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P.J.J. Ferket
A.A. Reusken

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Department of Mathematics and Computing Science
Eindhoven University of Technology
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The Netherlands
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P.J.J. Ferket
A.A. Reusken

Eindhoven University of Technology,
Department of Mathematics and Computing Science,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

Abstract

The local defect correction method by Hackbusch and the fast adaptive composite grid method by McCormick are presented for a model situation. The main differences and similarities between both methods are discussed. It is shown that for suitable choices of the components the iterates in the local defect correction iteration and in the fast adaptive composite grid iteration are the same.

A.M.S. Classifications: 65N22, 65N50
Keywords : local defect correction, fast adaptive composite grid method.

1 Introduction

Many practical *boundary value problems* produce solutions that contain several *high activity regions*. In these regions the solution varies much more rapidly than in the remaining part of the domain. This behaviour of the solution may be caused by the differential operator itself, by the forcing term in the differential equation, by the boundary conditions or by an irregular boundary (e.g. a re-entrant corner).

In order to obtain a numerical approximation of the solution, the boundary value problem can be discretized on a *uniform grid*. Due to the large variations of the solution in the high activity regions, a relatively small grid size is required there to obtain a sufficiently accurate approximation of the solution. Outside the high activity regions the behaviour of the solution is much more smooth. Therefore a (much) larger grid size seems to be sufficient in that part of the domain. However, if we discretize on a uniform grid, the grid size is small everywhere. Because of the large number of grid points in the uniform grid, the system of algebraic equations that results from the discretization process will be relatively large. Furthermore, the numerical approximation has to be stored at each grid point. So it is clear that approximating the continuous solution on a single uniform grid is *computationally inefficient* for boundary value problems that produce solutions that contain high activity regions.

Instead, the solution can be approximated using *several uniform grids with different grid sizes that cover different parts of the domain* [3, 5, 8, 11]. At least one grid should cover the entire domain. The grid size of this *global coarse grid* is chosen in agreement with the smooth behaviour of the solution outside the high activity regions. Besides a global grid several *local grids* are used that are uniform too. Each of them covers only a (small) part of the domain and contains a high activity region. The grid size of each of these grids is chosen in agreement with the behaviour of the solution in the corresponding high activity region. In this way every part of the domain is covered by a (locally) uniform grid whose grid size is in agreement with the behaviour of the continuous solution in that part of the domain. This refinement strategy is known as *local uniform grid refinement*. The solution is approximated on a *composite grid* which is the union of the uniform subgrids.

In [5] Hackbusch introduced a *local defect correction method* for approximating the continuous solution on a composite grid. In this iterative process a basic global discretization is improved by local discretizations defined in the subdomains. At every step this iterative process yields a discrete approximation of the continuous solution on the composite grid.

Another discretization and solution method on composite grids is the *fast adaptive composite grid method* by McCormick [7, 8, 9]. This method uses the global and local uniform grids both to define the composite grid problem and to solve this discrete problem. The method requires that the discrete problem on the composite grid is given explicitly. Approximations of the solution of this discrete problem are computed by solving (discrete) problems on the global and local grids.

The main issue of this paper is to compare the local defect correction iteration and the fast adaptive composite grid iteration. Therefore both methods are presented for the same model situation. In Section 2 this model situation is described. In Section 3 the local defect correction iteration as introduced by Hackbusch in [5] is described for the model situation. In Section 4 the fast adaptive composite grid method as introduced by McCormick in [9] is described for the model situation. This iteration is written in a form that is similar to the form of the local defect correction iteration in Section 3. The main differences and similarities

between both methods are discussed in Section 5. Our main result is that for suitable choices of the components the iterates in the local defect correction iteration and in the fast adaptive composite grid iteration are the same.

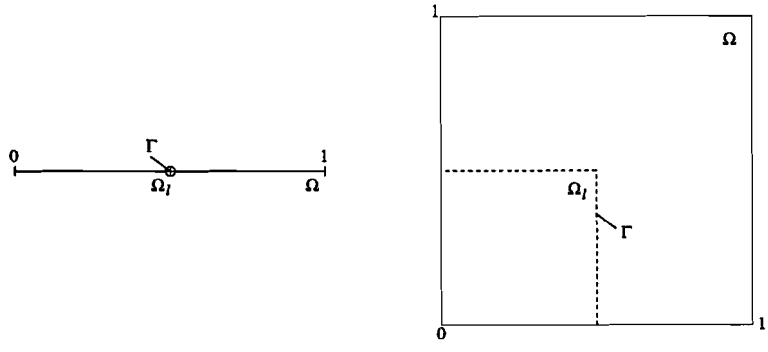

Figure 1: Ω, Ω_l and Γ in $I\!R^1$ and $I\!R^2$.

2 Model Situation

We consider *Dirichlet boundary value problems*

$$\begin{aligned} \mathcal{L}U &= f \quad \text{in } \Omega, \\ U &= g \quad \text{at } \partial\Omega, \end{aligned} \tag{2.1}$$

with $\Omega = (0,1)$ in the one dimensional case and $\Omega = (0,1) \times (0,1)$ in the two dimensional case, $\partial\Omega$ the boundary of Ω and $\mathcal{L}$ a scalar linear elliptic second-order differential operator. Dirichlet boundary conditions are chosen for ease of notation. We assume that problem (2.1) is such that the continuous solution varies very rapidly in some (small) part of the domain, which is contained in the region $\Omega_l \subset \Omega$. In the remaining part of the domain the continuous solution is assumed to behave much more smoothly. In the two dimensional case the region Ω_l is assumed to be rectangular. The boundary $\partial\Omega_l$ of Ω_l consists of two parts. A part that coincides with $\partial\Omega$ and a remaining part. The latter part is called the *interface* $\Gamma = \partial\Omega_l \backslash \partial\Omega$. We note that we may have $\partial\Omega_l \cap \partial\Omega = \{\emptyset\}$, in which case the interface Γ coincides with $\partial\Omega_l$.

In order to compute a numerical approximation of the solution U we discretize (2.1) with respect to some discretization grid using *finite differences*. We assume that the finite difference matrices that appear in the following sections are all regular.

In the local defect correction iteration two *uniform* grids are used, a *global* one and a *local* one. The *global coarse grid* Ω^H is a uniform grid with grid size H that covers the domain Ω. The local fine grid Ω_l^h is a uniform grid with grid size h that covers the region Ω_l (see Figure 2). The *space of grid functions* on Ω^H (Ω_l^h) is denoted by $\mathcal{F}^H$ ($\mathcal{F}_l^h$). Since the continuous solution varies (much) more rapidly in Ω_l than in the remainder of Ω, a (much) smaller grid size is needed in Ω_l than in the remainder of Ω to provide the required level of resolution: $h < H$.

We assume that the interface Γ coincides with grid points of Ω^H in the one dimensional case and with grid lines of Ω^H in the two dimensional case. Also we assume that all grid points of $\Omega^H \cap \Omega_l$ belong to Ω_l^h. We note that Ω_l^h does not contain grid points on the interface Γ. These fine interface grid points generate the fine interface grid Γ^h (see Figure 2). The coarse interface grid Γ^H is build up by all coarse interface grid points $\mathbf{x} \in \Omega^H \cap \Gamma$. We note that in the one dimensional case we have $\Gamma^H = \Gamma^h$. The corresponding spaces of grid functions are denoted by $\mathcal{F}_\Gamma^h$ and $\mathcal{F}_\Gamma^H$.

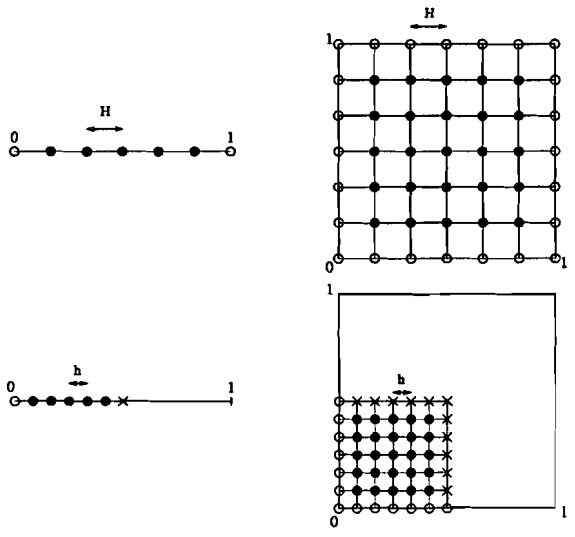

Figure 2: Examples of global coarse grids Ω^H and local fine grids Ω_l^h in $I\!R^1$ and $I\!R^2$. •: grid point of Ω^H, Ω_l^h respectively, ○: boundary point; ×: grid point of Γ^h.

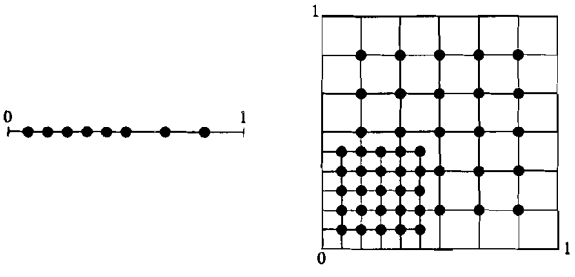

Figure 3: Examples of composite grids in $I\!R^1$ and $I\!R^2$.

