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by

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TH-Report 71-E-17

February 1971

Submitted in partial fulfillment of the requirements
for the degree of Ir. (M.Sc.) at the Eindhoven University of Technology.
The work was carried out in the Measurement and Control Group under
directorship of Prof.dr.ir. P. Eykhoff.
Advisor ir. A.J.W. van den Boom.

ISBN 90 6144 017 3

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Summary.

A discrete process, the output of which is disturbed by additive noise, is considered. The use of classical regression analysis for estimating the parameters of the process leads to - even asymptotically - biased estimates. To overcome this problem iterative schemes, based on the Gauss-Markov estimator, are discussed.

To achieve good results with these schemes one has to estimate, in general, more parameters than strictly necessary for describing the process and the additive noise.

Two schemes are derived for estimating a minimum number of parameters when a priori information is available. The results obtained with the different estimation schemes are good.

Contents.

1. Introduction.

One can give many reasons for trying to estimate the parameters of a process, for example:

1. Consider an industrial process, that one wants to control. To find an optimal regulater one has to know the dynamical behaviour of that process. This dynamical behaviour is fully described by the differential or difference equation, which gives the relation between the input and the output of that process.
2. Also in economics one has to know the parameters of the economical processes, if one wants to know what the result will be of some change in, for example, the economical behaviour of the gouvernment.
3. In medical science it can be very useful to know the parameters of the biological processes. The parameters of the dynamics of the arteries, for example, indicate something about the physical condidition of the arteries, which can be of interest for further research.

We see that there are many fields in which parameter estimation (and identification) can be applied.

In most cases (even when we are able to build a mathematical model of the process) the whole situation is too complex to handle. So we have to reduce the complexity of our model using a priori information about the process and we have to use our physical intuition to achieve an approximated model of the process, which can be handled mathematically.

A survey of system identification and parameter estimation was given by Åström & Eykhoff (3), in which paper 213 references are presented. In that paper several methods for solving estimation problems are given.

Our work is based on the least-squares estimator (L.S.-estimator). Under certain conditions, this estimator can work well if we know the order of difference equation, which describes the process. In the past different ways of determining the order of the process were given (Åström (1), Woodside (14)). We will assume, that the order of the process is known.

Starting from the least-squares estimator, we will derive several estimation schemes to estimate the parameters of the following set of equations:

$$\begin{cases} y_k = \sum\limits_{i=0}^{p} b_i u_{k-i} - \sum\limits_{i=1}^{q} a_i y_{k-i} + e_k \\ e_k = \sum\limits_{i=1}^{s} c_i \xi_{k-i} - \sum\limits_{i=1}^{r} d_i e_{k-i} + \xi_k, \end{cases}$$

in which ξ_k is assumed to be a white noise sample uncorrelated with the input sequence $\{u_k\}_1^N$.
For the different estimation schemes we will make assumptions for the values of p, q, s and r.
The different schemes are presented in chapter 2.
In chapter 3 we will give the results, obtained with the schemes discussed in this paper.
There will be some suggestions for future work, using the schemes given here (chapter 4).

In all computer programs the input and output data are generated by the digital computer.
In the simulation a transient will appear in the sequence $\{y_k\}_1^N$.
As the algorithms are derived for stationary signals, the first $10\times m$ ($m=p+q+r+s+1$) sample pairs are not used for the estimation of the parameters.

To get an idea about the performance of the estimators we compute a number of runs with different data, starting from the same initial conditions for the estimates of the parameters. Then we calculate the average and the standard deviation of the obtained estimates. These quantaties give an impression of the quality of the estimation scheme.

2. Some estimation schemes.

2.1 Least-squares estimation scheme.

Consider a discrete process P for which the relation between input and output can be described by the following difference equation (D.E.):

$$x_k = \sum_{i=0}^{p} b_i u_{k-i} - \sum_{i=1}^{q} a_i x_{k-i}. \tag{2.1}$$

Let the output x_k be disturbed by an additive noise signal n_k (see fig 2.1), viz;

$$y_k = x_k + n_k, \tag{2.2}$$

where y_k is the observable disturbed output of the process.

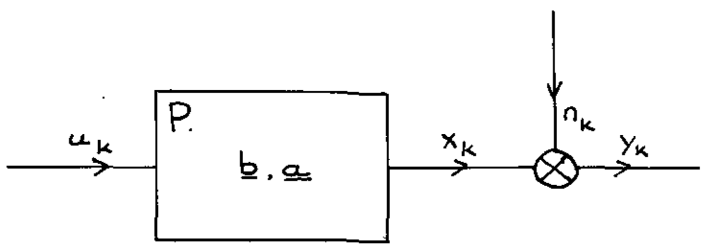


Fig 2.1 : A linear process. The output is disturbed with additive noise.

We want to have a relation between y_k and u_k, because these are the two signals we can observe.
From eq.(2.2) follows:

$$y_k + \sum_{i=1}^{q} a_i y_{k-i} = x_k + \sum_{i=1}^{q} a_i x_{k-i} + n_k + \sum_{i=1}^{q} a_i n_{k-i}. \tag{2.3}$$

Eq.(2.1) and eq.(2.3) give us the desired relation, viz:

$$y_k = \sum_{i=0}^{p} b_i u_{k-i} - \sum_{i=1}^{q} a_i y_{k-i} + n_k + \sum_{i=1}^{q} a_i n_{k-i}. \tag{2.4}$$

Suppose the sequences $\{u_k\}_1^N$ and $\{y_k\}_1^N$ are available ($N \gg p+q+1$)

If we assume that $q \geq p$, which is not necessary, we can write down the following set of equations:

$$
\begin{array}{l}
y_{q+1} = b_0 u_{q+1} + b_1 u_q + \ldots + b_p u_{q+1-p} - a_1 y_q - a_2 y_{q-1} - \ldots - a_q y_1 + e_{q+1} \\
y_{q+2} = b_0 u_{q+2} + b_1 u_{q+1} + \ldots + b_p u_{q+2-p} - a_1 y_{q+1} - a_2 y_q - \ldots - a_q y_2 + e_{q+2} \\
\vdots \\
y_N = b_0 u_N + b_1 u_{N-1} + \ldots + b_p u_{N-p} - a_1 y_{N-1} - a_2 y_{N-2} - \ldots - a_q y_{N-q} + e_N
\end{array}
\qquad (2.5)
$$

in which $e_k = n_k + a_1 n_{k-1} + a_2 n_{k-2} + \ldots + a_q n_{k-q}$ $\quad k=q+1,\ldots,N$ $\quad$ (2.6)

e_k is called the equation error.

The set of equations (2.5) can be written in matrix-notation:

$$\underline{y} = \Omega(u,y)\underline{b}' + \underline{e}, \qquad (2.7)$$

with $\underline{y}^T = (y_{q+1}, y_{q+2}, \ldots, y_N)$,

$\underline{b}'^T = (b_0, b_1, \ldots, b_p, -a_1, -a_2, \ldots, -a_q)$,

$\underline{e}^T = (e_{q+1}, e_{q+2}, \ldots, e_N)$, $\qquad$ (2.8)

$$
\Omega(u,y) = (U \vdots Y) = \left[\begin{array}{cccc|cccc}
u_{q+1} & u_q & \ldots & u_{q+1-p} & y_q & y_{q-1} & \ldots & y_1 \\
u_{q+2} & u_{q+1} & \ldots & u_{q+2-p} & y_{q+1} & y_q & \ldots & y_2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
u_N & u_{N-1} & \ldots & u_{N-p} & y_{N-1} & y_{N-2} & \ldots & y_{N-q}
\end{array}\right]
$$

Premultiplying eq.(2.7) with $\{\Omega^T(u,y)\Omega(u,y)\}^{-1}\Omega^T(u,y)$ gives:

$$\{\Omega^T(u,y)\Omega(u,y)\}^{-1}\Omega^T(u,y)\underline{y} = \underline{b}' + \{\Omega^T(u,y)\Omega(u,y)\}^{-1}\Omega^T(u,y)\underline{e}.$$

If we call $\underline{\beta}'$ the estimate of $\underline{b}'$, with

$$\underline{\beta}' = \{\Omega^T(u,y)\Omega(u,y)\}^{-1}\Omega^T(u,y)\underline{y}, \qquad (2.9)$$

then we have the same scheme as the least squares estimation scheme (Deutsch(5), Goldberger(8)).

We will prove, that in general $\underline{\beta}'$ is an asymptotically biased estimator of $\underline{b}'$ (Shaw(11), Evers(6)).

We suppose that u_k and n_k are samples of two mutually uncorrelated stationary stochastic processes and that $E\{u_k\}=0$ and/or $E\{n_k\}=0$. Combining eq.(2.9) with eq.(2.7) gives:

$$\underline{\beta}'=\{\Omega^T(u,y)\Omega(u,y)\}^{-1}\Omega^T(u,y)(\Omega(u,y)\underline{b}'+\underline{e})$$
$$=\underline{b}'+\{\Omega^T(u,y)\Omega(u,y)\}^{-1}\Omega^T(u,y)\underline{e}.$$

In general it is very difficult to calculate the expected value of $\underline{\beta}'-\underline{b}'$ for any value of the sequence length of $\{u_k\}_1^N$ and $\{y_k\}_1^N$.

It is possible to calculate $\lim\limits_{N\to\infty} E\{\underline{\beta}'-\underline{b}'\}$, in which $E\{c\}$ stands for taking the expected value of c.

Therefor we have to use two theorems for the limit in probability (Goldberger (8)), viz:

$\underline{a}$ if $\operatorname*{plim}\limits_{N\to\infty}(x_N)=c$, with c deterministic, then $\lim\limits_{N\to\infty} E\{x_N\}=c$.

$\underline{b}$ if the elements of A_N and B_N converge in probability, then

$$\operatorname*{plim}_{N\to\infty}(A_N^{-1}B_N)=\operatorname*{plim}_{N\to\infty}(A_N)^{-1}\operatorname*{plim}_{N\to\infty}(B_N)\ .$$

Let $P=\Omega^T(u,y)\Omega(u,y)$ then

$$\operatorname*{plim}_{N\to\infty}(\frac{1}{N-q}P)=\Gamma(\text{a positive definite matrix}).$$

$$\operatorname*{plim}_{N\to\infty}\left(\frac{\Omega^T(u,y)\underline{e}}{N-q}\right)=\operatorname*{plim}_{N\to\infty}\{\frac{1}{N-q}(\sum_{i=q+1}^{N} u_ie_i,\sum_{i=q}^{N-1} u_ie_{i+1},\dots\dots,\sum_{i=q+1-p}^{N-p} u_ie_{i+p},$$
$$\sum_{i=q}^{N-1} y_ie_{i+1},\dots\dots\dots\dots,\sum_{i=1}^{N-q} y_ie_{i+q})^T\}$$
$$=(\psi_{ue}(0),\psi_{ue}(1),\dots,\psi_{ue}(p),\psi_{ye}(1),\dots\dots,\psi_{ye}(q))^T$$
$$=(\ 0\ ,\ 0\ ,\dots,\ 0\ ,\psi_{ye}(1),\dots\dots,\psi_{ye}(q))^T$$

So $\operatorname*{plim}\limits_{N\to\infty}(\underline{\beta}'-\underline{b}')=\Gamma^{-1}(0,0,\dots,0,\psi_{ye}(1),\dots\dots,\psi_{ye}(q))^T$.

In general $\operatorname*{plim}\limits_{N\to\infty}(\underline{\beta}'-\underline{b}')$ is unequal to zero.

Only if $\psi_{ye}(k)=0$ for $1\leq k\leq q$ then $\Gamma^{-1}(0,0,\dots,0,\psi_{ye}(1),\dots,\psi_{ye}(q)=\underline{0}$,

hence the estimate $\underline{\beta}'$ of $\underline{b}'$ is asymptotically unbiased.

This is the case when $\{e_k\}_{-\infty}^{\infty}$ is a white noise sequence.

As $e_k=n_k+a_1n_{k-1}+\dots\dots+a_qn_{k-q}$, the sequence $\{n_k\}_1^N$ must be noise, which is derived from white noise by a filter, having the same backward parameters as the process (see fig 2.2).

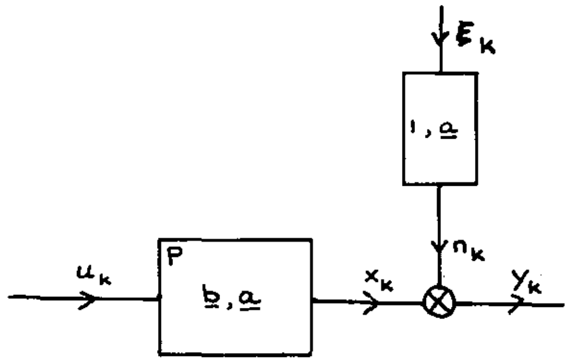


fig 2.2: Linear process, whereof the parameters can be estimated unbiasedly.

Remark: The backward parameters are those parameters, which are working on the output of the process; the forward parameters are those parameters, which are working on the input of the process.

We suppose that $\{e_k\}_1^N$ is a white noise sequence.
There are different ways of evaluating eq.(2.9) (Westenberg(13)). Two of them we will give here.

1. The explicit way:
   We fill up the matrix $\Omega(u,y)$ and the vector $\underline{y}$. We get a set of p+q+1 equations with p+q+1 unknown parameters by calculating

   $$\{\Omega^T(u,y)\Omega(u,y)\}^{-1} \text{ and } \Omega^T(u,y)\underline{y}$$

   Now we can calculate $\underline{\beta}'$, but we have only an estimate of $\underline{b}'$ after N samples.

2. The implicit way:
   When we use this way of evaluating eq.(2.9), we get an estimate of $\underline{b}'$ after each pair of input-output samples.

   Define: $P_k^{-1}=\Omega_k^T(u,y)\Omega_k(u,y)$ and $S_k=\Omega_k^T(u,y)\underline{y}_k$, with

   $$\underline{y}_k^T=(y_{q+1},y_{q+2},\dots\dots\dots,y_{q+k})$$

and

$$\Omega_k(u,y)=[U\,\vdots\,Y]=\left[\begin{array}{ccc|ccc} u_{q+1} & \cdots\cdots & u_{q+1-p} & y_q & \cdots\cdots & y_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ u_{q+k} & \cdots\cdots & u_{q+k-p} & y_{q+k-1} & \cdots\cdots & y_k \end{array}\right]$$

$$\Omega_{k+1}(u,y) = \left[\begin{array}{c} \Omega_k(u,y) \\ \hline \underline{\omega}_{k+1}^T \end{array}\right],$$

with $\underline{\omega}_{k+1}^T=(u_{k+1+q},u_{k+q},\ldots\ldots,u_{k+1-p+q},y_{k+q},y_{k-1+q},\ldots\ldots,y_{k+1})$,

and $\underline{y}_{k+1} = \left[\begin{array}{c} \underline{y}_k \\ \hline y_{k+1+q} \end{array}\right]$.

$$P_{k+1}^{-1}=\Omega_{k+1}^T(u,y)\Omega_{k+1}(u,y)=\Omega_k^T(u,y)\Omega_k(u,y)+\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T$$

$$=P_k^{-1}+\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T. \qquad (2.10)$$

Postmultiplying eq.(2.10) with P_k and premultiplying with P_{k+1}

gives: $$P_k=P_{k+1}+P_{k+1}\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T P_k \qquad (2.11)$$

$$P_k\underline{\omega}_{k+1}=P_{k+1}\underline{\omega}_{k+1}+P_{k+1}\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}$$

$$=P_{k+1}\underline{\omega}_{k+1}\{1+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}$$

$$P_k\underline{\omega}_{k+1}\{1+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k=P_{k+1}\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T P_k \qquad (2.12)$$

Eq(2.12) and eq.(2.11) give:

$$P_{k+1}=P_k-P_k\underline{\omega}_{k+1}\{1+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k. \qquad (2.13)$$

$S_{k+1}=S_k+\underline{\omega}_{k+1}y_{k+1}$;

$$\underline{\beta}'_{k+1}=P_{k+1}S_{k+1}$$

$$=(P_k-P_k\underline{\omega}_{k+1}\{1+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k)(S_k+\underline{\omega}_{k+1}y_{k+1})$$

$$=\underline{\beta}'_k+P_k\underline{\omega}_{k+1}y_{k+1}-P_k\underline{\omega}_{k+1}\{1+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k S_k-$$
$$-P_k\underline{\omega}_{k+1}\{1+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}y_{k+1}$$

$$\underline{\beta}'_{k+1}=\underline{\beta}'_k+P_k\underline{\omega}_{k+1}y_{k+1}(1-\{1+\underline{\omega}^T_{k+1}P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}^T_{k+1}P_k\underline{\omega}_{k+1})-$$

$$-P_k\underline{\omega}_{k+1}\{1+\underline{\omega}^T_{k+1}P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}^T_{k+1}\underline{\beta}'_k$$

$$\underline{\beta}'_{k+1}=\underline{\beta}'_k-P_k\underline{\omega}_{k+1}\{1+\underline{\omega}^T_{k+1}P_k\underline{\omega}_{k+1}\}^{-1}(\underline{\omega}^T_{k+1}\underline{\beta}'_k-y_{k+1}) \qquad (2.14)$$

Eq.(2.13) and eq.(2.14) are the iterative formulas for the normal least squares estimation scheme.

If we know the characteristics of the equation error, given by its covariance matrix Σ ($\Sigma=E\{\underline{ee}^T\}$) then we can make an asymptotic unbiased estimate of $\underline{b}'$.

Goldberger(8) gives that the best linear unbiased L.S. estimator of $\underline{b}'$ is given by

$$\underline{\beta}'=\{\Omega^T(u,y)\Sigma^{-1}\Omega(u,y)\}^{-1}\Omega^T(u,y)\Sigma^{-1}\underline{y} \qquad (2.15)$$

Eykhoff(7) has worked this out, assuming that:

$$\Sigma^{-1}=D^TD,$$

in which D^T is a lower triangular matrix.

This gives for eq.(2.15):

$$\underline{\beta}'=\{(D\Omega(u,y))^T(D\Omega(u,y))\}^{-1}(D\Omega(u,y))^T(D\underline{y}).$$

We see that D represents a "noise-whitening" filter, applied on the sequences $\{u_k\}_1^N$ and $\{y_k\}_1^N$. This estimator is called the Gauss-Markov estimator.

In practice, we do not know D or Σ and in the following sections we will give estimation schemes, which approximate this Gauss-Markov estimator.

2.2 Approximated Markov estimators.

Consider the following equation:

$$y_k=\sum_{i=0}^{p} b_iu_{k-i}-\sum_{i=1}^{q} a_iy_{k-i}+n_k+\sum_{i=1}^{q} a_in_{k-i} \qquad (2.4)$$

Define the following shift-operator (z-operator):

$$u_{k-j}=z^{-j}u_k$$

$$y_{k-i}=z^{-i}y_k.$$

Now equation (2.4) becomes in z-notation :

$$(1+A)y_k=(b_0+B)u_k+(1+A)n_k, \tag{2.16}$$

with $A=a_1z^{-1}+a_2z^{-2}+........+a_qz^{-q}$,

and $B=b_1z^{-1}+b_2z^{-2}+........+b_pz^{-p}$.

