Analysis of both kinematically and statically admissible velocity fields in plane strain compression

ANALYSIS OF BOTH KINEMATICALLY AND STATICALLY ADMISSIBLE VELOCITY FIELDS IN PLANE STRAIN COMPRESSION.

P.C. Veenstra
H. Hijink

October 1978 PT-Rapport Nr. 0439

---

1. Introduction

The co-ordinate system used to describe a situation of plane strain compression (upsetting) of a specimen between parallel flat dies is defined in fig. 1.1.

Fig. 1.1.

The upper die moves with the velocity $-\frac{1}{2}\dot{U}_0$ with respect to the plane of symmetry $z = 0$, while the lower die has the velocity $+\frac{1}{2}\bar{U}_0$. In the frictionless case the velocity field is given by

$$\left.\begin{aligned} \dot{U}_x &= \frac{x}{h}\dot{U}_0 \\ \dot{U}_y &= 0 \\ \dot{U}_z &= -\frac{z}{h}\dot{U}_0 \end{aligned}\right\} \qquad (1.1)$$

The effect of friction, which presents itself by bulging of the specimen, can be accounted for by introducing

$$\dot{U}_x = \dot{U}_x\ (x,z) \tag{1.2}$$

The function must be symmetrical with respect to z = 0 and assume there a maximum value. Moreover the function must be antisymmetrical with respect to x = 0.

The present study aims to trace those velocity fields which both stabilize or minimize the system with respect to power and satisfy equilibrium conditions in case of non strain-hardening material.

2. The calculus

In the analysis reduced (dimensionless) quantities are used

$$\left.\begin{array}{l} x^* = \frac{x}{h}\ ;\ z^* = \frac{z}{h}\ ;\ b^* = \frac{b}{h} \\ \sigma^*_{ij} = \frac{\sigma_{ij}}{\bar{\sigma}}\ \ ;\ \dot{\varepsilon}^*_{ij} = \dot{\varepsilon}_{ij}\ \frac{h}{\dot{U}_0} \\ \dot{U}^*_i = \frac{\dot{U}_i}{\dot{U}_0} \end{array}\right\} \tag{2.1}$$

For the sake of simplicity the asterisks will be omitted.

Introduce the velocity field

$$\dot{U}_x = x\ \{P_0 + f\ (x,z,P_i)\} \tag{2.2}$$

where P_i are free parameters.

The relation must satisfy the conditions

$$\left.\begin{array}{l} \dot{U}_x\ (x,o,P_i) = \max \\ \dot{U}_x\ (x,z,P_i) = \dot{U}_x\ (x,-z,P_i) \\ \dot{U}_x\ (x,z,P_i) = -\dot{U}_x\ (-x,z,P_i) \end{array}\right\} \tag{2.3}$$

The continuity of the material flow requires

$$\tfrac{1}{2}\, x = \int_0^{\frac{1}{2}} \dot{U}_x \, dz \qquad (2.4)$$

from which follows that

$$P_0 = 1 - 2 \int_0^{\frac{1}{2}} f\,(x,z,P_i)\, dz \qquad (2.5)$$

It is denoted

$$J = \int_0^{\frac{1}{2}} f\,(x,z,P_i)\, dz \qquad (2.6)$$

and hence

$$P_0 = 1 - 2J \qquad (2.7)$$

From eqs. 2.2 and 2.7 it is derived

$$\dot{\varepsilon}_{xx} = 1 - 2J + f\,(x,z,P_i) + x\,\frac{\partial f}{\partial x} \qquad (2.8)$$

Because of the situation of plane strain it is found

$$-\dot{U}_z = \int \dot{\varepsilon}_{xx}\, dz = (1-2J)z + \int f(x,z,P_i)dz + x\int \frac{\partial f}{\partial x}\, dz \qquad (2.9)$$

which must satisfy the boundary conditions

$$\left.\begin{aligned} \dot{U}_z &= 0 && \text{for } z = 0 \\ \dot{U}_z &= -\tfrac{1}{2} && \text{for } z = \tfrac{1}{2} \end{aligned}\right\} \qquad (2.10)$$