In the fast adaptive composite grid iteration a *composite* grid is used. The *global composite grid* Ω_c is a nonuniform grid that covers the domain Ω. It is the union of the global coarse grid Ω^H and the local fine grid Ω_l^h (see Figure 3). The space of grid functions on the composite grid is denoted by $\mathcal{F}_c$.

3 Local Defect Correction Iteration (LDC)

In the *local defect correction iteration* the global coarse grid Ω^H and the local fine grid Ω_l^h are used to compute a numerical approximation of the continuous solution U of (2.1). At each iteration step a discrete problem on Ω^H and one on Ω_l^h is defined and solved. The local defect correction iteration was introduced by Hackbusch in [5].

First we discretize (2.1) with respect to the global coarse grid Ω^H. At each grid point $\mathbf{x} \in \Omega^H$ the differential operator in (2.1) is replaced by a *finite difference approximation.* This yields the *basic coarse grid problem*:

$$L^H u^H = f^H \text{ on } \Omega^H, \tag{3.1}$$

with $u^H, f^H \in \mathcal{F}^H$ and $L^H : \mathcal{F}^H \rightarrow \mathcal{F}^H$. The Dirichlet boundary values in (2.1) are incorporated in f^H (see Example 3.1).

After solving (3.1) boundary value problem (2.1) is discretized with respect to the local fine grid Ω_l^h. In this discretization process Dirichlet boundary values are needed at all grid points $\mathbf{x} \in \Gamma^h$. Therefore, we restrict the solution of (3.1) to the coarse interface grid Γ^H:

$$(r_\Gamma u^H)(\mathbf{x}) = u^H(\mathbf{x}) \quad \mathbf{x} \in \Gamma^H. \tag{3.2}$$

In (3.2) $r_\Gamma : \mathcal{F}^H \rightarrow \mathcal{F}_\Gamma^H$ is a *trivial injection* that restricts grid functions on Ω^H to Γ^H. Dirichlet boundary conditions at grid points $\mathbf{x} \in \Gamma^h \backslash \Gamma^H$ result from *interpolating* the values in (3.2). We note that in the one dimensional case this interpolation can be omitted. The finite difference approximations used on the local fine grid may differ from those used on the global coarse grid. The *basic local fine grid problem* is given by

$$L_l^h u_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma u^H \text{ on } \Omega_l^h, \tag{3.3}$$

with $u_l^h, f_l^h \in \mathcal{F}_l^h$, $L_l^h : \mathcal{F}_l^h \rightarrow \mathcal{F}_l^h$, $L_\Gamma^h : \mathcal{F}_\Gamma^h \rightarrow \mathcal{F}_l^h$ and $p_\Gamma : \mathcal{F}_\Gamma^H \rightarrow \mathcal{F}_\Gamma^h$. *Natural* Dirichlet boundary conditions on $\partial\Omega_l \cap \partial\Omega$ are incorporated into f_l^h. The incorporation of the *artificial* Dirichlet boundary conditions on Γ in the system is given explicitly by the term $-L_\Gamma^h p_\Gamma r_\Gamma u^H$.

Example 3.1
Consider the one dimensional Poisson problem:

$$\begin{aligned} -U_{xx} &= f(x) \quad 0 < x < 1, \\ U(0) &= a, \\ U(1) &= b. \end{aligned}$$

Suppose that $H = 1/8$, $\Omega_l = (0, 1/4)$, and $h = 1/32$.
Define

$$\begin{aligned} x_i &:= i * H \quad i = 1, \ldots, 7, \\ y_j &:= j * h \quad j = 1, \ldots, 7. \end{aligned}$$

If we use central differences we get the following basic global coarse grid problem (3.1):

$$\frac{1}{H^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} u^H(x_1) \\ \\ \vdots \\ \\ u^H(x_7) \end{pmatrix} = \begin{pmatrix} f(x_1) + \frac{a}{H^2} \\ f(x_2) \\ \vdots \\ f(x_6) \\ f(x_7) + \frac{b}{H^2} \end{pmatrix}.$$

If in the discretization process on the local fine grid also central differences are used we get the following basic local fine grid problem (3.3):

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} u_l^h(y_1) \\ \\ \vdots \\ \\ u_l^h(y_7) \end{pmatrix} = \begin{pmatrix} f(y_1)+\frac{a}{h^2} \\ f(y_2) \\ \vdots \\ \\ f(y_7) \end{pmatrix} - \frac{1}{h^2}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} u^H(x_2)$$

□

By solving the local fine grid problem (3.3) we aim at improving the approximation of the continuous solution U in the region Ω_l. However, the Dirichlet boundary conditions on Γ^h result from the basic global coarse grid problem and the approximation u_l^h can be no more accurate than the approximation u^H *at the interface.* In general, local phenomena cause the approximations $u^H(\mathbf{x})$ to be relatively inaccurate at all grid points $\mathbf{x} \in \Omega^H$. Therefore the results of this simple two step process usually do not achieve an accuracy that is in agreement with the added resolution (see e.g. [4], [5]). In the local defect correction iteration coarse and fine processing steps are reused to quickly obtain such accuracy.

In the first step of the local defect correction iteration the approximation u_l^h (from (3.3)) is used to *update the global coarse grid problem* (3.1). The right hand side of (3.1) is updated at grid points that are part of a *local coarse subgrid* Ω_s^H. This is a uniform grid with grid size H that covers a region $\Omega_s \subseteq \Omega_l$. The interface between Ω_s and Ω is denoted by Γ_s (see Figure 4). We assume that this interface Γ_s coincides with grid points of Ω^H in the one

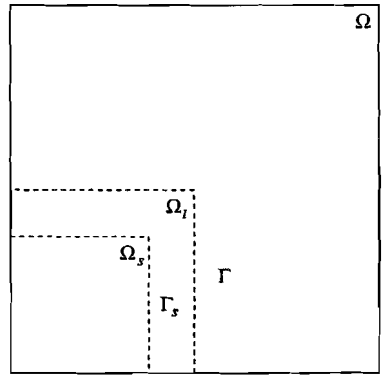

Figure 4: Example of the regions Ω, Ω_l, and Ω_s in $\mathbb{R}^2$.

dimensional case and with grid lines of Ω^H in the two dimensional case. Further we assume that a constant *dist* exists such that

$$\forall \mathbf{x} \in \Gamma_s \; : \quad \min\{|\mathbf{x}-\mathbf{y}| \mid \mathbf{y} \in \Gamma\} = dist. \tag{3.4}$$

If $dist = 0$ then $\Omega_s = \Omega_l$, $\Gamma_s = \Gamma$, and $\Omega_s^H = \Omega_l^H$. The *local coarse grid* Ω_l^H is a uniform grid with grid size H that covers the local region Ω_l. The space of grid functions on Ω_l^H is denoted by $\mathcal{F}_l^H$.

The updated global coarse grid problem is given by

$$L^H \bar{u}^H = \bar{f}^H \tag{3.5a}$$

with

$$\bar{f}^H(\mathbf{x}) := \begin{cases} (L_l^H r_l u_l^h)(\mathbf{x}) + (L_\Gamma^H r_\Gamma u^H)(\mathbf{x}) & \mathbf{x} \in \Omega_s^H \\ f^H(\mathbf{x}) & \mathbf{x} \in \Omega^H \backslash \Omega_s^H \end{cases} . \tag{3.5b}$$

The operators $L_l^H : \mathcal{F}_l^H \rightarrow \mathcal{F}_l^H$ and $L_\Gamma^H : \mathcal{F}_\Gamma^H \rightarrow \mathcal{F}_l^H$ are coarse grid analogues of L_l^h and L_Γ^h in (3.3) and they satisfy:

$$(L^H w^H)(\mathbf{x}) = (L_l^H w_l^H)(\mathbf{x}) + (L_\Gamma^H w_\Gamma^H)(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^H \tag{3.6}$$

for all grid functions $w^H \in \mathcal{F}^H$, $w_l^H \in \mathcal{F}_l^H$, and $w_\Gamma^H \in \mathcal{F}_\Gamma^H$ that satisfy

$$\begin{aligned} w_l^H(\mathbf{x}) &= w^H(\mathbf{x}) \qquad \mathbf{x} \in \Omega_l^H, \\ w_\Gamma^H(\mathbf{x}) &= w^H(\mathbf{x}) \qquad \mathbf{x} \in \Gamma^H. \end{aligned}$$

The *trivial injection* r_l is used to restrict grid functions on Ω_l^h to Ω_l^H:

$$(r_l u_l^h)(\mathbf{x}) := u_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^H. \tag{3.7}$$

We note that this is possible since we have assumed that $\Omega_l^h \cap \Omega_l^H = \Omega_l^H$.
We define the characteristic function χ by

$$(\chi w)(\mathbf{x}) := \begin{cases} w(\mathbf{x}) & \mathbf{x} \in \Omega_s^H \\ 0 & \mathbf{x} \in \Omega^H \backslash \Omega_s^H \end{cases} . \tag{3.8}$$

Then we can rewrite (3.5a), (3.5b) as follows:

$$L^H \bar{u}^H = f^H + \chi(L_l^H r_l u_l^h + L_\Gamma^H r_\Gamma u^H - f^H). \tag{3.9}$$

So the right hand side of the global coarse grid problem is *corrected* by the *defect* of a *local fine grid approximation.* (Hence the name *local defect correction iteration*). Once we have solved (3.9) we can update the local fine grid problem:

$$L_l^h \bar{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma \bar{u}^H. \tag{3.10}$$

The approximations $\bar{u}^H$ and $\bar{u}_l^h$ of U are used to define an approximation of U on the composite grid:

$$\bar{u}_c(\mathbf{x}) := \begin{cases} \bar{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \bar{u}^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases} .$$

In [4] an error analysis for this approximation that results after one local defect correction step is given.