Suppose that $n_k=\sum_{i=1}^{s_0} g_i\xi_{k-i}-\sum_{i=1}^{r} d_in_{k-i}+\xi_k$,

or in z-notation:

$$(1+D)n_k=(1+G)\xi_k \text{ or } n_k=\frac{1+G}{1+D}\xi_k. \tag{2.17}$$

Eq.(2.16) and eq.(2.17) give:

$$(1+A)y_k=(b_0+B)u_k+\frac{(1+A)(1+G)}{(1+D)}\xi_k \tag{2.18}$$

where
$$\frac{(1+A)(1+G)}{(1+D)}\xi_k=e_k \tag{2.19}$$

Suppose we can write eq.(2.19) as

$$\frac{1}{(1+D')}\xi_k=e_k, \tag{2.20}$$

with $D'=d_1'z^{-1}+d_2'z^{-2}+.......+d_{r_0}'z^{-r_0}$,

then there are a number of algorithms we can use for estimating $\underline{b}'$ and the coefficients of D', viz:

The explicit algorithm of Clarke,

The iterative scheme given by Hastings-James & Sage,

The first extended matrix method.

When we have an estimate of the coefficients of D', then we can approximate the elements of Σ^{-1} and also the elements of the matrix D, defined in the preceding section, viz:

$$\tilde{D}=\begin{bmatrix} 1 & & & & \\ \delta_1(k) & 1 & & & \bigcirc \\ \delta_2(k) & \delta_1(k) & 1 & & \\ \vdots & & & \ddots & \\ \delta_{r_0}(k) & & & & \ddots \\ & \ddots & & & \\ \bigcirc & & \delta_{r_0}(k) & \cdots\cdots & 1 \end{bmatrix} \quad (k\times k)$$

Remark: We will denote the k-th estimate of the i-th element of $\underline{d}'$ as $\delta_i(k)$, and the k-th estimate of $\underline{d}'$ as $\underline{\delta}_k$.

2.2.1. The explicit algorithm of Clarke (4).

Suppose the sequences $\{u_k\}_1^N$ and $\{y_k\}_1^N$ are available.

With these sequences a least squares estimate $\underline{\beta}_0'$ of $\underline{b}'$ can be made using the explicit algorithm described in section 2.1 of this chapter.

With this $\underline{\beta}_0'$ and the given sequences we can compute $\hat{\underline{e}}_0$, using

$$\hat{\underline{e}}_0=\underline{y}-\Omega(u,y)\underline{\beta}_0'. \tag{2.21}$$

Eq.(2.20) can be written in matrix notation;viz:

$$\underline{e}=-E\underline{d}+\underline{\xi}, \tag{2.22}$$

with $\underline{e}^T=(e_{q+1},e_{q+2},\dots\dots,e_N)$,

$\underline{d}^T=(d_1',d_2',\dots\dots\dots,d_{r_0}')$,

$\underline{\xi}^T=(\xi_{q+1},\xi_{q+2},\dots\dots,\xi_N)$,

and ξ_k is a white noise sample.

$$E=\begin{bmatrix} e_q & e_{q-1} & \dots\dots\dots\dots e_{q+1-r_0} \\ e_{q+1} & e_q & \dots\dots\dots\dots e_{q+2-r_0} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ e_{N-1} & e_{N-2} & \dots\dots\dots\dots e_{N-r_0} \end{bmatrix}$$

Suppose that $\hat{\underline{e}}_0$ is a rather good approximation of $\underline{e}$. Now we can estimate a $\underline{\delta}^1$, which is a consistent estimate of $\underline{d}^1$ of the following equation

$$\hat{\underline{e}}_0=-\hat{E}\underline{d}^1+\underline{\xi}' \tag{2.23}$$

$\underline{d}^1$ of eq.(2.23) need not to be equal to $\underline{d}$ of eq.(2.22).

ξ_k' is a sample of a white noise sequence which is not the same as that one of which ξ_k is a sample.

We hope that $\underline{\delta}^1$ is a rather good approximation of $\underline{d}$.

The estimate will be given by $\underline{\delta}^1=-\{\hat{E}^T\hat{E}\}^{-1}\hat{E}^T\hat{\underline{e}}_0$.

Now we filter the sequences $\{u_k\}_1^N$ and $\{y_k\}_1^N$ with the estimate $\underline{\delta}^1$ in the following way:

$$u_k^*=u_k+\sum_{i=1}^{r_0}\delta_i^1 u_{k-i}, \text{ and}$$

$$y_k^*=y_k+\sum_{i=1}^{r_0}\delta_i^1 y_{k-i}.$$

And we get the sequences $\{u_k^*\}_{1+r_0}^N$ and $\{y_k^*\}_{1+r_0}^N$.

Using this sequences a new L.S. estimate $\underline{\beta}_1'$ of $\underline{b}'$ is obtained.
With this new estimate and the sequences $\{u_k\}_1^N$ and $\{y_k\}_1^N$ $\underline{\hat{e}}_1$ is computed from $\underline{\hat{e}}_1=\underline{y}-\Omega(u,y)\underline{\beta}_1'$.

If the loss-function, defined by $V=\sum_{i=q+1}^{N}\hat{e}_i^2$, is smaller for the last sequence $\{\hat{e}_k\}_{q+1}^N$ than for the previous one, then we make a new estimate of $\underline{d}^1$ and continue the procedure given above. If the loss-function is not smaller for the last sequence then the algorithm is stopped and the previous estimate of $\underline{b}'$ is taken as the estimate for the process-parameters.

We see that after some runs through the procedure this scheme gives an estimate of the process-parameters as well as an estimate of the backward parameters of the (equivalent)noise filter (2.20).

2.2.2. The iterative scheme given by Hastings-James & Sage.

Hastings-James & Sage (9) suggested the following iterative scheme.
After each estimate $\underline{\beta}_k'$ of $\underline{b}'$ based on k+q input-output pairs, an estimate of the equation error is computed from $\hat{e}_k=y_k-\underline{\omega}_k^T\underline{\beta}_k'$. With this $\hat{e}_k$ a least squares estimate $\underline{\delta}_k'$ of $\underline{d}'$ is made.
Then the input and output signals are filtered with $\underline{\delta}_k'$.
We obtain the filtered vector $\underline{\omega}_{k+1}^*$ in the following way:

$$\underline{\omega}_{k+1}^*=\underline{\omega}_{k+1}+\delta_1(k)\underline{\omega}_k+\delta_2(k)\underline{\omega}_{k-1}+\dots\dots+\delta_{r_0}(k)\underline{\omega}_{k+1-r_0} \quad (2.24)$$

in which $\underline{\omega}_i$ is the same vector as we defined in the normal least squares estimation scheme (see eq.(2.13) and eq.(2.14)).

In the same way we obtain a y^*_{k+1} from:

$$y^*_{k+1}=y_{k+1}+\delta_1(k)y_k+\delta_2(k)y_{k-1}+\dots\dots\dots+\delta_{r_0}(k)y_{k+1-r_0}.$$

We see that the sequences $\{u^*_k\}_1^N$ and $\{y^*_k\}_1^N$ are no longer stationary because $\underline{\delta}_k$ alters with k; so we may not use the normal L.S. scheme for estimating $\underline{b}'$.

In the beginning we will have poor estimates of $\underline{d}$, and the filtering will not be as good as we would like to have it. For this reason we want to make the influence of new input-output pairs greater than the influence of older input-output data.

This can be achieved if we minimize the following function with respect to $\underline{\beta}'_k$:

$$V=\sum_{i=q+1}^{k}\rho^{k-i}f_i^2, \text{ with } 0<<\rho<1 \text{ and } f_i=y^*_i-\underline{\omega}^*_i\underline{\beta}_k.$$

Then the algorithm becomes (see Appendix A):

$$\begin{cases} P_{k+1}=\frac{1}{\rho}(P_k-P_k\underline{\omega}^*_{k+1}\{\rho+\underline{\omega}^{*T}_{k+1}P_k\underline{\omega}^*_{k+1}\}^{-1}\underline{\omega}^{*T}_{k+1}P_k) & (2.25)\\ \underline{\beta}'_{k+1}=\underline{\beta}'_k-P_k\underline{\omega}^*_{k+1}\{\rho+\underline{\omega}^{*T}_{k+1}P_k\underline{\omega}^*_{k+1}\}^{-1}(\underline{\omega}^{*T}_{k+1}\underline{\beta}'_k-y^*_{k+1}), & (2.26)\end{cases}$$

and for estimating $\underline{d}'$ in an analogous way:

$$\begin{cases} PE_{k+1}=\frac{1}{\nu}(PE_k-PE_k\underline{\hat{e}}_{k+1}\{\nu+\underline{\hat{e}}^T_{k+1}PE_k\underline{\hat{e}}_{k+1}\}^{-1}\underline{\hat{e}}^T_{k+1}PE_k) & (2.27)\\ \underline{\delta}'_{k+1}=\underline{\delta}'_k-PE_k\underline{\hat{e}}_{k+1}\{\nu+\underline{\hat{e}}^T_{k+1}PE_k\underline{\hat{e}}_{k+1}\}^{-1}(\underline{\hat{e}}^T_{k+1}\underline{\delta}'_k-\hat{e}_{k+1}), & (2.28)\end{cases}$$

in which $\underline{\hat{e}}^T_k=(\hat{e}_{k-1},\hat{e}_{k-2},\dots\dots,\hat{e}_{k-r_o})$.

2.2.3. The first extended matrix method.

Smets(12) suggested to combine eq.(2.17) and eq.(2.19), viz:

$$(1+A)y_k=(b_0+B)u_k+\frac{1}{(1+D')}\xi_k. \qquad (2.29)$$

We can write this equation as follows:

$$y_k=\sum_{i=0}^{p}b_iu_{k-i}-\sum_{i=1}^{q}a_iy_{k-i}-\sum_{i=1}^{r_0}d'_ie_{k-i}+\xi_k, \qquad (2.30)$$

or in matrix notation: $\underline{y}=\Omega(u,y,e)\underline{b}^*+\underline{\xi}$, (2.31) with $\underline{b}^{*}=\left[\begin{array}{c}\underline{b}' \\ \hline \underline{d}'\end{array}\right]$ and

$$\Omega(u,y,e)=[U\vdots Y\vdots E]=\left[\begin{array}{ccc|ccc|ccc} u_{q+1} & \cdots\cdots & u_{q+1-p} & y_q & \cdots\cdots & y_1 & e_q & \cdots\cdots & e_{q+1-r_0} \\ \cdot & & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ u_N & \cdots\cdots & n_{N-p} & y_{N-1} & \cdots & y_{N-q} & e_{N-1} & \cdots & e_{N-r_0} \end{array}\right]$$

Premultiplying eq.(2.31) with $\{\Omega^T(u,y,e)\Omega(u,y,e)\}^{-1}\Omega^T(u,y,e)$ gives:

$$\{\Omega^T(u,y,e)\Omega(u,y,e)\}^{-1}\Omega^T(u,y,e)\underline{y}=\underline{b}^{*}+\{\Omega^T(u,y,e)\Omega(u,y,e)\}^{-1}\Omega^T(u,y,e)\underline{\xi}.$$

Analogous to section 2.1 we can prove that

$$\underline{\beta}^{*}=\{\Omega^T(u,y,e)\Omega(u,y,e)\}^{-1}\Omega^T(u,y,e)\underline{y}$$

is a consistent estimate of $\underline{b}^{*}$ as plim $(\Omega^T(u,y,e)\underline{\xi})=0$, as $\underline{\xi}$ is a white noise sequence.

Unfortunately we do not know the elements of sub-matrix E. It is possible, when we use an iterative scheme, to calculate an estimate of the elements of E after each estimate of $\underline{b}^{*}$.

So the estimation scheme becomes:

$$\underline{\beta}^{*}=\{\Omega^T(u,y,\hat{e})\Omega(u,y,\hat{e})\}^{-1}\Omega^T(u,y,\hat{e})\underline{y}.$$

In the beginning we have bad estimates of the elements of E so we have to use again a weighting-factor. We can use now the algorithm given by eq.(2.25) and eq.(2.26) to estimate $\underline{b}^{*}$, viz:

$$\begin{cases} P_{k+1}=\frac{1}{\rho}(P_k-P_k\underline{\omega}^{*}_{k+1}\{\rho+\underline{\omega}^{*T}_{k+1}P_k\underline{\omega}^{*}_{k+1}\}^{-1}\underline{\omega}^{*T}_{k+1}P_k) & (2.25) \\ \underline{\beta}^{*}_{k+1}=\underline{\beta}^{*}_{k}-P_k\underline{\omega}^{*}_{k+1}\{\rho+\underline{\omega}^{*T}_{k+1}P_k\underline{\omega}^{*}_{k+1}\}^{-1}(\underline{\omega}^{*T}_{k+1}\underline{\beta}^{*}_{k}-y_{k+1}) & (2.26) \end{cases}$$

In this algorithm $\underline{\omega}^{*T}_{k+1}$ becomes $(\underline{\omega}^{T}_{k+1},\underline{\hat{e}}^{T}_{k+1})$.

Smets(12) has shown that there is a strong analogy between this method and the method given by Clarke.

<u>Remark:</u> The problem of the two iterative schemes given before and the following schemes are the weighting-factors ρ and ν. These factors give a lower bound to the values of the elements of the matrices P and PE, so that after a number of iterations the covariance of the estimates doesn't decrease anymore.

If we increase these factors in an exponential way when we are estimating then the covariance will tends to a lower bound, which is not as great as in the case when we keep the weighting-factors constant. When we bring the weighting-factors after a number of iterations to 1 then the covariance of the estimates will decrease to zero if the length of the input and output signals goes to infinity.

2.2.4. The second extended matrix method.

Young(15) has also suggested an extended matrix method (Smets (12)). Suppose that eq.(2.19) can be written as follows:

$$(1+C')\xi_k=e_k, \tag{2.32}$$

with $C'=c_1'z^{-1}+c_2'z^{-2}+\dots\dots+c_{s0}'z^{-s0}$.

We combine eq.(2.18) and eq.(2.32):

$$(1+A)y_k=(b_0+B)u_k+(1+C')\xi_k, \text{ or}$$

$$y_k=\sum_{i=0}^{p} b_i u_{k-i}-\sum_{i=1}^{q} a_i y_{k-i}+\sum_{i=1}^{s0} c_i'\xi_{k-1}+\xi_k \tag{2.33}$$

This equation can be written in matrix-notation:

$$\underline{y}=\Omega(u,y,\xi)\underline{b}^*+\underline{\xi}, \tag{2.34}$$

with $\underline{y}^T=(y_{q+1},y_{q+2},\dots\dots\dots y_N)$,

$$\underline{b}^*=\begin{bmatrix}\underline{b}'\\ \hline \underline{c}'\end{bmatrix},$$

$$\underline{\xi}^T=(\xi_{q+1},\xi_{q+2},\dots\dots,\xi_N),$$

$$\Omega(u,y,\xi)=[U\vdots Y\vdots \Xi]=\left[\begin{array}{cc|cc|cc} u_{q+1} & \dots\dots u_{q+1-p} & y_q & \dots\dots y_1 & \xi_q & \dots\dots \xi_{q+1-s0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_N & \dots\dots u_{N-p} & y_{N-1} & \dots y_{N-q} & \xi_{N-1} & \dots\dots \xi_{N-s0} \end{array}\right]$$

Premultiplying eq.(2.34) with $\{\Omega^T(u,y,\xi)\Omega(u,y,\xi)\}^{-1}\Omega^T(u,y,\xi)$ gives:

$$\{\Omega^T(u,y,\xi)\Omega(u,y,\xi)\}^{-1}\Omega^T(u,y,\xi)\underline{y}=\underline{b}^*+\{\Omega^T(u,y,\xi)\Omega(u,y,\xi)\}^{-1}\Omega^T(u,y,\xi)\underline{\xi}.$$

Now in analogy to the first extended matrix method

$$\underline{\beta}^{*}=\{\Omega^{T}(u,y,\xi)\Omega(u,y,\xi)\}^{-1}\Omega^{T}(u,y,\xi)\underline{y}$$

is a consistent estimate of $\underline{b}^{*}$ as plim $(\Omega^{T}(u,y,\xi)\underline{\xi})=\underline{0}$.
We do not know the elements of sub-matrix Ξ, so we have to replace these elements by their estimates.
These estimates are given by

$$\hat{\xi}_k=y_k-\underline{\omega}_k^{*T}\underline{\beta}_k^{*}, \tag{2.35}$$

with $\underline{\omega}_k^{*T}=(u_k,u_{k-1},\ldots,u_{k-p},y_{k-1},\ldots,y_{k-q},\hat{\xi}_{k-1},\ldots,\hat{\xi}_{k-s_0})$.
Again we have to introduce a weighting-factor, because in the beginning we have bad estimates of ξ_i.

The estimation scheme becomes now:

$$\underline{\beta}^{*}=\{\Omega^{T}(u,y,\hat{\xi})\Omega(u,y,\hat{\xi})\}^{-1}\Omega^{T}(u,y,\hat{\xi})\underline{y},$$

or in an iterative way with eq.(2.25) and eq.(2.26):

$$\begin{cases} P_{k+1}=\frac{1}{\rho}(P_k-P_k\underline{\omega}_{k+1}^{*}\{\rho+\underline{\omega}_{k+1}^{*T}P_k\underline{\omega}_{k+1}^{*}\}^{-1}\underline{\omega}_{k+1}^{*T}P_k) & (2.25)\\ \underline{\beta}_{k+1}^{*}=\underline{\beta}_k^{*}-P_k\underline{\omega}_{k+1}^{*}\{\rho+\underline{\omega}_{k+1}^{*T}P_k\underline{\omega}_{k+1}^{*}\}^{-1}(\underline{\omega}_{k+1}^{*T}\underline{\beta}_k^{*}-y_{k+1}) & (2.26)\end{cases}$$

2.3 Schemes for estimating a minimum number of parameters.

Åström, Bohlin and Wensmark(1) have proved, that each linear process can be described by the following equation:

$$(1+A^{*})y_k=(b_0+B^{*})u_k+(1+C)\xi_k,$$

in which A^{*}, B^{*} and C are polynomials in z^{-1}.
This is easy to see: multiply eq(2.18) with (1+D) and define:

$$(1+A)(1+D)=(1+A^{*})$$
$$(b_0+B)(1+D)=(b_0+B^{*})$$
$$(1+A)(1+G)=(1+C)$$

When we use this process-description we see, that we have to estimate more parameters than when we were able to estimate the parameters of the noise filter $\frac{(1+C)}{(1+D)}$.

We have seen that untill now there are schemes to estimate the parameters of a moving-average(second extended matrix method) or an auto-regressive model(Clarke, Hastings-James & Sage, first extended matrix method) of the noise filter. These models will have, in general, more parameters that are significant than the description of the noise filter by $\frac{(1+C)}{(1+D)}$.