Next it can be calculated

$$\dot{\varepsilon}_{xz} = \tfrac{1}{2} \left(\frac{\partial \dot{U}_x}{\partial z} + \frac{\partial \dot{U}_z}{\partial x} \right) \qquad (2.11)$$

and in plane strain

$$\dot{\bar{\varepsilon}} = \frac{2}{\sqrt{3}} \left(\dot{\varepsilon}_{xx}^{\,2} + \dot{\varepsilon}_{xz}^{\,2} \right)^{\frac{1}{2}} \qquad (2.12)$$

from which according to Levy-von Mises is derived

$$\tau_{xz} = \frac{2}{3} \frac{\dot{\varepsilon}_{xz}}{\dot{\bar{\varepsilon}}} \qquad (2.13)$$

Through this the frictional stress in the contact plane is known as

$$\tau_0 = \frac{2}{3} \frac{\dot{\varepsilon}_{xz}(x,\tfrac{1}{2},P_i)}{\dot{\bar{\varepsilon}}(x,\tfrac{1}{2},P_i)} \qquad (2.14)$$

From eqs. 2.2. and 2.7 the relative sliding velocity in the contact plane is found to be

$$\dot{U}_{xo} = x\,\{1 - 2J + f\,(x,\tfrac{1}{2},P_i)\} \qquad (2.15)$$

Since physically it must hold that

$$\dot{U}_{xo} \geq 0 \qquad \text{for} \qquad x \geq 0$$

the kinematical constraint is

$$f\,(0,\tfrac{1}{2},P_i) - 2J \geq -1 \qquad (2.16)$$

The reduced press force is defined as

$$F^* = \frac{F}{b\,.\,w\,.\,\bar{\sigma}} \qquad (2.17)$$

where w is the width of the specimen and hence the product b.w is the surface contact area.
Because of equilibrium it must hold for any value of z

$$F^* = \frac{\{\sigma_{AVE}\}_z}{\bar{\sigma}} \qquad (2.18)$$

where

$$\{\sigma_{AVE}\}_z = \frac{1}{b/2} \int_0^{b/2} \sigma_z \, dx \qquad (2.19)$$

On the other hand the power dissipated in the process equals

$$P = F\,\dot{U}_0 = F_d\,\dot{U}_0 + F_f\,\dot{U}_0 \qquad (2.20)$$

where F_d represents the contribution of deformation to the press force, whereas F_f does so with respect to friction.

Through this it can readily be shown to hold in dimensionless notation

$$F_d = \frac{4}{b} \int_{x=0}^{b/2} \int_{z=0}^{1/2} \dot{\varepsilon} \, dx \, dz \qquad *) \qquad (2.21)$$

and

$$F_f = \frac{4}{b} \int_{0}^{b/2} |\tau_0| \, \dot{U}_{xo} \, dz \qquad *) \qquad (2.22)$$

which through eqs. 2.12, 2.13 and 2.15 can be expressed in terms of the co-ordinates and the parameters P_i. Thus it becomes possible to investigate whether the sum $F_d + F_f = F$ minimizes of stabilizes power as a function of the parameters.

The equations for equilibrium in plane strain are

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} &= - \frac{\partial \tau_{xz}}{\partial z} \\ \frac{\partial \sigma_z}{\partial z} &= - \frac{\partial \tau_{xz}}{\partial x} \end{aligned}\right\} \qquad (2.23)$$

Because of symmetry it must hold that $\dot{\varepsilon}_{xz} = 0$ for $z = 0$ and hence $\tau_{xz} = 0$. From the plasticity condition it follows that

$$\{\sigma_z\}_{z=0} = \{\sigma_x\}_{z=0} - \frac{2}{\sqrt{3}}$$

The boundary condition is

$$\{\sigma_x\}_{z=0} = 0 \qquad \text{for } x = b/2$$

and thus

$$\{\sigma_z\}_{z=0} = - \frac{2}{\sqrt{3}} + \int_{x}^{b/2} \left\{\frac{\partial \tau_{xz}}{\partial z}\right\}_{z=0} dx \qquad (2.24)$$

and next

$$\left\{\sigma_{z_{AVE}}\right\}_{z=0} = - \frac{2}{\sqrt{3}} + \frac{1}{b/2} \int_{0}^{b/2} \int_{x}^{b/2} \left\{\frac{\partial \tau_{xz}}{\partial z}\right\}_{z=0} dx \, dx \qquad (2.25)$$

The second equation of equilibrium renders

$$\{\sigma_z\}_{z=\frac{1}{2}} = - \int_{0}^{\frac{1}{2}} \left\{\frac{\partial \tau_{xz}}{\partial x}\right\}_{x} dz + \{\sigma_z\}_{z=0} \qquad (2.26)$$

---

*) Asterisks are omitted.