In the local defect correction iteration global problems like (3.9) and local problems like (3.10) are combined in the way described above. The iterative process is given below and it will be referred to as **LDC**.

LDC

Start: exact solution of the global problem

$$L^H u_0^H = f^H \text{ on } \Omega^H$$

exact solution of the local problem

$$L_l^h u_{l,0}^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma u_0^H \text{ on } \Omega_l^h$$

computation of a composite grid approximation

$$u_{c,0}(\mathbf{x}) := \begin{cases} u_{l,0}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_0^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases}$$

$i = 1, 2, \ldots$

a) computation of the right hand side of the global problem

$$\bar{f}^H := (1-\chi) f^H + \chi L_l^H r_l u_{l,i-1}^h + \chi L_\Gamma^H r_\Gamma u_{i-1}^H \tag{3.11a}$$

b) exact solution of the global problem

$$L^H u_i^H = \bar{f}^H \text{ on } \Omega^H \tag{3.11b}$$

c) exact solution of the local problem

$$L_l^h u_{l,i}^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma u_i^H \text{ on } \Omega_l^h \tag{3.11c}$$

d) computation of a composite grid approximation

$$u_{c,i}(\mathbf{x}) := \begin{cases} u_{l,i}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_i^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases} \tag{3.11d}$$

Remark 3.2
- The local defect correction iteration was introduced by Hackbusch in [5].
- The starting procedure of the local defect correction iteration is equal to the Local Uniform Grid Refinement method for the stationary case described in [11].
- In the local defect correction iteration it is not necessary to compute the composite grid approximation explicitly. (3.11d) is added for reasons that will become clear later on. □

Any fixed point $(\hat{u}^H, \hat{u}_l^h)$ of the iterative process (3.11) is characterized by the system (see [5])

$$\begin{array}{ll} L^H \hat{u}^H - \chi L_\Gamma^H r_\Gamma \hat{u}^H - \chi L_l^H r_l \hat{u}_l^h = (1-\chi) f^H & \text{on } \Omega^H, \\ L_l^h \hat{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma \hat{u}^H & \text{on } \Omega_l^h. \end{array} \tag{3.12}$$

In [5] it is shown that under certain conditions the local defect correction iteration converges with a contraction number $\sim H^\rho$, where $\rho > 0$ depends on the consistency orders of the

discretization processes on the global coarse grid and on the local fine grid. An essential assumption in the asymptotic convergence proof is that $dist > 0$ holds (see (3.4)). Numerical experiments in [5] indicate that also for $dist = 0$ the iterates converge very fast and that the convergence rate improves as $dist$ increases. We note that $dist$ cannot be chosen "too large", since the high activity region of the problem has to be contained in Ω_s. When d is chosen "too large" the quality of the resulting approximation deteriorates.

$(\hat{u}^H, \hat{u}_l^h)$ from (3.12) is an approximation of (U^H, U_l^h) with U^H, U_l^h the restriction of U from (2.1) to Ω^H, Ω_l^h respectively.
Let d^H be the local discretization error of the discretization process on the global coarse grid:

$$d^H := L^H U^H - f^H. \tag{3.13}$$

Let d_l^h be the local discretization error of the discretization process on the local fine grid:

$$d_l^h := L_l^h U_l^h + L_\Gamma^h U_\Gamma^h - f_l^h \tag{3.14}$$

where U_Γ^h is the restriction of U to Γ^h. Similarly U_Γ^H and U_l^H are restrictions of U to Γ^H and Ω_l^H respectively. In the following theorem we give expressions for the errors $U^H - \hat{u}^H$ and $U_l^h - \hat{u}_l^h$.

Theorem 3.3
The limit value $(\hat{u}^H, \hat{u}_l^h)$ of the local defect correction iteration satisfies:

$$\begin{aligned}(L^H - \chi(L_\Gamma^H - L_l^H r_l (L_l^h)^{-1} L_\Gamma^h p_\Gamma) r_\Gamma)(U^H - \hat{u}^H) = \\ (1-\chi) d^H + \chi L_l^H r_l (L_l^h)^{-1} d_l^h + \chi L_l^H r_l (L_l^h)^{-1} L_\Gamma^h (p_\Gamma U_\Gamma^H - U_\Gamma^h)\end{aligned} \tag{3.15}$$

$$L_l^h (U_l^h - \hat{u}_l^h) = d_l^h + L_\Gamma^h (p_\Gamma U_\Gamma^H - U_\Gamma^h) - L_\Gamma^h p_\Gamma r_\Gamma (U^H - \hat{u}^H) \tag{3.16}$$

Proof:
According to (3.2), (3.6), (3.7), (3.8) and (3.13) we have

$$L^H U^H - \chi L_\Gamma^H r_\Gamma U^H - \chi L_l^H r_l U_l^h = (1-\chi) f^H + (1-\chi) d^H.$$

According to (3.2) and (3.14) we have

$$L_l^h U_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma U^H + d_l^h + L_\Gamma^h (p_\Gamma U_\Gamma^H - U_\Gamma^h).$$

In combination with (3.12) we get

$$L^H (U^H - \hat{u}^H) - \chi L_\Gamma^H r_\Gamma (U^H - \hat{u}^H) - \chi L_l^H r_l (U_l^h - \hat{u}_l^h) = (1-\chi) d^H$$

$$L_l^h (U_l^h - \hat{u}_l^h) = d_l^h + L_\Gamma^h (p_\Gamma U_\Gamma^H - U_\Gamma^h) - L_\Gamma^h p_\Gamma r_\Gamma (U^H - \hat{u}^H).$$

If we substitute the second of these two equations into the first one, we obtain (3.15) □

From (3.15) it is clear that the error $U^H - \hat{u}^H$ is determined by the following quantities (see also [5]):

(1) d^H in $\Omega^H \backslash \Omega_s^H$;

(2) d_l^h in Ω_s^H;

(3) $p_\Gamma U_\Gamma^H - U_\Gamma^h$ on Γ (interpolation error of p_Γ).

The error $U^H - u^H$ (u^H from (3.1)) depends on d^H in Ω_s^H, where the local discretization error is assumed to be relatively large (recall that the continuous solution varies very rapidly in the high activity region which is contained in Ω_s). According to **(1)** and **(2)** the error $U^H - \hat{u}^H$ does *not* depend on d^H in Ω_s^H, but on d_l^h instead. The value of the latter local discretization error in a grid point $\mathbf{x} \in \Omega_s^H$ is in general smaller than the value of the former error. Thus $\hat{u}^H$ may be expected to be a better approximation of U^H than u^H. Numerical experiments in [5] show that the discretization error $U^H - \hat{u}^H$ depends on *dist* in (3.4). For some model problems the optimal choice is a small $dist > 0$ (e.g. $d = H$). For some other model problems the discretization error appears to be optimal for $dist = 0$.

Remark 3.4
Since we have assumed that all grid points $\mathbf{x} \in \Omega_l^H$ are grid points of Ω_l^h too, we have for the *trivial injection* r_l:

$$(r_l U_l^h)(\mathbf{x}) - U^H(\mathbf{x}) = U(\mathbf{x}) - U(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega_l^H.$$

So the interpolation error of r_l is zero. If the interpolation error of the restriction is not zero then the error $U^H - \hat{u}^H$ also depends on this interpolation error. □

Remark 3.5
If the discretization processes on the global coarse grid and on the local fine grid are given then the iterates in (3.11) depend on the choice for the prolongation operator p_Γ and on the value of *dist*. □

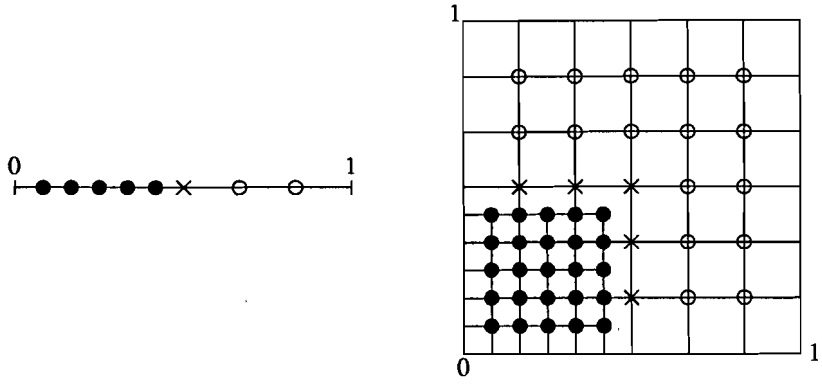

Figure 5: Examples of composite grids in $\mathbb{R}^1$ and $\mathbb{R}^2$; $\circ$: grid point in Ω_C, $\bullet$: grid point in Ω_l^h, $\times$: grid point in Γ^H.

