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Abstract.

In this paper the notion of Dirac basis will be introduced. It is the continuous pendant of the discrete basis for Hilbert spaces. The introduction of this new notion is closely connected to a theory of generalized functions. Here De Graaf's theory will be employed. It is based on the triplet $S_{X,A} \subset X \subset T_{X,A}$ where X is a Hilbert space. In a well specified way any member of $T_{X,A}$ can be expanded with respect to a Dirac basis. Both the introduction of Dirac bases and a new interpretation of Dirac's bracket notion will lead to a mathematical rigorization of various aspects of Dirac's formalism for quantum mechanics. This rigorization goes much beyond earlier proposals.

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0. Introduction

In the preface to his book on the foundations of quantum mechanics Von Neumann says that Dirac's formalism is 'scarcely to be surpassed in brevity and elegance' but that it 'in no way satisfies the requirements of mathematical rigour'. As far as we know no paper on Dirac's formalism mathematically foundates the bold way in which Dirac treats the continuous spectrum of a self-adjoint operator. Most papers on this subject only solve the so-called generalized eigenvalue problem. But Dirac's formalism contains more aspects.

In this section we shall treat parts of an interpretation of the formalism in terms of our distribution theory. It consists of the definition of ket and bra space, of the definition of Dirac basis (the pendant of the orthogonal basis in Hilbert spaces), of Parseval's identity, of the 'orthogonality' of the elements of a Dirac basis, of the Fourier expansion of kets, and of matrices, all in the spirit of Dirac.

In a subsequent paper we shall show that for each self-adjoint operator T there is a Dirac basis which consists of eigenkets. Thus all points in the spectrum of a self-adjoint operator can be dealt with in the same manner. Moreover, we then shall show that for each point λ in the spectrum of T with multiplicity m there exist m independent eigenkets. Of course, Hilbert spaces are too small to fulfil these wishes. Therefore, it is natural to look for spaces which extend Hilbert spaces and which have a structure comparable to Hilbert space structure. The trajectory spaces $T_{X,A}$ turn out to be successful candidates.

In Dirac's formalism the hermitean conjugate of the ket space, the so-called bra space, is in 1-1 correspondence with the ket space. Dirac supposes a pairing between bra and ket space. In distribution theory this can never be the case. We solve this problem by a new interpretation of Dirac's bracket notion.

1. Analyticity spaces and trajectory spaces

We shall employ the theory of generalized functions as introduced by De Graaf, [G]. In this section we want to give the features of the theory [G] which are relevant for this paper.

In a Hilbert space X consider the evolution equation

$$(1.1) \qquad \frac{du}{dt} = -A\,u \quad , \quad t > 0 ,$$

where A is a nonnegative unbounded self-adjoint operator. A solution F of (1.1) is called a trajectory if F satisfies

$$(1.2.i) \qquad \forall_{t>0} \; : F(t) \in X$$

$$(1.2.ii) \quad \forall_{t>0} \forall_{\tau>0} \; : e^{-\tau A} F(t) = F(t+\tau) .$$

We emphasize that $\lim_{t \downarrow 0} F(t)$ does not necessarily exist in X-sense. The complex vector space of all trajectories is denoted by $T_{X,A}$. The space $T_{X,A}$ is considered as a space of generalized functions in [G]. The Hilbert space X is embedded in $T_{X,A}$ by means of the linear mapping

$$\mathrm{emb}(f) : t \mapsto e^{-tA} f \quad , \quad f \in X .$$

The analyticity space $S_{X,A}$ is defined as the dense linear subspace of X consisting of smooth elements of the form $e^{-tA}\omega$ where $\omega \in X$ and $t > 0$. Hence $S_{X,A} = \bigcup_{t>0} e^{-tA}(X) = \bigcup_{n \in \mathbb{N}} e^{-\frac{1}{n}A}(X)$. For each $f \in S_{X,A}$ there exists $\tau > 0$ such that $e^{\tau A} f \in S_{X,A}$. Furthermore, for each $F \in T_{X,A}$ we have $F(t) \in S_{X,A}$ for all $t > 0$. The analyticity space $S_{X,A}$ is the test function space in [G].