Body equilibrium requires that

$$\left\{\sigma_{z_{AVE}}\right\}_{z=\frac{1}{2}} = \left\{\sigma_{z_{AVE}}\right\}_{z=0}$$

and hence

$$\frac{1}{b/2} \int_0^{b/2} \int_0^{1/2} \left\{\frac{\partial \tau_{xz}}{\partial x}\right\}_x dz\, dx = 0 \qquad (2.27)$$

Now all conditions for a kinematically admissible velocity field which both minimizes power and satisfies equilibrium are defined in the eqs. 2.21, 2.22, 2.29 and 2.31.

It can be shown that all velocity fields which are developable in power series, like the parabolic field

$$\dot{U}_x = x\,\{P_0 - Pz^2\}$$

or the cosine field, the exponential field, etc., neither satisfy the condition of minimum power nor equilibrium.
The first field encountered which satisfies power conditions is the hyper elliptic field

$$\dot{U}_x = x\left[A + B\,\{1 - (2Pz)^m\}^{1/m}\right]$$

However from this class of velocity fields only the elliptic field proves to satisfy the conditions of equilibrium with respect to $z = 0$. In order to satisfy body equilibrium the field has to be modified, as will be discussed later.

3. The elliptic velocity field

$$\dot{U}_x = x\left[A + B\sqrt{1 - (2Pz)^2}\right] \qquad (3.1)$$

From eq. 2.6 it follows

$$J = \int_0^{\frac{1}{2}} \sqrt{1-(2Pz)^2}\, dz = \frac{1}{4}\left\{\sqrt{1-P^2} + \frac{1}{P}\,\text{arc sin}\, P\right\} \qquad (3.2)$$

and thus through eqs. 2.7 and 2.1

$$\dot{U}_x = x\left[1 + B\left\{\sqrt{1-(2Pz)^2} - 2J\right\}\right] \tag{3.3}$$

$$\dot{\varepsilon}_{xx} = 1 + B\left\{\sqrt{1-(2Pz)^2} - 2J\right\} \tag{3.4}$$

$$-\dot{U}_z = z\left[1 + B\left\{\frac{1}{2}\sqrt{(1-2Pz)^2} - 2J\right\}\right] + \frac{B}{4P}\text{ arc sin } 2Pz \tag{3.5}$$

which satisfies the boundary conditions at z = 0 and at z = ½.

$$\left.\begin{aligned} &\frac{\partial \dot{U}_z}{\partial x} = 0 \\ &\frac{1}{2}\frac{\partial \dot{U}_x}{\partial z} = \dot{\varepsilon}_{xz} = -\,2BP^2\frac{xz}{\sqrt{(1-(2Pz)^2}} \end{aligned}\right| \tag{3.6}$$

$$\dot{\bar{\varepsilon}} = \frac{2}{\sqrt{3}}\left[\left[1 + B\left\{\sqrt{1-(2Pz)^2} - 2J\right\}\right]^2 + \frac{[2BP^2xz]^2}{1-(2Pz)^2}\right]^{\frac{1}{2}} \tag{3.7}$$

$$\tau_{xz} = -\frac{2}{\sqrt{3}}BP^2\frac{xz}{\left[\left\{1-(2Pz)^2\right\}\left\{1+B(\sqrt{1-(2Pz)^2} - 2J)\right\}^2+\left\{2PB^2xz\right\}^2\right]^{\frac{1}{2}}} \tag{3.8}$$