4 Fast Adaptive Composite Grid Iteration (FAC)

In the fast adaptive composite grid method [7, 8, 9] the discrete problem on the composite grid has to be specified a priori. It is not an implicit result of the iterative process as in the local defect correction iteration.

In our model situation boundary value problem (2.1) is discretized on the composite grid Ω_c (see Figure 5). At all grid points $\mathbf{x} \in \Omega_c$ the differential operator in (2.1) is replaced by a finite difference approximation. We denote the resulting *composite grid problem* by

$$L_c u_c = f_c \quad \text{on } \Omega_c, \tag{4.1}$$

with $u_c, f_c \in \mathcal{F}_c$ and $L_c : \mathcal{F}_c \to \mathcal{F}_c$.
The composite grid is partitioned in the following way (see Figure 5)

$$\Omega_c = \Omega_C \cup \Gamma^H \cup \Omega_l^h \tag{4.2}$$

where Ω_C consists of all grid points of the global coarse grid Ω^H that are not part of the region Ω_l or the interface Γ.
We assume that at all grid points $\mathbf{x} \in \Omega_C$ the same finite difference formula is used as in the discretization process on the global coarse grid in (see (3.1)). Define the *trivial injection* $r_c : \mathcal{F}_c \to \mathcal{F}^H$ by

$$(r_c w_c)(\mathbf{x}) = w_c(\mathbf{x}) \quad \mathbf{x} \in \Omega^H \tag{4.3}$$

for all grid functions $w_c \in \mathcal{F}_c$. Then we have:

$$(L_c u_c)(\mathbf{x}) = (L^H r_c u_c)(\mathbf{x}) \quad \mathbf{x} \in \Omega_C, \tag{4.4}$$

$$f_c(\mathbf{x}) = f^H(\mathbf{x}) \quad \mathbf{x} \in \Omega_C, \tag{4.5}$$

with L^H and f^H as in (3.1).
Further we assume that at all grid points $\mathbf{x} \in \Omega_l^h$ the same finite difference formula is used as in the discretization process on the local fine grid in Section 3 (see (3.3)). Define the *trivial injection* $r_{cl} : \mathcal{F}_c \to \mathcal{F}_l^h$ by

$$(r_{cl} w_c)(\mathbf{x}) = w_c(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h \tag{4.6}$$

for all grid functions $w_c \in \mathcal{F}_c$. Then we have:

$$(L_c u_c)(\mathbf{x}) = (L_l^h r_{cl} u_c)(\mathbf{x}) + (L_\Gamma^h \hat{p}_\Gamma r_\Gamma r_c u_c)(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h, \tag{4.7}$$

$$f_c(\mathbf{x}) = f_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h, \tag{4.8}$$

with L_l^h, L_Γ^h, f_l^h and r_Γ as in (3.3) and $\hat{p}_\Gamma : \mathcal{F}_\Gamma^H \rightarrow \mathcal{F}_\Gamma^h$ an interpolation operator on the interface Γ.

Remark 4.1
The interpolation operator $\hat{p}_\Gamma$ (which is part of a *discretization process*) may be different from the interpolation operator p_Γ in (3.3) (which is part of a *solution process*). □

Next we give examples of composite grid problems for one and two dimensional Poisson problems.

Example 4.2
Consider the one dimensional Poisson problem:

$$\begin{array}{rcll} -U_{xx} & = & f(x) & 0 < x < 1, \\ U(0) & = & a, & \\ U(1) & = & b. & \end{array}$$

Let $H = 1/8$, $\Omega_l = (0, 1/4)$, $h = 1/32$ and

$$\begin{array}{rcll} x_i & := & i * H & i = 1, \ldots, 7, \\ y_j & := & j * h & j = 1, \ldots, 7, \end{array}$$

as in Example 3.1.
In this case the interface Γ consists of one point, x_2. At this interface point the following finite difference approximation can be used:

$$-U_{xx}(x_2) \doteq [2U(x_2) - \frac{2}{5}U(x_2 + H) - \frac{8}{5}U(x_2 - h)]/hH.$$

This scheme corresponds to a first order approximation of $-U_{xx}$ at a 'nonuniform grid point' with $H = 4h$. At all other grid points central differences can be used. Then the following composite grid problem results:

$$\frac{1}{hH}\begin{pmatrix} 8 & -4 & & & & & & & \\ -4 & 8 & -4 & & & & & & \\ & & \ddots & & & & & & \\ & & -4 & 8 & -4 & & & & \\ & & & -8/5 & 2 & -2/5 & & & \\ & & & & -1/4 & 1/2 & -1/4 & & \\ & & & & & & \ddots & & \\ & & & & & & -1/4 & 1/2 & -1/4 \\ & & & & & & & -1/4 & 1/2 \end{pmatrix} \begin{pmatrix} u_c(y_1) \\ u_c(y_2) \\ \vdots \\ u_c(y_7) \\ u_c(x_2) \\ u_c(x_3) \\ \vdots \\ u_c(x_6) \\ u_c(x_7) \end{pmatrix} = \begin{pmatrix} f(y_1) + \frac{a}{h^2} \\ f(y_2) \\ \vdots \\ f(y_7) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_6) \\ f(x_7) + \frac{b}{H^2} \end{pmatrix}$$

Compare this composite grid problem with the discrete problems in Example 3.1. □

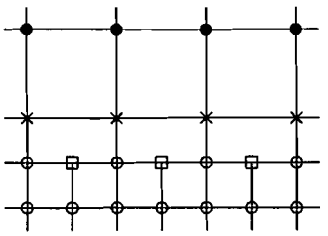

Figure 6: •: grid point of Ω_C; ×: grid point of Γ^H; o: standard grid point of Ω_l^h; □: nonstandard grid point of Ω_l^h.

Example 4.3
Consider the Poisson problem on a square

$$\begin{array}{ll} \Delta u(x,y) = f(x,y) & 0 < x < 1,\ 0 < y < 1, \\ u(0,y) = g(0,y),\ u(1,y) = g(1,y) & 0 \le y \le 1, \\ u(x,0) = g(x,0),\ u(x,1) = g(x,1) & 0 \le x \le 1. \end{array}$$

Suppose that we have a local refinement region with $h = H/2$ (see Figure 6).
At grid points of Ω_C (see Figure 6) the standard five point stencil

$$1/H^2 \begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}$$

can be used.
At standard grid points of Ω_l^h (see Figure 7) the standard five point stencil

$$1/h^2 \begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}$$

can be used.
At nonstandard grid points of Ω_l^h (see Figure 7) the following six point stencil can be used:

$$1/h^2 \begin{pmatrix} -1/2 & & -1/2 \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}.$$

This scheme corresponds to a linear interpolation operator $\hat{p}_\Gamma$ on the interface.
At grid points of Γ^H the following stencil can be used:

$$1/Hh \begin{pmatrix} & -2/3 & \\ -1/2 & 3 & -1/2 \\ & -4/3 & \end{pmatrix}.$$

This scheme corresponds to a first order approximation of $-\Delta U$ at a 'nonuniform grid point' with $h = H/2$. □

In the remainder of this section we consider the composite grid problem (4.1) as given. In the fast adaptive composite grid method approximations of u_c from (4.1) are computed in an iterative way. At each iteration step a discrete problem on the uniform global coarse grid and one on the uniform local fine grid are solved exactly and the resulting solutions are used to improve the current iterate.

