i.e. zero strain rate. Another free boundary condition results from the balance of momentum.

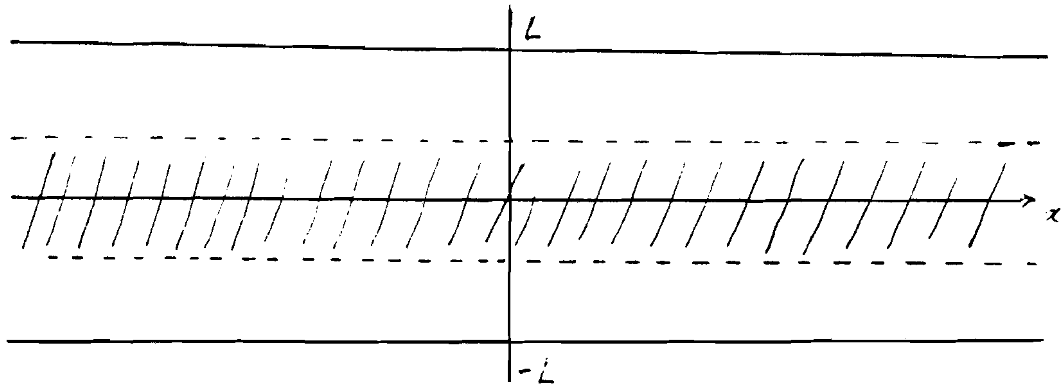
Here we consider an incompressible Bingham fluid flowing between two parallel plates. Again x denotes the coordinate along the direction of motion and y the coordinate in the direction perpendicular to the plates. So in the representative xy-plane the velocity has the form $\vec{v} = (v(y,t), 0)$ and the equation of motion in the viscous region equals

(V.5) $$\rho\,\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} + \eta\,\frac{\partial^2 v}{\partial y^2} \;,\quad \frac{\partial p}{\partial y} = 0\,.$$

Consequently,

(V.6) $$\frac{\partial p}{\partial x} = -f(t) \;,\quad f(t) > 0$$

which we assume to be given.



Let $y = \pm s(t)$ be the equation for the free boundary of the rigid core. Then the velocity field satisfies the parabolic equation

$$\rho \frac{\partial v}{\partial t} - \eta \frac{\partial^2 v}{\partial y^2} = f(t)$$

in the regions $-L < y < -s(t)$ and $s(t) < y < L$. By symmetry we only need to consider the upper half layer. We impose the no-slip condition

$$v(L,t) = 0 \quad , \quad t > 0$$

and some initial condition

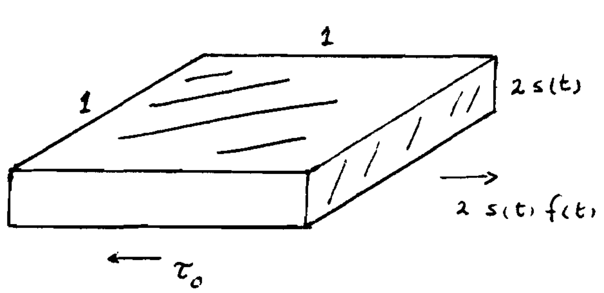
$$v(y, 0) = v_0(y)$$

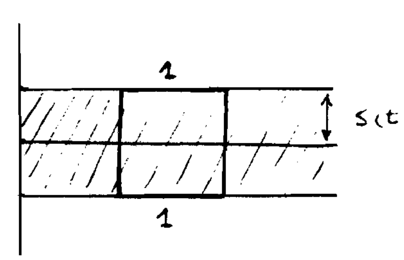
such that $v_0(y)$ is constant for $0 < y < s(0)$, $v_0(L) = 0$.

Since $\sigma = 0$ at the free boundary

$$\text{(V.7)} \qquad \frac{\partial v}{\partial y} \Big|_{x = s(t)} = 0 .$$

For the second free boundary condition we apply Newton's law. Consider a portion of the rigid core situated between two unit squares parallel to the plates





The driving force equals $2\, s(t)\, f(t) - 2\,\tau_0$. The mass of the portion equals $2\, s(t)\, \rho$. Hence

(V.8) $$2\,s(t)\,\rho\,\frac{\partial v}{\partial t}\Big|_{x=s(t)} = 2\,s(t)\,f(t) - 2\tau_0$$

or equivalently

(V.8') $$\frac{\partial v}{\partial t}\Big|_{x=s(t)} = \frac{1}{\rho}\,f(t) - \frac{\tau_0}{\rho\,s(t)}\,.$$

The free boundary conditions are neither of Cauchy nor of Stefan type.

As a special case we consider stationary solutions for $f(t) = f_0$ with $f_0 > \tau_0 / L$. Then $\frac{\partial v}{\partial t} = 0$ and so by (V.8), $s(t) = \tau_0 / f_0$. The equations reduces to

(V.9) $$\frac{d^2 v}{dy^2} = \frac{1}{\eta} f_0 \quad , \quad \frac{\tau_0}{f_0} < y < L$$

$$v(L) = 0$$

$$\frac{dv}{dy}\left[\frac{\tau_0}{f_0}\right] = 0\,.$$

Hence

$$v(y) = -\frac{f_0}{\eta}\,(L-y)\,(L+y-\frac{2\tau_0}{f_0}) \quad , \quad \frac{\tau_0}{y_0} < y < L\,.$$

For the non-stationary problem we take $w = \frac{\partial v}{\partial y}$. Then for w we get

(V.10)

(i) $$\rho\,\frac{\partial w}{\partial t} - \eta\,\frac{\partial^2 w}{\partial y^2} = 0\,,$$

(ii) $$\frac{\partial w}{\partial y}\,(L,t) = -\frac{1}{\eta}\,f(t)\,,$$

(iii) $$w(y,0) = v_0{}'(y)\,,$$

(iv) $$w(s(t),t) = 0\,,$$

(v) $$\eta\,\frac{\partial w}{\partial y}\,(s(t),t) = -\tau_0 / s(t)\,.$$

(This is a free boundary problem with Cauchy data on the free boundary.)

Explanation.

We have $\rho\,\frac{\partial v}{\partial t} - \eta\,\frac{\partial^2 v}{\partial y^2} = f(t)$. So at $y = s(t)$

$$f(t) - \frac{\tau_0}{s(t)} - \eta \frac{\partial w}{\partial y} \Big|_{y=s(t)} = f(t)$$

so that

$$\eta \frac{\partial w}{\partial y} \Big|_{y=s(t)} = - \frac{\tau_0}{s(t)} \,.$$

Another transformation is the following

$$z = \frac{\partial v}{\partial t} = \frac{\eta}{\rho} \frac{\partial w}{\partial y} + \frac{1}{\rho} f$$

yielding the Stefan type problem

(V.11)

(i) $$\rho \frac{\partial z}{\partial t} - \eta \frac{\partial^2 z}{\partial y^2} = \dot{f} \quad , \quad s(t) < y < L \, , \; t > 0 \, ,$$

(ii) $$z(L,t) = 0 \quad , \quad t > 0 \, ,$$

(iii) $$z(y,0) = \frac{\eta}{\rho} v_0'' + \frac{1}{\rho} f(0) \quad , \quad s(0) < y < L$$

(iv) $$z(s(t),t) = (f(t) - \frac{\tau_0}{s(t)}) / \rho \quad , \quad t > 0$$

(v) $$\frac{\partial z}{\partial y} (s(t),t) = \frac{1}{\eta} \frac{\tau_0}{s(t)} \dot{s}(t) \quad , \quad t > 0 \, .$$

Eindhoven, 6-2-1990

Appendix A

Some concepts and theorems of vector analysis in $\mathbb{R}^2$ and $\mathbb{R}^3$

A.1. Curve.

$$\underline{x} = \underline{x}(t) = (x_1(t)\,,\, x_2(t)\,,\, x_3(t))\,.$$

The curve is called *smooth* if $x_1(t)$, $x_2(t)$ and $x_3(t)$ are differentiable. A smooth curve has a *tangent*

$$\underline{\dot{x}}(t) = (\dot{x}_1(t)\,,\, \dot{x}_2(t)\,,\, \dot{x}_3(t))\,.$$

A.2. Surface.

$$\underline{x} = \underline{x} = (x_1(u,v)\,,\, x_2(u,v)\,,\, x_3(u,v))\,,\ \underline{t} = \frac{\dot{x}(t)}{|\dot{x}(t)|}\,.$$

The surface is said to be smooth if $x_1(u,v)$, $x_2(u,v)$ and $x_3(u,v)$ are differentiable. The tangent surface at $\underline{x} = \underline{a}$ is given by

$$\underline{y} = \underline{a} + \lambda \frac{\partial x}{\partial u}(\underline{a}) + \mu \frac{\partial x}{\partial v}(\underline{a})$$

or correspondingly

$$(\underline{y}\,,\, \frac{\partial x}{\partial u}(a) \times \frac{\partial x}{\partial v}(a)) = (\underline{a}\,,\, \frac{\partial x}{\partial u}(\underline{a}) \times \frac{\partial x}{\partial v}(\underline{a}))\,.$$

So the *normal* of the surface at $\underline{x} = \underline{a}$ equals

$$\underline{n}(\underline{a}) = \pm \frac{\dfrac{\partial x}{\partial u}(\underline{a}) \times \dfrac{\partial x}{\partial v}(\underline{a})}{\left|\dfrac{\partial x}{\partial u}(\underline{a}) \times \dfrac{\partial x}{\partial v}(\underline{a})\right|}\,.$$

Special case $z = z(x,y)$, $\underline{x} = (x, y, z(x,y))$.
Take $u = x$ and $v = y$. Then

$$\frac{\partial x}{\partial u} = (1, 0, \frac{\partial z}{\partial x}),\ \frac{\partial x}{\partial v} = (0, 1, \frac{\partial z}{\partial y})\,.$$

So

$$\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} = \left[-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right]\,.$$

A.3. **Scalar field.**

Let $\Omega \subset \mathbb{R}^3$. A function ϕ from Ω in $\mathbb{R}$ is said to be a scalar field. Notation: $\phi(\underline{x})$ or $(\underline{x}, \phi(\underline{x}))$. Fysical examples: temperature, mass density, pressure. Surfaces of the form $\phi(\underline{x}) = c$ are called *equiscalar* (or equipotential) surfaces.