2.3.1. The third extended matrix method.

Consider the following equation:

$$(1+A)y_k=(b_0+B)u_k+\frac{(1+C)}{(1+D)}\xi_k.$$

We can write this equation as follows:

$$y_k=\sum_{i=0}^{p} b_i u_{k-i}-\sum_{i=1}^{q} a_i y_{k-i}+\sum_{i=1}^{s} c_i \xi_{k-i}-\sum_{i=1}^{r} d_i e_{k-i}+\xi_k,$$

or in matrix-notation:

$$\underline{y}=\Omega(u,y,\xi,e)\underline{b}^*+\underline{\xi},$$

with $\underline{b}^{*T}=(\underline{b}'^T,\underline{c}'^T)=(\underline{b}^T,-\underline{a}^T,\underline{c}^T,-\underline{d}^T)$,

and $\Omega(u,y,\xi,e)=[U\vdots Y\vdots \Xi\vdots E]=$

$$=\left[\begin{array}{ccc|ccc|ccc|ccc}
u_{q+1} & \cdots & u_{q+1-p} & y_q & \cdots & y_1 & \xi_q & \cdots & \xi_{q+1-s} & e_q & \cdots & e_{q+1-r} \\
\cdot & & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot & & \cdot \\
u_N & \cdots & u_{N-p} & y_{N-1} & \cdots & y_{N-q} & \xi_{N-1} & \cdots & \xi_{N-s} & e_{N-1} & \cdots & e_{N-r}
\end{array}\right]$$

A consistent estimate of $\underline{b}^*$ is given by:

$$\underline{\beta}^*=\{\Omega^T(u,y,\xi,e)\Omega(u,y,\xi,e)\}^{-1}\Omega^T(u,y,\xi,e)\underline{y}.$$

Now we do not know the elements of Ξ and E, so we have to replace them by their estimates given by:

$$\hat{e}_k=y_k+\sum_{i=1}^{q}\alpha_i(k)y_{k-i}-\sum_{i=0}^{p}\beta_i(k)u_{k-i}, \text{ and}$$

$$\hat{\xi}_k=\hat{e}_k+\sum_{i=1}^{r}\delta_i(k)\hat{e}_{k-i}-\sum_{i=1}^{s}\gamma_i(k)\hat{\xi}_{k-i}.$$

The scheme becomes then:

$$\underline{\beta}^*=\{\Omega^T(u,y,\hat{\xi},\hat{e})\Omega(u,y,\hat{\xi},\hat{e})\}^{-1}\Omega^T(u,y,\hat{\xi},\hat{e})\underline{y}.$$

We have to introduce again a weighting-factor, as we use an iterative scheme, starting with bad estimates of the equation errors and the white noise samples.
The iterative formulas we have to use are given again by eq.(2.25) and eq.(2.26); viz:

$$\begin{cases} P_{k+1}=\frac{1}{\rho}(P_k-P_k\underline{\omega}_{k+1}\{\rho+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k) & (2.25) \\ \underline{\beta}_{k+1}^*=\underline{\beta}_k^*-P_k\underline{\omega}_{k+1}\{\rho+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}(\underline{\omega}_{k+1}^T\underline{\beta}_k^*-y_{k+1}), & (2.26) \end{cases}$$

with $\underline{\omega}_k^T=(u_k,u_{k-1},\ldots\ldots,u_{k-p},y_{k-1},\ldots\ldots,y_{k-q},\hat{\xi}_{k-1},\ldots\ldots,\hat{\xi}_{k-s},\hat{e}_{k-1},\ldots,\hat{e}_{k-r})$.

2.3.2. An other approach to the problem.

Suppose we know the sequence of equation errors. We can easily see, that if we subtract e_k from y_k we get a set of equations of the following type:

$$y_k-e_k=\sum_{i=0}^{p} b_i u_{k-i}-\sum_{i=1}^{q} a_i y_{k-i}.$$

In this case we need only a set of p+q+1 input-output pairs to calculate the parameters of the process, because we haven't any uncertainty in the equations at all.
In practical cases we don't know the sequence $\{e_k\}_{q+1}^{N}$. We can only estimate this sequence.
In the following we will describe a scheme for estimating the parameters of the following equation:

$$(1+A)y_k=(b_0+B)u_k+\frac{(1+C)}{(1+D)}\xi_k, \qquad (2.36)$$

in which $\frac{(1+C)}{(1+D)}\xi_k=e_k$ (2.37)

and $C=c_1z^{-1}+\ldots\ldots\ldots+c_sz^{-s}$,

$D=d_1z^{-1}+\ldots\ldots\ldots+d_rz^{-r}$.

Let's first look at eq.(2.37).
We rewrite this equation in the following way:

$$(1+D)e_k=(1+C)\xi_k, \text{ or}$$

$$\underline{e}=\Xi\underline{c}-E\underline{d}+\underline{\xi}=\Omega(\xi,e)\underline{c}'+\underline{\xi}.$$

If we know the sequence $\{e_k\}_{q+1}^N$, then this is an equation of the same type as eq.(2.33) except for the control term $(b_0+B)u_k$. So we can use the second extended matrix method for estimating the noise parameters $\underline{c}$ and $\underline{d}$.
Assume we have $\{e_k\}_{q+1}^N$ and rather good estimates for the sequence $\{\xi_k\}_{q+1}^N$ and the parameters of the noise filter, then we can give a prediction for e_{N+1} by using the following equation:

$$e^*_{N+1}=-\sum_{i=1}^{r}\delta_i e_{N+1-i}+\sum_{i=1}^{s}\gamma_i\hat{\xi}_{N+1-i}.$$

By the given assumptions $e_{N+1}-e^*_{N+1}$ will appproximate the white noise sample ξ_{N+1}.
When we subtract e^*_k from y_k and $e_k-e^*_k$ approximate the white noise sample quite well, than we get a set of equations of which the equation errors are nearly white noise samples, so that we can estimate the process-parameters asymptotical unbiased.
Unfortunately we don't have the sequence $\{e_k\}_{q+1}^N$ available, so that we have to use estimates of e_k.
Define $\underline{\gamma}_k^{'T}=(\gamma_1,\gamma_2,\dots\dots,\gamma_s,-\delta_1,-\delta_2,\dots\dots,-\delta_r)$ after k iterations.
Schematically this iterative estimation scheme becomes:

1. $\underline{\beta}_k'$ and $\underline{\gamma}_k'$ and $(\hat{\xi}_{k-1},\hat{\xi}_{k-2},\dots\dots,\hat{\xi}_{k-s},\hat{e}_{k-1},\dots\dots\dots,\hat{e}_{k-r})$ are known.
2. compute $\hat{e}_k$ using the equation:

$$\hat{e}_k=y_k+\sum_{i=1}^{q}\alpha_i(k)y_{k-i}-\sum_{i=0}^{p}\beta_i(k)u_{k-i},$$

   in which $\alpha_i(k)$ is the estimate of a_i after k iterations.
3. estimate $\underline{\gamma}_k'$, using the second extended matrix method.
4. calculate $\hat{\xi}_k$ by using the equation:

$$\hat{\xi}_k=\hat{e}_k+\sum_{i=1}^{r}\delta_i(k)\hat{e}_{k-i}-\sum_{i=1}^{s}\gamma_i(k)\hat{\xi}_{k-i}.$$

5. Make a prediction of e_{k+1}, using the equation:

$$e^*_{k+1}=-\sum_{i=1}^{r}\delta_i(k)\hat{e}_{k+1-i}+\sum_{i=1}^{s}\gamma_i(k)\hat{\xi}_{k+1-i}$$

6. When u_{k+1} and y_{k+1} become available estimate $\underline{\beta}'_{k+1}$ with eq.(2.25) and eq.(2.26) and subtracting e^*_{k+1} from y_{k+1}.
7. go to point 2.

Remark: In eq.(2.25) and eq.(2.26) the $\underline{\omega}^*_{k+1}$ vector becomes:

$\underline{\omega}^{*T}_{k+1}=(u_{k+1},u_k,\ldots\ldots,u_{k+1-p},y_k,\ldots\ldots\ldots,y_{k+1-q})$, and the part $(\underline{\omega}^{*T}_{k+1}\underline{\beta}'_k-y^*_{k+1})$ becomes $(\underline{\omega}^{*T}_{k+1}\underline{\beta}'_k-y_{k+1}+e^*_{k+1})$.

3. Experimental results and discussion.

3.1. General remarks.

Consider the following process:

$$(1+A)y_k=(b_0+B)u_k+e_k. \qquad (3.1)$$

In the following the process-parameters are chosen as; viz:

$$(1+A)=1-1.5z^{-1}+0.7z^{-2}$$

$$(b_0+B)=0+1.0z^{-1}+0.5z^{-2}.$$

In the z-plane the poles of this process are 0.75±0.36j and the zero is-0.5 (see fig 3.1).

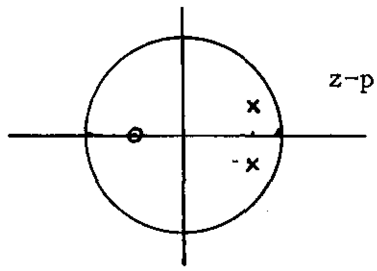


fig 3.1: poles and zero of $\frac{z^{-1}+0.5z^{-2}}{1-1.5z^{-1}+0.7z^{-2}}$.

We call $\frac{b_0+B}{1+A}=H(z^{-1})$.

As input in all programs is chosen a white noise signal with a rectangular amplitude distribution between -1 and +1.

If the white input signal has a power of σ_u^2 then the power of the output , σ_x^2, is given by:

$$\sigma_x^2=\frac{\sigma_u^2}{2\pi j}\oint_{|z|=1} H(z)H(z^{-1})\frac{dz}{z}, \qquad \text{(Jury (10))}.$$

so the ratio $\frac{\sigma_x^2}{\sigma_u^2}$ becomes:

$$\frac{\sigma_x^2}{\sigma_u^2}=\frac{1}{2\pi j}\oint_{|z|=1} H(z)H(z^{-1})\frac{dz}{z}$$

Åström, Jury & Agniel(2) give in their paper a fast method to calculate this integral.

For the given process the ratio $\frac{\sigma_u^2}{\sigma_x^2}$ becomes 18.8.

In each following section we will define the properties of the equation error sequence $\{e_k\}_{1+q}^{N}$.

The results given in the following sections are the averages over ten runs of 1000 iterations, unless in the tables an other number of runs is given. Also an estimate of the standard deviation, based on those ten runs, of the estimates is given.

Remark I : In all tables we will give first the averages of the estimated values of the parameters over ten runs. Their standard deviations are given immediately below the averages.

Remark II : Consider a stochastic process $z_k=\mu+x_k$, in which μ is constant and x_k is a stochastic variable with $E\{x_k\}=0$.

Suppose the estimate of μ is given by $\bar{z}=\sum_{i=1}^{N} z_i$.

The confidence interval for μ is given by

$$\bar{z}-t_\nu(\tfrac{1}{2}\alpha)s/\sqrt{n}<\mu<\bar{z}+t_\nu(\tfrac{1}{2}\alpha)s/\sqrt{n},$$

with s^2 for the estimate of the variance of z and ν for for the degree of freedom.

In our case z_i is the estimated value of a parameter in the i-th run. We have 10 runs, so $\nu=9$.

When we want the confidence interval, with a chance of 5% that μ lies outside this interval, then $t_9(\tfrac{1}{2}\alpha)=2.26$.

So we get:

$$\bar{z}-0.7s<\mu<\bar{z}+0.7s.$$

Remark III: The programs are written in Algol 60. The listings and the papertapes of the programs are available at the Eindhoven University of Technology, Group Measurement and Control.

3.2 The algorithm of Hastings-James & Sage.

We used the equations 2.25-2.28 given in section 2.2.2. viz:

$$\begin{cases} P_{k+1}=\frac{1}{\rho}(P_k-P_k\underline{\omega}^*_{k+1}\{\rho+\underline{\omega}^{*T}_{k+1}P_k\underline{\omega}^*_{k+1}\}^{-1}\underline{\omega}^{*T}_{k+1}P_k) & (2.25) \\ \underline{\beta}'_{k+1}=\underline{\beta}'_k-P_k\underline{\omega}^*_{k+1}\{\rho+\underline{\omega}^{*T}_{k+1}P_k\underline{\omega}^*_{k+1}\}^{-1}(\underline{\omega}^{*T}_{k+1}\underline{\beta}'_k-y^*_{k+1}) & (2.26) \end{cases}$$

$$\begin{cases} PE_{k+1}=\frac{1}{\nu}(PE_k-PE_k\hat{\underline{e}}_{k+1}\{\nu+\hat{\underline{e}}^T_{k+1}PE_k\hat{\underline{e}}_{k+1}\}^{-1}\hat{\underline{e}}^T_{k+1}PE_k) & (2.27) \\ \underline{\delta}_{k+1}=\underline{\delta}_k-PE_k\hat{\underline{e}}_{k+1}\{\nu+\hat{\underline{e}}^T_{k+1}PE_k\hat{\underline{e}}_{k+1}\}^{-1}(\hat{\underline{e}}^T_{k+1}\underline{\delta}_k-\hat{e}_{k+1}) & (2.28) \end{cases}$$

Evers(6) already wrote a procedure for calculating eq.(2.25)-eq.(2.26) and eq.(2.27)-eq.(2.28). This procedure has been optimized.
We studied the performance of this algorithm as function of the noise power and the weighting-factors.
The following equation errors were simulated:

$$e_k=\frac{\lambda\xi_k}{1+D},$$

with $D=-z^{-1}+0.2z^{-2}$ and ξ_k a sample of a white noise sequence with a rectangular amplitude distribution between -1 and +1.
As $e_k=(1+A)n_k$, the noise filter has the following transfer function:

$$G(z^{-1})=\frac{1}{(1+D)(1+A)}.$$

So it has the same poles as the process plus two poles at +.725 and +.275.

The ratio σ_n^2/σ_ξ^2 is in this case 98.52.
The D.E. of which we want to estimate the parameters becomes:

$$y_k=\sum_{i=0}^{2} b_i u_{k-i}-\sum_{i=1}^{2} a_i y_{k-i}-\sum_{i=1}^{2} d_i e_{k-i}+\xi_k,$$

in which $\sigma_u^2=\sigma_\xi^2$.

For a good idea how the algorithm works, we look to the results for $\lambda=1$ ($\frac{S}{N}=\sigma_x^2/\sigma_n^2=0.2$) and the weighting-factors $\rho=\nu=0.9913$ (Table 3.3).

$\rho=\nu=0.9913$ implies that after 528 iterations only 1% of the first error output of the model is taken into account.
We see that the algorithm converges to such values, that the real values of the parameters lie in the confidence intervals of the estimates.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.556	.752	+.049	1.077	.521	-0.995	.270
	.056	.082	.088	.098	.049	.162	.157
200	-1.529	.730	+.038	1.048	.514	-1.060	.284
	.056	.079	.076	.082	.022	.113	.131
300	-1.519	.720	+.020	1.030	.505	-1.068	.286
	.042	.061	.067	.069	.013	.092	.086
400	-1.514	.715	+.025	1.028	.504	-1.003	.235
	.028	.038	.048	.046	.012	.066	.090
500	-1.509	.708	+.014	1.015	.501	-0.971	.190
	.017	.024	.030	.034	.017	.101	.080
600	-1.499	.700	-.001	1.004	.504	-0.985	.201
	.017	.020	.024	.027	.018	.076	.061
700	-1.502	.703	+.001	1.008	.501	-0.950	.172
	.012	.017	.016	.017	.016	.077	.058
800	-1.498	.702	-.011	0.990	.489	-0.960	.158
	.022	.022	.024	.031	.020	.057	.052
900	-1.498	.708	-.003	0.998	.493	-0.977	.204
	.017	.019	.017	.018	.018	.066	.052
1000	-1.502	.702	+.001	1.007	.498	-0.985	.196
	.016	.016	.016	.018	.016	.090	.087

Table 3.1

Algorithm of Hastings-James & Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=3.2$ (+5 dB). $\rho=.9913$. $\nu=.9913$.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.565	.722	+.022	1.053	.510	-0.925	.244
	.056	.056	.054	.085	.080	.132	.152
200	-1.538	.706	+.022	1.038	.516	-1.000	.272
	.032	.036	.033	.049	.043	.085	.127
300	-1.525	.698	-.001	1.020	.509	-1.015	.270
	.028	.030	.032	.035	.027	.070	.052
400	-1.513	.696	+.022	1.024	.509	-0.962	.234
	.026	.028	.028	.036	.028	.059	.080
500	-1.517	.700	+.006	1.012	.505	-0.948	.193
	.026	.034	.030	.043	.034	.092	.062
600	-1.504	.697	-.010	1.007	.515	-0.971	.207
	.025	.034	.030	.042	.035	.077	.066
700	-1.505	.700	-.006	1.010	.506	-0.945	.182
	.020	.029	.023	.024	.030	.080	.072
800	-1.502	.702	-.023	0.983	.482	-0.957	.166
	.031	.036	.040	.051	.039	.061	.062
900	-1.500	.714	-.005	0.999	.489	-0.973	.206
	.030	.034	.031	.033	.038	.062	.052
1000	-1.504	.704	+.001	1.014	.496	-0.983	.196
	.028	.028	.031	.038	.035	.083	.080

Table 3.2

Algorithm of Hastings-James & Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.8$ (-1 dB). $\rho=.9913$. $\nu=.9913$

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.624 .047	.762 .053	+.035 .114	1.105 .190	.507 .171	-0.849 .102	.232 .158
200	-1.592 .034	.737 .048	+.043 .068	1.084 .119	.526 .088	-0.938 .078	.274 .141
300	-1.578 .028	.726 .032	-.003 .071	1.050 .068	.510 .051	-0.962 .082	.274 .072
400	-1.540 .034	.702 .046	+.050 .058	1.057 .066	.516 .061	-0.918 .061	.252 .096
500	-1.551 .035	.710 .048	+.009 .065	1.027 .089	.507 .075	-0.909 .096	.205 .062
600	-1.533 .023	.704 .042	-.021 .062	1.025 .082	.534 .079	-0.943 .077	.225 .082
700	-1.523 .035	.700 .038	-.011 .051	1.027 .050	.513 .060	-0.918 .076	.198 .091
800	-1.524 .034	.704 .045	-.045 .082	0.975 .094	.467 .079	-0.936 .063	.181 .075
900	-1.517 .046	.719 .048	-.003 .065	1.013 .068	.486 .084	-0.957 .056	.226 .055
1000	-1.517 .039	.705 .041	+.007 .068	1.038 .081	.496 .079	-0.969 .082	.208 .082

Table 3.3

Algorithm of Hastings-James & Sage.

d_1=-1, d_2=.2. $\frac{S}{N}$=.2 (-7 dB). ρ=.9913. ν=.9913.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.672 .049	.806 .042	+.043 .225	1.106 .226	.456 .258	-0.805 .096	.245 .167
200	-1.649 .037	.785 .024	+.068 .143	1.093 .198	.494 .150	-0.886 .088	.279 .139
300	-1.641 .032	.775 .024	+.007 .127	1.076 .132	.497 .080	-0.897 .082	.273 .070
400	-1.599 .028	.740 .039	+.111 .100	1.125 .124	.511 .117	-0.869 .057	.260 .092
500	-1.606 .037	.744 .042	+.026 .128	1.045 .186	.500 .154	-0.859 .090	.219 .065
600	-1.588 .026	.734 .031	-.024 .119	1.046 .160	.547 .152	-0.893 .065	.250 .087
700	-1.575 .045	.725 .032	-.006 .094	1.071 .094	.520 .120	-0.873 .075	.222 .097
800	-1.576 .037	.724 .035	-.050 .112	0.997 .148	.445 .144	-0.878 .054	.196 .071
900	-1.572 .044	.742 .040	+.006 .114	1.038 .125	.462 .159	-0.917 .045	.257 .066
1000	-1.564 .041	.722 .042	+.036 .116	1.098 .140	.490 .149	-0.926 .069	.243 .091

Table 3.4

Algorithm of Hastings-James & Sage.

d_1=-1, d_2=.2. $\frac{S}{N}$=.05 (-13 dB). ρ=.9913. ν=.9913.