Let $\tilde{u}_c$ be an approximation of u_c. Inserting $\tilde{u}_c$ into the system $L_c u_c - f_c = 0$ we obtain the *composite grid defect*

$$d_c := f_c - L_c \tilde{u}_c. \tag{4.9}$$

This defect vanishes if and only if $\tilde{u}_c$ is equal to u_c.
The exact correction $v_c = u_c - \tilde{u}_c$ satisfies

$$L_c v_c = d_c. \tag{4.10}$$

The composite grid defect d_c is restricted to the global coarse grid and to the local fine grid:

$$d^H := \tilde{r}_c d_c, \tag{4.11}$$

$$d_l^h := r_{cl} d_c. \tag{4.12}$$

In (4.11) we have used a restriction operator $\tilde{r}_c : \mathcal{F}_c \to \mathcal{F}^H$. We assume that $\tilde{r}_c$ is the trivial injection at $\Omega_c \backslash \Omega_l^h$:

$$(\tilde{r}_c w_c)(\mathbf{x}) = w_c(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h \tag{4.13}$$

for all grid functions $w_c \in \mathcal{F}_c$. In (4.12) r_{cl} is the trivial injection from (4.6).
First an approximation $v^H \in \mathcal{F}^H$ of v_c is computed by solving the following global coarse grid problem

$$L^H v^H = d^H, \tag{4.14}$$

with L^H as in (3.1).
Then an approximation $v_l^h \in \mathcal{F}_l^h$ of v_c is computed. The approximation v^H of v_c that results from (4.14) is used to define Dirichlet boundary conditions on the interface in the following local fine grid problem (cf. (3.3))

$$L_l^h v_l^h = d_l^h - L_\Gamma^h \hat{p}_\Gamma r_\Gamma v^H, \tag{4.15}$$

with L_l^h, L_Γ^h, r_Γ as in (3.3) and $\hat{p}_\Gamma$ as in (4.7).
The approximation v_l^h from (4.15) is used to improve the approximation $\tilde{u}_c$ of u_c at grid points of Ω_l^h:

$$\bar{u}_c(\mathbf{x}) := \tilde{u}_c(\mathbf{x}) + v_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h. \tag{4.16}$$

At grid points $\mathbf{x} \in \Omega_c \backslash \Omega_l^h$ the approximation v^H from (4.14) is used to improve the approximation $\tilde{u}_c$:

$$\bar{u}_c(\mathbf{x}) := \tilde{u}_c(\mathbf{x}) + v^H(\mathbf{x}) \text{ at } \mathbf{x} \in \Omega_c \backslash \Omega_l^h. \tag{4.17}$$

The fast adaptive composite grid iteration is an iterative process that combines local and global discrete problems in the way described above. At each iteration step an approximation of u_c is computed. The iterative process is given below and it will be referred to as **FAC**.

FAC

Start: Initial composite grid approximation $u_{c,0}$ given.

$i = 1, 2, \ldots$

a1) computation of the composite grid defect

$$d_c := f_c - L_c u_{c,i-1} \tag{4.18a}$$

a2) restriction of the composite grid defect to the global coarse grid

$$d^H := \tilde{r}_c d_c \tag{4.18b}$$

a3) restriction of the composite grid defect to the local fine grid

$$d_l^h := r_{cl} d_c \tag{4.18c}$$

b) exact solution of the global problem

$$L^H v^H = d^H \text{ on } \Omega^H \tag{4.18d}$$

c) exact solution of the local problem

$$L_l^h v_l^h = d_l^h - L_\Gamma^h \hat{p}_\Gamma r_\Gamma v^H \text{ on } \Omega_l^h \tag{4.18e}$$

d) correction of the composite grid approximation

$$u_{c,i}(\mathbf{x}) := u_{c,i-1}(\mathbf{x}) + v_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h \tag{4.18f}$$

$$u_{c,i}(\mathbf{x}) := u_{c,i-1}(\mathbf{x}) + v^H(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h \tag{4.18g}$$

Remark 4.4
- Iteration (4.18) was introduced by McCormick in [9] as the *fast adaptive composite grid method in its delayed correction form.*
- The fixed point of this iterative process is given by the exact solution of the composite grid problem (4.1).
- The composite grid defect is equal to zero at all grid points that do not belong to the interface for $i \geq 2$. □

FAC is not applicable to nonlinear problems, since the correction equation (4.10) is only valid if L_c is linear. In case L_c is nonlinear, the correction equation can be written in the FAS-form (see e.g. [2])

$$L_c(\tilde{u}_c + v_c) - L_c \tilde{u}_c = d_c. \tag{4.19}$$

We compute approximations of $\tilde{u}_c + v_c = u_c$ on the uniform global coarse grid and on the uniform local fine grid. First we transfer the correction equation (4.19) to the uniform global coarse grid:

$$L^H w^H = \tilde{r}_c d_c + L^H \hat{r}_c \tilde{u}_c. \tag{4.20}$$

The restriction operator $\tilde{r}_c$ is used to restrict composite grid defects. The restriction operator $\hat{r}_c : \mathcal{F}_c \to \mathcal{F}^H$ is used to restrict composite grid approximations and may be different from $\tilde{r}_c$. We assume that $\hat{r}_c$ is the trivial injection at $\Omega_c \backslash \Omega_l^h$:

$$(\hat{r}_c w_c)(\mathbf{x}) = w_c(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h \tag{4.21}$$

for all grid functions $w_c \in \mathcal{F}_c$.
Next we transfer the correction equation (4.19) to the local fine grid (cf. (4.15)):

$$L_l^h w_l^h = r_{cl} d_c + L_l^h r_{cl} \tilde{u}_c - L_\Gamma^h \hat{p}_\Gamma r_\Gamma (w^H - \hat{r}_c \tilde{u}_c). \tag{4.22}$$

The approximations w^H and w_l^h are used to compute a new approximation of u_c:

$$\bar{u}_c(\mathbf{x}) := \begin{cases} w_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ w^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases} .$$

The considerations above lead to the *full fast adaptive composite grid iteration* (**FFAC**).

FFAC

Start: Initial composite grid approximation $u_{c,0}$ given.
$i = 1, 2, \ldots$

a1) computation of the defect on the composite grid

$$d_c := f_c - L_c u_{c,i-1} \tag{4.23a}$$

a2) restriction of the composite grid defect to the global coarse grid

$$d^H := \tilde{r}_c d_c \tag{4.23b}$$

a3) restriction of the composite grid defect to the local fine grid

$$d_l^h := r_{cl} d_c \tag{4.23c}$$

b) exact solution of the global problem

$$L^H w^H = d^H + L^H \hat{r}_c u_{c,i-1} \text{ on } \Omega^H \tag{4.23d}$$

c) exact solution of the local problem

$$L_l^h w_l^h = d_l^h + L_l^h r_{cl} u_{c,i-1} - L_\Gamma^h \hat{p}_\Gamma r_\Gamma (w^H - \hat{r}_c u_{c,i-1}) \text{ on } \Omega_l^h \tag{4.23e}$$

d) correction of the composite grid approximation

$$u_{c,i}(\mathbf{x}) := w_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h \tag{4.23f}$$
$$u_{c,i}(\mathbf{x}) := w^H(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h \tag{4.23g}$$

Lemma 4.5
For *linear* problems the iterative processes **FAC** and **FFAC** are equivalent, i.e. the iterates $u_{c,i}$ in (4.18) and (4.23) are the same provided that the starting vectors $u_{c,0}$ in **FAC** and **FFAC** are the same.

Proof:
Suppose that the $(i-1)$-th iterate $u_{c,i-1}$ in **FAC** is the same as in **FFAC**. We show that $u_{c,i}$ in **FAC** is the same as in **FFAC**.
Using (4.21) we can write (4.23g) as follows:

$$u_{c,i}(\mathbf{x}) := u_{c,i-1}(\mathbf{x}) + (w^H - \hat{r}_c u_{c,i-1})(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h.$$

From (4.23d) we have

$$L^H(w^H - \hat{r}_c u_{c,i-1}) = d^H.$$

Combination of these two equations yields

$$u_{c,i}(\mathbf{x}) = u_{c,i-1}(\mathbf{x}) + ((L^H)^{-1} d^H)(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h$$

in **FFAC**.
From (4.18d) and (4.18g) we have

$$u_{c,i}(\mathbf{x}) = u_{c,i-1}(\mathbf{x}) + ((L^H)^{-1} d^H)(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h$$

in **FAC**.
If $u_{c,i-1}$ in **FAC** is the same as in **FFAC**, then d^H in **FAC** is the same as in **FFAC**. From the latter two equations it follows that $u_{c,i}(\mathbf{x})$ in **FAC** is the same as in **FFAC** at all grid points $\mathbf{x} \in \Omega_c \backslash \Omega_l^h$.
Using (4.6)) we can write (4.23f) as follows:

$$u_{c,i}(\mathbf{x}) := u_{c,i-1}(\mathbf{x}) + (w_l^h - r_{cl} u_{c,i-1})(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h.$$

From (4.23e) we have

$$L_l^h(w_l^h - r_{cl} u_{c,i-1}) = d_l^h - L_\Gamma^h \hat{p}_\Gamma r_\Gamma (w^H - \hat{r}_c u_{c,i-1}).$$

Combination of these two equations yields

$$u_{c,i}(\mathbf{x}) = u_{c,i-1}(\mathbf{x}) + ((L_l^h)^{-1}(d_l^h - L_\Gamma^h \hat{p}_\Gamma r_\Gamma (w^H - \hat{r}_c u_{c,i-1})))(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h$$

in **FFAC**.
From (4.18e) and (4.18f) we have

$$u_{c,i}(\mathbf{x}) = u_{c,i-1}(\mathbf{x}) + ((L_l^h)^{-1}(d_l^h - L_\Gamma^h \hat{p}_\Gamma r_\Gamma (v^H)))(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h$$

in **FAC**.
Since d^H in **FAC** is the same as in **FFAC** we have from (4.18d) and (4.23d) that

$$v^H = w^H - \hat{r}_c u_{c,i-1}.$$

From the latter three equations it follows that $u_{c,i}(\mathbf{x})$ in **FAC** is the same as in **FFAC** at all grid points $\mathbf{x} \in \Omega_l^h$. □