A scalar field is said to be *differentiable* at $\underline{x} \in \Omega$ if there exists a linear functional $L(\underline{x}) : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\phi(\underline{x}+\underline{h}) - \phi(\underline{x}) = L(\underline{x})\,\underline{h} + o(|\underline{h}|)\,.$$

If $(\cdot\,,\cdot)$ denotes the Euclidean inner product we can write

$$L(\underline{x})\,\underline{h} = \nabla\phi(\underline{x})\cdot\underline{h}$$

for some vector $\nabla\phi(x) \in \mathbb{R}^3$. The mapping

$$\nabla\phi : \Omega \to \mathbb{R}^3$$

is called the *gradient* of ϕ. We have in cartesian coordinates

$$(\nabla\phi)\,(\underline{a}) = \left[\frac{\partial\phi}{\partial x_1}(\underline{a}),\ \frac{\partial\phi}{\partial x_2}(\underline{a}),\ \frac{\partial\phi}{\partial x_3}(\underline{a})\right],\ \underline{a} \in \Omega\,.$$

Let $\underline{v} \in \mathbb{R}^3$ with $|\underline{v}| = 1$. Then the directional derivative

$$\lim_{t\to 0}\frac{\phi(x+t\,v) - \phi(x)}{t}$$

equals

$$\nabla\phi(\underline{x})\cdot\underline{v}\,.$$

It follows that the directional derivative is maximal or minimal if $\underline{v} = \lambda\,\nabla\phi(\underline{x})$. So $\nabla\phi(\underline{x})$ points in the direction of maximal increase or decrease.

Consider the equiscalar surface $\phi(\underline{x}) = \phi(\underline{a})$. Taking $z = z(x,y)$ we obtain

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x} = 0\ ,\quad \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y} = 0\,.$$

So the tangent plane at $\underline{x} = \underline{a}$ is spanned by

$$\left[-\frac{\partial\phi}{\partial z}(\underline{a}), 0, \frac{\partial\phi}{\partial x}(\underline{a})\right] \text{ and } \left[0, -\frac{\partial\phi}{\partial z}(\underline{a}), \frac{\partial\phi}{\partial y}(\underline{a})\right]$$

with normal $\nabla\phi(\underline{a})\,/\,|\nabla\phi(\underline{a})|$.

So the equation for the tangent plane is given by

$$\nabla\phi(\underline{a})\cdot(\underline{y} - \underline{a}) = 0\,.$$

A.4. **Vector field.**

Let $\Omega \subset \mathbb{R}^3$. A vector field is a function from Ω into $\mathbb{R}^3$. Notation $\underline{v}(\underline{x})$ or $(\underline{x}, \underline{v}(\underline{x}))$. The second notation suggests that at each point of Ω a vector is attached. Fysical examples: electro-magnetic field, velocity field of a fluid. In particular for each scalar field ϕ, the gradient $\nabla\phi$ is a vector field. A curve $\underline{x}(t)$ with the property that

$$\dot{\underline{x}}(t) = \lambda\, \underline{v}(\underline{x}(t))$$

is called a *stream line* or *field line*.

A vector field $\underline{v}$ is said to be differentiable at $\underline{x} \in \Omega$ if there exists a linear mapping $A(\underline{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\underline{v}(\underline{x}+\underline{h}) - \underline{v}(\underline{x}) = A(\underline{x})\underline{h} + o(|\underline{h}|)\,.$$

A.5. **Operations on vector fields.**

Given a vector field $\underline{v} = (v_1, v_2, v_3)$ which is differentiable in a region $\Omega \subset \mathbb{R}^3$.

<u>curl</u> $\underline{v} \mapsto$ curl $\underline{v}$

$$\text{curl } \underline{v} = \nabla \times \underline{v} = \left[\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3},\, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1},\, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right]$$

- $\text{curl } (\underline{v} + \underline{w}) = \text{curl } \underline{v} + \text{curl } \underline{w}$
- $\text{curl } (\phi\, \underline{v}) = \phi \text{ curl } \underline{v} + \nabla\phi \times \underline{v}$
- $\text{curl } (\nabla\phi) = 0$

Divergence. $\underline{v} \to \text{div } \underline{v}$

$$\text{div } \underline{v} = \nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

- $\text{div } (\underline{v} + \underline{w}) = \text{div } \underline{v} + \text{div } \underline{w}$
- $\text{div } (\phi\, \underline{v}) = \phi \text{ div } \underline{v} + \nabla\phi \cdot \underline{v}$
- $\text{div } (\underline{v} \times \underline{w}) = \underline{w} \cdot \text{curl } \underline{v} - \underline{v} \cdot \text{curl } \underline{w}$
- $\text{div curl } \underline{v} = 0.$

Further for a twice differentiable scalar field (vector field) $\phi(\underline{v})$ we have

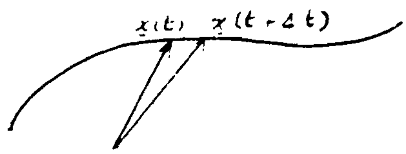
$$\Delta\phi = \text{div}(\nabla\phi) = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$$

$$\Delta\underline{v} = (\Delta v_1, \Delta v_2, \Delta v_3)$$

- $\nabla(\text{div}\,\underline{v}) = \text{curl}\,(\text{curl}\,\underline{v}) + \Delta\,\underline{v}$
- $\nabla(\Delta\phi) = \Delta(\nabla\phi)$
- $\text{curl}\,(\Delta\,\underline{v}) = \Delta\,(\text{curl}\,\underline{v})$
- $\text{div}\,(\Delta\,\underline{v}) = \Delta\,(\text{div}\,\underline{v})$.

A.6. **Line-integrals, surface integrals.**

Given a smooth curve K, $\underline{x} = \underline{x}(t)$, $a \le t \le b$



Then $\Delta s \approx |x(t+\Delta t) - x(t)|$

$\approx |\dot{x}(t)|\ \Delta t$

In the limit $\Delta t \downarrow 0$ we get

$ds = |\dot{x}(t)|\ dt$

length of K : $\int_K ds = \int_a^b |\dot{x}(t)|\ dt$

For a scalar field ϕ we define

$$\int_K \phi\ ds = \int_a^b \phi(\underline{x}(t))\ \underline{\dot{x}}(t)\ dt\ .$$

Let $\underline{t}(\underline{x})$ denote the normalized tangent vector at $\underline{x} \in K$, $\underline{t} = \dfrac{\dot{\underline{x}}}{|\dot{\underline{x}}|}$. Then for a vector field $\underline{v}$ we have by taking $\phi(x) = (\underline{v}(\underline{x}), \underline{t}(\underline{x}))$

$$\int_K (\underline{v}\cdot\underline{t})\,ds = \int_a^b (\underline{v}(\underline{x}(t))\cdot\underline{\dot{x}}(t))\,dt\ .$$

Important special case.

Take $\underline{v} = \nabla\phi$. Then

$$((\nabla\phi)\,\underline{x}(t)\cdot\underline{\dot{x}}(t)) = \frac{d}{dt}\,(\phi(\underline{x}(t))$$

so that

$$\int_K (\nabla\phi\cdot\underline{t})\,ds = \phi(\underline{x}_e) - \phi(\underline{x}_s)$$

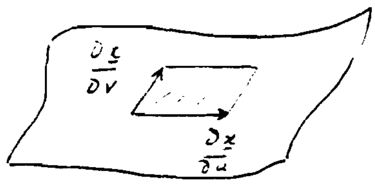
where $\underline{x}_e = \underline{x}(b)$ is the endpoint of K and $\underline{x}_s = \underline{x}(a)$ its startingpoint.
In particular, if K is a closed curve then

$$\oint_K (\nabla\phi\cdot\underline{t})\,ds = 0\ .$$

Gradient fields are said to be *conservative*. For instance if a force field $\mathbf{F} = \nabla\phi$, the energy needed to go from $\underline{p}$ to $\underline{q}$ does not depend on the road being followed.

Let there be given a smooth surface S,

$$\underline{x} = \underline{x}(u,v) \quad , \quad (u,v) \in P \subset \mathbb{R}^2$$



$$d\sigma = \left| \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right| dudv$$

For a scalar field ϕ we define

$$\iint_S \phi \; d\sigma = \iint_P \phi(\underline{x}(u,v)) \left| \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right| dudv \; .$$

If the surface S is given by $z = z(x,y)$, we get

$$\iint_S \phi \; d\sigma = \iint_P \phi(x,y,z(x,y)) \sqrt{1 + \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2} \; dxdy \; .$$

Let $\underline{w}$ be a vector field. Suppose the surface S is orientable, i.e. the normal $\underline{n} : S \rightarrow \mathbb{R}^3$

$$\underline{n} = \pm \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \Big/ \left| \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right|$$

can be taken continuous on S. So the plus or minus sign is fixed by giving the direction of the normal in- one point of S. Let S thus be given an orientation. Then the scalar field $(\underline{w} \cdot \underline{n})$ is well defined on S. We have

$$\iint_S (\underline{w} \cdot \underline{n}) d\sigma = \pm \iint_P (\underline{w} \cdot \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}) \; dudv$$

$\uparrow$ depending on the orientation of S .