9 runs.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.691 .052	.816 .047	+.157 .492	1.508 1.122	.636 .769	-0.776 .101	.240 .155
200	-1.667 .040	.790 .036	+.154 .279	1.370 .830	.658 .481	-0.857 .099	.278 .122
300	-1.662 .040	.787 .034	+.002 .242	1.245 .529	.579 .296	-0.872 .071	.275 .063
400	-1.614 .041	.746 .048	+.201 .202	1.262 .266	.582 .293	-0.842 .052	.273 .081
500	-1.629 .040	.757 .039	+.035 .251	1.097 .382	.526 .308	-0.835 .081	.232 .060
600	-1.616 .038	.751 .032	-.091 .252	1.084 .324	.612 .298	-0.872 .054	.259 .081
700	-1.596 049	.735 .033	-.051 .209	1.098 .203	.542 .231	-0.852 .058	.236 .086
800	-1.602 .039	.739 .027	-.192 .334	0.889 .368	.358 .275	-0.867 .048	.220 .070
900	-1.589 .052	.744 .046	-.036 .241	1.025 .236	.435 .303	-0.893 .042	.272 .064
1000	-1.587 .041	.731 .043	+.023 .243	1.142 .260	.470 .272	-0.905 .087	.254 .093

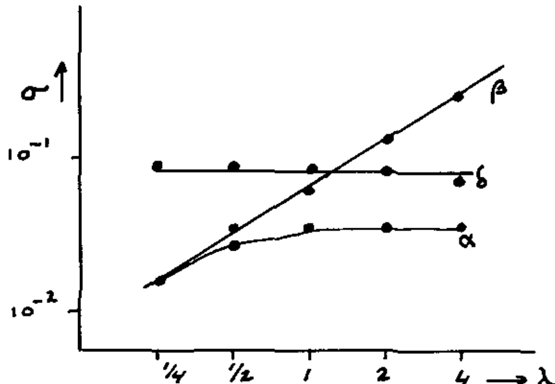
Table 3.5
Algorithm of Hastings-James & Sage.
$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.0125$ (-19 dB). $\rho=.9913$. $\nu=.9913$.

We see, that after approximately 300 iterations the standard deviation does not decrease anymore.

3.2.1. Dependency on noise power.

We give here the results for $\lambda=.25,.5,1,2$ and 4, which means for $\frac{S}{N}$ resp. 3.2, .8, .2, .05 and .0125 or 5, -1, -7, -13 and -19 dB.
We keep ρ and ν constant(0.9913). See for the results Table 3.1-3.5.
We see that, when we increase the noise power, the standard deviations of the estimates β_0, β_1 and β_2 grow linearly with λ(proportional with the square root of $\frac{S}{N}$), while the standard deviations of the α's go to a constant value

We see that the standard deviation of the δ's remain nearly constant (see graph 3.1).



Graph 3.1.

Furthermore we see that, when we increase the noise power, $(\underline{b}'-\underline{\beta}'_k)$ becomes larger for constant k.
We would say, that when we increase the noise power, the speed of concergency decreases. We weren't able to prove this, but the results give a strong indication in that direction.

3.2.2. Dependency on weighting-factors.

Next we changed the weighting-factors. We choose λ=4(S/N=.0125 or -19 dB) and gave ρ and ν the following values: 1, .995, .990, .985, .980 and .975, which means a decrease of the influence of the model output errors to 1% in resp. ∞, 917, 458, 305, 228 and 181 samples (table 3.6-3.11).
We see, that when we decrease the weighting-factors, the standard deviation of the β's increases, while the means of the estimated values are going to vary more and more around the true values.
The means of the estimates of the δ's are getting better and better, when we decrease the weighting factors, while the standard deviation becomes a little bit larger.
With the α's something strange is going on. First the standard deviation is getting smaller and then it grows again, with a minimum for the weighting-factor .990. $a_i-\bar{\alpha}_i$ decreases constantly.
We see that for $\rho=\nu=.980$ the point is reached, where for λ=4. the true value of the backward parameters are beginning to lie in the confidence interval of 95%.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.698 .094	.820 .090	+.142 .395	1.103 .638	.354 .581	-0.802 .158	.301 .094
200	-1.675 .079	.796 .070	+.073 .332	1.109 .426	.486 .350	-0.834 .122	.287 .079
300	-1.667 .076	.787 .063	+.083 .261	1.120 .366	.457 .275	-0.838 .100	.282 .073
400	-1.664 .073	.785 .054	+.020 .209	1.066 .337	.449 .273	-0.853 .097	.288 .063
500	-1.655 .070	.778 .053	+.024 .201	1.059 .285	.450 .228	-0.854 .092	.288 .055
600	-1.651 .069	.775 .053	+.026 .192	1.046 .270	.460 .207	-0.853 .088	.285 .057
700	-1.648 .071	.772 .054	+.001 .180	1.042 .232	.468 .176	-0.856 .078	.281 .052
800	-1.644 .073	.769 .055	+.002 .174	1.047 .207	.453 .161	-0.855 .068	.279 .051
900	-1.642 .072	.767 .054	-.007 .163	1.029 .198	.446 .146	-0.856 .067	.272 .049
1000	-1.640 .070	.764 .053	-.010 .140	1.024 .168	.458 .122	-0.858 .067	.275 .050

Table 3.6

Algorithm of Hastings-James & Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.0125$ (-19 dB). $\rho=1$. $\nu=1$.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.676 .092	.817 .088	+.138 .367	1.102 .605	.449 .538	-0.806 .158	.300 .097
200	-1.665 .077	.787 .065	+.062 .351	1.100 .379	.508 .320	-0.843 .115	.286 .084
300	-1.653 .071	.774 .055	+.084 .266	1.124 .344	.461 .258	-0.848 .086	.280 .076
400	-1.643 .069	.766 .046	-.026 .228	1.018 .342	.454 .286	-0.869 .086	.289 .060
500	-1.625 .059	.754 .042	-.001 .231	1.012 .298	.454 .221	-0.869 .083	.289 .044
600	-1.621 .060	.753 .047	+.003 .220	0.990 .309	.472 .204	-0.866 .088	.279 .064
700	-1.618 .067	.749 .047	-.058 .216	1.003 .240	.496 .185	-0.876 .060	.270 .050
800	-1.606 .070	.738 .049	-.034 .213	1.030 .214	.447 .222	-0.874 .045	.268 .061
900	-1.602 .062	.734 .046	-.050 .242	0.978 .277	.433 .196	-0.876 .060	.247 .054
1000	-1.595 .052	.731 .036	-.036 .197	0.983 .237	.490 .176	-0.886 .065	.264 .072

Table 3.7

Algorithm of Hastings-James &Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.0125$ (-19 dB). $\rho=.995$. $\nu=.995$.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.667 .103	.790 .099	+.285 .293	1.335 .542	.580 .483	-0.865 .173	.334 .071
200	-1.639 .083	.765 .062	+.147 .354	1.193 .322	.584 .411	-0.857 .136	.286 .100
300	-1.633 .081	.758 .062	+.052 .214	1.080 .388	.463 .253	-0.883 .059	.253 .082
400	-1.630 .082	.757 .048	-.024 .322	1.063 .396	.525 .292	-0.899 .079	.291 .066
500	-1.594 .046	.728 .033	+.104 .332	1.062 .430	.477 .297	-0.873 .107	.285 .052
600	-1.602 .063	.748 .051	+.024 .238	1.097 .421	.560 .279	-0.868 .119	.285 .082
700	-1.595 .082	.741 .058	-.075 .271	1.145 .213	.665 .214	-0.885 .049	.263 .055
800	-1.577 .084	.720 .066	-.022 .239	1.084 .296	.504 .364	-0.883 .046	.242 .090
900	-1.587 .071	.742 .052	-.123 .365	0.938 .423	.414 .334	-0.908 .063	.246 .045
1000	-1.574 .050	.731 .040	-.093 .245	1.028 .270	.551 .330	-0.926 .071	.272 .064

Table 3.8

Algorithm of Hastings-James & Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.0125$ (-19 dB). $\rho=0.990$. $\nu=0.990$.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.662 .102	.787 .098	+.265 .313	1.327 .526	.609 .455	-0.875 .174	.337 .077
200	-1.627 .084	.755 .059	+.145 .408	1.167 .378	.599 .487	-0.859 .134	.282 .113
300	-1.627 .079	.752 .064	+.006 .238	1.044 .455	.458 .319	-0.890 .047	.297 .089
400	-1.619 .085	.746 .056	-.050 .410	1.038 .468	.544 .339	-0.896 .075	.286 .063
500	-1.582 .038	.721 .036	+.117 .420	1.035 .525	.474 .375	-0.877 .107	.287 .055
600	-1.598 .065	.749 .060	-.018 .307	1.109 .495	.588 .329	-0.859 .139	.282 .094
700	-1.588 .078	.739 .062	-.096 .290	1.171 .254	.713 .261	-0.888 .044	.262 .067
800	-1.564 .080	.710 .072	-.003 .265	1.104 .329	.503 .427	-0.889 .070	.239 .117
900	-1.578 .065	.741 .051	-.140 .436	0.909 .500	.390 .382	-0.916 .081	.242 .064
1000	-1.558 .058	.722 .061	-.075 .329	1.0681 .359	.605 .409	-0.933 .082	.266 .080

Table 3.9

Algorithm of Hastings-James & Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.0125$ (-19 dB). $\rho=.985$. $\nu=.985$.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.657	.783	+.246	1.318	.632	-0.883	.341
	.101	.098	.344	.519	.439	.177	.086
200	-1.616	.746	+.154	1.143	.604	-0.859	.280
	.086	.062	.468	.472	.574	.136	.126
300	-1.625	.750	-.035	1.022	.461	-0.892	.297
	.078	.068	.281	.526	.382	.058	.105
400	-1.612	.738	-.059	1.021	.561	-0.889	.282
	.087	.065	.466	.517	.377	.078	.066
500	-1.578	.720	+.119	1.005	.472	-0.882	.294
	.043	.047	.495	.603	.441	.106	.063
600	-1.599	.752	-.064	1.126	.608	-0.847	.282
	.071	.071	.389	.547	.359	.154	.104
700	-1.588	.739	-.103	1.193	.745	-0.891	.269
	.074	.066	.308	.325	.302	.052	.086
800	-1.556	.704	+.023	1.133	.511	-0.893	.242
	.080	.079	.294	.355	.471	.093	.142
900	-1.574	.742	-.157	0.887	.377	-0.921	.242
	.063	.053	.493	.564	.412	.099	.088
1000	-1.549	.715	-.049	1.121	.658	-0.935	.264
	.073	.081	.413	.448	.469	.095	.099

Table 3.10

Algorithm of Hastings-James & Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.0125$ (-19 dB). $\rho=.980$. $\nu=.980$.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.652	.779	+.230	1.307	.650	-0.891	.345
	.101	.098	.378	.521	.435	.181	.097
200	-1.608	.740	+.168	1.118	.601	-0.859	.279
	.088	.069	.529	.577	.661	.139	.135
300	-1.626	.750	-.073	1.006	.468	-0.890	.296
	.079	.075	.330	.589	.436	.078	.125
400	-1.609	.734	-.062	1.006	.573	-0.879	.281
	.089	.071	.506	.556	.411	.083	.075
500	-1.579	.722	+.111	0.975	.470	-0.887	.302
	.057	.061	.565	.676	.498	.106	.074
600	-1.604	.756	-.108	1.142	.621	-0.835	.287
	.079	.081	.482	.595	.383	.166	.114
700	-1.589	.741	-.104	1.212	.766	-0.893	.281
	.073	.071	.331	.409	.342	.069	.108
800	-1.552	.700	+.051	1.168	.527	-0.896	.248
	.082	.085	.322	.380	.506	.116	.166
900	-1.571	.742	-.176	0.867	.342	-0.924	.243
	.067	.057	.538	.624	.436	.115	.108
1000	-1.543	.711	-.020	1.177	.707	-0.936	.264
	.089	.098	.488	.526	.517	.110	.120

Table 3.11

Algorithm of Hastings-James & Sage.

$d_1=-1$, $d_2=.2$. $\frac{S}{N}=.0125$ (-19 dB). $\rho=.975$. $\nu=.975$.

With this high noise level we should weight the squares of the model errors with approximately .985 to get good values for the averages of the estimated parameters.
We saw that for lower noise levels, with constant ρ and ν, the speed of convergency seemed to be greater, so that we expect, that for lower noise levels we can use weighting-factors greater than .985. Therefore we should have to estimate the noise power and then given a value to the weighting-factors depending on this noise power.
In general we see that the speed of convergency increases when we decrease the weighting-factors. So we can also start with for example $\rho=\nu=.975$ and bring this value slowly or after a number of iterations to 1.

Remark I : In this algorithm we calculate after each estimate of $\underline{d}$ a whole new vector $\underline{\omega}^*$ and y^* (see eq.(2.24)) and after each estimate of $\underline{b}'$ a whole new $\underline{\hat{e}}_k$ and $\hat{e}_k$ with $\hat{e}_{k-i}=y_{k-i}-\underline{\omega}^T_{k-i}\underline{\beta}'_k$.

Remark II : We start estimating $\underline{d}$ when we know the first $\hat{e}$.

3.3. The extended matrix methods.

We wrote a program with which it is possible to define the dimension of the process and noise filter parameter-vector as well as the dimension of the vector with the estimates of the the process and noise filter parameters.
One can generate now the following set of D.E.'s:

$$\begin{cases} y_k=\sum\limits_{i=0}^{p} b_i u_{k-i}-\sum\limits_{i=1}^{q} a_i y_{k-i}+e_k \\ e_k=\sum\limits_{i=1}^{s} c_i \xi_{k-i}-\sum\limits_{i=1}^{r} d_i e_{k-i}+\xi_k, \end{cases}$$

and estimate the parameters of the following model:

$$\begin{cases} y_k=\sum\limits_{i=0}^{ps} \beta_i u_{k-i}-\sum\limits_{i=1}^{qs} \alpha_i y_{k-i}+e'_k \\ e'_k=\sum\limits_{i=1}^{ss} \gamma_i \xi'_{k-i}-\sum\limits_{i=1}^{rs} \delta_i e'_{k-i}+\xi'_k \end{cases}$$

If one takes ss equal to zero then one has the first extended matrix method. If rs is chosen equal to zero the algorithm becomes the second extended matrix method. When rs and ss are both taken unequal to zero one uses the third extended matrix method. For all the following results we simulated the following set of D.E.'s:

$$\begin{cases} y_k=1.5y_{k-1}-.7y_{k-2}+u_{k-1}+0.5u_{k-2}+e_k \\ e_k=0.5e_{k-1}+.3\lambda\xi_{k-1}+\lambda\xi_k. \end{cases}$$

The corresponding noise filter has a power gain of 43.62.
We have now the possibility to examine the effect of estimating a wrong number of noise parameters on the estimates of the process-parameters.
First we will show that it is necessary to estimate some noise parameters.
We simulated the given set of D.E.'s with $\lambda=1$(S/N=.432 or -3.7 dB) and estimated only the process-parameters. The results are given in table 3.12 and graph 3.2. We can see, that all estimates are more or less biased. This agrees completely with the theory (see section 2.1).

Remark : As suggested in section 2.2.3 in this program one has the possibility to increase the weighting-factor to 1. This is done in an exponential way; viz:

$$\rho_{k+1}=\rho_k+(1-\rho_k)\Delta\rho=(1-\Delta\rho)\rho_k+\Delta\rho.$$

3.3.1. The first extended matrix method.

With this algorithm we approximate the equation errors by an autoregressive model; viz:

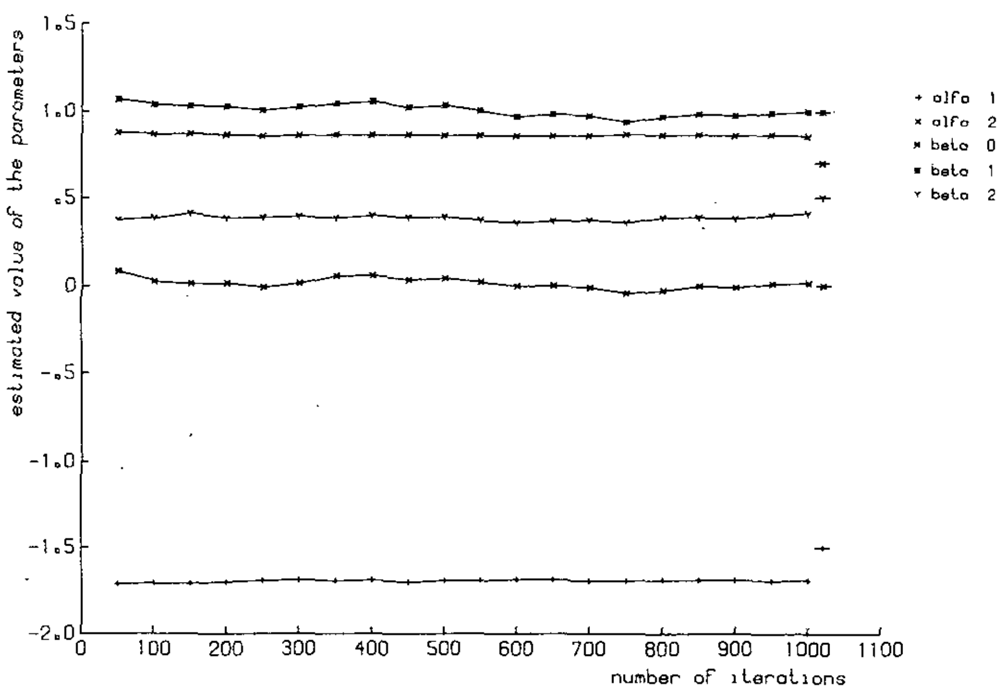
$$e_k=-d_1'e_{k-1}-d_2'e_{k-2}-\dots-d_{r_o}'e_{k-r_o}+\xi_k', \tag{3.2}$$

or $e_k=-D'e_k+\xi_k'$.