In $\mathcal{T}_{X,A}$ the topology can be described by the seminorms

$$(1.3) \qquad F \mapsto \|F(t)\| \quad , \quad F \in \mathcal{T}_{X,A} .$$

Because of property (1.2.ii), the space $\mathcal{T}_{X,A}$ is a Frechet space with this topology. In $\mathcal{S}_{X,A}$ we take the inductive limit topology. We note that this inductive limit is not strict. A set of seminorms is produced in [G] which generates the inductive limit topology.

The pairing $<\cdot\,,\,\cdot>$ between $\mathcal{S}_{X,A}$ and $\mathcal{T}_{X,A}$ is defined by

$$(1.4) \qquad <g,F> := (e^{\tau A} g\,,\,F(\tau)) \quad , \quad g \in \mathcal{S}_{X,A},\ F \in \mathcal{T}_{X,A} .$$

Here $(\cdot\,,\,\cdot)$ denotes the inner product on X. Definition (1.4) makes sense for $\tau > 0$ sufficiently small. Due to the trajectory property it does not depend on the choice of τ. The spaces $\mathcal{S}_{X,A}$ and $\mathcal{T}_{X,A}$ are reflexive in the given topologies.

The space $\mathcal{S}_{X,A}$ is nuclear if and only if A generates a semigroup of Hilbert-Schmidt operators on X. Then A has an orthonormal basis of eigenvectors v_k, $k \in \mathbb{N}$, with eigenvalues λ_k. In addition, for all $t > 0$ the series $\sum_{k=1}^{\infty} e^{-\lambda_k t}$ converges. It can be shown that $f \in \mathcal{S}_{X,A}$ if and only if there exists $\tau > 0$ such that

$$(1.5) \qquad (f,v_k) = \mathcal{O}(e^{-\lambda_k \tau})$$

and $F \in \mathcal{T}_{X,A}$ if and only if

$$(1.6) \qquad <v_k,F> = \mathcal{O}(e^{\lambda_k t})$$

for all $t > 0$.

For examples of these spaces, see [G], [EG], [EGP] and [E_{II}].

2. Dirac basis

In this paper $T_{X,A}$ will be a nuclear trajectory space. So in X there exists an orthonormal basis $(v_k)_{k\in\mathbb{N}}$ of eigenvectors of A such that $A\,v_k = \lambda_k\, v_k$ and $\forall_{t>0} : \sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty$. For convenience we take the eigenvalues ordered, $0 \le \lambda_1 \le \lambda_2 \le \dots$.

(2.1) Definition.

The quadruple $(M,\mu,G,T_{X,A})$ is called a Dirac basis in $T_{X,A}$ if the following conditions are satisfied

(i) M is a σ-finite measure space with nonnegative measure μ.

(ii) $G : \alpha \mapsto G_\alpha$ is a mapping from M into $T_{X,A}$.

(iii) The functions $\alpha \mapsto \langle v_k, G_\alpha\rangle$ are μ-measurable.

(iv) $$\int_M \langle v_k, G_\alpha\rangle \overline{\langle v_\ell, G_\alpha\rangle}\, d\mu_\alpha = \delta_{k\ell} \quad , \quad k,\ell \in \mathbb{N}\,.$$

Here $\delta_{k\ell}$ is the Kronecker delta.

Example. An orthonormal basis $(u_k)_{k\in\mathbb{N}}$ in a Hilbert space X is expressed by $(\mathbb{N},\mu,u,X)$ where μ is the counting measure and u is the mapping $u : \mathbb{N} \to X : k \mapsto u_k$.

(2.2) Notation.

The Dirac basis $(M,\mu,G,T_{X,A})$ will be denoted by $(G_\alpha)_{\alpha\in M}$.

Remark. It is natural to call two Dirac bases $(G_\alpha)_{\alpha\in M}$ and $(H_\alpha)_{\alpha\in M}$ equivalent if the functions $\alpha \mapsto \langle v_k, G_\alpha\rangle$ and $\alpha \mapsto \langle v_k, H_\alpha\rangle$ are members of the same equivalence class in $L_2(M,\mu)$. It is appropriate to reserve the notion of Dirac basis for the equivalence class only. However, here we will stick to Definition (2.1).