Thus the frictional stress is

$$\tau_0 = -\frac{1}{\sqrt{3}}BP^2\frac{x}{\left[(1-P^2)\left\{1+\frac{B}{2}\left(\sqrt{1-P^2}-\frac{1}{P}\text{ arc sin P}\right)\right\}^2 + \left\{BP^2x\right\}^2\right]^{\frac{1}{2}}} \tag{3.9}$$

The relative sliding velocity follows from eq. 3.3

$$\dot{U}_{xo} = x\left[1 + \frac{B}{2}\left(\sqrt{1-P^2} - \frac{1}{P}\text{ arc sin P}\right)\right] \tag{3.10}$$

Hence the kinematical constraint is

$$B \leq \frac{2}{\frac{1}{P}\text{ arc sin P} - \sqrt{1-P^2}} = B_{MAX} \tag{3.11}$$

From eqs. 3.9 and 3.10 follows the contribution of friction to the reduced press force

$$F_f = \frac{2}{\sqrt{3}}\cdot\frac{1}{b/2}\int_0^{b/2}\frac{BP^2\left[1+\frac{B}{2}\left(\sqrt{1-P^2} - \frac{1}{P}\text{ arc sin P}\right)\right]x^2}{\left[(1-P^2)\left\{1+\frac{B}{2}\left(\sqrt{1-P^2}-\frac{1}{P}\text{ arc sin P}\right)\right\}^2+\left\{BP^2x\right\}^2\right]^{\frac{1}{2}}}dx \tag{3.12}$$

The contribution of deformation is

$$F_d = \frac{4}{\sqrt{3}} \cdot \frac{1}{b/2} \int_{x=0}^{b/2} \int_{z=0}^{1/2} \left[\left\{ 1+B\left(\sqrt{1-(2Pz)^2}-2J\right) \right\}^2 + \frac{\{2BP^2xz\}^2}{1-(2Pz)^2} \right]^{\frac{1}{2}} dx\, dz \qquad (3.13)$$

Numerical variational analysis of $F = F_f + F_d$ shows that this functional minimizes with respect to the parameters B and P. A typical example is shown in fig. 3.1.

Fig. 3.1

From this it is clear that the minimum is extremely flat, which implies that it is impossible to determine the value B_{opt} sufficiently accurate. The same holds even more for P_{opt}.
However, when now considering the conditions for equilibrium, it follows from eq. 3.8 that

$$\left\{ \frac{\partial \tau_{xz}}{\partial z} \right\}_{z=0} = - \frac{2}{\sqrt{3}} \; \frac{BP^2 x}{1 + B\ (1-2J)} \qquad (3.13)$$

and hence according to eq. 2.24

$$\{\sigma_z\}_{z=0} = - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \; \frac{BP^2}{1 + B\ (1-2J)} \left\{ \left(\frac{b}{2}\right)^2 - x^2 \right\} \qquad (3.14)$$

Next it is found through eq. 2.25

$$\left\{ \sigma_{AVE} \right\}_{z=0} = - \frac{2}{\sqrt{3}} \left\{ 1 + \frac{1}{3} \; \frac{BP^2}{1 + B\ (1-2J)} \left(\frac{b}{2}\right)^2 \right\} \qquad (3.15)$$

The condition

$$F_{min} = \left\{\sigma_{z_{AVE}}\right\}_{z=0}$$

yields

$$B = \frac{3\,(\frac{1}{2}\sqrt{3}\,F_{min} - 1)}{P\left(\frac{b}{2}\right)^2 - 3\,(\frac{1}{2}\sqrt{3}\,F_{min} - 1)(1 - 2J)} \qquad (3.15)$$

This is visualized in fig. 3.1 where the curve according to eq. 3.15 intersects the curve corresponding to F in a well defined way. From the numerical analysis of F it appears that $P \approx 1$ and that variation of this parameter has only a minor influence on F_{min}. Using estimated values for P as obtained from the numerical analysis and applying eq. 3.15 the table 3.1 results which is plotted in fig. 3.2.

Fig. 3.2.