In our case $1+D'=(1-.5z^{-1})/(1+.3z^{-1})=1-.8z^{-1}+.24z^{-2}-.072z^{-3}+\dots..$
For the simulation and estimation is chosen: $\lambda=1$(S/N=.432 or -3.7 dB) $\rho_0=.9913$ and $\Delta\rho=.001$.

number of iterations	α_1	α_2	β_0	β_1	β_2
100	-1.709 .027	.859 .044	+.024 .103	1.042 .116	.386 .110
200	-1.701 .030	.857 .026	+.012 .065	1.029 .084	.386 .058
300	-1.690 .031	.850 .027	+.014 .072	1.026 .089	.394 .052
400	-1.690 .029	.853 .021	+.059 .055	1.057 .108	.398 .072
500	-1.693 .015	.851 .018	+.042 .106	1.035 .128	.392 .085
600	-1.687 .011	.849 .015	-.004 .077	0.968 .100	.357 .070
700	-1.695 .011	.852 .019	-.011 .071	0.974 .094	.375 .050
800	-1.689 .016	.853 .014	-.029 .083	0.968 .090	.390 .044
900	-1.688 .018	.849 .018	-.011 .078	0.974 .089	.380 .063
1000	-1.689 .013	.846 .014	+.012 .040	0.998 .073	.410 .064

Table 3.12
Extended matrix method. Only process parameters estimated. Coloured equation error.
$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



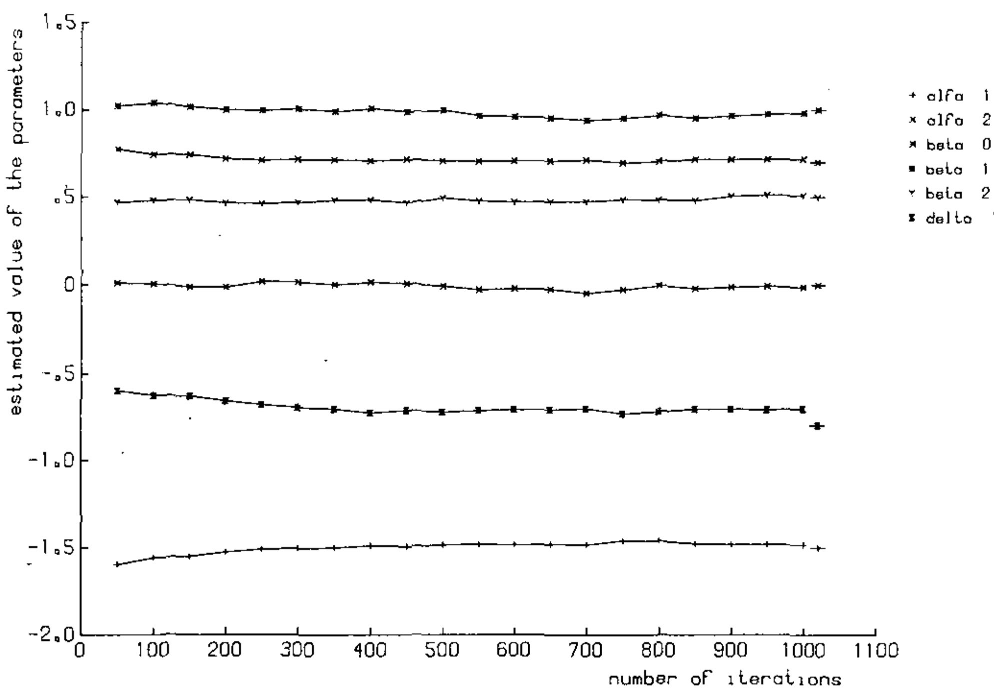
Graph 3.2, corresponding with table 3.12

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1
100	-1.558 .080	.742 .073	+.005 .064	1.038 .089	.482 .141	-.625 .139
200	-1.523 .044	.719 .037	-.012 .076	0.998 .071	.468 .083	-.657 .087
300	-1.503 .044	.715 .037	+.015 .066	1.004 .092	.470 .085	-.696 .075
400	-1.488 .045	.708 .039	+.014 .065	1.006 .074	.483 .064	-.726 .057
500	-1.482 .040	.707 .038	-.006 .054	0.997 .081	.496 .076	-.721 .041
600	-1.482 .052	.708 .051	-.020 .054	0.960 .099	.472 .061	-.706 .062
700	-1.482 .048	.713 .046	-.044 .079	0.939 .094	.476 .039	-.702 .054
800	-1.463 .046	.704 .039	-.003 .064	0.967 .082	.482 .062	-.719 .057
900	-1.479 .040	.715 .033	-.012 .062	0.965 .077	.506 .053	-.703 .038
1000	-1.481 .041	.718 .029	-.011 .040	0.981 .051	.509 .065	-.700 .051

Table 3.13.

First extended matrix method.
Only one backward parameter estimated.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=0.001.



Graph 3.3, corresponding with table 3.13.

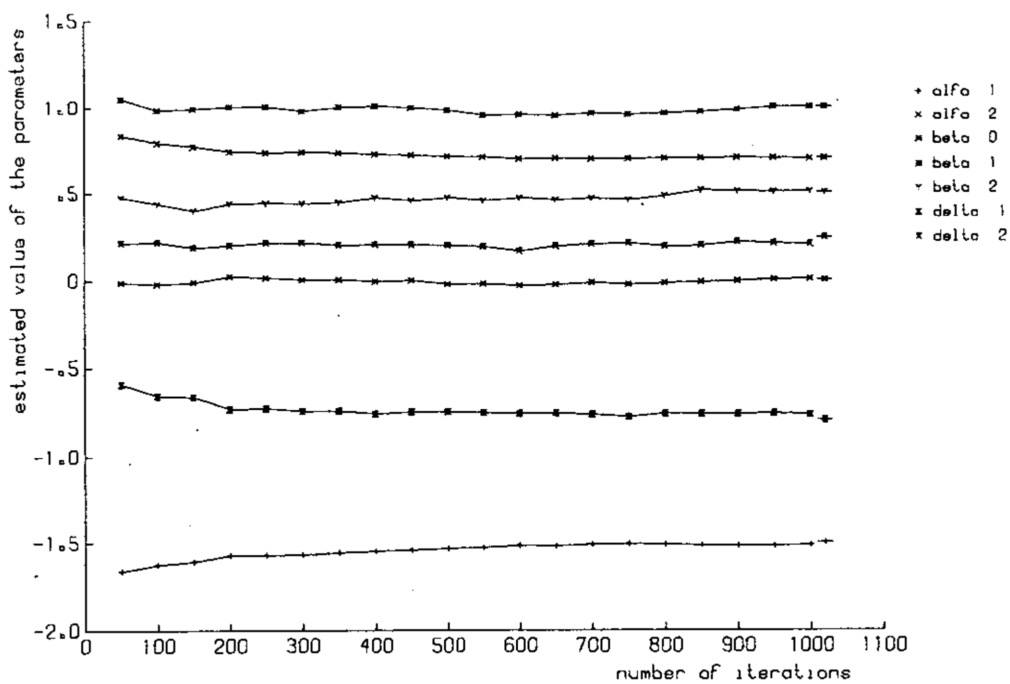
number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2
100	-1.623 .038	.827 .044	-.019 .101	0.983 .118	.438 .127	-.661 .166	.217 .119
200	-1.569 .037	.741 .046	+.023 .085	1.004 .122	.441 .075	-.735 .112	.202 .108
300	-1.566 .039	.737 .038	+.004 .088	0.977 .105	.442 .075	-.745 .088	.214 .069
400	-1.547 .031	.723 .033	-.007 .063	1.007 .101	.472 .076	-.762 .055	.202 .045
500	-1.533 .038	.710 .040	-.022 .047	0.980 .101	.471 .089	-.751 .059	.199 .074
600	-1.515 .046	.698 .051	-.031 .067	0.952 .085	.470 .052	-.763 .065	.164 .076
700	-1.510 .028	.695 .030	-.015 .078	0.960 .074	.467 .040	-.768 .072	.202 .072
800	-1.511 .022	.699 .015	-.016 .069	0.958 .086	.481 .065	-.762 .047	.189 .100
900	-1.517 .023	.701 .021	-.007 .038	0.982 .062	.507 .053	-.765 .053	.213 .072
1000	-1.512 .034	.696 .033	+.005 .049	0.998 .055	.507 .062	-.770 .068	.200 .066

Table 3.14

First extended matrix method.

Two backward noise-parameters estimated.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



Graph 3.4, corresponding with table 3.14.

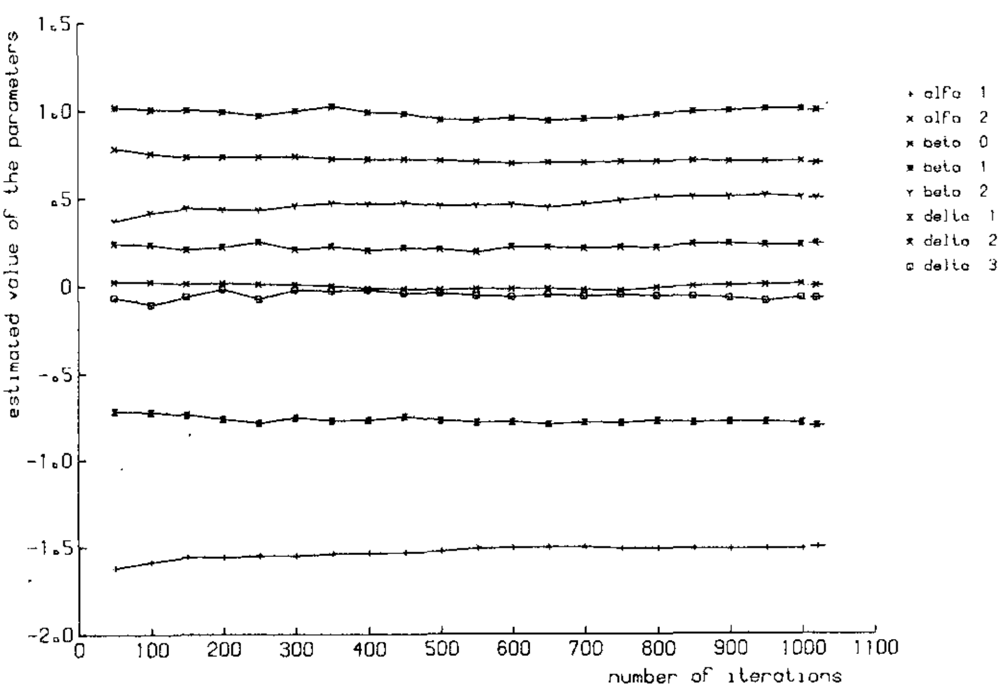
number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	δ_2	δ_3
100	-1.583 .045	.756 .051	+.025 .135	1.006 .172	.419 .173	-.720 .111	.238 .186	-.104 .127
200	-1.558 .050	.735 .046	+.017 .095	0.994 .144	.436 .085	-.760 .097	.225 .150	-.018 .085
300	-1.551 .049	.737 .048	+.008 .056	0.999 .091	.457 .069	-.752 .086	.209 .073	-.022 .061
400	-1.535 .058	.722 .053	-.014 .053	0.989 .094	.466 .079	-.764 .081	.205 .079	-.024 .085
500	-1.525 .062	.714 .060	-.023 .055	0.945 .106	.455 .078	-.768 .105	.210 .107	-.043 .075
600	-1.503 .051	.697 .048	-.019 .079	0.956 .089	.462 .045	-.776 .086	.224 .091	-.063 .076
700	-1.502 .038	.701 .036	-.024 .071	0.952 .091	.468 .058	-.778 .079	.215 .099	-.059 .043
800	-1.510 .029	.708 .029	-.012 .054	0.975 .073	.505 .038	-.771 .068	.218 .082	-.060 .049
900	-1.514 .041	.707 .040	+.003 .042	0.995 .051	.507 .051	-.776 .060	.241 .075	-.070 .058
1000	-1.511 .037	.710 .034	+.010 .052	1.008 .057	.503 .060	-.783 .046	.232 .067	-.069 .031

Table 3.15

First extended matrix method.

3 backward noise parameters estimated.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



Graph 3.5, corresponding with table 3.15.

When we increase the number of coefficients of D', that we estimate, we see that the quality of the estimates of the a's, the b's and the d's becomes better and better (see table 3.13-3.15 and graph 3.3-3.5).

The equation error ξ'_k in equation (3.2) is equal to:

$$-d'_{r_0+1}e_{k-r_0-1}-\dots\dots+\xi_k.$$

When we increase r_0 then the equation error will approximate a white noise sample better and better, so intuitively we expect that for large r_0 the results will be better as for smaller r_0. This can be observed in the tables and the graphs.

We see from the results, that, when we estimate one d-parameter, the estimates of the process-parameters are less biased, while the standard deviation of the α's becomes about 3 times as great as in the case where we did not estimate any d-parameter. Furthermore we see that the true value of d_1 lies outside the confidence interval of 95% of the estimated value.

When we increase the number of estimated d-parameters we see that the estimates of the process-parameters do not change very much, while the estimated values of the d-parameters become better and better.

3.3.2. The second extended matrix method.

For this algorithm we approximate the dynamical behaviour of the equation errors by a moving-average model; viz:

$$e_k=c'_1\xi'_{k-1}+c'_2\xi'_{k-2}+\dots\dots c'_{s_0}\xi'_{k-s_0}+\xi'_k \qquad (3.3)$$

or $e_k=C'\underline{\xi}'_k+\xi'_k$.

In our case $1+C'=(1+.3z^{-1})/(1-.5z^{-1})=1+.8z^{-1}+.4z^{-2}+.2z^{-3}+\dots\dots$

For the simulation is chosen again: $\lambda=1$(S/N=.432 or -3.7 dB) and $\rho_0=.9913$ and $\Delta\rho=.001$.

Again we see that, when we increase the number of coefficients of C' that we estimate, the quality of the estimates of the a-, b- and c-parameters are becomming better and better(see table 3.16-3.19 and graph 3.6-3.9).

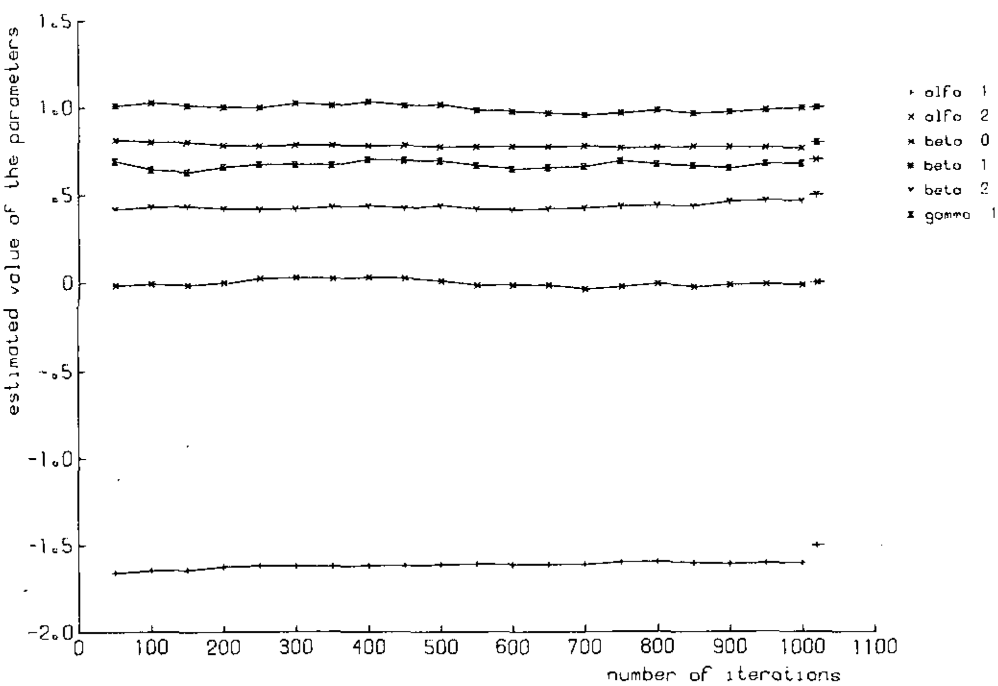
number of iterations	α_1	α_2	β_0	β_1	β_2	γ_1
100	-1.643 .034	.804 .037	-.009 .060	1.026 .090	.429 .129	.642 .177
200	-1.621 .031	.784 .032	-.008 .079	1.001 .084	.418 .081	.660 .060
300	-1.620 .024	.786 .016	+.028 .078	1.023 .100	.419 .090	.673 .098
400	-1.615 .025	.781 .023	+.029 .088	1.033 .085	.436 .074	.702 .056
500	-1.611 .019	.776 .025	+.006 .067	1.016 .093	.439 .083	.693 .041
600	-1.612 .018	.776 .027	-.016 .064	0.975 .099	.414 .068	.646 .061
700	-1.609 .021	.776 .024	-.040 .081	0.954 .095	.421 .043	.659 .079
800	-1.598 .021	.769 .022	-.010 .062	0.980 .068	.437 .056	.672 .080
900	-1.607 .020	.772 .019	-.014 .059	0.972 .065	.461 .044	.648 .050
1000	-1.602 .020	.767 .018	-.014 .033	0.995 .040	.465 .053	.677 .053

Table 3.16

Second extended matrix method.

One forward noise-parameter estimated.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



Graph 3.6, corresponding with table 3.16.

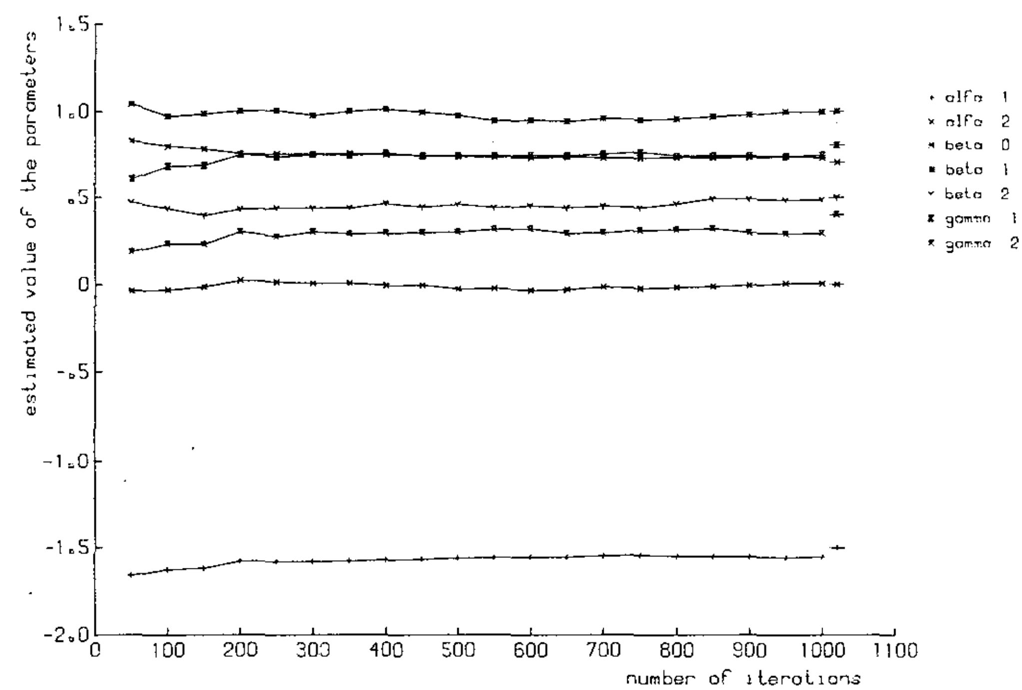
number of iterations	α_1	α_2	β_0	β_1	β_2	γ_1	γ_2
100	-1.625	.794	-.029	0.973	.435	.683	.237
	.042	.041	.096	.111	.126	.166	.173
200	-1.578	.751	+.023	1.003	.434	.746	.306
	.040	.043	.089	.118	.073	.106	.139
300	-1.579	.750	+.007	0.981	.438	.747	.306
	.035	.033	.089	.102	.062	.087	.059
400	-1.572	.745	-.006	1.011	.465	.751	.299
	.025	.029	.068	.103	.071	.065	.059
500	-1.559	.732	-.026	0.977	.458	.738	.305
	.028	.033	.052	.102	.089	.076	.065
600	-1.555	.728	-.033	0.951	.451	.741	.322
	.032	.039	.071	.087	.040	.084	.066
700	-1.547	.724	-.015	0.958	.448	.746	.300
	.024	.025	.082	.071	.046	.084	.052
800	-1.549	.727	-.018	0.956	.462	.736	.315
	.021	.019	.067	.084	.069	.058	.053
900	-1.553	.728	-.007	0.980	.489	.737	.300
	.020	.019	.038	.058	.051	.055	.057
1000	-1.552	.726	+.005	0.997	.487	.741	.297
	.034	.031	.048	.050	.058	.073	.056

Table 3.17

Second extended matrix method.