In our second paper [EG_{II}] for certain measure spaces M we will make a canonical choice out of the equivalence class of Dirac bases $(G_\alpha)_{\alpha\in M}$.

In this section we shall prove that any $f \in S_{X,A}$ can be expanded with respect to a Dirac basis. So the name of basis is proper.

Let $(G_\alpha)_{\alpha\in M}$ be a Dirac basis in $T_{X,A}$ with respect to μ. Then from Definition (2.1) it follows that the functions $\alpha \mapsto \langle v_k, G_\alpha\rangle$ are square μ-integrable. Denote their equivalence classes in $L_2(M,\mu)$ by φ_k, $k \in \mathbb{N}$. Then it follows that $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal system in $L_2(M,\mu)$. Let Y denote the Hilbert subspace of $L_2(M,\mu)$ spanned by the system $(\varphi_k)_{k\in\mathbb{N}}$ and define the linear operator U from X into Y by

$$U v_k = \varphi_k \qquad k \in \mathbb{N}\,.$$

Then U becomes a unitary operator. Moreover, $U(S_{X,A}) = S_{Y,B}$ and $U(T_{X,A}) = T_{Y,B}$, where $B = U A U^*$.

Let $f \in S_{X,A}$. Then $U f = \sum_{k=1}^{\infty} (f,v_k)\varphi_k$ and

$$(2.3) \qquad \langle f,G_\alpha\rangle = \sum_{k=1}^{\infty} (f,v_k)\langle v_k,G_\alpha\rangle\,.$$

So $\alpha \mapsto \langle f,G_\alpha\rangle$ is an L_2-representant of $U f$. Thus we derive

$$(2.4) \qquad (f,v_k)_X = \int_M \langle f,G_\alpha\rangle\overline{\langle v_k,G_\alpha\rangle}\, d\mu_\alpha$$

and more generally for $f,g \in S_{X,A}$

$$(2.4') \qquad (f,g)_X = \int_M \langle f,G_\alpha\rangle\langle g,G_\alpha\rangle\, d\mu_\alpha\,.$$

Now let $h \in S_{X,A}$. Then $h = e^{-tA} f$ for some $f \in S_{X,A}$ and $t > 0$. Consider the following formal computation

$$\begin{aligned} h &= \sum_{k=1}^{\infty} e^{-\lambda_k t} (f,v_k)\, v_k \\ &= \sum_{k=1}^{\infty} e^{-\lambda_k t} \left(\int_M <f,G_\alpha> \overline{<v_k,G_\alpha>} d\mu_\alpha \right) v_k \\ &\overset{(*)}{=} \sum_{k=1}^{\infty} <f,G_\alpha> \left(\sum_{k=1}^{\infty} e^{-\lambda_k t} \overline{<v_k,G_\alpha>}\, v_k \right) d\mu_\alpha \\ &= \int_M <f,G_\alpha> G_\alpha(t)\, d\mu_\alpha \,. \end{aligned}$$

The only problem in the above computation is the equality (*). We shall prove that summation and integration may be interchanged.

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_M e^{-\lambda_k t} |<f,G_\alpha> \overline{<v_k,G_\alpha>}|\, d\mu_\alpha \\ &\le \frac{1}{2} \sum_{k=1}^{\infty} e^{-\lambda_k t} \int_M \left(|<f,G_\alpha>|^2 + |<v_k,G_\alpha>|^2 \right) d\mu_\alpha \\ &= \frac{1}{2} \left(\| f \|^2 + 1 \right) \sum_{k=1}^{\infty} e^{-\lambda_k t} \,. \end{aligned}$$

By the Fubini-Tonelli theorem, equality (*) has been verified. It leads to the following theorem.

(2.5) Theorem.