Remark 4.6
- **FFAC** is applicable to nonlinear problems whereas **FAC** is not.
- For linear problems the **FFAC** iterates $u_{c,i}$ are independent of $\hat{r}_c$. For nonlinear problems the **FFAC** iterates $u_{c,i}$ depend on the choice for the restriction operator $\hat{r}_c$. □

Now we rewrite **FFAC** in a form that is very similar to the form in which the local defect correction iteration was presented in the previous section.
According to (4.23a), (4.23b) and (4.23d) the global coarse grid problem in the i-th step in **FFAC** is given by

$$L^H w^H = \tilde{r}_c f_c - \tilde{r}_c L_c u_{c,i-1} + L^H \hat{r}_c u_{c,i-1}.$$

Since $\hat{r}_c$ satisfies (4.21) we have (cf. (4.4))

$$(L_c u_{c,i-1})(\mathbf{x}) = (L^H \hat{r}_c u_{c,i-1})(\mathbf{x}) \quad \mathbf{x} \in \Omega_C.$$

Using this and (4.13) we get that this global coarse grid problem satisfies

$$(L^H w^H)(\mathbf{x}) = (\tilde{r}_c f_c)(\mathbf{x}) \quad \mathbf{x} \in \Omega_C.$$

Define the characteristic function $\tilde{\chi}$ by

$$(\tilde{\chi} w)(\mathbf{x}) := \begin{cases} w(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \cup \Gamma^H \\ 0 & \mathbf{x} \in \Omega_C \end{cases}. \tag{4.24}$$

Then (4.23d) can be written as follows:

$$L^H w^H = \tilde{r}_c f_c + \tilde{\chi}(L^H \hat{r}_c u_{c,i-1} - \tilde{r}_c L_c u_{c,i-1}).$$

According to (4.23a), (4.23c) and (4.23e) the local fine grid problem in the i-th iteration step in **FFAC** is given by

$$L_l^h w_l^h = r_{cl} f_c - r_{cl} L_c u_{c,i-1} + L_l^h r_{cl} u_{c,i-1} - L_\Gamma^h \hat{p}_\Gamma r_\Gamma w^H + L_\Gamma^h \hat{p}_\Gamma r_\Gamma \hat{r}_c u_{c,i-1}.$$

Since r_{cl} satisfies (4.6) and $\hat{r}_c$ satisfies (4.21) we have (cf. (4.7))

$$r_{cl} L_c u_{c,i-1} = L_l^h r_{cl} u_{c,i-1} + L_\Gamma^h \hat{p}_\Gamma r_\Gamma \hat{r}_c u_{c,i-1}.$$

Thus the local fine grid problem is equal to

$$L_l^h w_l^h = r_{cl} f_c - L_\Gamma^h \hat{p}_\Gamma r_\Gamma w^H.$$

The rewritten fast adaptive composite grid iteration is given below.

FFAC

Start: Initial composite grid approximation $u_{c,0}$ given.
$i = 1, 2, \ldots$

a) computation of the right hand side of the global problem

$$\tilde{f}^H := \tilde{r}_c f_c + \tilde{\chi}(L^H \hat{r}_c u_{c,i-1} - \tilde{r}_c L_c u_{c,i-1}) \tag{4.25a}$$

b) exact solution of the global problem

$$L^H w^H = \bar{f}^H \text{ on } \Omega^H \tag{4.25b}$$

c) exact solution of the local problem

$$L_l^h w_l^h = r_{cl} f_c - L_\Gamma^h \hat{p}_\Gamma r_\Gamma w^H \text{ on } \Omega_l^h \tag{4.25c}$$

d) correction of the composite grid approximation

$$u_{c,i}(\mathbf{x}) := w_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h \tag{4.25d}$$

$$u_{c,i}(\mathbf{x}) := w^H(\mathbf{x}) \quad \mathbf{x} \in \Omega_c \backslash \Omega_l^h. \tag{4.25e}$$

We note that for *linear* problems the iterates in (4.25) do *not* depend on the choice for the restriction operator $\hat{r}_c$.

Finally we take $u_{c,0}$ equal to the zero grid function on Ω_c in (4.25) and we regard the first iteration step in (4.25) as a starting procedure. The resulting fast adaptive composite grid iteration is denoted by **FFAC(0)**. As in **LDC** the discrete solution of the global problem (resp. of the local problem) at the i-th iteration step is denoted by u_i^H (resp. $u_{l,i}^h$).

FFAC(0)

Start: exact solution of the global problem

$$L^H u_0^H = \tilde{r}_c f_c \text{ on } \Omega^H$$

exact solution of the local problem

$$L_l^h u_{l,0}^h = r_{cl} f_c - L_\Gamma^h \hat{p}_\Gamma r_\Gamma u_0^H \text{ on } \Omega_l^h$$

computation of the composite grid approximation

$$u_{c,0}(\mathbf{x}) := \begin{cases} u_{l,0}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_0^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases}$$

$i = 1, 2, \ldots$

a) computation of the right hand side of the global problem

$$\bar{f}^H := \tilde{r}_c f_c + \tilde{\chi}(L^H \hat{r}_c u_{c,i-1} - \tilde{r}_c L_c u_{c,i-1}) \tag{4.26a}$$

b) exact solution of the global problem

$$L^H u_i^H = \bar{f}^H \text{ on } \Omega^H \tag{4.26b}$$

c) exact solution of the local problem

$$L_l^h u_{l,i}^h = r_{cl} f_c - L_\Gamma^h \hat{p}_\Gamma r_\Gamma u_i^H \text{ on } \Omega_l^h \tag{4.26c}$$

d) computation of a composite grid approximation

$$u_{c,i}(\mathbf{x}) := \begin{cases} u_{l,i}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_i^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h \end{cases} \tag{4.26d}$$

In the next section we compare the fast adaptive composite grid iteration **FFAC(0)** with the local defect correction iteration **LDC** from Section 3. The iterates in **FFAC(0)** are the same as the iterates in the fast adaptive composite grid method in its delayed correction form (as presented by McCormick in [9]) if the problem under consideration is linear and if the initial approximation in the latter method is chosen equal to the zero grid function on the composite grid.

Remark 4.7
If the discretization processes on the global coarse grid and on the local fine grid are given then the results of iteration (4.26) depend on the discretization process at the interface grid points (reflected in L_c in (4.26a)), on the choice for the prolongation operator $\hat{p}_\Gamma$ and on the choice for the restriction operator $\tilde{r}_c$. For nonlinear problems the results also depend on the choice for the restriction operator $\hat{r}_c$. □

5 Comparison of LDC and FAC

The local defect correction iteration and the fast adaptive composite grid iteration have been described in the previous sections as *discretization and solution methods* for a boundary value problem whose solution contains a *high activity region.* In this section we discuss the main *differences* and *similarities* between both methods.

From (3.11) and (4.26) it is clear that *computationally* the iterative processes are very similar. At each iteration step a global coarse grid right hand side is updated in a local region using information from the previous iteration step; in the fast adaptive composite grid iteration this 'updating region' is in general larger than in the local defect correction iteration. A discrete problem on the global coarse grid Ω^H is solved exactly. The resulting solution is used to define a discrete problem on the local fine grid Ω_l^h and this discrete problem is solved exactly. The solutions of the discrete problems in this iteration step are used to compute a new approximation of the continuous solution U of (2.1) on the composite grid.

The starting point for the fast adaptive composite grid iteration is a composite grid problem $L_c u_c = f_c$ that results from discretizing (2.1) on the composite grid Ω_c. This discrete problem has to be defined a priori; it is not an implicit result of the iterative process. The composite grid is composed of the uniform grids Ω^H and Ω_l^h. At grid points on the interface Γ between these two uniform grids, the composite grid is locally nonuniform. In the fast adaptive composite grid method discretization takes place on a nonuniform grid. At each iteration step an approximation of u_c is computed by solving discrete problems on the uniform grids Ω^H and Ω_l^h (not on the composite grid!). Often solving discrete problems on uniform grids is much easier than solving discrete problems on nonuniform grids.

In the local defect correction iteration the discretization process and the solution process are coupled. The 'limit discrete problem'

$$\begin{array}{ll} L^H \hat{u}^H - \chi L_\Gamma^H r_\Gamma \hat{u}^H - \chi L_l^H r_l \hat{u}_l^h = (1-\chi) f^H & \text{on } \Omega^H \\ L_l^h \hat{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma \hat{u}^H & \text{on } \Omega_l^h \end{array} \tag{5.1}$$

is an implicit result of the iterative process. At each iteration step an approximation of this limit discrete problem is computed. This discrete problem consists of two coupled discrete problems on uniform grids. In the approximations the unknowns in the second and third term on the left hand side of the global problem in (5.1) are replaced by approximations from the previous iteration step (see (3.11a), (3.11b) and (3.11c)). By exactly solving these discrete problems one obtains an approximation of $(\hat{u}^H, \hat{u}_l^h)$. In the local defect correction iteration, we only discretize with respect to *uniform* grids (in contrast with the composite defect correction method). In the solution process only discrete problems on uniform grids have to be solved (as in the fast adaptive composite grid iteration).