Two forward noise-parameters estimated.

$\frac{S}{N}=.432$ (-3.7 dB). $\rho_0=.9913$. $\Delta\rho=.001$.



Graph 3.7, corresponding with table 3.17.

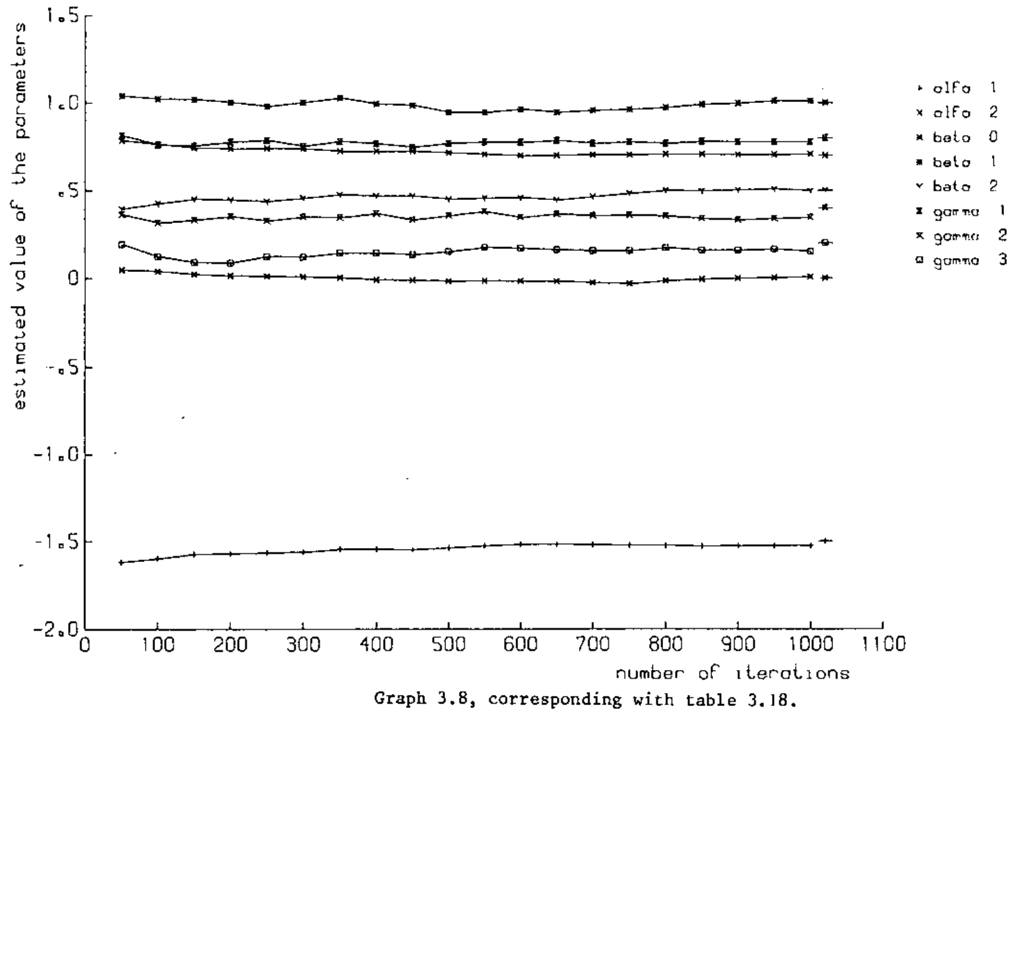
number of iterations	α_1	α_2	β_0	β_1	β_2	γ_1	γ_2	γ_3
100	-1.597 .046	.764 .047	+.039 .141	1.020 .192	.424 .161	.760 .139	.316 .215	.124 .223
200	-1.568 .049	.740 .043	+.015 .102	1.001 .157	.444 .096	.777 .099	.351 .145	.087 .219
300	-1.559 .047	.740 .043	+.009 .060	1.002 .095	.457 .075	.753 .099	.350 .110	.121 .120
400	-1.543 .047	.725 .041	-.008 .054	0.992 .096	.466 .078	.767 .082	.368 .106	.141 .114
500	-1.536 .046	.717 .045	-.019 .059	0.947 .109	.450 .080	.769 .104	.359 .119	.149 .106
600	-1.517 .030	.699 .031	-.017 .077	0.961 .088	.459 .048	.774 .079	.347 .086	.171 .049
700	-1.518 .025	.704 .023	-.025 .069	0.956 .089	.463 .059	.768 .064	.358 .081	.156 .058
800	-1.521 .023	.707 .017	-.016 .053	0.975 .071	.502 .039	.769 .062	.355 .061	.173 .065
900	-1.525 .029	.704 .029	-.001 .042	0.997 .048	.503 .055	.778 .052	.332 .067	.160 .093
1000	-1.524 .030	.707 .028	+.007 .052	1.009 .056	.498 .059	.779 .046	.346 .065	.151 .059

Table 3.18

Second extended matrix method.

3 forward noise-parameters estimated.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



Graph 3.8, corresponding with table 3.18.

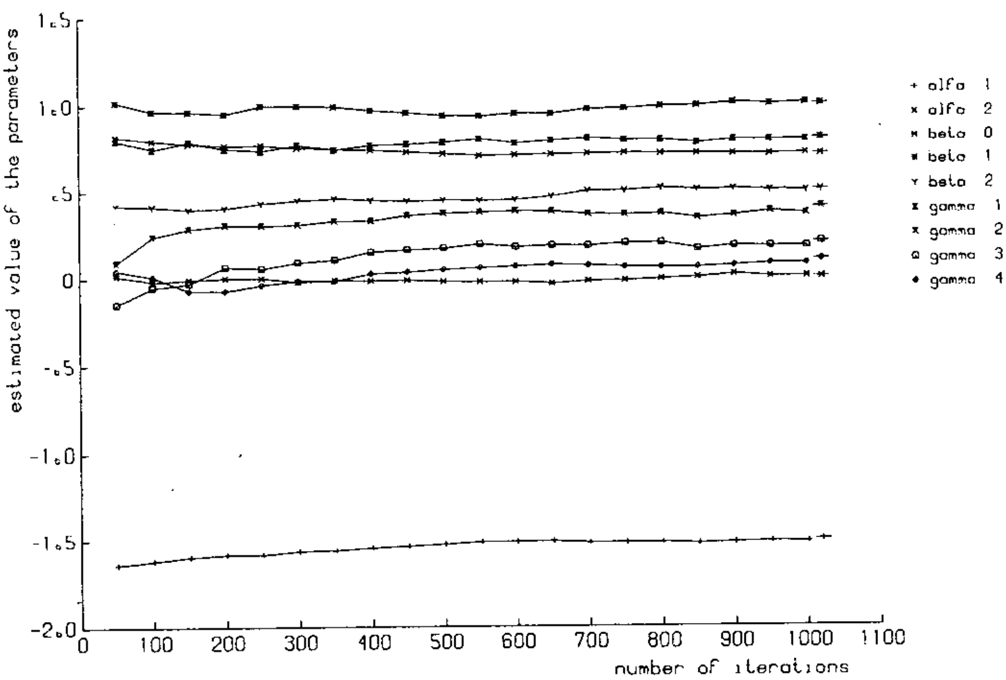
number of iterations	α_1	α_2	β_0	β_1	β_2	γ_1	γ_2	γ_3	γ_4
100	-1.621 .046	.788 .046	-.018 .124	0.966 .169	.411 .103	.739 .178	.241 .183	-.052 .258	+.012 .166
200	-1.587 .049	.757 .050	+.003 .044	0.951 .104	.402 .080	.738 .075	.304 .089	+.067 .188	-.071 .136
300	-1.566 .036	.744 .040	-.019 .064	0.996 .112	.448 .096	.760 .097	.307 .099	+.092 .112	-.014 .086
400	-1.547 .040	.729 .041	-.014 .057	0.969 .092	.447 .082	.760 .089	.327 .101	+.151 .078	+.028 .060
500	-1.523 .048	.708 .047	-.019 .099	0.938 .092	.448 .059	.779 .100	.368 .108	+.170 .083	+.051 .081
600	-1.509 .022	.701 .023	-.023 .062	0.947 .075	.447 .048	.769 .042	.381 .066	+.175 .065	+.066 .097
700	-1.515 .010	.706 .013	-.020 .054	0.975 .073	.499 .042	.799 .054	.361 .061	+.182 .076	+.069 .045
800	-1.517 .017	.705 .020	-.009 .052	0.987 .051	.510 .038	.789 .045	.363 .048	+.199 .084	+.058 .055
900	-1.517 .027	.702 .026	+.017 .048	1.007 .051	.505 .056	.789 .064	.352 .056	+.177 .051	+.066 .053
1000	-1.514 .038	.706 .033	+.003 .055	1.006 .069	.495 .068	.790 .047	.362 .071	+.175 .096	+.077 .058

Table 3.19

Second extended matrix method.

4 forward noise-parameters estimated.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



Graph 3.9, corresponding with table 3.19.

When we increase s_o, then, as in the previous section, the equation error will approximate a white noise sample better and better, so intuitively we expect that for large s_o the results will be better than for smaller s_o.
Furthermore we see that we have to estimate more c-parameters than d-parameters in the previous section to obtain results with the same quality.
This can be explained intuitively. The values of the d'-parameters are going to small values rather quickly(the 4th d'-parameter is equal to +.0216), while the c'-parameters tend to zero much slower (the 7th c'-parameter is equal to .025). So when estimating a fixed number of noise parameters in both cases the equation error in the previous section is less coloured then when estimating c'-parameters. When we compare the standard deviation of the noise parameters in both cases we see that they are of the same order of magnitude.

3.3.3. The third extended matrix method.

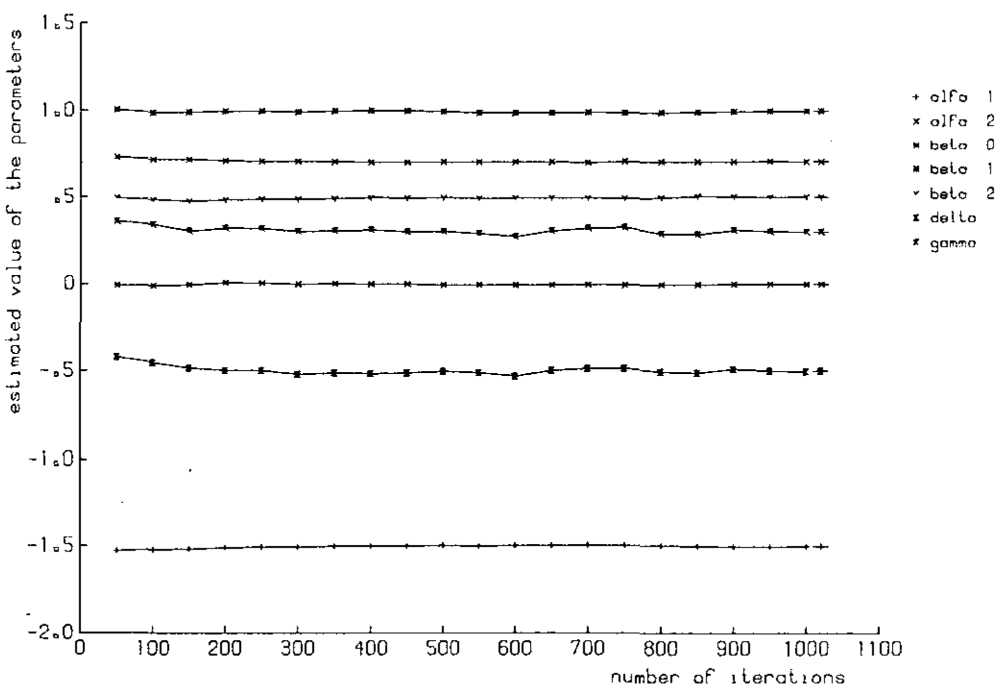
With this algorithm we estimated the parameters of the process and of the equation error, using the correct dimensions.
When we look at table 3.21 and graph 3.11 we see (with $\lambda=1$, S/N=.432 or -3.7 dB; ρ_0=.9913 and $\Delta\dot{\rho}$=.001) that the process- and noise filter-parameters are estimated extremely well. We have estimated now two noise filter-parameters to obtain good estimates, while with the first and second extended matrix method and estimating resp. 3 and 4 noise filter-parameters the results are a little bit worse. The computation time however is much longer.
Now we will look to the properties of the algorithm with relation to the noise power and the value of $\Delta\rho$.
We gave λ the values .25, 1, 4 and 16, wich means for S/N resp: 6.92,.432, .0269 and .0017 or resp 8.3, -3.7, .15.7 and -27.7 dB.
For all these values of λ we kept ρ_0 and $\Delta\rho$ constant (resp. .9913 and .001).
The results are to be found in table 3.20-3.23 and graph 3.10-3.13.
We see, in analogy with the algorithm of Hastings-James & Sage, that the standard deviation of the estimates of the forward parameters of the process grows linearly with λ.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.519 .019	.714 .025	-.009 .024	0.989 .028	.487 .029	-.452 .172	.342 .169
200	-1.507 .021	.706 .025	+.006 .020	0.999 .030	.488 .022	-.496 .140	.322 .131
300	-1.505 .019	.703 .019	+.000 .021	0.992 .025	.490 .022	-.519 .083	.301 .093
400	-1.499 .015	.697 .014	-.002 .016	1.001 .023	.498 .019	-.517 .060	.311 .073
500	-1.497 .012	.696 .013	-.005 .011	0.995 .024	.498 .022	-.502 .063	.301 .104
600	-1.497 .017	.697 .020	-.008 .016	0.988 .021	.496 .015	-.531 .085	.270 .126
700	-1.494 .008	.695 .009	-.003 .019	0.991 .017	.496 .010	-.485 .061	.320 .105
800	-1.497 .013	.699 .011	-.004 .017	0.990 .021	.498 .019	-.504 .076	.287 .112
900	-1.501 .008	.700 .007	-.002 .010	0.996 .015	.503 .014	-.487 .064	.310 .082
1000	-1.500 .010	.701 .011	+.000 .013	0.999 .014	.502 .018	-.502 .060	.300 .095

Table 3.20.

Third extended matrix method.

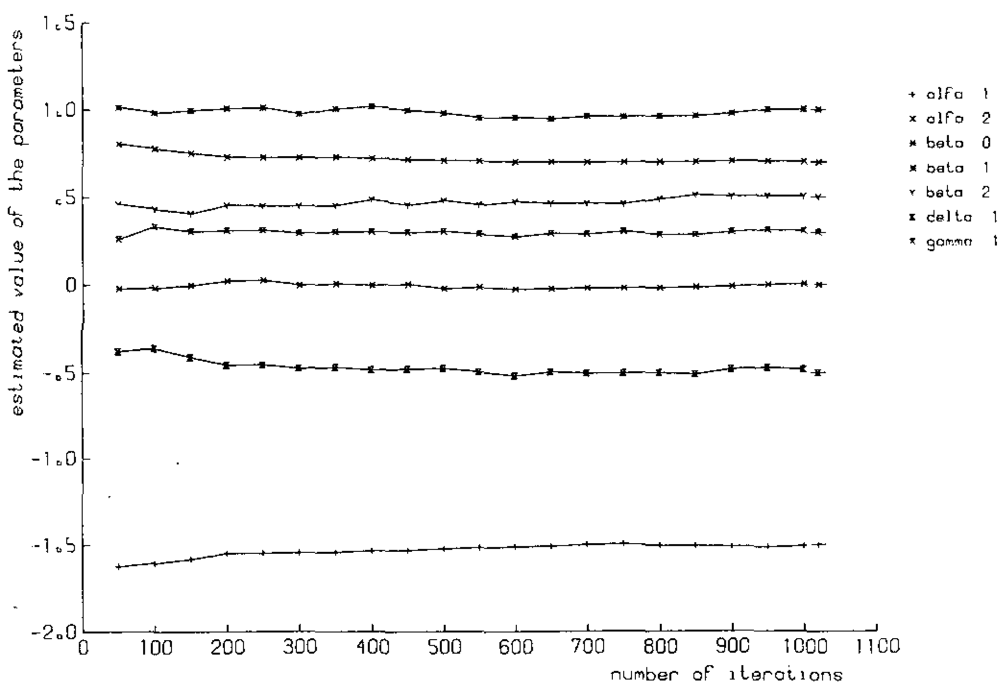
$\frac{S}{N}$=6.92 (8.3 dB). ρ_0=.9913. $\Delta\rho$=.001.