Let $h \in S_{X,A}$. Then

$$h = \int_M \langle h, G_\alpha \rangle \, G_\alpha \, d\mu_\alpha$$

by which we mean

$$e^{-tA} h = \int_M \langle h, G_\alpha \rangle \, G_\alpha(t) \, d\mu_\alpha \quad , \quad t > 0 \, ,$$

or equivalently

$$(*) \qquad h = \int_M \langle e^{\tau A} h, G_\alpha \rangle \, G_\alpha(\tau) \, d\mu_\alpha$$

where $\tau > 0$ has to be chosen so small that $e^{\tau A} h \in S_{X,A}$. Moreover, (*) does not depend on the choice of $\tau > 0$.

Of course it is nice to introduce a more general concept of basis. But then the natural question arises whether there exist Dirac basis other than orthonormal basis. In the remaining part of this section we show that there is an abundance of Dirac bases in $T_{X,A}$.

Let U be a unitary operator from the Hilbert space X onto a space $L_2(M,\mu)$, where μ is a σ-finite nonnegative measure on the measure space M. The operator U maps the orthonormal basis $(v_k)_{k\in\mathbb{N}}$ onto an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of $L_2(M,\mu)$. In each equivalence class φ_k we take a representant $\hat{\varphi}_k$. Now let $t > 0$ be fixed. Since the series $\sum_{k=1}^{\infty} e^{-\lambda_k t}$ converges, the expression

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \, |\varphi_k|^2$$

represants an element of $L_1(M,\mu)$. Since this series has positive terms there exists a null set N_t such that

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} |\hat{\varphi}_k(\alpha)|^2 < \infty \quad , \quad \alpha \in M \backslash N_t \, .$$

Now put $N = \bigcup_{n \in \mathbb{N}} N_{\frac{1}{n}}$. Then N is a null set with respect to μ. If we define $\tilde{\varphi}_k \in \varphi_k$ by

$$\tilde{\varphi}_k(\alpha) = \hat{\varphi}_k(\alpha) \quad , \quad \alpha \in M \backslash N$$

$$\tilde{\varphi}_k(\alpha) = 0 \quad , \quad \alpha \in N$$

we get the following example of a Dirac basis.

(2.6) Theorem.

For $\alpha \in M$ define G_α by

$$G_\alpha : t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t} \overline{\tilde{\varphi}_k(\alpha)} \, v_k \, .$$

Then $(G_\alpha)_{\alpha \in M}$ is a well-defined Dirac basis in $T_{X,A}$.

Proof. If $\alpha \in N$, then $G_\alpha = 0$. So let $\alpha \in M \backslash N$. Then for all $n \in \mathbb{N}$

$$\sum_{k=1}^{\infty} e^{\frac{1}{n}\lambda_k} \overline{\tilde{\varphi}_k(\alpha)} \, v_k \in X \, .$$

Let $t > 0$. Then there exists $n \in \mathbb{N}$ with $0 < \frac{1}{n} < t$ and thus we obtain

$$\sum_{k=K_1}^{K_2} e^{-2t\lambda_k} |\tilde{\varphi}_k(\alpha)|^2 \leq e^{-2(t-\frac{1}{n})\lambda_k} \sum_{k=K_1}^{\infty} e^{-\frac{2}{n}\lambda_k} |\tilde{\varphi}_k(\alpha)|^2 \, .$$

Hence

$$\forall_{t>0} : \sum_{k=1}^{\infty} e^{-t\lambda_k} \overline{\tilde{\varphi}_k(\alpha)} \, v_k \in X \quad \text{and} \quad G_\alpha \in T_{X,A} \, .$$

Since $\langle v_k, G_\alpha \rangle = \tilde{\varphi}_k(\alpha)$ and since $\tilde{\varphi}_k \in U v_k$ it follows that

$$\int_M \tilde{\varphi}_k \overline{\tilde{\varphi}_\ell}\, d\mu = (U v_k, U v_\ell) = \delta_{k\ell} \quad , \quad k,\ell \in \mathbb{N} .$$

So $(G_\alpha)_{\alpha\in M}$ is a Dirac basis. □

3. Dirac's bra and ket space

For the ket space we take a nuclear trajectory space $T_{X,A}$. The choice of A and X depends on the quantum mechanical system under consideration. But we do not wish to bore the reader with the mathematical and possibly physical problems around this choice.