A.7. **Gauss' integral relation.**

Let Ω be a bounded domain in $\mathbb{R}^3$ with piecewise smooth, orientable boundary $\partial\Omega$. Let $\underline{n}$ denote the outwardly directed normal on S. Then for a differentiable vector field $\underline{w} : \Omega \rightarrow \mathbb{R}^3$

$$\iiint_\Omega \text{div} \; \underline{w} \; d\underline{x} = \iint_{\partial\Omega} (\underline{w} \cdot \underline{n}) d\sigma \; .$$

Consequences.

- $$\iiint_\Omega \frac{\partial w_j}{\partial x_j}\, dx = \iint_{\partial\Omega} w_j\, n_j\, d\sigma$$

- $$\iiint_\Omega \nabla \phi\, dx = \iint_{\partial\Omega} \phi \underline{n}\, d\sigma$$

- $$\iiint_\Omega \operatorname{rot} \underline{w}\, dx = \iint_{\partial\Omega} (\underline{n} \times \underline{w})\, d\sigma$$

- $$\iiint_\Omega \Delta\phi\, dx = \iiint_{\partial\Omega} (\nabla \phi \cdot \underline{n})\, d\sigma\, .$$

Gauss formula also holds for bounded regions $\Omega \subset \mathbb{R}^2$ and differentiable vector fields $\underline{w} : \Omega \rightarrow \mathbb{R}^2$.
To this end define

$$\tilde{\Omega} = \Omega \times [0,1] \subset \mathbb{R}^3$$

$$\underline{\tilde{w}} = (w_1, w_2, 0)\, .$$

Then $\underline{n} = (\underline{n}, \underline{n}_2, 0)$ and

$$\iint_\Omega \left(\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} \right) d\underline{x} = \iiint_\Omega \operatorname{div} \tilde{w}\, d\underline{x} =$$

$$= \iint_{\partial\Omega} (\tilde{w} \cdot \tilde{n})\, d\sigma = \int_{\partial\Omega} (\underline{w} \cdot \underline{n})\, ds\, .$$

Example.
Consider a fluid with mass density $\rho(\underline{x},t)$ which flows with velocity $\underline{v}(\underline{x},t)$ through a surface S which is the boundary of a bounded region Ω.
The amount of fluid streaming out of Ω during a unit of time Δt can be approximated by

$$(\iint_S (\rho\, \underline{v} \cdot \underline{n})\, d\sigma)\, \Delta t\, .$$

Further, the change of mass of the volume Ω

$$\iiint_\Omega (\rho(\underline{x}, t+\Delta t) - \rho(\underline{x}, t)\, d\underline{x} \approx \left[\iiint_\Omega \frac{\partial \rho}{\partial t}\, d\underline{x} \right] \Delta t\, .$$

Thus we find

$$\iiint_\Omega \frac{\partial \rho}{\partial t}\, d\underline{x} = - \iint_S (\rho\, \underline{v} \cdot \underline{n})\, d\sigma$$

and applying Gauss divergence theorem yields

$$\iiint_{\Omega} \left[\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underline{v}) \, d\underline{x} = 0 \, .\right.$$

Since Ω is an arbitrarily taken region in $\mathbb{R}^3$ we end up with the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underline{v} = 0 \, .$$

In the special case that ρ is constant we deal with an incompressible fluid yielding div $\underline{v} = 0$.

A.8. **Stokes' integral relation.**

Let S be a piecewise smooth, orientable surface in $\mathbb{R}^3$ with a piecewise smooth, simply connected, closed curve K as its boundary. Let $\underline{v}$ be a differentiable vector field on S

$$\iint_S (\operatorname{rot} \underline{v} \cdot \underline{n}) \, d\sigma = \oint_K (\underline{a} \cdot \underline{t}) \, ds$$

where $\underline{t}$ and $\underline{n}$ fit together in a counterclockwise manner. If we take $S \subset \mathbb{R}^2$, we get Green's integral formula

$$\oint_K (v_1 t_1 + v_2 t_2) \, ds = \iint_S \left[\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right] dx_1 \, dx_2 \, .$$

Example.

An electromagnetic field has two components

the electric field $\underline{E}(\underline{x},t)$

the magnetic induction $\underline{B}(\underline{x},t)$.

In a region Ω without currents, $\underline{E}$ and $\underline{B}$ satisfy the following integral relation

$$\oint_K (\underline{E} \cdot \underline{t}) \, ds = -\frac{d}{dt} \iint_S (\underline{B} \cdot \underline{n}) \, d\sigma \quad \text{(Faraday's law)}$$

$$\oint_K (\underline{B} \cdot \underline{t}) \, ds = \varepsilon_0 \mu_0 \frac{d}{dt} \iint_S (\underline{E} \cdot \underline{n}) \, d\sigma \quad \text{(Maxwell's law)}$$

where ε_0 denotes the permittivity and μ_0 the permeability of vacuum. Further any piecewise smooth, orientable surface S in Ω with piecewise smooth closed boundary K may be taken. Applying Stokes we obtain Maxwell's equations

$$\operatorname{rot} \underline{E} = -\frac{\partial B}{\partial t} \quad , \quad \operatorname{rot} \underline{B} = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \quad \text{in } \Omega \, .$$

A.9. **Green's identities.**

Let Ω be a bounded region in $\mathbb{R}^3$ such that $\partial\Omega$ is piecewise smooth and orientable. By $\underline{n}$ the outwardly directed normal on $\partial\Omega$ is denoted.

Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, i.e. u is a scalar field from $\overline{\Omega}$ in $\mathbb{R}$ such that $u : \Omega \to \mathbb{R}$ is twice differentiable and $u : \partial\Omega \to \mathbb{R}$ is differentiable. Applying Gauss divergence theorem to Δu

yields

$$\iiint_{\Omega} \Delta u \, d\underline{x} = \iint_{\partial\Omega} (\nabla u \cdot \underline{n}) \, d\sigma .$$

In stead of $\nabla u \cdot \underline{n}$ one writes $\dfrac{\partial u}{\partial n}$, called the normal derivative of u on $\partial\Omega$.

Green's first identity.

Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $v \in C^1(\Omega)$.

$$\iiint_{\Omega} (v \, \Delta u + (\nabla u \cdot \nabla v)) \, d\underline{x} = \iint_{\partial\Omega} v \, \frac{\partial u}{\partial n} \, d\sigma .$$

(Observe that $\nabla \cdot (v \, \nabla u) = v \, \Delta u + \nabla u \cdot \nabla v$.)

Green's second identity.

Let $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

$$\iiint_{\Omega} (v \, \Delta u - u \, \Delta v) \, dx = \iint_{\partial\Omega} \left[v \, \frac{\partial u}{\partial n} - u \, \frac{\partial v}{\partial n} \right] d\sigma .$$

For a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ which is harmonic, i.e. $\Delta u = 0$ in Ω, it follows that

$$\iint_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma = 0 \quad \text{and} \quad \iiint_{\Omega} |\nabla u|^2 \, d\tau = \iint_{\partial\Omega} u \, \frac{\partial u}{\partial n} \, d\sigma .$$

We apply these results to the boundary problems:

Dirichlet: $\Delta u = -f$ in Ω , $u = g$ on $\partial\Omega$

Neumann: $\Delta u = -f$ in Ω , $\dfrac{\partial u}{\partial n} = h$ on $\partial\Omega$

with respect to the uniqueness of their solutions.

- $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$:

  $$\iiint_{\Omega} |\nabla u|^2 \, dx = 0$$

  whence $\nabla u = 0$, i.e. u is constant in Ω.
  Since $u = 0$ on $\partial\Omega$, we obtain $u \equiv 0$.

- $\Delta u = 0$ in Ω , $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

  We derive similarly that $u \equiv c$, c a constant. So the solution of the Neumann problem is unique up to a constant.

Appendix B

Flow of a Newtonian fluid

Each continuum theory is based on two systems of laws

- universal laws of balance, e.g. balance of mass, balance of momentum.
- constitutive laws or material laws, which are specific for the considered material, e.g. Hooke's law for a linearly elastic medium, Ohm's law for a conductor.

B.1. **Balance of mass.**

Consider a material body B which occupies at each time t a bounded region $\Omega(t)$ of $\mathbb{R}^3$. Each point $P \in B$ is at time t described by a vector $\underline{x} = \underline{x}(P,t)$. Now conservation of mass says

$$\int_{\Omega(t)} \rho(\underline{x}, t)\, d\underline{x} = \int_{\Omega(\tau)} \rho(\underline{x}, \tau)\, d\underline{x}$$

where $\rho(\underline{x},t)$ denotes the mass density of B at time t. So we see that

$$\frac{d}{dt}\Big(\int_{\Omega(t)} \rho(\underline{x},t)\, d\underline{x}\Big) = 0 \, .$$

From this equation one can deduce the so called *continuity equation*

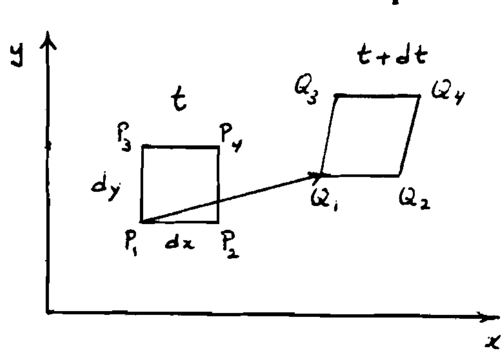
$$\frac{d\rho}{\partial t} + \operatorname{div} \rho\, \underline{v} = 0$$

where $\underline{v} = \dot{\underline{x}}$ denotes the velocity ($\underline{v}(P,t) = \dot{\underline{x}}(P,t)$). In the special case of an incompressible homogeneous fluid, where $\rho(\underline{x},t) = \rho_0$, a constant, we obtain the *incompressibility condition*

$$\operatorname{div} \underline{v} = 0 \, .$$

For an incompressible medium both mass and volume (not the shape of the volume) are conserved.