In the remainder of this section we assume that the discretization processes on the global coarse grid and on the local fine grid are given. Then the results of the local defect correction iteration depend on the choice for the prolongation operator p_Γ and on the value of *dist* (see Remark 3.5). The results of the fast adaptive composite grid iteration depend on the finite difference scheme that is used at the interface grid points and on the choice for the prolongation operator $\hat{p}_\Gamma$, the restriction operator $\tilde{r}_c$ and the initial composite grid approximation $u_{c,0}$. As stated in Section 2 we consider linear problems and we assume that the finite difference operators L^H, L_l^H, L_l^h and L_c are nonsingular. It will be shown below that for suitable choices of the components the local defect correction iterates and the fast adaptive composite grid iterates are the same.

First we consider the limit value $(\hat{u}^H, \hat{u}_l^h)$ of the local defect correction iteration with $dist = 0$. This limit value satisfies (5.1) with χ given by

$$(\chi w)(\mathbf{x}) = \begin{cases} w(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ 0 & \mathbf{x} \in \Omega^H \backslash \Omega_l^H \end{cases} . \tag{5.2}$$

At grid points $\mathbf{x} \in \Omega_l^H$ two approximations exist: $\hat{u}^H(\mathbf{x})$ and $\hat{u}_l^h(\mathbf{x})$. We show that these approximations are identical.

Lemma 5.1
The limit value $(\hat{u}^H, \hat{u}_l^h)$ of the local defect correction iteration with $dist = 0$ satisfies

$$\hat{u}^H(\mathbf{x}) = \hat{u}_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^H. \tag{5.3}$$

Proof:
According to (5.1) and (5.2) we have

$$(L^H \hat{u}^H)(\mathbf{x}) - (L_\Gamma^H r_\Gamma \hat{u}^H)(\mathbf{x}) - (L_l^H r_l \hat{u}_l^h)(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega_l^H.$$

According to (3.2) and (3.6) we have

$$(L^H \hat{u}^H)(\mathbf{x}) = (L_l^H \hat{u}^H|_{\Omega_l^H})(\mathbf{x}) + (L_\Gamma^H r_\Gamma \hat{u}^H)(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^H.$$

Thus

$$L_l^H \hat{u}^H|_{\Omega_l^H} = L_l^H r_l \hat{u}_l^h.$$

Since we have assumed that L_l^H is regular (see Section 2) we have that

$$\hat{u}^H(\mathbf{x}) = r_l \hat{u}_l^h(\mathbf{x}) \overset{(3.7)}{=} \hat{u}_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^H.$$

□

In case $dist = 0$ the coupled system (5.1) can be written as follows

$$\begin{array}{ll} \hat{u}^H(\mathbf{x}) = r_l \hat{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ (L^H \hat{u}^H)(\mathbf{x}) = f^H(\mathbf{x}) & \mathbf{x} \in \Omega^H \backslash \Omega_l^H \\ L_l^h \hat{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma r_\Gamma \hat{u}^H & \text{on } \Omega_l^h. \end{array} \tag{5.4}$$

Define $\hat{u}_c \in \mathcal{F}_c$ by

$$\hat{u}_c(\mathbf{x}) := \begin{cases} \hat{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \hat{u}^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \backslash \Omega_l^h = \Omega^H \backslash \Omega_l^H \end{cases} . \tag{5.5}$$

Define

$$\tilde{\Gamma}^H := \{\mathbf{x} \in \Omega_l^H \mid \text{distance}(\mathbf{x}, \Gamma) = H\}$$

The space of grid functions on $\tilde{\Gamma}^H$ is denoted by $\mathcal{F}_{\tilde{\Gamma}}^H$
We introduce L_o^H and $L_{\tilde{\Gamma}}^H$ such that (cf. (3.6))

$$(L^H \hat{u}^H)(\mathbf{x}) = (L_o^H \hat{u}^H|_{\Omega^H \backslash \Omega_l^H})(\mathbf{x}) + (L_{\tilde{\Gamma}}^H \hat{u}^H|_{\tilde{\Gamma}^H})(\mathbf{x}) \quad \mathbf{x} \in \Omega^H \backslash \Omega_l^H. \tag{5.6}$$

Note that by Lemma 5.1 we have $\hat{u}^H|_{\tilde{\Gamma}^H} = \hat{u}_l^h|_{\tilde{\Gamma}^H}$.
Let the *trivial injection* $r_{\tilde{\Gamma}} : \mathcal{F}_l^h \to \mathcal{F}_{\tilde{\Gamma}}^H$ be defined by

$$(r_{\tilde{\Gamma}} w_l^h)(\mathbf{x}) := w_l^h(\mathbf{x}) \quad \mathbf{x} \in \tilde{\Gamma}^H \tag{5.7}$$

for all $w_l^h \in \mathcal{F}_l^h$.
Then (5.4) can be written as follows

$$\begin{array}{ll} \hat{u}^H(\mathbf{x}) = r_l \hat{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ L_o^H \hat{u}^H|_{\Omega^H \setminus \Omega_l^H} + L_{\tilde{\Gamma}}^H r_{\tilde{\Gamma}} \hat{u}_l^h = f^H|_{\Omega^H \setminus \Omega_l^H} & \text{on } \Omega^H \setminus \Omega_l^H \\ L_l^h \hat{u}_l^h + L_\Gamma^h p_\Gamma r_\Gamma \hat{u}^H = f_l^h & \text{on } \Omega_l^h. \end{array} \tag{5.8}$$

Thus the composite grid function $\hat{u}_c$ is the solution of a composite grid problem.

Lemma 5.2
The composite grid approximation $\hat{u}_c$ from (5.5) satisfies:

$$\hat{L}_c \hat{u}_c = \hat{f}_c \tag{5.9}$$

with

$$\hat{L}_c = \begin{pmatrix} L_l^h & L_\Gamma^h p_\Gamma r_\Gamma \\ L_{\tilde{\Gamma}}^H r_{\tilde{\Gamma}} & L_o^H \end{pmatrix}, \tag{5.10}$$

$$\hat{u}_c = \begin{pmatrix} \hat{u}_l^h \\ \hat{u}^H|_{\Omega^H \setminus \Omega_l^H} \end{pmatrix}, \tag{5.11}$$

$$\hat{f}_c = \begin{pmatrix} f_l^h \\ f^H|_{\Omega^H \setminus \Omega_l^H} \end{pmatrix}. \tag{5.12}$$

Proof:
Follows easily from (5.8). □

Example 5.3
Consider the one dimensional Poisson problem from Example 3.1. The corresponding composite grid problem (5.9) is given by:

$$\frac{1}{hH} \begin{pmatrix} 8 & -4 & & & & & & & & & & \\ -4 & 8 & -4 & & & & & & & & & \\ & -4 & 8 & -4 & & & & & & & & \\ & & -4 & 8 & -4 & & & & & & & \\ & & & -4 & 8 & -4 & & & & & & \\ & & & & -4 & 8 & -4 & & & & & \\ & & & & & -4 & 8 & -4 & & & & \\ & & & -\frac{1}{4} & & & & \frac{1}{2} & -\frac{1}{4} & & & \\ & & & & & & & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \\ & & & & & & & & & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{u}_c(y_1) \\ \hat{u}_c(y_2) \\ \hat{u}_c(y_3) \\ \hat{u}_c(y_4) \\ \hat{u}_c(y_5) \\ \hat{u}_c(y_6) \\ \hat{u}_c(y_7) \\ \hat{u}_c(x_2) \\ \hat{u}_c(x_3) \\ \vdots \\ \hat{u}_c(x_6) \\ \hat{u}_c(x_7) \end{pmatrix} = \begin{pmatrix} f(y_1) + \frac{a}{h^2} \\ f(y_2) \\ f(y_3) \\ f(y_4) \\ f(y_5) \\ f(y_6) \\ f(y_7) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_6) \\ f(x_7) + \frac{b}{H^2} \end{pmatrix}$$

□

Composite grid problem (5.9) can be used as starting point for the fast adaptive composite grid iteration. Note that in this case $\hat{p}_\Gamma = p_\Gamma$ (see Remark 4.1).