Oncemore, the eigenvalues of A are denoted by λ_k and its eigenvectors by v_k, $k \in \mathbb{N}$. Using Dirac's bracket notation, we denote the elements of $T_{X,A}$ by $|G\rangle$. The label G is to be chosen such that it expresses best the properties of the ket $|G\rangle$ which are relevant in the context.

Each ket $|G\rangle$ can be regarded as a trajectory

$$|G\rangle : t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t}\, \overline{\langle v_k, |G\rangle\rangle}\, v_k .$$

To $|G\rangle$ uniquely corresponds the bra $\langle G|$ defined by

$$\langle G| : t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle v_k, |G\rangle\rangle\, v_k .$$

The bra space is anti-isomorphic to the ket space.

(3.1) Definition.

The expression $\langle F \mid G\rangle$, called the bracket of $\langle F|$ and $|G\rangle$, denotes the complex valued function

$$\langle F \mid G\rangle : t \mapsto \overline{\langle\langle |F\rangle(t), |G\rangle\rangle} \quad , \quad t > 0 .$$

We note that the bracket $\langle F \mid G\rangle$ is well-defined because $|F\rangle(t) \in S_{X,A}$ for all $t > 0$. It extends to an analytic function on the open right half plane, and it can even be considered as an almost periodic distribution on the imaginary axis.

Let $f \in S_{X,A}$. Since $S_{X,A}$ can be embedded in $T_{X,A}$, to f there corresponds the ket $|f\rangle$ and the bra $\langle f|$. We note that for $f \in S_{X,A}$, $|f\rangle(-\tau)$ and $\langle f|(-\tau)$ makes sense for $\tau > 0$ sufficiently small. In addition, we obtain $\langle f \mid G\rangle(-\tau) = \overline{\langle\langle |f\rangle(-\tau), |G\rangle\rangle}$.

To emphasize this nice property of the members of $S_{X,A}$, the kets and the bras corresponding to the elements of $S_{X,A}$ are called test kets and test bras. We mention the following relations

(3.2.i) $\quad \langle F(t) \mid G\rangle(0) = \langle F \mid G(t)\rangle(0) = \langle F \mid G\rangle(t)$

(3.2.ii) $\quad \langle F \mid G\rangle = \overline{\langle G \mid F\rangle}$.

4. A mathematical interpretation of some aspects of Dirac's formalism

In this section we assume that $(|\alpha\rangle)_{\alpha\in M}$ is a Dirac basis in $T_{X,A}$ with respect to the σ-finite nonnegative measure μ on M. By Theorem (2.4) any test ket $|g\rangle$ can be written as

$$(4.1) \qquad |g\rangle(0) = \int_M \langle \alpha \mid g\rangle(-\tau) \ |\alpha\rangle(\tau) \ d\mu_\alpha$$

where the expression (4.1) does not depend on the choice of $\tau > 0$ and τ has to be chosen so small that $|g\rangle(-\tau) \in S_{X,A}$.

(4.2) Theorem.

Let $|f\rangle$ be a test ket. Then

$$|f\rangle = \int_M \langle\alpha \mid f\rangle(0) \; |\alpha\rangle \, d\mu_\alpha$$

i.e.

$$|f\rangle(t) = \int_M \langle\alpha \mid f\rangle(-\tau) \; |\alpha\rangle(t+\tau) \, d\mu_\alpha \quad , \quad t > -\tau ,$$

where $\tau \geq 0$ has to be chosen so small that $|f\rangle(-\tau) \in S_{X,A}$.

Proof. Let $t > -\tau$ and put $g = |f\rangle(t)$. Then we have seen that

$$g = |g\rangle(0) = \int_M \langle\alpha \mid g\rangle(-t-\tau) \; |\alpha\rangle(t+\tau) \, d\mu_\alpha .$$

From the identity $\langle\alpha \mid g\rangle(-t-\tau) = \langle\alpha \mid f\rangle(-\tau)$ the assertion follows. □

Parseval's identity is an immediate consequence of the definition of Dirac basis.

$$(4.3) \qquad \|f\|^2 = \int_M |\langle\alpha \mid f\rangle(0)|^2 \, d\mu_\alpha \quad , \quad f \in S_{X,A} .$$