Heuristic two-dimensional explanation.



$\underline{v} = (v_x, v_y)$

At time t we have $P_1 = (x,y)$, $P_2 = (x+dx,\, y)$, $P_3 = (x, y+dy)$ $P_4 = (x+dx,\, y+dy)$.

At time $t + dt$ we have $Q_1 = (x + v_x(x,y)\,dt\,,\; y + v_y(x,y)\,dt)$

$$Q_2 \approx (x + dx + v_x(x+dx,y)\,dt\,,\; y + v_y(x+dx,y)\,dt)\,,$$

$$Q_3 \approx (x + v_x(x,\, y+dy)\,dt\,,\; y + dy + v_y(xy+dy)\,dt)\,,$$

$$Q_4 \approx (x + dx + v_x(x+dx,\, y+dy)\,dt\,,\; y + dy + v_y(x+dx,\, y+dy)\,dt)\,.$$

Volume at time t : $dx\,dy$
Volume at time $t + dt$: $\approx (Q_2\overrightarrow{Q_1}\;)_x\,(Q_3\overrightarrow{Q_1}\;)_y =$

$$= (dx + (v_x(x+dx,\, y) - v_x(x,y))\,dt) \cdot (dy + (v_y(x,y+dy) - v_y(x,y))\,dt$$

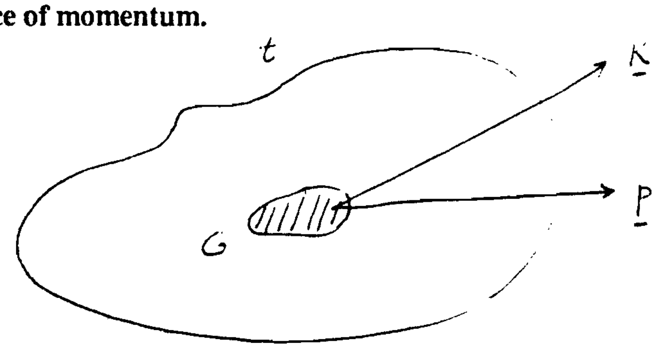
$$\approx dxdy + \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right] dt + O(dt^2)\,.$$

Conservation of volume yields

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0\,.$$

B.2.

Balance of momentum.



At time t, let $\underline{k}$ denote the total force on a material unit volume and let $\underline{p}$ denote the total momentum of the volume. Then we have

$$\underline{k} = \dot{\underline{p}} \quad \text{(Newton's law)}$$

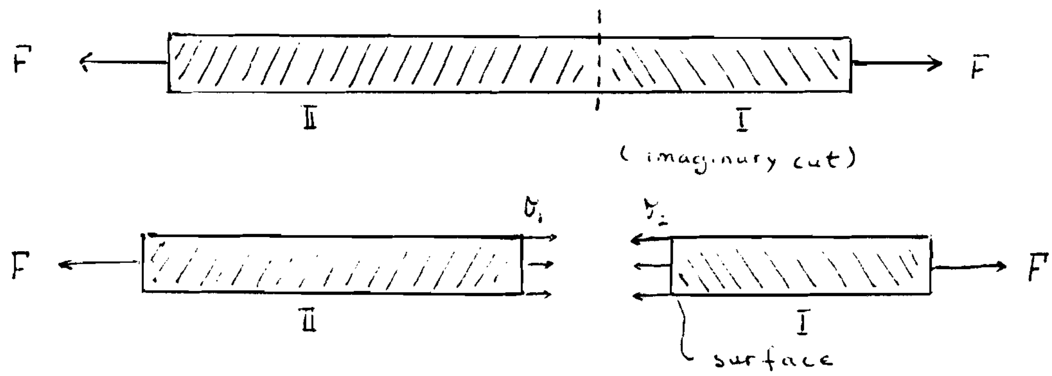
which can be worked out to the following form

$$\operatorname{div} T + \underline{b} = \rho\, \dot{\underline{v}}\,,$$

the equation of motion.

One dimensional explanation.

Consider a bar on which on both end points a force F is imposed



σ_1 : force per unit surface (= stress) which is exerted on part II by part I

σ_2 : force per unit surface which is exerted on part I by part II.

So balance of forces yields

$$\sigma_1 S = P \,, \quad \sigma_2 S = P$$

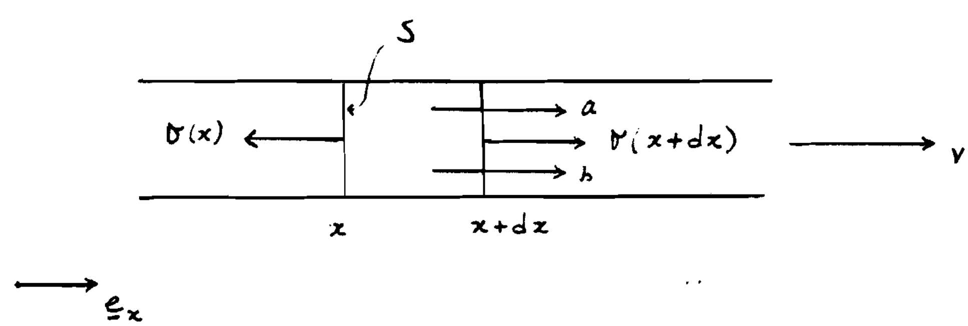
and so

$$\sigma_1 = \sigma_2 = \frac{P}{S}$$

σ is called the normal stress:

(action = reaction!)

Consider a one-dimensional "unit-volume"



σ : stress , b : volume force (e.g. mass) , $a = \dot{v}$.

Total force in the x-direction on a unit element dx

$$K = -\sigma(x)S + \sigma(x+dx)S + b\,S\,dx\ .$$

According to Newton's law ($K = m\,a$) we have

$$(\sigma(x+dx) - \sigma(x))\,S + b\,S\,dx = m\,a = \rho\,S\,dx \cdot a$$

from which we derive the equation of motion

$$\frac{d\sigma}{dx} + b = \rho\,a\ .$$

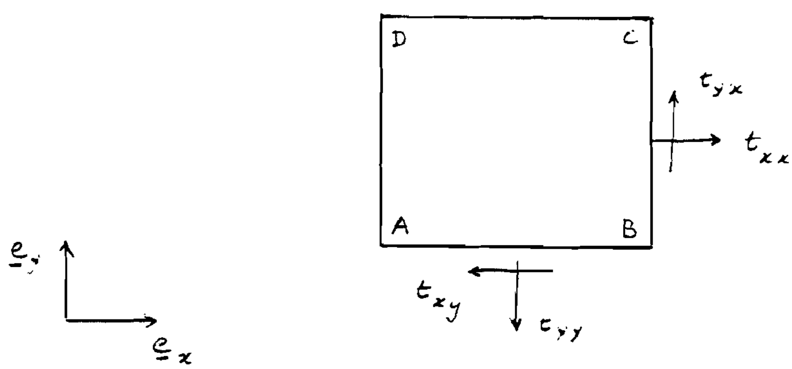
Suppose

- volume force is zero (or negligible) : $b = 0$
- motion is stationary (or quasi-stationary) : $a = 0$

Then we have

$$\sigma'(x) = \frac{d\sigma}{dx} = 0\ .$$

Two dimensional analogue of the previous equation ($\underline{b} = \underline{a} = \underline{0}$)



Stresses are forces per unit-surface, e.g. total force on the BC-surface

$$(t_{xx}\,\underline{e}_x + t_{yx}\,\underline{e}_y)\,dy\,dz\ .$$

From the balance of moments we obtain the symmetry relation

$$t_{xy} = t_{yx}\ .$$

Balancing the forces in the x-direction yields

$$t_{xx}(x+dx,\,y)\,dy\,dz - t_{xx}(x,y)\,dy\,dz +$$

$$+\,t_{xy}(x,\,y+dy)\,dx\,dz - t_{xy}(x,y)\,dx\,dz =$$

$$= \left[\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y}\right] dx\,dy\,dz = 0$$

and similarly in the y-direction

$$\left[\frac{\partial t_{yx}}{\partial x} + \frac{\partial t_{yy}}{\partial y} \right] dx\, dy\, dz = 0 \, .$$

Thus we obtain the balance equations

$$\left[\begin{array}{l} \dfrac{\partial t_{xx}}{\partial x} + \dfrac{\partial t_{xy}}{\partial y} = 0 \\ \dfrac{\partial t_{yx}}{\partial x} + \dfrac{\partial t_{yy}}{\partial y} = 0 \end{array} \right.$$

t_{xx}, t_{yy} are called normal stresses
$t_{xy} = t_{yx}$ are called shear stresses.

Recapitulation.
For a two-dimensional flow of an incompressible fluid with a homogeneous mass density without volume forces or acceleration there are the unknowns

$$v_x \, , \, v_y \, , \, t_{xx} \, , \, t_{xy} \, , \, t_{yy} \text{ and } t_{zz}$$

satisfying the equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} = 0$$

$$\frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0 \, .$$

In order to determine the unknowns we need three more equations.