Theorem 5.4
Assume that $\tilde{r}_c$, f^H and f_l^h satisfy the following relation:

$$(\tilde{r}_c f_l^h)(\mathbf{x}) = f^H(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^H. \tag{5.13}$$

Then **FFAC(0)** for composite grid problem (5.9) and **LDC** with $dist = 0$ are equivalent, i.e. the iterates $u_{c,i}$ in (4.26) and (3.11) are the same.
Proof:
Since we consider iteration (4.26) for problem (5.9), the prolongation operator $\hat{p}_\Gamma$ in (4.26) is equal to the prolongation operator p_Γ in (3.11).
From (4.13), (5.12) and (5.13) it follows that

$$\tilde{r}_c \hat{f}_c = f^H.$$

From (4.6) and (5.12) it follows that

$$r_{cl} \hat{f}_c = f_l^h.$$

Thus $u_{c,0}$ in **LDC** is the same as in **FFAC(0)**.
Suppose that $u_{c,i-1}$ in **LDC** is the same as $u_{c,i-1}$ in **FFAC(0)**. We show that $u_{c,i}$ in **LDC** is the same as $u_{c,i}$ in **FFAC(0)**. It is sufficient to show that $u_i^H(\mathbf{x})$ in **LDC** is the same as $u_i^H(\mathbf{x})$ in **FFAC(0)**. Note that $\Omega^H \backslash \Omega_l^H = \Omega_c \backslash \Omega_l^h$.
For $\bar{f}^H$ in (3.11a) we have

$$\begin{aligned} \bar{f}^H &= (1-\chi) f^H + \chi L_l^H r_l u_{l,i-1}^h + \chi L_\Gamma^H r_\Gamma u_{i-1}^H \\ &= (1-\chi) f^H + \chi L^H r_c u_{c,i-1} \\ &= L^H r_c u_{c,i-1} + (1-\chi)(f^H - L^H r_c u_{c,i-1}) \end{aligned}$$

where r_c is the trivial injection from (4.3).
Define

$$v_{\mathbf{L}}^H := (1-\chi)(f^H - L^H r_c u_{c,i-1}).$$

Then u_i^H in **LDC** is given by

$$u_i^H = r_c u_{c,i-1} + (L^H)^{-1} v_{\mathbf{L}}^H.$$

At $\mathbf{x} \in \Omega_l^H$ we have

$$v_{\mathbf{L}}^H(\mathbf{x}) = 0$$

by definition of χ (see (5.2).
At $\mathbf{x} \in \Omega_C$ we have

$$\begin{aligned} v_{\mathbf{L}}^H(\mathbf{x}) &= f^H(\mathbf{x}) - (L^H r_c u_{c,i-1})(\mathbf{x}) \\ &= f^H(\mathbf{x}) - (L^H u_{i-1}^H)(\mathbf{x}) \\ &= 0 \end{aligned}$$

which follows from the global problems in (3.11).
At $\mathbf{x} \in \Gamma^H$ we have

$$\begin{aligned} v_{\mathbf{L}}^H(\mathbf{x}) &= f^H(\mathbf{x}) - (L^H r_c u_{c,i-1})(\mathbf{x}) \\ &= (\tilde{r}_c \hat{f}_c)(\mathbf{x}) - (L^H r_c u_{c,i-1})(\mathbf{x}) \end{aligned}$$

which follows from (4.13) and (5.12).
For $\bar{f}^H$ in (4.26a) we have (see **FFAC** in (4.25))

$$\begin{aligned}\bar{f}^H &= \tilde{r}_c\hat{f}_c + \tilde{\chi}(L^H\hat{r}_c u_{c,i-1} - \tilde{r}_c\hat{L}_c u_{c,i-1}) \\ &= L^H\hat{r}_c u_{c,i-1} + \tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}).\end{aligned}$$

where we have used that $(L^H\hat{r}_c u_{c,i-1})(\mathbf{x}) = (\tilde{r}_c\hat{L}_c u_{c,i-1})(\mathbf{x}) \quad \mathbf{x} \in \Omega_C$.
Define

$$v_{\mathbf{F}}^H := \tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}).$$

Then u_i^H in **FFAC(0)** is given by

$$u_i^H = \hat{r}_c u_{c,i-1} + (L^H)^{-1} v_{\mathbf{F}}^H.$$

At $\mathbf{x} \in \Omega_C$ we have

$$\begin{aligned}v_{\mathbf{F}}^H(\mathbf{x}) &= (\tilde{r}_c\hat{f}_c)(\mathbf{x}) - (\hat{L}_c u_{c,i-1})(\mathbf{x}) \\ &= (\tilde{r}_c\hat{f}_c)(\mathbf{x}) - (L^H u_{i-1}^H)(\mathbf{x}) \\ &= 0\end{aligned}$$

which follows from the definition of $\hat{L}_c$ and from the global problems in iteration (4.26).
At $\mathbf{x} \in \Omega_l^h$ we have

$$\begin{aligned}\hat{f}_c(\mathbf{x}) - (\hat{L}_c u_{c,i-1})(\mathbf{x}) &= (r_{cl}\hat{f}_c)(\mathbf{x}) - (L_l^h u_{l,i-1}^h)(\mathbf{x}) - (L_\Gamma^h p_\Gamma r_\Gamma u_{i-1}^H)(\mathbf{x}) \\ &= 0\end{aligned}$$

which follows from the definition of $\hat{L}_c$ and the local problems in iteration (4.26).
Thus at $\mathbf{x} \in \Omega_l^H$ we have

$$v_{\mathbf{F}}^H(\mathbf{x}) = (\tilde{r}_c(\hat{f}_c - \hat{L}_c u_{c,i-1}))(\mathbf{x}) = 0.$$

At $\mathbf{x} \in \Gamma^H$ we have

$$\begin{aligned}v_{\mathbf{F}}^H(\mathbf{x}) &= (\tilde{r}_c\hat{f}_c)(\mathbf{x}) - (\tilde{r}_c\hat{L}_c u_{c,i-1})(\mathbf{x}) \\ &= (\tilde{r}_c\hat{f}_c)(\mathbf{x}) - (L^H r_c u_{c,i-1})(\mathbf{x})\end{aligned}$$

which follows from (4.3), (4.13) and the definition of $\hat{L}_c$.
Thus we have $v_{\mathbf{F}}^H = v_{\mathbf{L}}^H$.
Since r_c and $\hat{r}_c$ both are trivial injections at $\Omega_c \backslash \Omega_l^h$ we have

$$(r_c u_{c,i-1})(\mathbf{x}) = (\hat{r}_c u_{c,i-1})(\mathbf{x}) \quad \mathbf{x} \in \Omega^H \backslash \Omega_l^H.$$

Now it follows that $u_i^H(\mathbf{x})$ in **LDC** is the same as $u_i^H(\mathbf{x})$ in **FFAC(0)** at all grid points $\mathbf{x} \in \Omega^H \backslash \Omega_l^H$. □

Theorem 5.4 holds for the local defect correction iteration with *dist* = 0. In [5] an asymptotic convergence proof of the local defect correction iteration is given for *dist* > 0. The condition *dist* > 0 is important for the convergence analysis because results concerning "interior regularity" are used. In practice, we expect that for a large class of problems the local defect correction iterates even for *dist* = 0 converge fast (see e.g. [5] and [4]).

We note that for $dist = 0$ the limit value of the local defect correction iteration does not depend on $f^H(\mathbf{x})$ at $\mathbf{x} \in \Omega_l^H$ (see e.g. Lemma 5.2). Given $\tilde{r}_c$ and f_l^h we can define $f^H(\mathbf{x})$ at $\mathbf{x} \in \Omega_l^H$ by

$$f^H(\mathbf{x}) := (\tilde{r}_c f_l^h)(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^H.$$

In this way relation (5.13) is satisfied automatically.

Theorem 5.4 states that (for linear problems) the iterates in the local defect correction iteration with $dist = 0$ (introduced by Hackbusch in [5]) are the same as in the fast adaptive composite grid method (introduced by McCormick in [9]) for composite grid problem (5.9) with the zero grid function on Ω_c as initial composite grid approximation. We note that global problems in the local defect correction iteration may still be different from global problems in the fast adaptive composite grid iteration.

Theorem 5.5
Assume that (5.13) holds and that the restriction operator $\hat{r}_c$ in (4.26) is equal to the trivial injection r_c from (4.3).
Then all steps in **FFAC(0)** for composite grid problem (5.9) and in **LDC** with $dist = 0$ are identical.
Proof:
It is sufficient to show that $\bar{f}^H$ in **LDC** (denoted by $\bar{f}_{\mathbf{L}}^H$) is the same as $\bar{f}^H$ in **FFAC(0)** (denoted by $\bar{f}_{\mathbf{F}}^H$).
From the proof of Theorem 5.4 we have

$$\begin{aligned} \bar{f}_{\mathbf{L}}^H &= L^H r_c u_{c,i-1} + v^H, \\ \bar{f}_{\mathbf{F}}^H &= L^H \hat{r}_c u_{c,i-1} + v^H, \end{aligned}$$

with

$$v^H(\mathbf{x}) = \begin{cases} 0 & \mathbf{x} \in \Omega_l^H \\ 0 & \mathbf{x} \in \Omega_C \\ (\tilde{r}_c \hat{f}_c)(\mathbf{x}) - (L^H r_c u_{c,i-1})(\mathbf{x}) & \mathbf{x} \in \Gamma^H \end{cases}.$$

Since $\hat{r}_c = r_c$ it follows that $\bar{f}_{\mathbf{L}}^H = \bar{f}_{\mathbf{F}}^H$. □

Theorem 5.5 is an extension of Theorem 5.5. Theorem 5.4 states that under certain conditions the fast adaptive composite grid method and the local defect correction method yield the same iterates. Theorem 5.5 states that under an extra condition for the restriction operator $\hat{r}_c$ both methods become completely equivalent, i.e. all steps in these methods are identical.

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