From Theorem (4.1) we obtain that for every ket $|F\rangle$ and every $t > 0$

$$(4.4) \qquad |F\rangle(t) = e^{-(t-\tau)A} \, |F\rangle(\tau) = \int_M \langle\alpha \mid F\rangle(\tau) \; |\alpha\rangle(t-\tau) \, d\mu_\alpha .$$

The integral expression (4.4) does not depend on the choice of τ, $0 < \tau < t$ and the integral converges absolutely in X. The ket $|F\rangle$ can thus be expressed by

$$|F\rangle \ : \ t \mapsto \int_M \langle\alpha \mid F\rangle(\tau) \ \ |\alpha\rangle(t-\tau)\ d\mu_\alpha$$

independent of the choice of $0 < \tau < t$.

(4.5) Definition.

By the expression $\int_M \langle\alpha \mid F\rangle \ |\alpha\rangle\, d\mu_\alpha$ is meant the trajectory

$$t \mapsto \int_M \langle\alpha \mid F\rangle(\tau) \ \ |\alpha\rangle\,(t-\tau)\ d\mu_\alpha \ .$$

It follows that

(4.6) Theorem.

$$|F\rangle \ = \int_M \langle\alpha \mid F\rangle \ |\alpha\rangle\, d\mu_\alpha \ .$$

The integral converges in $T_{X,A}$.

Consider the following identity

$$\langle\alpha \mid \beta\rangle(t) \ = \sum_{k=1}^{\infty} \ e^{-\lambda_k t} \langle\alpha \mid v_k\rangle(0) \ \langle v_k \mid \beta\rangle(0)$$

where $\alpha,\beta \in M$. Following Section 2 there exists a unitary operator U from X onto a Hilbert subspace Y of $L_2(M,\mu)$ such that the function $\alpha \mapsto \langle\alpha \mid v_k\rangle(0)$ is a representant of $U v_k$. Moreover, $U(S_{X,A}) = S_{Y,B}$ and $U(T_{X,A}) = T_{Y,B}$ where $B = U A U^*$.

Let δ_β, $\beta \in M$, denote the trajectory

$$\delta_\beta \;:\; t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t} < v_k \mid \beta > (0)\, U\, v_k \,.$$

Then $\delta_\beta = U \mid \beta > \in T_{Y,B}$. Thus it is clear that

$$< U f, \delta_\beta > = < \beta \mid f > (0) \quad , \quad f \in S_{X,A} \,.$$

Since the function $\beta \mapsto < \beta \mid f > (0)$ is a representant of $U f$, the functional δ_β is an evaluation functional, popularly speaking a Dirac delta function. Since $\delta_\beta(\alpha,t) = < \delta_\beta(t), \delta_\alpha > = < \alpha \mid \beta > (t)$ the generalization of the orthonormality relations for discrete basis as suggested by Dirac

$$(4.7) \qquad < \alpha \mid \beta > = \delta_\beta(\alpha)$$

admits the interpretation $< \alpha \mid \beta > (t) = \delta_\beta(\alpha, t)$.

Many mathematicians have tried to interprete Dirac's formalism of quantum mechanics. We mention the following references.
[An], [Bö], [Hi], [Ja], [Me] and [Ro].

5. Matrices with respect to Dirac bases

In [G], Ch. V, it is shown that $T_{X \otimes X, A \boxplus A}$ is a completion of the algebraic tensor product $T_{X,A} \otimes_a T_{X,A}$. Here $A \boxplus A$ denotes the nonnegative self-adjoint operator $A \otimes I + I \otimes A$ which satisfies

$$(5.1.\text{i}) \qquad (A \boxplus A)(v_k \otimes v_\ell) = (\lambda_k + \lambda_\ell)(v_k \otimes v_\ell) \quad , \quad k, \ell \in \mathbb{N} \,.$$

$$(5.1.\text{ii}) \qquad e^{-t(A \boxplus A)} = e^{-tA} \otimes e^{-tA} \quad , \quad t > 0 \,.$$

It easily follows that the space $T_{X\otimes X, A\boxplus A}$ is nuclear. The simplest elements of $T_{X\otimes X, A\boxplus A}$ are given by $F \otimes G$ where $F,G \in T_{X,A}$. We note that

$$F \otimes G : t \mapsto F(t) \otimes G(t) \quad , \quad t > 0 .$$

In Dirac's bracket notation these elements are expressed by $|G\rangle\langle F|$. Now let $(|\alpha\rangle)_{\alpha\in M}$ be a Dirac basis in $T_{X,A}$ with respect to the measure μ. Then replacing M by $M\times M$ we find the following result.