Table 3.1

b	P_{EST}	F_{MIN}	F_f	F_d	B	bB
1	0.98	1.332	0.084	1.248	3.126	3.13
2	0.98	1.585	0.228	1.358	1.501	3.00
4	0.99	2.135	0.687	1.448	0.744	2.98
6	0.99	2.702	1.206	1.496	0.496	2.98
8	0.99	3.227	1.767	1.510	0.373	2.98
10	0.99	3.848	2.303	1.545	0.298	2.98
12	0.99	4.423	2.766	1.657	0.249	2.99
16	0.99	5.586	3.695	1.891	0.187	2.99

It is concluded that

1. Though neither the quantity F_f nor F_d is a linear function of the compression ratio b, the minimum reduced press force F_{min} and hence minimum power in the system behaves virtually linearly.

This agrees with the slip line solution after Prandtl [1] and the solution proposed by Unksow [2], though the latter refers to rotational symmetry.

2. As the compression ratio increases the effect of friction becomes increasingly dominant.
3. Most probably there exists a hyperbolic relationship between the shape factor B of the velocity field and the compression ratio b

$$B \cdot b = 3 \qquad (3.16)$$

Now, when introducing this relation in eq. 3.15 more precise values for the parameter P can be found as a function of b.
This is not important for power, but it proves to be relevant for the stress distribution. Performing the calculations the results as listed in table 3.2 are found.

		Table 3.2				
b	P	F_{MIN}	B	B_{MAX}	1-P	b(1-P)
1	n	o s	o l u t	i o n		
2	0.9830	1.585	1.5000	1.5657	0.0170	0.0340
4	0.9907	2.135	0.7500	1.5247	0.0093	0.0372
6	0.9936	2.702	0.5000	1.4770	0.0064	0.0384
8	0.9955	3.277	0.3750	1.4306	0.0045	0.0360
10	0.9963	3.848	0.3000	1.4263	0.0037	0.0370
12	0.9968	4.423	0.2500	1.4128	0.0032	0.0384
16	0.9975	5.586	0.1875	1.3487	0.0025	0.0400
						0.0373

From this it is concluded:

1. The hyperbolic relation 3.16 implies that the shape factor P in case of minimum power and equilibrium is also controlled by a hyperbolic relation

$$b\ (1-P) = \frac{1}{3^3} = 0.037 \qquad (3.17)$$

2. In case that $b < 1.8$ no solution is found because the kinematical constraint eq. 3.11 is violated.
   The conditions for minimum power and equilibrium cannot simultaneously be satisfied. Probably a kind of mechanical instability is present.

The results obtained up to now make it possible to formulate the elliptic velocity field explicitely in terms of the compression ratio b.
Consequently the stress distribution according to eqs. 2.13, 2.14, 2.24 and 2.26 resp. can be calculated for any state of compression.
However, since τ_{xz} according to eq. 3.8 is a steady function of x, $\{\sigma_z\}_{z=\frac{1}{2}}$ turns out to be less in absolute value than the corresponding value $\{\sigma_z\}_{z=0}$. For this reason $\left\{\sigma_{z_{AVE}}\right\}_{z=\frac{1}{2}} \neq \left\{\sigma_{z_{AVE}}\right\}_{z=0}$, which violates the conditions for body equilibrium.

4. The modified elliptic velocity field

The problem that the elliptic field does not satisfy body equilibrium can be met by modifying the field in such a way that the shear stress is a non steady function of x.
This is achieved if the frictional shear stress vanishes at the edge ($x = b/2$, $z = 1/2$) as well as in the plane of symmetry ($x = 0$).
The modified elliptic velocity field which satisfies this requirement is defined by

$$\dot{U}_x = x \left[1 + B\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^n \left\{\sqrt{1 - (2Pz)^2} - 2J\right\}\right] \qquad (4.1)$$

where

$$\begin{cases} n \geq o \\ \mu \approx 1 \end{cases}$$

The quantity μ is introduced in order to avoid instability of the computation at the very edge $x = b/2$. Its physical meaning is that at this edge some frictional stress is assumed to be present, for instance of a Coulumb nature.
Following the same procedure as in the previous sections it is found

$$\dot{\varepsilon}_{xx} = 1 + B\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^{n-1}\left\{\left(\frac{b}{2}\right)^2 - (2n+1)(\mu x)^2\right\}\left\{\sqrt{1-(2Pz)^2} - 2J\right\} \quad (4.2)$$