B.3. **Constitutive (= material) equations.**

These equations characterise the specific medium. They are often determined on experimental/empirical grounds.
The starting point for a *Newtonian fluid* is that the stresses are proportional to the velocity gradients:

$$t_{ij} \sim \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \, .$$

(We have to take the symmetric part of the velocity gradient since $t_{ij} = t_{ji}$.)

A fluid can be taken to be incompressible. A *compressible* medium under hydrostatic pressure ($t_{xx} = t_{yy} = t_{zz} = -p$) gets a smaller volume (p increases). For a compressible medium the hydrostatic pressure is related to the normal stresses as follows

(*) $$p = -\frac{1}{3}\,(t_{xx} + t_{yy} + t_{zz})\,.$$

However for an incompressible we do not have this constitutive relation: hydrostatic pressure is an additional essential unknown.
Therefore the stresses are replaced by the so called *deviatoric* stresses τ_{ij}:

$$t_{xx} = -p + \tau_{xx}$$

$$t_{yy} = -p + \tau_{yy}$$

$$t_{zz} = -p + \tau_{zz} \quad \text{and} \quad t_{xy} = \tau_{xy}\,.$$

Since $\tau_{zz} = -(\tau_{xx} + \tau_{yy})$, we deal with the unknowns

$$v_x\,,\ v_y\,,\ \tau_{xx}\,,\ \tau_{yy}\,,\ \tau_{xy} \quad \text{and} \quad p\,.$$

We need three constitutive equations. For a Newtonian fluid they have the following simple form

$$\tau_{xx} = 2\eta\,\frac{\partial v_x}{\partial x}\,,\quad \tau_{yy} = 2\eta\,\frac{\partial v_y}{\partial y}$$

$$\tau_{xy} = \eta\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right]$$

where η is the viscosity (= material constant).

Recapitulation.

For the unknowns v_x, v_y, τ_{xx}, τ_{xy}, τ_{yy} and p we have found the following equations

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0\,,$$

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\,,$$

$$\frac{\partial \tau_{xy}}{\partial x} - \frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} = 0\,,$$

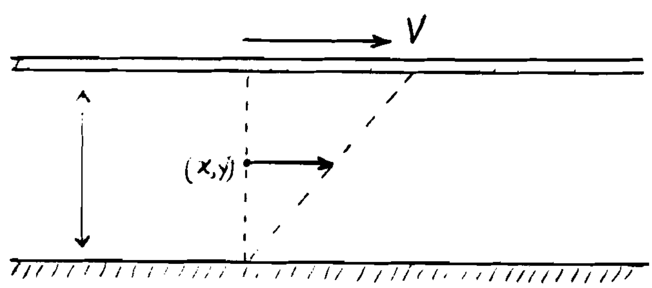
$$\tau_{xx} = 2\eta\,\frac{\partial v_x}{\partial x}\,,$$

$$\tau_{yy} = 2\eta\,\frac{\partial v_y}{\partial y}\,,$$

$$\tau_{xy} = \eta\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right].$$

B.4.

Flow between two plates.



Given two plates at $y = 0$ and $y = d$, the plate at $y = 0$ remains at rest while the other plate at $y = d$ move in the x-direction with a constant velocity V. Between these plates there is a flow of a Newtonian fluid with viscosity η and with a homogeneous mass density ρ_0.

Assumptions.

- the flow is stationary
- the flow is laminar
- the volume forces are negligible
- no z-dependence
- the pressure gradient has no x-component

Then the first two assumptions yield $\underline{v}(x,y) = v(x,y)\,\underline{e}_x$, i.e. $v_x = v(x,y)$ and $v_y = 0$. So from the incompressibility condition we derive

$$\frac{\partial v_x}{\partial x} = 0 \quad \Rightarrow \quad v_x = v(y)\,.$$

The constitutive equations are given by

$$\tau_{xx} = \tau_{yy} = 0 \quad \left[\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial y} = 0\right]$$

$$\tau_{xy} = \eta\, v'(y)\,.$$

The equations of motion are given by

$$\eta\, v''(y) = 0 \quad \left[\frac{\partial p}{\partial x} = 0\right]$$

$$\frac{dp}{dy} = 0 \quad \Rightarrow \quad p = p_0 \quad \text{(constant)}\,.$$

We obtain

$$v(y) = C_1 y + C_2 .$$

Introducing the no slip conditions

$$v_x(0) = 0 \quad \text{and} \quad v_x(d) = V$$

it follows that $C_1 = \dfrac{V}{d}$ and $C_2 = 0$. Hence

$$v(y) = \frac{V}{d} y .$$

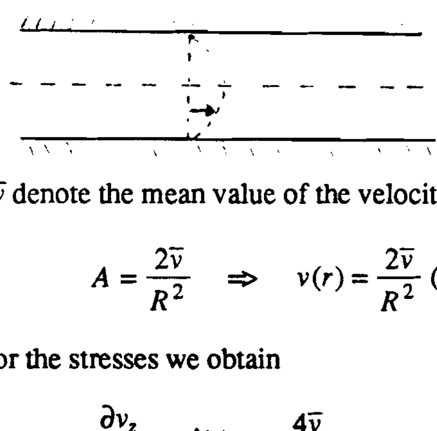
For the corresponding stresses we find

$$\tau_{xy} = t_{xy} = \eta \frac{V}{d}, \quad \tau_{xx} = \tau_{yy} = 0 \;\Rightarrow\; t_{xx} = t_{yy} = -p_0 .$$

Remark. This kind of simple results do not hold any longer for more complex (non-linear) constitutive equations.

B.5. **Poiseuille flow, Darcy's law.**

Darcy's law describes the flow of a fluid through a porous medium. Although the law possesses a more general validity, we restrict here to a *rigid* porous medium. Clearly, if the fluid flows through the pores of the medium it encounters resistance which is due to the viscosity of the fluid and, more importantly, due to the surface tensions in the fluid. For our very heuristic derivation of Darcy's law we consider first the Poiseuille flow through a circular pipe.



As a solution we obtain

$$\underline{v} = v(r)\,\underline{e}_z$$

met

$$v(r) = A(R^2 - r^2).$$

Let $\bar{v}$ denote the mean value of the velocity $v(r)$, then we have

$$A = \frac{2\bar{v}}{R^2} \quad \Rightarrow \quad v(r) = \frac{2\bar{v}}{R^2}(R^2 - r^2) .$$

So for the stresses we obtain

$$\frac{\partial v_z}{\partial r} = v'(r) = -\frac{4\bar{v}}{R^2} r \quad \Rightarrow \quad \tau_{rz} = \tau_{zr} = -\frac{4\eta}{R^2}\bar{v}\, r .$$

The equations of motion are in cylindrical coordinates

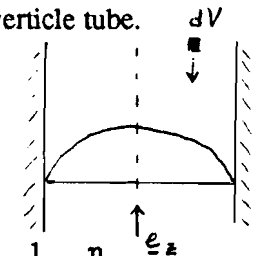
$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{d}{dr} r\, t_{rz} + b_z = \rho\, a_z = 0$$

$$\frac{\partial p}{\partial r} = 0 \quad \Rightarrow \quad p = p(z)\,.$$

Hence

$$\frac{-dp}{dz} + b_z = -\frac{1}{r}\,\frac{d}{dr}\,(r\,\tau_{zr}) = \frac{8\eta}{R^2}\,\bar{v}\,.$$

Consider a verticle tube.



Then

$$b_z = -\rho g$$

and herewith

$$-\frac{dp}{dz} - \rho\,g = \frac{8\eta}{R^2}\,\bar{v}.$$

Define $\dfrac{8\eta}{R^2} = \dfrac{1}{S} = \dfrac{\eta}{k}$

with S the permeability (S increases if η decreases or R increases)
and with k the porosity of the medium.

Then we obtain

$$\bar{v} = -\frac{k}{\eta}\,\frac{d}{dz}\,(p(z) + \rho\,g\,z)\,.$$

This equation can be generalized to *Darcy's law*

$$\underline{v} = -\frac{k}{\eta}\,\nabla\,(p + \rho\,g\,z)\,.$$

Appendix C

Sobolev spaces

Consider the closed interval [0,1].

By $C([0,1])$ we mean the vector space of all continuous functions on the interval [0,1]. Defining for every $f \in C([0,1])$

$$\|f\|_{\infty,0} = \max_{t\in[0,1]} |f(t)| = \sup_{t\in(0,1)} |f(t)|$$

$C([0,1])$ becomes a Banach space, i.e. a complete normed space. By $C^0([0,1])$ we denote the vector space of all restrictions to the open interval (0,1) of functions in $C([0,1])$. Put differently, $C^0([0,1])$ consists of all uniformly continuous functions on (0,1).

For every $k \in \mathbb{N}$ define $C^k([0,1])$ as the vector space of all k times continuously differentiable functions on (0,1) for which $f^{(k)} \in C^0([0,1])$. For $f \in C^k([0,1])$ we have $f^{(j)} \in C^0([0,1])$ $j = 0,1,\ldots,k$ because

$$f(x) = f(0) + x f'(0) + \cdots + \frac{x^{k-1}}{(k-1)!} f^{(k-1)}(0) + \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} f^{(k)}(t)\, dt\ .$$

So a suitable norm on $C^k([0,1])$ is given by

$$\|f\|_{\infty,k} = \sum_{j=0}^{k} \sup_{t\in(0,1)} |f^{(j)}(t)|\ .$$

By introducing other norms on $C^0([0,1])$ we arrive at completions of $C^0([0,1])$, e.g. Banach spaces of (equivalence classes of) measurable functions. Classical are the L_p-spaces where the norm is given by

$$\|f\|_{p,0} = (\int_0^1 |f(t)|^p\, dt)^{1/p} \quad , \quad 1 \le p < \infty\ .$$

The corresponding completions are denoted by $L_p([0,1])$. They do not consists of functions, although we often treat them as if they do.