(5.2) Theorem.

The elements $|\beta\rangle\langle\alpha|$, $\alpha,\beta \in M$ establishes a Dirac basis in $T_{X\otimes X, A\boxplus A}$ with respect to the measure $\mu \otimes \mu$ on $M \times M$.

Proof. First observe that the functions $(\alpha,\beta) \mapsto \langle v_\ell \mid \beta\rangle(0)\langle\alpha \mid v_k\rangle(0)$ are measurable, because $(|\alpha\rangle)_{\alpha\in M}$ is a Dirac basis (cf. Definition (2.1)). Moreover, it easily follows that

$$\int_{M\times M} \langle v_\ell \mid \beta\rangle(0)\langle\beta \mid v_m\rangle(0)\langle\alpha \mid v_k\rangle(0)\langle v_n \mid \alpha\rangle(0)\, d\mu_\alpha\, d\mu_\beta = \delta_{\ell m}\delta_{kn}$$

for all $k,\ell,m,n \in \mathbb{N}$. □

It leads to the following definition.

(5.3) Definition.

Let $R \in T_{X\otimes X, A\boxplus A}$, and let $(\alpha,\beta) \in M\times M$. Then we define the function $[R]_{\alpha\beta} : (0,\infty) \mapsto \mathbb{C}$ by

$$[R]_{\alpha\beta} : t \mapsto \langle R(t) , |\alpha\rangle\langle\beta|\rangle_{X\otimes X} .$$

This definition makes sense because $\mathcal{R}(t) \in S_{X\otimes X, A\boxplus A}$ for all $t > 0$. It follows that $[\mathcal{R}]_{\alpha\beta}(t) = \langle\alpha \mid \mathcal{R}(t) \mid \beta\rangle$ because $\mathcal{R}(t)$ can be seen as a continuous linear mapping from $T_{X,A}$ into $S_{X,A}$ by [G], Ch. VI. For $\mathcal{B} \in S_{X\otimes X, A\boxplus A}$ the functions $[\mathcal{B}]_{\alpha\beta}$ can be extended to a larger domain. For such $\mathcal{B}$ there exists $\tau > 0$ such that $e^{\tau(A\boxplus A)}\mathcal{B} \in S_{X\otimes X, A\boxplus A}$. Hence $[\mathcal{B}]_{\alpha\beta}(t)$ is well-defined for $t \geq -\tau$. Now a reformulation of Theorem (2.4) leads to the following result.

(5.4) Lemma.

Let $\mathcal{B} \in S_{X\otimes X, A\boxplus A}$. Then

$$\mathcal{B} = \int_{M\times M} [\mathcal{B}]_{\alpha\beta}(-\tau) \; (|\alpha\rangle\langle\beta|)(\tau) \, d\mu_\alpha \, d\mu_\beta$$

or, equivalently, for all $t > -\tau$

$$e^{-t(A\boxplus A)}\mathcal{B} = \int_{M\times M} [\mathcal{B}]_{\alpha\beta}(-\tau) \; (|\alpha\rangle\langle\beta|) \; (t+\tau) \, d\mu_\alpha \, d\mu_\beta$$

where $\tau \geq 0$ has to be chosen so small that $e^{\tau(A\boxplus A)}\mathcal{B} \in S_{X\otimes X, A\boxplus A}$.