$$-\dot{U}_z = z + B\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^{n-1}\left\{\left(\frac{b}{2}\right)^2 - (2n+1)(\mu x)^2\right\}\left\{\frac{1}{2} z\sqrt{1-(2Pz)^2} + \frac{1}{4P}\arcsin 2Pz - 2Jz\right\} \quad (4.3)$$

$$\frac{1}{2}\frac{\partial \dot{U}_x}{\partial z} = -2BP^2\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^n \frac{xz}{\sqrt{1-(2Pz)^2}} \quad (4.4)$$

$$\frac{1}{2}\frac{\partial \dot{U}_z}{\partial x} = B\mu^2 n\, x\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^{n-2}\left\{3\left(\frac{b}{2}\right)^2 - (2n+1)(\mu x)^2\right\}\left\{\frac{1}{2} z\sqrt{1-(2Pz)^2} + \frac{1}{4P}\arcsin 2Pz - 2Jz\right\} \quad (4.5)$$

From these results obtained and through eqs. 2.11, 2.12 and 2.13 the quantities $\dot{\varepsilon}$ and τ_{xz} can be calculated .

For $z = \frac{1}{2}$ the frictional stress is found to be

$$\tau_o = -\frac{1}{\sqrt{3}} BP^2 \frac{\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^n x}{\left[(1-P^2)\left\{1+\frac{B}{2}\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^{n-1}\left\{\left(\frac{b}{2}\right)^2 - (2n+1)(\mu x)^2\right\}\right. \right.} \; * \quad (4.6)$$

$$* \; \overline{\left.\left.\left\{\sqrt{1-P^2} - \frac{1}{P}\arcsin P\right\}\right\}^2 + \left\{BP^2\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^n x\right\}^2\right]^{1/2}}$$

As a matter of fact $\tau_o = 0$ for $x = 0$, whereas it assumes a minor value for $x = \frac{b}{2}$ if μ is sufficiently close to one, as will be shown later in table 5.2.

The relative sliding velocity in the contact plane follows from eq. 4.1.

$$\dot{U}_{xo} = x\left[1 + \frac{B}{2}\left\{\left(\frac{b}{2}\right)^2 - (\mu x)^2\right\}^n\left\{\sqrt{1-P^2} - \frac{1}{P}\arcsin P\right\}\right] \quad (4.7)$$

The kinematical constraint is

$$B \leq \frac{2}{\left(\frac{b}{2}\right)^{2n} \left\{\frac{1}{P} \text{ arc sin } P - \sqrt{1-P^2}\right\}} = B_{max} \tag{4.8}$$

Firstly it has been investigated whether the parameter n has a significant influence on the power dissipated in the system as well as on the optimum values of B and P.
As shown in fig. 4.1. this proves to be not the case. If the parameter n is varied over a wide range as shown, minimum power varies about only 5%. However, the parameter is quite relevant for the ratio of the contribution of friction and the contribution of deformation to power. Physically this is the very reason that the parameter n takes care of body equilibrium.
In order to find the n-values which satisfy the condition for body equilibrium in the integral eq. 2.27

$$\Delta\sigma_{z_{AVE}} = \frac{1}{b/2} \int_0^{b/2} \int_0^{1/2} \left\{\frac{\partial\tau_{xz}}{\partial x}\right\}_x dz\, dx = 0 \tag{4.9}$$

the roots for n are solved.
The procedure is fully numerical, i.e. first τ_{xz} is calculated from the relations as derived before, next the function is numerically derivated with respect to x and subsequently integrated with respect to z and x. It is assumed that the hyperbolic relations 3.16 and 3.17 may be applied, whereas μ = 0.999 is introduced.
The results are listed in table 4.1.