Graph 3.10, corresponding with table 3.20.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	−1.601 .039	.777 .047	−.018 .101	0.983 .135	.433 .139	−.366 .160	.332 .170
200	−1.546 .044	.729 .048	+.023 .084	1.008 .117	.456 .097	−.455 .147	.311 .160
300	−1.544 .045	.729 .044	−.003 .086	0.974 .098	.450 .070	−.474 .114	.293 .105
400	−1.533 .024	.721 .027	−.005 .072	1.019 .089	.488 .067	−.483 .085	.301 .080
500	−1.522 .039	.709 .040	−.023 .050	0.980 .100	.479 .087	−.474 .097	.305 .119
600	−1.509 .043	.702 .047	−.027 .072	0.955 .089	.476 .052	−.516 .106	.273 .142
700	−1.499 .030	.698 .029	−.020 .067	0.960 .070	.464 .043	−.503 .075	.288 .097
800	−1.504 .022	.703 .018	−.018 .061	0.964 .073	.488 .061	−.499 .100	.285 .130
900	−1.510 .028	.706 .026	−.011 .040	0.978 .058	.503 .056	−.481 .085	.301 .094
1000	−1.508 .042	.703 .034	+.002 .046	1.000 .051	.504 .059	−.482 .085	.308 .092

Table 3.21.
Third extended matrix method.
$\frac{S}{N}$=.432 (−3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



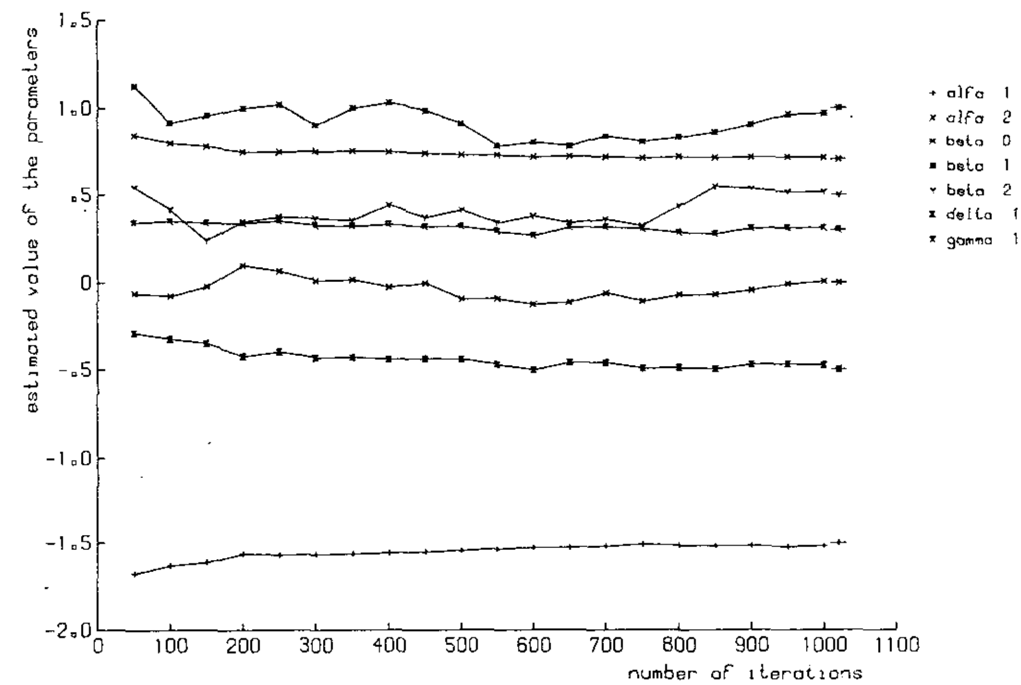
Graph 3.11, corresponding with table 3.21.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.627 .057	.793 .065	-.077 .405	0.915 .470	.415 .494	-.327 .173	.349 .183
200	-1.564 .058	.740 .064	+.098 .328	0.997 .446	.347 .321	-.425 .181	.337 .158
300	-1.567 .044	.744 .047	+.011 .350	0.901 .396	.364 .243	-.432 .104	.323 .091
400	-1.558 .049	.741 .050	-.025 .274	1.030 .397	.439 .265	-.440 .115	.329 .093
500	-1.542 .047	.726 .046	-.092 .192	0.912 .396	.412 .347	-.441 .130	.317 .155
600	-1.526 .045	.714 .046	-.124 .268	0.797 .345	.378 .194	-.502 .115	.269 .153
700	-1.520 .046	.712 .041	-.058 .298	0.834 .288	.359 .164	-.460 .102	.317 .120
800	-1.516 .031	.710 .027	-.074 .276	0.826 .333	.430 .249	-.490 .085	.281 .122
900	-1.517 .041	.710 .035	-.044 .158	0.906 .248	.533 .201	-.470 .095	.308 .089
1000	-1.515 .057	.706 .045	+.005 .202	0.969 .218	.517 .220	-.473 .104	.311 .094

Table 3.22.

Third extended matrix method.

$\frac{S}{N}$=.0269 (-15.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



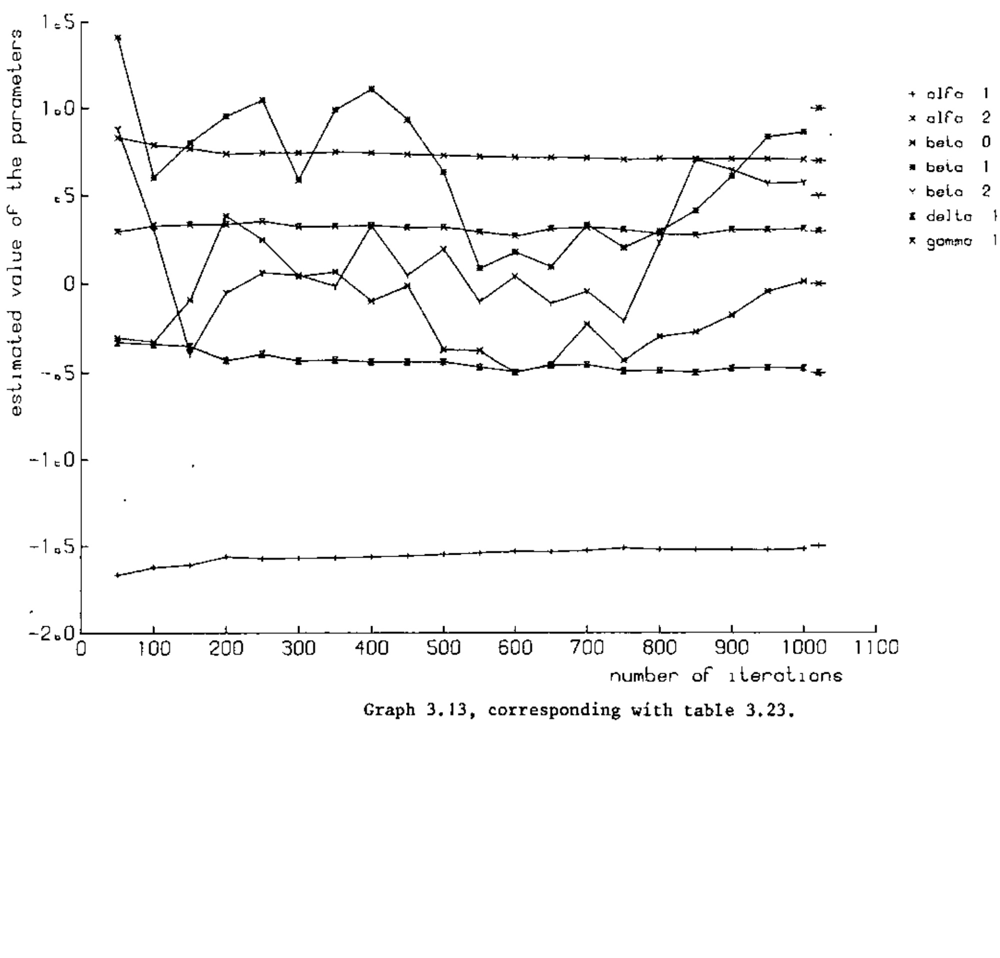
Graph 3.12, corresponding with table 3.22.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.623 .064	.788 .075	-.329 1.621	0.603 1.956	+.304 1.875	-.342 .155	.330 .180
200	-1.562 .068	.737 .072	+.386 1.302	0.953 1.754	-.052 1.266	-.426 .190	.338 .165
300	-1.568 .045	.744 .048	+.042 1.395	0.589 1.566	+.047 .913	-.429 .101	.325 .093
400	-1.561 .056	.743 .055	-.098 1.093	1.111 1.574	+.331 1.020	-.434 .121	.333 .094
500	-1.546 .049	.729 .046	-.370 .767	0.634 1.573	+.196 1.399	-.435 .131	.320 .158
600	-1.530 .043	.718 .043	-.497 1.070	0.179 1.382	+.040 .796	-.495 .113	.272 .154
700	-1.525 .048	.716 .043	-.230 1.184	0.335 1.139	-.043 .653	-.451 .105	.320 .119
800	-1.519 .036	.712 .032	-.297 1.097	0.299 1.322	+.235 .991	-.485 .083	.283 .118
900	-1.518 .043	.711 .037	-.180 .631	0.613 .987	+.645 .802	-.470 .095	.308 .087
1000	-1.516 .057	.707 .045	-.013 .806	0.863 .870	+.576 .883	-.471 .106	.312 .097

Table 3.23.

Third extended matrix method.

$\frac{S}{N}$=.0017 (-27.7 dB). ρ_0=.9913. $\Delta\rho$=.001.



Graph 3.13, corresponding with table 3.23.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.605 .045	.778 .050	-.021 .099	0.982 .118	.447 .127	-.341 .193	.364 .169
200	-1.548 .045	.730 .052	+.028 .084	1.004 .122	.452 .084	-.445 .165	.333 .142
300	-1.545 .043	.729 .041	+.003 .090	0.975 .102	.453 .075	-.464 .103	.316 .091
400	-1.531 .034	.719 .036	-.009 .074	1.012 .104	.484 .076	-.479 .091	.322 .088
500	-1.515 .047	.704 .048	-.031 .049	0.971 .109	.480 .093	-.471 .116	.311 .156
600	-1.497 .062	.694 .062	-.040 .079	0.939 .091	.473 .055	-.546 .130	.254 .163
700	-1.488 .037	.691 .033	-.013 .091	0.957 .078	.468 .054	-.491 .087	.326 .123
800	-1.496 .036	.702 .023	-.019 .092	0.957 .097	.496 .093	-.522 .132	.273 .168
900	-1.516 .032	.711 .023	-.006 .044	0.995 .066	.522 .073	-.446 .123	.348 .107
1000	-1.509 .051	.704 .040	+.010 .081	1.010 .086	.513 .090	-.474 .106	.333 .111

Table 3.24.

Third extended matrix method.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=0.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.601 .039	.777 .047	-.018 .101	0.983 .135	.433 .139	-.366 .160	.332 .170
200	-1.546 .044	.729 .048	+.023 .084	1.008 .117	.456 .097	-.455 .147	.311 .160
300	-1.544 .045	.729 .044	-.003 .086	0.974 .098	.450 .070	-.474 .114	.293 .105
400	-1.533 .024	.721 .027	-.005 .072	1.019 .089	.488 .067	-.483 .085	.301 .080
500	-1.522 .039	.709 .040	-.023 .050	0.980 .100	.479 .087	-.474 .097	.305 .119
600	-1.509 .043	.702 .047	-.027 .072	0.955 .089	.476 .052	-.516 .106	.273 .142
700	-1.499 .030	.698 .029	-.020 .067	0.960 .070	.464 .043	-.503 .075	.288 .097
800	-1.504 .022	.703 .018	-.018 .061	0.964 .073	.488 .061	-.499 .100	.285 .130
900	-1.510 .028	.706 .026	-.011 .040	0.978 .058	.503 .056	-.481 .085	.301 .094
1000	-1.508 .042	.703 .034	+.002 .046	1.000 .051	.504 .059	-.482 .085	.308 .092

Table 3.25.

Third extended matrix method.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.001.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.603 .038	.778 .047	-.018 .098	0.984 .133	.434 .138	-.366 .160	.328 .170
200	-1.556 .043	.736 .048	+.019 .081	1.006 .108	.447 .090	-.446 .143	.305 .153
300	-1.551 .038	.733 .042	-.001 .080	0.978 .097	.444 .071	-.468 .101	.286 .102
400	-1.543 .024	.727 .028	-.001 .059	1.007 .089	.468 .066	-.475 .071	.290 .073
500	-1.535 .026	.720 .030	-.010 .043	0.990 .092	.466 .074	-.471 .078	.294 .096
600	-1.528 .027	.715 .032	-.014 .054	0.974 .087	.467 .059	-.492 .084	.278 .114
700	-1.523 .020	.712 .025	-.013 .050	0.974 .074	.464 .045	-.484 .071	.284 .094
800	-1.522 .014	.712 .019	-.013 .048	0.974 .074	.474 .043	-.485 .068	.282 .095
900	-1.522 .019	.711 .023	-.011 .041	0.977 .065	.483 .044	-.481 .066	.287 .082
1000	-1.519 .025	.709 .026	-.005 .037	0.985 .055	.486 .042	-.482 .064	.290 .077

Table 3.26.

Third extended matrix method.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.005.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.612 .043	.785 .050	-.019 .091	0.992 .113	.447 .124	-.343 .179	.348 .148
200	-1.578 .040	.753 .047	+.012 .072	1.001 .087	.434 .069	-.410 .142	.320 .111
300	-1.571 .032	.747 .040	+.001 .076	0.983 .087	.437 .068	-.434 .094	.300 .080
400	-1.562 .029	.740 .036	-.003 .057	0.996 .087	.449 .063	-.442 .088	.301 .074
500	-1.554 .027	.734 .033	-.007 .046	0.989 .085	.451 .062	-.445 .085	.298 .089
600	-1.548 .029	.729 .035	-.011 .050	0.978 .082	.454 .057	-.463 .093	.284 .100
700	-1.544 .023	.726 .029	-.009 .050	0.978 .076	.456 .048	-.450 .085	.299 .096
800	-1.542 .020	.725 .025	-.010 .049	0.977 .077	.462 .043	-.458 .080	.288 .088
900	-1.540 .021	.723 .027	-.009 .041	0.981 .069	.472 .046	-.454 .077	.295 .080
1000	-1.538 .022	.721 .025	-.005 .038	0.985 .060	.475 .045	-.458 .071	.294 .073

Table 3.27.

Third extended matrix method.

$\frac{S}{N}$=.432 (-3.7 dB). ρ_0=.9913. $\Delta\rho$=.025.

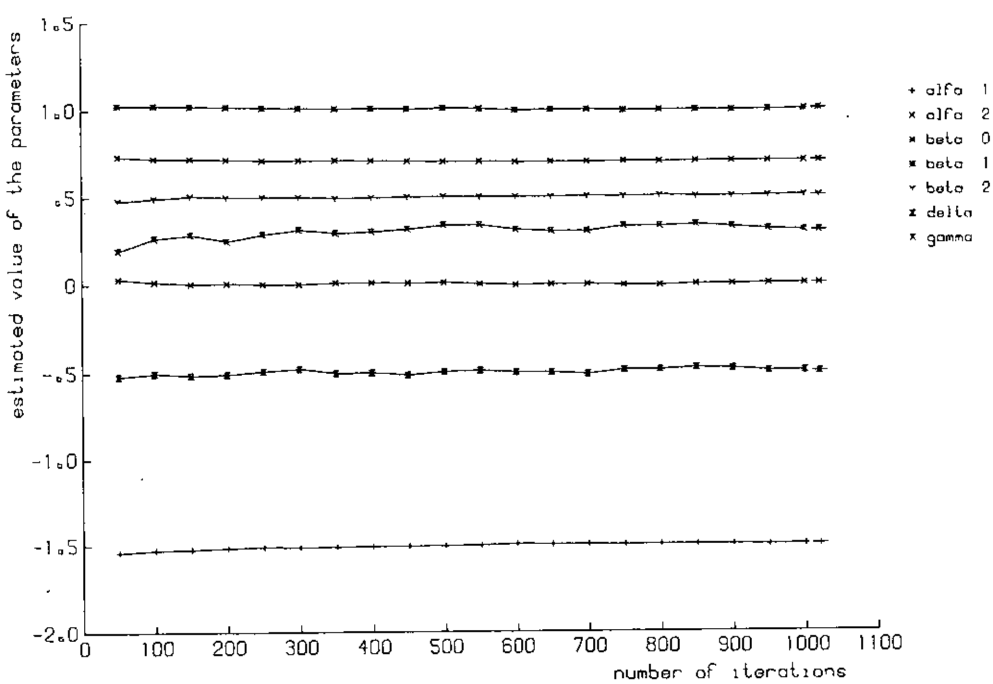
The standard deviation of the estimates of the backward parameters tends to a constant value. Furthermore the standard deviations of the estimates of the c- and d-parameters remain constant. Also we see again that the averages of the estimates of the a-parameters are good for all λ, while the averages of the b-parameters are getting worse when we increase λ. Also the averages of the estimates of the c- and d-parameters are getting worse when λ grows, but we have the impression that for large values of λ the algorithm had not achieved the final values for these parameters, because there is still a trend to better values.
For looking to what happens if we change $\Delta\rho$, we gave it the values 0, .001, .005 and .025, which means that the difference between and 1 is reduced to half the value of the difference when we start the algorithm after resp. ∞, 698, 128 and 27 iterations. For λ we have chosen 1 and for ρ_0:.9913.
The results are given in table 3.24-3.27. We see in the first place that the standard deviation becomes smaller when we increase ρ to 1. When we increase $\Delta\rho$ we see that in general the means of the estimates become worse, while the standard deviation tends to a certain lower bound.

3.4. Equation error correction.

The results given in this section are obtained with the scheme described in section 2.3.2. In that scheme, one has to use two weighting-factors;viz: ρ for the part where we estimate $\underline{b}'$ and ν for the part where we estimate $\underline{c}$ and $\underline{d}$.
We let them grow to one in an exponential way. For all the results given $\rho_0=\nu_0=.9913$ and $\Delta\rho=\Delta\nu=.001$.
The process given in the beginning of paragraph 3.3 is simulated again with λ equal to $\frac{1}{4}$, 1, 4 and 16.
The results are given in table 3.28-3.31 and graph 3.14-3.17.
When we compare these results with those obtained with the third extended matrix method, we see that for $\lambda=\frac{1}{4}$ the averages and standard deviations are nearly the same. For all higher noise levels the results obtained with the third extended matrix method are better than those obtained with the equation error correction.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.527 .024	.719 .039	+.014 .023	1.024 .027	.493 .040	-.502 .103	.261 .156
200	-1.515 .022	.712 .033	+.004 .011	1.014 .019	.497 .025	-.508 .132	.246 .121
300	-1.512 .013	.708 .022	+.000 .016	1.007 .014	.498 .015	-.476 .105	.311 .085
400	-1.507 .015	.705 .016	+.009 .013	1.008 .026	.496 .020	-.500 .086	.299 .110
500	-1.503 .014	.699 .014	+.008 .015	1.007 .021	.499 .019	-.493 .049	.334 .066
600	-1.496 .014	.695 .012	-.006 .012	0.993 .021	.498 .017	-.499 .048	.307 .062
700	-1.499 .014	.697 .015	-.003 .011	0.995 .021	.500 .012	-.511 .063	.298 .094
800	-1.498 .010	.698 .010	-.009 .018	0.993 .019	.502 .012	-.484 .078	.326 .117
900	-1.500 .012	.699 .011	-.004 .019	0.992 .021	.496 .017	-.480 .061	.321 .104
1000	-1.500 .011	.699 .008	-.000 .009	0.996 .020	.502 .017	-.493 .053	.299 .088

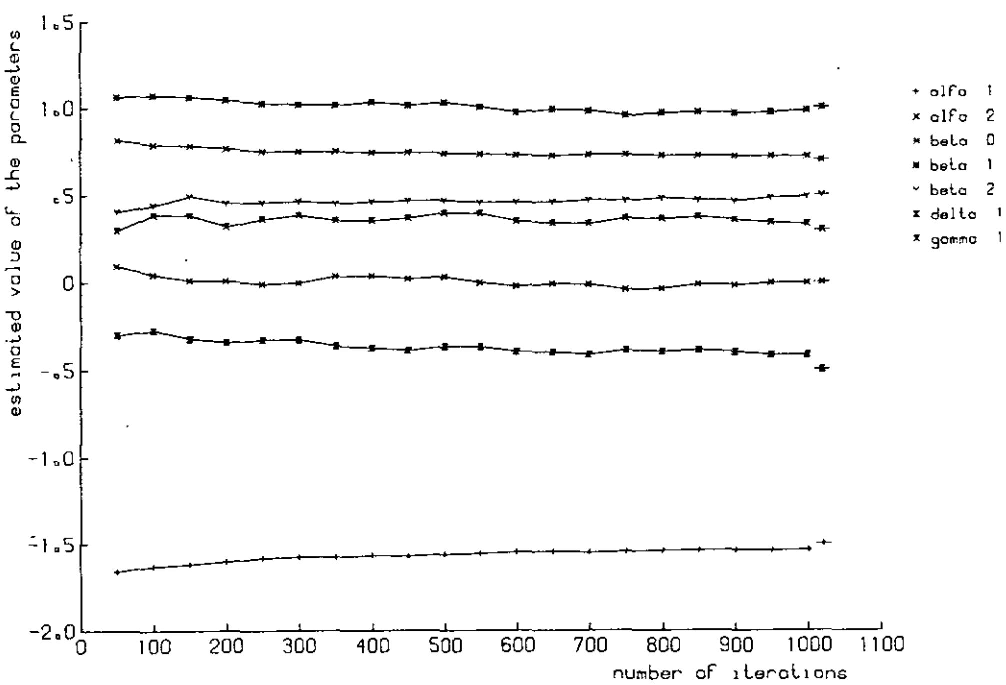
Table 3.28.
Equation error correction.
$\lambda=\frac{1}{4}$ (S/N=6.92 or 8.3 dB)
$\rho_0=\nu_0=.9913$. $\Delta\rho=\Delta\nu=.001$.