Correspondingly we introduce on $C^k([0,1])$ the norms

$$\|f\|_{p,k} = \sum_{j=0}^{k} (\int_0^1 |f^{(j)}(t)|^p\, dt)^{1/p} \quad , \quad 1 \le p < \infty\ .$$

The corresponding completions are the Sobolev spaces $W^{p,k}([0,1])$. We have the Sobolev-imbedding

$$W^{p,k}([0,1]) \hookrightarrow C^{k-1}([0,1])$$

i.e.

$$W^{p,k}([0,1] \subset C^{k-1}([0,1])$$

and

$$\exists_{C>0}\ \forall_{f\in W^{p,k}([0,1])} :\ \|f\|_{\infty,k-1} \le C\, \|f\|_{p,k}\ .$$

The Banach space $W^{p,k}([0,1])$ admits the following characterisation:

$W^{p,k}([0,1])$ is the Banach of all functions
$f \in C^{k-1}([0,1])$ for which the $(k-1)-$th derivative $f^{(k-1)}$
is absolutely continuous and has a generalized derivative
that belongs to $L_p([0,1])$.

As an explanation of the above statement we consider the case $k = 1$, the general case can then the obtained by induction.

Definition.
A function $f \in C^0([0,1])$ is said to be absolutely continuous if there is a (Lebesgue) integrable function g on $(0,1)$ such that

$$f(x) = f(0) + \int_0^x g(t)\, dt\ .$$

g is called the generalized derivative of f and is determined up to a function which is zero almost everywhere.

Lemma.
Let $f \in C^1([0,1])$. Then for every $x \in (0,1)$

$$|f(x)| \le 2\, \|f\|_{p,1}\ .$$

Proof.
Since $f(x) = f(0) + \int_0^x f'(t)\, dt$, we get the estimate

$$|f(0)| \le \int_0^1 |f(x)|\, dx + \int_0^1 (\int_0^x |f'(t)|\, dt)\, dx$$

$$\overset{\text{Hölder}}{\le} (\int_0^1 |f(x)|^p\, dx)^{1/p} + (\int_0^1 |f'(t)|^p\, dt)^1 = \|f\|_{p,1}\ .$$

So

$$|f(x)| \le |f(0)| + \int_0^x |f'(t)|\, dt \le$$

$$\leq \|f\|_{p,1} + \|f\|_{p,1} = 2\,\|f\|_{p,1} .$$

Corollary.
For any $f \in C^1([0,1])$ we have $\|f\|_{\infty,0} \leq 2\,\|f\|_{p,1}$.

Now let (f_n) be a Cauchy sequence in $C^1([0,1])$ with respect to the norm $\|\cdot\|_{p,1}$. Then (f_n) is a Cauchy sequence in $C^0([0,1])$ because

$$\|f_n - f_m\|_{\infty,0} \leq 2\,\|f_n - f_m\|_{p,1} .$$

So there is $f \in C^0([0,1])$ such that

$$\|f_n - f\|_{\infty,0} \to 0 \quad (n \Rightarrow \infty) .$$

Since

$$\|f_n{}' - f_m{}'\|_{p,0} \leq \|f_n - f_m\|_{p,1}$$

the sequence $(f_n{}')$ is Cauchy in $L_p([0,1])$ and hence converges to some $g \in L_p([0,1])$.

We have

$$f_n(x) = f_n(0) + \int_0^x f_n{}'(t)\,dt$$

and so in the limit $n \to \infty$

$$f(x) = f(0) + \int_0^x g(t)\,dt .$$

It follows that f is absolutely continuous with generalized derivative $g \in L_p([0,1])$. Moreover

$$\|f_n - f\|_{p,1} = \|f_n - f\|_{p,0} + \|f_n{}' - g\|_{p,0} .$$

The Sobolev space $W^{p,k}([0,1])$ can be introduced in a different way.
To this end we introduce the space $C_c^\infty((0,1))$ consisting of all infinitely differentiable functions on (0,1) with a compact support within (0,1). $C_c^\infty((0,1))$ is a vector space with a topology defined such that

$$\phi_n \to \phi \quad \text{in} \quad C_c^\infty((0,1))$$

means that there exists a compact set $K \subset (0,1)$ such that

$$\forall_n : \ \mathrm{supp}\,(\phi_n) \subset K$$

and

$$\forall_{j \in \mathbb{N}}: \ \sup_{t \in K} |\phi_n^{(j)}(t) - \phi^{(j)}(t)| \to 0 \quad (n \to \infty).$$

In the sequel we write $D((0,1))$ instead of $C_c^\infty((0,1))$.
By $D^*((0,1))$ we denote the dual space of $D((0,1))$, i.e. the vector space of all continuous linear functionals on $D((0,1))$. The elements of $D^*((0,1))$ are called **distributions** or generalized functions.
We introduce the concept of distributional derivative.
For each $j \in \mathbb{N}$ the linear mapping

$$D^j : \phi \mapsto \phi^{(j)}$$

is continuous on $D((0,1))$. Consequently for every continuous linear functional L on $D((0,1))$, $L \circ D^j$ is a continuous linear functional. Now we define

$$\hat{D}^j L = (-1)^j L \circ D^j .$$

Let f be an integrable function on $[0,1]$. Then f determines a continuous linear functional $\hat{f}$ on $D((0,1))$ through

$$\hat{f}(\phi) = \int_0^1 f(t)\, \phi(t)\, dt$$

and so

$$(\hat{D}^j \hat{f})(\phi) = (-1)^j \int_0^1 f(t)\, \phi^{(j)}(t)\, dt .$$

In particular, for $f \in C^j([0,1])$

$$(\hat{D}^j \hat{f})(\phi) = (-1)^j \int_0^1 f(t)\, \phi^{(j)}(t)\, dt$$

$$= \int_0^1 f^{(j)}(t)\, \phi(t)\, dt$$

and therefore

$$\hat{D}^j \hat{f} = (D^j f)\hat{} .$$

We conclude that the mapping $\hat{D}^j$ yields a generalization of the classical differential operator D^j. In this new terminology we have the following characterization result

$$f \in W^{p,k}([0,1]) \ \Rightarrow \ f \in L_p([0,1])$$

and

$$\exists_{g_1, \ldots, g_k \in L_p([0,1])} : \hat{D}^j \hat{f} = \hat{g}_j$$

or put differently

$$(-1)^j \int_0^1 f(t)\, \phi^{(j)}(t)\, dt = \int_0^t g_j(t)\, \phi(t)\, dt \ .$$

The Sobolev spaces $W^{2,k}((0,1))$ are Hilbert spaces with inner products

$$(f,g)_k = \sum_{j=0}^{k} \int_0^1 f^{(j)}(t)\, \overline{g^{(j)}(t)}\, dt$$

where $f^{(k)}$ denote the k-th generalized derivative of f.

Consider the space $C^1_{per}([0,1])$ consisting of all $f \in C^1([0,1])$ with $f(0) = f(1)$ and $f'(0) = f'(1)$. Each $f \in C^1_{per}([0,1])$ has an absolutely convergent Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}} a_n\, e^{2\pi i n x} \quad , \quad x \in \mathbb{R}$$

with

$$a_n = \int_0^1 f(t)\, e^{-2\pi i n t}\, dt \quad , \quad n \in \mathbb{Z} \ .$$

Put

$$e_n(x) = (1 + 4\pi^2 n^2)^{-\frac{1}{2}}\ e^{2\pi i n x} \quad , \quad x \in \mathbb{R} \ , \ n \in \mathbb{Z} \ .$$

Then $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal set in $W^{2,1}([0,1])$. Consequently we have

Corollary.
Let $W^{2,1}_{per}([0,1])$ denote the closed subspace of $W^{2,1}([0,1])$ in which $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis. Then $f \in W^{2,1}_{per}([0,1])$ if and only if $f \in W^{2,1}([0,1])$ and $f(0) = f(1)$.

Let l denote the continuous linear functional

$$l(f) = f(1) - f(0) \quad , \quad f \in W^{2,1}([0,1]) \ .$$

Then we have

$$W^{2,1}_{per}([0,1]) = \ker(l) \ .$$

Since $W^{2,1}([0,1])$ is a Hilbert space, there exists $f_0 \in W^{2,1}([0,1])$ such that

$$l(f) = (f, f_0)_{2,1} \ .$$

It follows that

$$W^{2,1}([0,1]) = W^{2,1}_{per}([0,1]) \oplus \langle f_0 \rangle \ .$$

Lemma.

$$f_0(x) = \frac{\sinh(x - \frac{1}{2})}{\sinh(½)} \ .$$

Proof.

By F_0 we denote the primitive of f_0 satisfying $F_0(0) = F_0(1) = 0$. Then

$$f(1) - f(0) = \int_0^1 f'(x)\, dx$$

and

$$f(1) - f(0) = \int_0^1 f_0(x) f(x) + f_0'(x) f'(x)\, dx$$

$$= \int_0^1 [-F_0(x) + F_0''(x)] f'(x)\, dx$$

so that

$$F_0'' - F_0 = -1 \quad , \quad F_0(0) = F_0(1) = 0 \, .$$

Hence

$$F_0(x) = \frac{\cosh 1 - 1}{\sinh 1} \sinh x - \cosh x + 1$$

and

$$F_0'(x) = f_0(x) = \frac{\sinh(x - \frac{1}{2})}{\sinh \frac{1}{2}} \ .$$

□

Now we turn to the more dimensional case.