TABLE 4.1

b	$n*10^4$	$3b^2$	$\Delta\sigma_{z_{AVE}} *10^2$
2	12	12	-2.280
4	48	48	-0.420
6	112	108	-0.106
8	195	102	-0.012
10	300	300	+0.018
12	440	432	+0.022
14	598	588	+0.010
16	775	768	-0.014
18	990	972	-0.049
[illegible]	[illegible]	[illegible]	[illegible]

It appears that the calculated values as shown in the second column can be very closely approximated by the relationship

$$n = 3b^2 \, 10^{-4} \tag{4.10}$$

In order to check its validity the relation is substituted into eq. 4.9 and the deviation from body equilibrium $\Delta\sigma_{AVE}$ thus obtained is listed in the last column.
It is clear that if equation 4.10 is applied the modified elliptic velocity field satisfies body equilibrium quite well.

5. Results

It is concluded that the modified elliptic velocity field satisfies requirements of minimum power, local equilibrium and body equilibrium if the relevant parameters are given by

$$\left.\begin{aligned} B &= 3/b \\ P &= 1 - \frac{1}{3^3 b} \\ n &= 3b^2 . \, 10^{-4} \end{aligned}\right\} \tag{5.1}$$

Since τ_{xz} now explicitely can be calculated as a function of the co-ordinates for a given value of the compression ratio b, through eqs. 2.24 and 2.26 the stress distributions in the plane of symmetry as well as in the contact surface can be computed.
The results are listed in table 5.1., whereas figs. 5.1, 5.2 and 5.3 visualize some stress distributions.
Finally in fig. 5.4 the stress at the point $\{x = 0 \;,\; z = 1/2\}$ is represented as a function of the compression ration. In most cases this value corresponds to the maximum tool load and for this reason it is of technological importance.

Conclusion

Apart from the fact that the modified elliptic velocity field is of theoretical interest because it explains the stress peaks at the edge of the specimen as observed in experiments [3], the introduction of this particular field does not greatly affect the stress situation as derived from the elliptic velocity field, if the compression ratio ranges in $2 \leq b^* \leq 10$.

In this case and in order to calculate the maximum stress on the tool the eq. 3.14 at the point $x = 0$ can be applied safely, as it overestimates slightly the maximum load in the contact plane.
Hence

$$\sigma_{z_{MAX}} \approx -\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \frac{BP^2}{1 + B(1 - 2J)} \left(\frac{b}{2}\right)^2 \qquad \Bigg\} \qquad 2 \leq b \leq 10$$

The relation is visualized by the dotted line in fig. 5.4.
However, when the compression ration increases and thus according to eq. 4.10 the parameter n increases, the influence of modifying the elliptic field on the stress situation becomes more and more significant as is clear from fig. 5.4 with respect to the maximum stress on the tool surface.

TABLE 5.2

b	$\frac{x}{b/2}$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
2	$\{\sigma_z\}_{z=0}$	1.79	1.78	1.76	1.73	1.69	1.63	1.56	1.47	1.38	1.27	1.15
	$\Delta\sigma_z$	0.26	0.25	0.23	0.17	0.14	0.12	0.10	0.09	0.08	0.06	+7.23
	$\{\sigma_z\}_{z=\frac{1}{2}}$	1.53	1.53	1.53	1.56	1.55	1.51	1.46	1.38	1.30	1.21	8.38
	τ_o	0.00	0.573	0.576	0.577	0.577	0.577	0.577	0.577	0.577	0.577	0.004
4	ID	2.61	2.60	2.56	2.48	2.38	2.25	2.08	1.89	1.67	1.42	1.15
		0.18	0.16	0.12	0.10	0.08	0.07	0.06	0.05	0.04	0.03	+3.97
		2.43	2.44	2.44	2.38	2.30	2.18	2.02	1.84	1.63	1.39	5.12
		0.00	0.524	0.562	0.571	0.574	0.575	0.576	0.576	0.576	0.577	0.003
6	ID	3.49	3.47	3.39	3.28	3.11	2.90	2.64	2.33	1.98	1.58	1.15
		0.13	0.11	0.08	0.07	0.05	0.04	0.04	0.03	0.03	0.02	+2.75
		3.34	3.36	3.31	3.21	3.06	2.86	2.60	2.30	1.95	1.56	3.90
		0.00	0.521	0.562	0.570	0.573	0.575	0.575	0.576	0.576	0.576	0.002
8	ID	4.43	4.39	4.29	4.13	3.89	3.59	3.23	2.80	2.30	1.75	1.15
		0.12	0.09	0.06	0.05	0.04	0.03	0.03	0.02	0.02	0.02	+2.11
		4.31	4.30	4.23	4.08	3.85	3.56	3.20	2.78	2.28	1.73	3.26
		0.00	0.527	0.564	0.571	0.574	0.575	0.576	0.576	0.576	0.576	0.002
10	ID	5.46	5.41	5.28	5.06	4.75	4.36	3.87	3.31	2.66	1.93	1.15
		0.10	0.07	0.05	0.04	0.03	0.03	0.02	0.02	0.02	0.01	+1.72
		5.36	5.34	5.23	5.02	4.72	4.33	3.85	3.29	2.64	1.92	2.87
		0.00	0.536	0.566	0.572	0.574	0.575	0.576	0.576	0.577	0.577	0.002
12	ID	6.63	6.57	6.40	6.12	5.72	5.22	4.60	3.88	3.05	2.13	1.15
		0.09	0.06	0.05	0.04	0.03	0.02	0.02	0.02	0.01	0.01	+1.45
		6.54	6.51	6.45	6.08	5.69	5.20	4.58	3.86	3.04	2.12	2.60
		0.00	0.544	0.568	0.573	0.575	0.576	0.576	0.577	0.577	0.577	0.001