Graph 3.14, corresponding with table 3.28.

number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.627 .042	.787 .066	+.044 .102	1.070 .119	.442 .126	-.273 .194	.387 .213
200	-1.598 .034	.765 .042	+.011 .050	1.044 .078	.456 .064	-.340 .182	.321 .152
300	-1.575 .018	.748 .025	-.001 .064	1.017 .061	.467 .045	-.324 .142	.386 .118
400	-1.566 .030	.745 .026	+.039 .052	1.032 .097	.463 .078	-.372 .109	.356 .124
500	-1.561 .031	.738 .028	+.032 .068	1.030 .088	.470 .083	-.364 .082	.397 .079
600	-1.546 .024	.727 .021	-.023 .053	0.974 .089	.457 .073	-.395 .066	.349 .076
700	-1.548 .022	.729 .023	-.013 .048	0.983 .089	.472 .043	-.413 .082	.335 .106
800	-1.537 .019	.723 .017	-.036 .073	0.971 .080	.484 .040	-.394 .098	.363 .134
900	-1.535 .018	.721 .015	-.017 .076	0.967 .085	.467 .062	-.398 .084	.358 .118
1000	-1.533 .017	.718 .011	-.003 .037	0.982 .079	.491 .062	-.415 .068	.333 .094

Table 3.29.
Equation error compensation.
$\lambda=1$ (S/N=.432 or -3.7 dB).
$\rho_0=\nu_0=.9913$. $\Delta\rho=\Delta\nu=.001$.



Graph 3.15, corresponding with table 3.29.

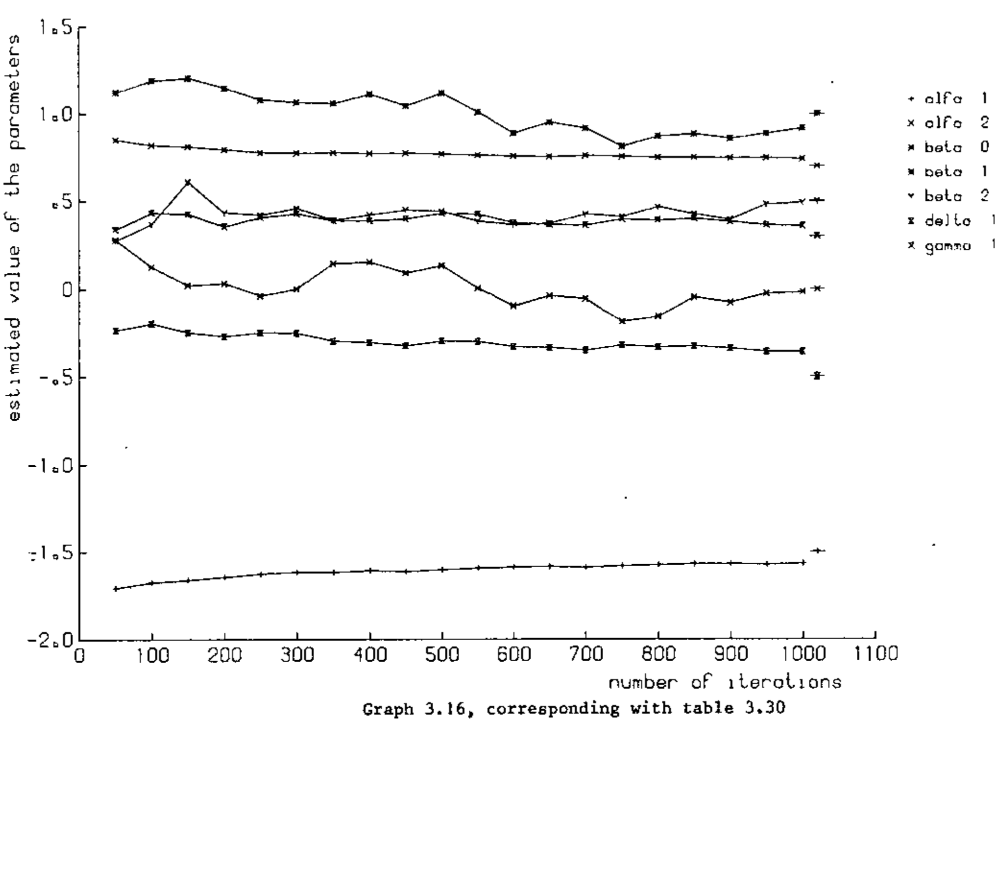
number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.673 .040	.820 .063	+.124 .449	1.188 .528	.369 .479	-.188 .217	.432 .235
200	-1.641 .031	.794 .035	+.033 .224	1.146 .322	.432 .208	-.265 .187	.356 .152
300	-1.613 .018	.775 .021	+.001 .254	1.065 .235	.457 .168	-.245 .163	.429 .137
400	-1.605 .028	.772 .025	+.152 .206	1.113 .372	.422 .284	-.302 .127	.388 .132
500	-1.601 .033	.770 .030	+.135 .275	1.118 .345	.441 .300	-.291 .101	.431 .097
600	-1.587 .025	.758 .021	-.090 .221	0.981 .352	.365 .278	-.327 .078	.377 .090
700	-1.589 .020	.761 .020	-.055 .198	0.916 .352	.424 .182	-.347 .091	.363 .118
800	-1.573 .021	.751 .018	-.149 .290	0.874 .315	.466 .153	-.329 .106	.390 .144
900	-1.568 .018	.745 .014	-.074 .303	0.857 .331	.395 .246	-.336 .091	.382 .127
1000	-1.566 .016	.741 .012	-.017 .149	0.918 .316	.491 .237	-.355 .074	.356 .098

Table 3.30.

Equation error compensation.

$\lambda=4$ (S/N=.0269 or -15.7 dB).

$\rho_0=\nu_0=.9913$. $\Delta\rho=\Delta\nu=.001$.



Graph 3.16, corresponding with table 3.30

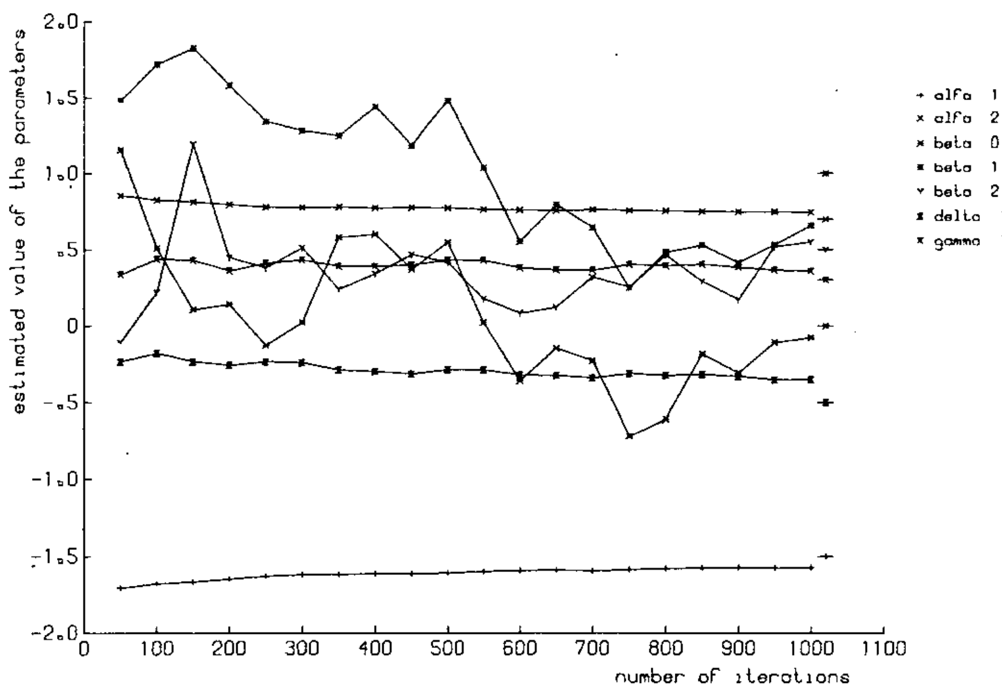
number of iterations	α_1	α_2	β_0	β_1	β_2	δ_1	γ_1
100	-1.678 .041	.824 .063	+.509 1.800	1.723 2.192	.217 1.914	-.180 .228	.437 .247
200	-1.644 .032	.796 .034	+.142 .922	1.586 1.307	.448 .834	-.257 .188	.361 .153
300	-1.616 .017	.778 .019	+.022 1.023	1.277 .930	.512 .678	-.239 .170	.433 .144
400	-1.608 .025	.775 .023	+.603 .823	1.437 1.461	.342 1.110	-.296 .135	.389 .137
500	-1.605 .033	.774 .029	+.549 1.096	1.480 1.360	.418 1.150	-.285 .106	.432 .102
600	-1.592 .025	.762 .020	-.360 .889	0.553 1.392	.083 1.086	-.320 .083	.380 .094
700	-1.594 .021	.765 .019	-.226 .794	0.647 1.403	.320 .743	-.340 .092	.366 .121
800	-1.578 .022	.754 .019	-.601 1.163	0.485 1.255	.464 .613	-.322 .106	.393 .145
900	-1.572 .019	.748 .016	-.306 1.216	0.412 1.321	.169 .981	-.330 .090	.384 .127
1000	-1.570 .016	.744 .013	-.077 .599	0.660 1.265	.551 .950	-.350 .074	.358 .098

Table 3.31.

Equation error compensation.

λ=16 (S/N=.0017 or -27.7 dB).

$\rho_0=\nu_0$=.9913. $\Delta\rho=\Delta\nu$=.001.



Graph 3.17, corresponding with table 3.31.

We see however that the estimates still tend to better values, and we expect that decreasing the weighting-factors will lead to better results.
The computation time for this algorithm is less then for the extended matrix method with the same number of parameters.
It is necessary to study the influence of the weighting-factors on the results to be able to say something about the accuracy of this algorithm.

4. Remarks and suggestions.

The accuracy of the algorithms is not the only feature of interest.
The time required to obtain an estimate is also important.
For the extended matrix methods we simulated 5 process-parameters and two equation error parameters.
On the EL-X8 computer of the Eindhoven University of Technology the following computation times are required for the simulation and estimation of 10 runs of 1000 iterations each (using the MCA system):

number of noise parameters	computation time in sec
0	587
1	745
2	933
3	1139
4	1393

In the equation error correction program we also tested the stability of the estimated process after 50 iterations. For this algorithm, estimating two equation error parameters, we need only 727 sec.

Looking at the results, we see that the third extended matrix method has proved to be the best of the schemes presented.

There are however some problems left.
When we use the third extended matrix method for estimating the process, given in this report, and two c- and two d-parameters, the results depend strongly upon the choice of ρ_0 and $\Delta\rho$. It is not quite clear whether one has to look for a relation between ρ_0 and S/N or for a relation between ρ_0 and λ. Further research in this direction would be very useful.
As already mentioned, the difference in the behaviour of the standard deviation of the α's and the β's might be a subject for further research.
It is the authors opinion that it is time now to apply the schemes presented in this and other papers in practical cases in order to get an impression of the direction in which the theoretical work must evolve.

List of symbols.

$\underline{u}$, $\underline{y}$, $\underline{e}$, ...	: vectors containing samples of input-signal, outputsignal,......
$\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$, $\underline{b}'$, $\underline{c}'$, $\underline{d}'$	: vectors containing the parameter-values to be estimated.
$\underline{\beta}'$, $\underline{\gamma}$, $\underline{\delta}$	: vectors containing the estimates.
A, B, C, D	: polynomials in z^{-1}, having the following structure $A=a_1z^{-1}+a_2z^{-2}+......+a_qz^{-q}$.
x_k, u_k, y_k,.....	: shorthand notation to represent the sampled values at time $t=k\tau$ of x, u, y,..
τ	: sample time.
E	: expectation operator.
$\Omega(u,y,e,...)$	: matrix having the structure $(U \vdots Y \vdots E \vdots ...)$
U, Y, E, ...	: matrices containing samples of u, y, e,...
ξ	: white noise.
n	: additive output noise.
e	: equation error.
u	: input signal.
x	: undisturbed output signal.
y	: disturbed output signal.
ρ, ν	: weighting-factors.
σ_u^2, σ_x^2,....	: power of u, x, ...
Ψ_{ab}	: correlation function of a and b.

Remark: most symbols are defined in the text itself.

Appendix A.

Exponential weighting of the model errors.

Define $V_k=\underline{e}_k^T\underline{e}_k$, in which

$\underline{e}_k^T=(\rho^{k-1}e_1,\rho^{k-2}e_2,\ldots\ldots,e_k)$, with

$e_i=\underline{\omega}_i^T\underline{\beta}_k'-y_i$, with

$\underline{\omega}_i^T=(u_{i+q},u_{i+q-1},\ldots\ldots,u_{i+q-p},y_{i+q-1},\ldots\ldots y_i)$ and

$\underline{\beta}_k'^T=(\beta_0,\beta_1,\ldots,\beta_p,\alpha_1,\alpha_2,\ldots,\alpha_q)$ after k iterations.

We suppose that $q\geq p$, which is not essential.

Now write $\rho^{k-i}e_i=\underline{\omega}_i^{*T}\underline{\beta}_k'-y_k^*$, so (in matrix-notation)

$\underline{e}_k=\Omega_k(u^*,y^*)\underline{\beta}_k'-\underline{y}_k^*$, with

$\Omega_k(u^*,y^*)^T=(\underline{\omega}_1^*\ \underline{\omega}_2^*\ \ldots\ \underline{\omega}_k^*)$,

$\underline{\omega}_i^*=\rho^{k-1}\underline{\omega}_i$,

and $y_i^*=\rho^{k-i}y_i$.

Now we get:

$$V_k=\{\Omega_k(u^*,y^*)\underline{\beta}_k'-\underline{y}_k^*\}^T\{\Omega_k(u^*,y^*)\underline{\beta}_k'-\underline{y}_k^*\}.$$

We want to minimize V_k with relation to $\underline{\beta}_k'$, so $\frac{\partial V_k}{\partial\underline{\beta}_k'}$ has to be zero

$$\frac{\partial V_k}{\partial\underline{\beta}_k'}=2\Omega_k^T(u^*,y^*)\times\{\Omega_k(u^*,y^*)\underline{\beta}_k'-\underline{y}_k^*\}$$

$$\Omega_k^T(u^*,y^*)\Omega_k(u^*,y^*)\underline{\beta}_k'=\Omega_k^T(u^*,y^*)\underline{y}_k^* \text{ or}$$

$$\underline{\beta}_k'=\{\Omega_k^T(u^*,y^*)\Omega_k(u^*,y^*)\}^{-1}\Omega_k^T(u^*,y^*)\underline{y}_k^* \tag{A.1}$$

Now call $\Omega_k^T(u^*,y^*)\Omega_k(u^*,y^*)=P_k^{-1}$ (A.2)

We can easy see that: $\Omega_{k+1}(u^*,y^*)=\begin{bmatrix}\rho\Omega_k(u^*,y^*)\\ \hline \underline{\omega}_{k+1}^T\end{bmatrix}$ and $\underline{y}_{k+1}^*=\begin{bmatrix}\underline{y}_k^*\\ \hline y_{k+1}\end{bmatrix}$, sothat

$$\begin{aligned}P_{k+1}^{-1}&=\{\rho\Omega_k^T(u^*,y^*)\ \underline{\omega}_{k+1}\}\{\rho\Omega_k^T(u^*,y^*)\ \underline{\omega}_{k+1}\}^T\\&=\rho^2\Omega_k^T(u^*,y^*)\Omega_k(u^*,y^*)+\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T\\&=\rho^2P_k^{-1}+\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T\end{aligned}$$

Premultiplying this equation with P_{k+1} and postmultiplying with P_k gives:

$$P_k=\rho^2 P_{k+1}+P_{k+1}\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T P_k \tag{A.3}$$

$$P_k\underline{\omega}_{k+1}=\rho^2 P_{k+1}\underline{\omega}_{k+1}+P_{k+1}\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}$$

$$=P_{k+1}\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}$$

$$P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k=P_{k+1}\underline{\omega}_{k+1}\underline{\omega}_{k+1}^T P_k \tag{A.4}$$

Combining eq.(A.3) with eq(A.4) gives:

$$P_{k+1}=\frac{1}{\rho^2}(P_k-P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k) \tag{A.5}$$

$$S_k=\Omega_k^T(u^*,y^*)\underline{y}_k^* \tag{A.6}$$

$$S_{k+1}=\{\rho\Omega_k^T(u^*,y^*)\ \underline{\omega}_{k+1}\}\{\rho\underline{y}_k^{*T}\ y_{k+1}\}^T$$

$$=\rho^2 S_k+\underline{\omega}_{k+1}y_{k+1}.$$

We can see from eq.(A.1), eq.(A.2) and eq.(A.6) that

$$\underline{\beta}'_{k+1}=P_{k+1}S_{k+1}$$

$$=\frac{1}{\rho^2}(P_k-P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k)(\rho^2 S_k+\underline{\omega}_{k+1}y_{k+1})$$

$$=P_kS_k-P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_kS_k+\frac{1}{\rho^2}P_k\underline{\omega}_{k+1}y_{k+1}-$$

$$-\frac{1}{\rho^2}P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}y_{k+1}$$

$$=\underline{\beta}'_k-P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T\underline{\beta}'_k+$$

$$+\frac{1}{\rho^2}P_k\underline{\omega}_{k+1}y_{k+1}(1-\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1})$$

$$=\underline{\beta}'_k-P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\underline{\omega}_{k+1}^T\underline{\beta}'_k+$$

$$+\frac{1}{\rho^2}P_k\underline{\omega}_{k+1}y_{k+1}(1-\rho^2\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}-$$

$$-\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\})$$

$$\underline{\beta}'_{k+1}=\underline{\beta}'_k-P_k\underline{\omega}_{k+1}\{\rho^2+\underline{\omega}_{k+1}^T P_k\underline{\omega}_{k+1}\}^{-1}(\underline{\omega}_{k+1}^T\underline{\beta}'_k-y_{k+1}) \tag{A.7}$$

Eq.(A.5) and eq.(A.7) are the formulas given in paragraph 2.2.2.

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