So the previous lemma gives an interpretation of the following heuristic formula

$$(5.5) \qquad \mathcal{B} = \int_{M\times M} [\mathcal{B}]_{\alpha\beta} \; |\alpha\rangle\langle\beta| \; d\mu_\alpha \, d\mu_\beta \, .$$

Hence it seems proper to call $[\mathcal{B}]$ the matrix of $\mathcal{B}$ with respect to the Dirac basis $(|\alpha\rangle)_{\alpha\in M}$. In addition, following [G], Kernel theorem (6.1), the ket $\mathcal{B}|F\rangle$ is a test ket for each ket $|F\rangle$. We mention the following identities.

$$(5.6) \qquad \mathcal{B}|F\rangle \; : \; t \mapsto \int_{M\times M} [\mathcal{B}]_{\alpha\beta}(-\tau) \; \langle\beta \mid F\rangle(\tau) \; |\alpha\rangle \; (t+\tau) \, d\mu_\alpha \, d\mu_\beta$$

for all $t > -\tau$, and

$$(5.7) \qquad <\alpha \mid B \mid F>(-\tau) = \int_M [B]_{\alpha\beta}(-\tau) <\beta \mid F>(\tau)\, d\mu_\beta .$$

(Cf. [E], p. 37.)

(5.8) Theorem.

Let $R \in T_{X \otimes X, A \boxplus A}$. Then

$$R = \int_{M\times M} [R]_{\alpha\beta} \; (|\alpha><\beta|)\, d\mu_\alpha\, d\mu_\beta$$

where the integral has to be understood in the following sense

$$(5.8') \qquad R : t \mapsto \int_{M\times M} [R]_{\alpha\beta}(\tau) \; (|\alpha><\beta|)(t-\tau)\, d\mu_\alpha\, d\mu_\beta .$$

The integrals do not depend on the choice of τ, $0 < \tau < t$ and exist as integrals of measurable functions from $M \times M$ into $X \otimes X$.

Following the above theorem it makes sense to call the expression $[R]$ defined in (5.2) the matrix of R.

Let $|f>$ be a test ket. Then by [G], Kernel Theorem (6.2), $R|f>$ is a ket. It is given by the formula

$$(5.9) \qquad R|f> : t \mapsto \int_{M\times M} [R]_{\alpha\beta}(\tau) <\beta \mid f>(-\tau) \; |\alpha>(t-\tau)\, d\mu_\alpha\, d\mu_\beta$$

and the "α-th component" of $R|f>$

$$(5.10) \qquad <\alpha \mid R \mid f> : t \mapsto \int_M [e^{-(t-\tau)A}R]_{\alpha\beta}(\tau) <\beta \mid f>(-\tau)\, d\mu_\beta .$$

Here $0 < \tau < t$ has to be taken so small that $|f\rangle(-\tau) \in S_{X,A}$. The proof of (5.9) and (5.10) is straightforward and is omitted. We note further that for all $0 < s < \tau$ we have

$$(5.10') \quad \langle\alpha\,|\,R\,|\,f\rangle : s \mapsto \int_M [R]_{\alpha\beta}(s) \, \langle\beta\,|\,f\rangle(-s) \, d\mu_\beta \, .$$

In [E] we have introduced a similar matrix notion for the continuous linear mappings from $S_{X,A}$ into itself and, also, from $T_{X,A}$ into itself. The following formulas have got a rigorous interpretation in [E], p. 38 - 40. Let $L : S_{X,A} \to S_{X,A}$ be continuous and linear. Then with L there can be associated the matrix $[L] : (\alpha,\beta) \mapsto [L]_{\alpha\beta}$, $(\alpha,\beta) \in M \times M$, such that

$$(5.11.1) \quad L = \int_{M\times M} [L]_{\alpha\beta} \, |\alpha\rangle\langle\beta| \, d\mu_\alpha \, d\mu_\beta \, ,$$

$$(5.11.2) \quad L|f\rangle = \int_{M\times M} [L]_{\alpha\beta} \, \langle\beta\,|\,f\rangle \, |\alpha\rangle \, d\mu_\alpha \, d\mu_\beta \quad , \quad |f\rangle \in S_{X,A} \, ,$$

$$(5.11.3) \quad [L_1 L_2]_{\alpha\beta} = \int_M [L_1]_{\alpha\gamma} \, [L_2]_{\gamma\beta} \, d\mu_\gamma \quad , \quad \alpha,\beta \in M \, .$$

In the same manner continuous linear mappings from $T_{X,A}$ into itself are treated.

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