Let Ω be a bounded region in $\mathbb{R}^n$.

Definition.

$C^0(\overline{\Omega})$ is the vector space consisting of all restrictions to Ω of continuous functions on $\overline{\Omega}$. Put differently, $C^0(\overline{\Omega})$ consists of all (bounded) uniformly continuous functions on Ω. The norm in $C^0(\overline{\Omega})$ is defined by

$$\|f\|_{\infty,0} = \sup_{x \in \Omega} |f(x)| \ ,$$

$C^0(\overline{\Omega})$ is a Banach space.

Let $D_j = \dfrac{\partial}{\partial x_j}$. Then for every multi-index $\alpha \in \mathbb{N}_0^n$,

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} .$$

Further we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition.

$C^k(\overline{\Omega})$ is the vector space of all functions f for which for all α with $|\alpha| \le k$ the derivative $D^\alpha f$ exists with

$$D^\alpha f \in C^0(\overline{\Omega}) .$$

The norm

$$\|f\|_{\infty,k} = \sum_{|\alpha| \le k} \|D^\alpha f\|_{\infty,0}$$

turns $C^k(\overline{\Omega})$ into a Banach space.

Besides we introduce

$$C^\infty(\overline{\Omega}) = \bigcap_{k=1}^{\infty} C^k(\overline{\Omega})$$

and

$$D(\Omega) = C_c^\infty(\Omega) = \{\phi \in C^\infty(\overline{\Omega}) \mid \operatorname{supp}\phi \subset \Omega \text{ compact}\} .$$

With the aid of the space $D(\Omega)$ we can introduce weak (generalized) derivatives for non-differentiable functions. First we introduce the topology of $D(\Omega)$.

– A sequence (ϕ_m) in $D(\Omega)$ is said to be convergent to $\phi \in D(\Omega)$ if

$$\exists_{\substack{K \text{ compact} \\ K \subset \Omega}} \forall_{n \in \mathbb{N}} : \operatorname{supp}\phi_m \subset K \quad \text{and}$$

$$\forall_{\alpha \in \mathbb{N}_0^n} : \sup_{x \in K} |(D^\alpha \phi_m)(x) - (D^\alpha \phi)(x)| \to 0 .$$

Every continuous linear functional on $D(\Omega)$ is called a *distribution*. For instance, the functional

$$l_{\alpha,x_0} \; : \; \phi \;\mapsto\; (D^\alpha \phi)(x_0)$$

with $\alpha \in \mathbb{N}_0^n$ and $x_0 \in \Omega$ is continuous on $D(\Omega)$. This functional is heuristically written as

$$l_{\alpha,x_0}(\phi) = (-1)^{|\alpha|} \int_\Omega \phi(x)\, \delta^{(\alpha)}(x - x_0)\, dx$$

and $\delta(x - x_0)$ is called a delta function.

Also, let $p \ge 1$ and let $f \in L_p(\Omega)$. Then the linear functional $\hat{f}$ is defined by

$$\hat{f}(\phi) = \int_\Omega \phi(x)\, f(x)\, dx \;\; , \quad \phi \in D(\Omega) .$$

The functional $\hat{f}$ is continuous. Thus for any $p \ge 1$, $L_p(\Omega)$ is *imbedded* into the dual space $D^*(\Omega)$. For $L \in D^*(\Omega)$ we write $L \in L_p(\Omega)$ meaning that $L = \hat{f}$ for some $f \in L_p(\Omega)$.

For every distribution L in $D^*(\Omega)$ its distributional derivative $\hat{D}^\alpha L$ is defined by

$$(\hat{D}^\alpha L)(\phi) = (-1)^{|\alpha|} L(D^\alpha \phi).$$

In particular for $f \in L_p(\Omega)$

$$(\hat{D}^\alpha \hat{f})(\phi) = (-1)^{|\alpha|} \int_\Omega (D^\alpha \phi)(x) f(x)\, dx .$$

The distribution $\hat{D}^\alpha \hat{f}$ is called the weak derivative of order α of f. It is a natural extension of the classical notion of derivative:

Take $\psi \in C^\infty(\overline{\Omega})$. Then

$$(\hat{D}^\alpha \hat{\psi})(\phi) = \int_\Omega \phi(x) (D^\alpha \psi)(x)\, dx .$$

Often one writes $D^\alpha f$ in stead of $\hat{D}^\alpha \hat{f}$ although f is not differentiable.

We come to the definition of the Sobolev spaces $W^{p,k}(\Omega)$.

Definition.

$$f \in W^{p,k}(\Omega) :\Longleftrightarrow f \in L_p(\Omega) \text{ and}$$

$$\forall_{\alpha \in \mathbb{N}_0^k, |\alpha| \le k} : \underset{\uparrow \text{ weak interpretation}}{D^\alpha f} \in L_p(\Omega) .$$

The norm in $W^{p,k}(\Omega)$ is given by

$$\|f\|_{p,k} = \Big(\sum_{|\alpha| \le k} \|D^\alpha f\|_{p,0}^p \Big)^{1/p} .$$

$W^{p,k}(\Omega)$ is a Banach space, in particular $W^{2,k}(\Omega)$ is a Hilbert space with inner product

$$(f,g)_{2,k} = \sum_{|\alpha| \le k} (D^\alpha f, D^\alpha g)_2 .$$

For $k = 1$ we have

$$(f,g)_{2,1} = \int_\Omega [f(x)\, g(x) + (\nabla f \cdot \nabla g)(x)]\, dx .$$

Remark. It can be proved (Serrin and Morrey, 1964) that $W^{p,k}(\Omega)$ is the completion of $C^k(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{p,k}$.

Finally some remarks on the so called Sobolev embedding theorems.
For the proof of these theorems conditions are needed on the shape of the region Ω. Most classical configurations such as balls, cones, cylinders and ellipsoides satisfy these conditions. We have

$$W^{k,p}(\Omega) \hookrightarrow C^l(\overline{\Omega}) \quad \text{with} \quad l < k - \frac{n}{p}$$

i.e.

$$W^{k,p}(\Omega) \subset C^l(\overline{\Omega})$$

and

$$\exists_{C_{k,l} > 0} \; \|f\|_{\infty,l} \leq C_{k,l} \, \|f\|_{p,k} \, .$$

Appendix D

The notion: weak solution

Let K be a linear second order differential operator given by

$$K\,u = \sum_{i,j=1}^{n} a_{ij}(x)\,D_i D_j\,u + \sum_{i=1}^{n} b_i(x)\,D_i\,u + c(x)\,u$$

where $D_i = \dfrac{\partial}{\partial x_i}$ and where u belongs to $C^2(\mathbb{R}^n)$.

Assuming that the coefficients $a_{ij}(x)$, $b_i(x)$ and $c(x)$ belong to $C^\infty(\mathbb{R}^n)$, K is a continuous linear mapping from $D(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ into itself. We can extend K to the distribution space $D^*(\mathbb{R}^n)$ in the following natural way

For every $L \in D^*(\mathbb{R}^n)$ define $\hat{K}\,L$ by

$$\hat{K}\,L = \sum_{i,j=1}^{n} a_{ij}(x)\,\hat{D}_i\,\hat{D}_j\,L + \sum_{i=1}^{n} b_i(x)\,\hat{D}_i\,L + c(x)\,L$$

i.e.

$$(\hat{K}\,L)\,(\phi) = L(K^*\,\phi) \quad , \quad \phi \in D(\mathbb{R}^n)$$

where K^* denotes the differential operator.

The operator K^* is called the formal adjoint of K. If $K = K^*$ then K is called (essentially) self-adjoint. E.g. the Laplacian Δ in $\mathbb{R}^n$ is self-adjoint.
Let $u,v \in C^2(\mathbb{R}^n)$. Then we have

$$v(K\,u) - u(K^*\,v) = \operatorname{div} P = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}$$

where the vector field P is defined by

$$P_i(x) = \sum_{j=1}^{n} [a_{ij}(x)\,v\,D_j\,u - u\,D_j(a_{ij}(x)\,v)] + b_i(x)\,u\,v\,.$$

So for a bounded region $\Omega \subset \mathbb{R}^n$ with piecewise smooth orientable boundary $\partial\Omega$ we have by Gauss' divergence theorem

$$\int_\Omega (v\,K\,u - u\,K^*\,u)\,dx = \int_{\partial\Omega} (P \cdot n)\,d\sigma\,.$$

This relation is called *Green's formula*. If $K = \Delta$. Green's formula become the second identity of Green

$$\int_{\Omega} (v\,\Delta u - u\,\Delta v)\,dx = \int_{\partial\Omega} (v\,\frac{\partial u}{\partial n} - u\,\frac{\partial v}{\partial n})\,d\sigma\,.$$

Consider the linear second order partial differential equation

$$(*) \qquad K\,u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\,D_i D_j\,u + \sum_{i=1}^{n} b_i(x)\,D_i\,u + c(x)\,u = f(x)\,.$$

Then u is called a *classical* or *strong* solution of (*) in a region $\Omega \subset \mathbb{R}^n$ if

i) $u \in C^2(\Omega)$,

ii) $K\,u = f$ in Ω.

A distribution $L \in D^*(\Omega)$ is called a *generalized* or *weak* solution of (*) if

$$\hat{K}\,L = \hat{f}$$

i.e.

for all $\phi \in D(\Omega)$:

$$L(K^*\,\phi) = \hat{f}(\phi) = \int_{\Omega} f(x)\,\phi(x)\,dx\,.$$

Theorem.
Any strong solution of (*) is a weak solution.