TABLE 5.2 CNTD

b	x / (b/2)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
14	ID	8.00	7.93	7.71	7.36	6.86	6.22	5.45	4.54	3.50	2.35	1.15
		0.08	0.06	0.04	0.03	0.02	0.02	0.02	0.01	0.01	0.006	+1.26
		7.92	7.87	7.67	7.33	6.84	6.20	5.43	4.53	3.49	2.34	2.41
		0.00	0.551	0.570	0.574	0.576	0.576	0.576	0.577	0.577	0.577	0.001
16	ID	9.64	9.55	9.29	8.84	8.22	7.42	6.45	5.32	4.04	2.62	1.15
		0.08	0.05	0.04	0.03	0.02	0.02	0.01	0.01	0.01	0.004	+1.12
		9.56	9.50	9.25	8.81	8.20	7.40	6.44	5.31	4.03	2.62	2.27
		0.00	0.557	0.572	0.575	0.576	0.576	0.577	0.577	0.577	0.577	0.001
18	ID	11.67	11.56	11.22	10.66	9.88	8.88	7.68	6.27	4.69	2.94	1.15
		0.08	0.05	0.03	0.02	0.02	0.01	0.01	0.01	0.006	0.003	+1.00
		11.59	11.51	11.19	10.64	9.86	8.87	7.67	6.26	4.63	2.94	2.15
		0.00	0.563	0.574	0.576	0.576	0.577	0.577	0.577	0.577	0.577	0.001
20	ID	14.21	14.07	13.64	12.94	11.96	10.71	9.20	7.46	5.49	3.34	1.15
		0.08	0.05	0.03	0.02	0.02	0.01	0.01	0.007	0.005	0.001	+0.90
		14.13	14.02	13.61	12.92	11.94	10.70	9.19	7.39	5.48	3.34	2.05
		0.00	0.567	0.575	0.576	0.577	0.577	0.577	0.577	0.577	0.577	0.001
22	ID	17.46	17.29	16.75	15.86	14.62	13.04	11.15	8.96	6.50	3.84	1.15
		0.08	0.05	0.03	0.02	0.02	0.01	0.008	0.006	0.004	0.000	+0.81
		17.38	17.24	16.72	15.84	14.60	13.03	11.14	8.95	6.50	3.84	1.96
		0.00	0.571	0.576	0.577	0.577	0.577	0.577	0.577	0.577	0.577	0.001
24	ID	21.71	21.48	20.80	19.66	18.08	16.08	13.67	10.90	7.81	4.49	1.15
		0.09	0.05	0.03	0.02	0.01	0.01	0.007	0.005	0.003	0.000	+0.72
		21.62	21.43	20.77	19.64	18.07	16.07	13.66	10.89	7.81	4.49	1.87
		0.00	0.573	0.576	0.577	0.577	0.577	0.577	0.577	0.577	0.577	0.001