Proof.
Let $u \in C^2(\Omega)$ with $K\,u = f$. Then for all $\phi \in D(\Omega)$

$$(\hat{K}\,\hat{u})\,(\phi) = \int_{\Omega} u(K^*\,\phi)\,dx$$

$$= \int_{\Omega} (K\,u)\,\phi\,dx - \int_{\partial\Omega} (P,n)\,d\sigma\,.$$

Since $\phi \in D(\Omega)$ we have $P \,\Big|_{\partial\Omega} = 0$.

Conversely,

Theorem.
Let $u \in C^2(\Omega)$ be a weak solution of (*) with $f \in C^0(\Omega)$. Then u is a strong solution.

Let for every $y \in \Omega$ the distribution $\delta_y \in D^*(\Omega)$ be defined by $\delta_y(\phi) = \phi(y)$, $\phi \in D(\Omega)$. A distribution $S_y \in D^*(\Omega)$ is called a *fundamental* solution of the differential operator K in Ω if

$$\hat{K}\,S_y = \delta_y$$

i.e.

$$S_y(K^* \phi) = \phi(y) \quad , \quad \phi \in D(\Omega) .$$

In many concrete cases S_y is represented by a smooth function with only a singularity at $x = y$. It is clear that S_y is in general not uniquely determined. Therefore boundary conditions have to be added.

Suppose $S_y = \hat{g}_y$, we mean

$$S_y(\phi) = \int_\Omega g_y(x)\, \phi(x)\, dx .$$

Then under certain conditions the solution of $K\, u = f$ is given by

$$u(x) = \int_\Omega g_y(x)\, \phi(y)\, dy .$$

Example. Take $K = -\Delta$. Then a fundamental solution in $\mathbb{R}^n$ is given by

$$g_y(x) = g(x-y)$$

with

$$g(x) = \frac{1}{2\pi} \log \frac{1}{|x|} \quad , \quad n = 2$$

$$g(x) = \frac{1}{(n-2)\,\sigma_n} \frac{1}{|x|^{n-2}} \quad , \quad n \geq 3$$

where

$$\sigma_n = \frac{n\, \pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}\, n+1)} .$$

So a solution of the Poisson equation (in case $n = 3$)

$$\Delta u = -f(x) \quad , \quad x \in \Omega$$

is given by

$$u(x) = \frac{1}{4\pi} \int_\Omega \frac{f(y)}{|x-y|}\, dy \quad , \quad x \in \Omega .$$

In order to satisfy the possible boundary conditions a harmonic function can be added to u.

Appendix E

Some fixed point theorems

Banach's contraction theorem.
Let (X,d) be a complete metric space and $T : X \to X$ a contractive mapping, i.e.

$$\exists_{\alpha, 0<\alpha<1} \; \forall_{x,y \in X,\, x \neq y} :$$
$$d(Tx, Ty) \leq \alpha\, d(x,y) .$$

Then there exists $x_0 \in X$ with $T x_0 = x_0$.

Proof.
Let $x \in X$ and define the sequence $x_n = T^n x$. If we can prove that the sequence (x_n) is convergent in X with limit x_0, then x_0 is a fixed point of T. Indeed,

$$x_0 = \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T(T^{n-1} x) - T x_0$$

where in the last step the continuity of T is used.
To prove that (x_n) is a convergence we show that it is a Caucy sequence.
Let $n > m$. Then

$$\begin{aligned} d(x_n, x_m) &= d(T^n x,\, T^m x) \leq \\ &\leq d(T^n x,\, T^{n-1} x) + \cdots + d(T^{m+1} x,\, T^m x) \leq \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)\, d(T x,\, x) \\ &\leq \frac{\alpha^m}{1-\alpha}\, d(T x,\, x) . \end{aligned}$$

Hence $d(x_n, x_m) \to 0$ if $n,m \to \infty$. □

Theorem.
Let (X,d) be a compact metric space and $T : X \to X$ a semi-contractive mapping, i.e.

$$\forall_{x,y \in X,\, x \neq y} : \; d(Tx, Ty) < d(x,y) .$$

Then T has a fixed point.

Proof.
The mapping T is continuous, and hence also $\phi : X \to \mathbb{R}^+$ defined by

$$\phi(x) = d(x, Tx) \quad , \quad x \in X ,$$

is continuous:

$$|d(x, Tx) - d(y, Ty)| \leq d(x,y) + d(Tx, Ty) .$$

So the set $\phi(X) = \{d(x, Tx) \mid x \in X\}$ is compact in $\mathbb{R}^+$ and so has a minimum. Suppose

$$d(x_0, T x_0) = \min \{d(x, Tx) \mid x \in X\} .$$

If $x_0 \neq T x_0$, then

$$d(T x_0, T^2 x_0) < d(x_0, T x_0)$$

which gives a contradiction. Conclusion $x_0 = T x_0$. □

Brouwer's fixed point theorem.
Let B denote the closed unit ball in $\mathbb{R}^n$ and $f : B \to B$ a continuous mapping. Then f has a fixed point in B.

This theorem has the following consequence.

Corollary.
Let K be a non-empty compact convex subset of a finite dimensional normed space X and let $f : K \to K$ be a continuous mapping. Then f possesses a fixed point in K.

Proof.
We can as well assume that $X = \mathbb{R}^d$. If $K = \{x \in \mathbb{R}^d \mid \|x\| \leq r\}$ then define $f_r(x) = \frac{1}{r} f(\frac{x}{r})$ and apply Brouwer's theorem.
Otherwise, take $r > 0$ so large that $K \subseteq B_r = \{x \in \mathbb{R}^d \mid \|x\| \leq r\}$.
Define $\phi : B \to K$ by

$$\phi(x) = y \quad \text{with} \quad \|x - y\| = \text{dist}(x,K) .$$

(Remark: y is uniquely determined.)

Then ϕ is continuous and $\phi(x) = x$ for all $x \in K$. So $f \circ \phi$ from B_r into $K \subset B_r$ is continuous. According to Brouwer's theorem $f \circ \phi$ has a fixed point $x \in B_r$. Since $(f \circ \phi)(x) = x$, $x \in K$ so that $\phi(x) = x$ and $f(x) = x$. □

Schauder's fixed point theorem.
Let E denote a closed bounded convex subset of a normed space X, and let $f : E \to E$ be a mapping with the property that

- $f(E)$ is relatively compact.

Then f has a fixed point in E.

Remark: If $f : X \to X$ is a compact mapping, then condition is satisfied for all E.

REFERENCES

To Chapter I

C. Baiocchi and A. Capelo, Variational and quasi-variational inequalities: applications to free boundary problems, John Wiley, 1984.

L.A. Caffarelli, The regularity of elliptic and parabolic free boundaries, Bull. Am. Math. Soc. **82** (1976), 616-618.

D. Kinderlehrer and G. Stampacchia, An introduction to variational inequalities and their applications. Ac. Press, New-York, 1980.

A.C. Mac Kie, The mathematics of electrochemical machining, J. Math. Anal. Appl. **117** (1986), 548-560.

To Chapter II

W.H. Green and G.A. Ampt, Studies on soil physics 1. The flow of air and water through soils, J. Agric. Sci. **4** (1911), 1-24.

M. Muskat, Two fluid systems in porous media, The encroachment of water into an oil sand, Physics **5** (1934), 250-264.

A. Fasano and M. Primicerio, Liquid flow in partially saturated porous media, J. Inst. Math. Appl. **23** (1979), 503-517.

To Chapter III

G. Lamé and B.P. Clapeyron, Memoire sur la solidification par refroidissement d'un globe liquide, Annales Chimie Physique **47** (1831), 250-256.

J. Stefan, Ueber die Theorie der Eisbildung, Monatshefte Mat. Phys. **1** (1890), 1-6.

J. Stefan, Ueber die Theorie der Eisbildung, insbesondere ueber die Eisbildung in Polarmeere, Annalen Physik, Chemie **42** (1891), 269-286.

J.F. Ciavaldini, Résolution numérique d'un problème de Stefan à deux phases, Thèse de 3 ème Cycle, Univ. de Rennes, 1972.

A. Friedman, Free boundary problems for parabolic equations I. Melting of solids, J. Math. Mech. **8** (1959), 499-517.

A. Friedman, The Stefan problem in several space variables, Trans. Am. Math. Soc. **132** (1968), 51-87.

S.D. Howison, Similarity solutions to the Stefan problem and the binary alloy problem, IMA J. Appl. Math. **40** (1988), 147-161.

O.A. Okinik, A method of solution of the general Stefan problem, Soviet Math. Dokl. **1** (1960), 1350-1354.

M. Primicerio, Mushy regions in phase change problems in K.H. Hoffman and R. Gorenflo (eds.). Applied nonlinear functional analysis: variational methods and ill-posed problems, Verlag Peter Lang, Frankfurt, 1983.

J.C.W. Rogers and A.E. Berger, Some properties of the non-linear semigroup for the problem $u_t - \Delta f(u) = 0$, Nonlinear Anal. Th. Appl. **8** (1984), 909-939.

To Chapter IV

A Fasano and M. Primicerio, New results on some classical parabolic free boundary problems, Appl. Math. **38** (1981), 439-460.

A. Fasano and M. Primicerio (eds.), Nonlinear diffusion problems, Lect. Notes in Math., no. **1224**, Springer-Verlag, Berlin 1986.

I. Stakgold, Reaction-diffusion problems in chemical engineering in Lect. Notes in Math., no. **1224**, 119-152.

To Chapter V

P. Chadwick, Continuum mechanics, G. Allen & Unwin Ltd. London, 1976.

G. Böhm, Non-Newtonian fluid mechanics, Series in applied mathematics and mechanics, North-Holland Amsterdam, 1987.