

A mathematical model for the spread of infectious diseases based on a system of partial differential equations

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A MATHEMATICAL MODEL FOR THE SPREAD OF
INFECTIOUS DISEASES BASED ON A SYSTEM
OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

Diseases which exhibit a spread due to interactions between a human population and its environment are referred to as Men-Environment-Men (M.E.M.) diseases. Examples are typhoid fever, infectious hepatitis B and cholera.

Appreciable changes of the environment, e.g., an earthquake or a heat wave, may lead to a high concentration of infective agents of these diseases within the environment. This may cause an epidemic. Negative impacts of an epidemic may be of an economic nature, but not in the least may concern the health of an entire human population.

It is therefore important to study the spatial spread in time of these diseases, and to develop a mathematical model describing this spread in an adequate manner. Having such a model, especially so-called threshold parameters are of interest, firstly in order to be able to distinguish between situations which do or do not lead to an epidemic and secondly to enable the authorities to take necessary precautionary actions. On the basis of our mathematical model, we can say something about the velocity of the spread of a disease, the possibility that an epidemic may or may not lead to an endemic.

In this report, space heterogeneous reaction-diffusion equations (RDE's) are discussed. Analytical results will be given for (special cases of) one- and two-dimensional space heterogeneous RDE's. Also numerical results in the one-dimensional case will be given. RDE's as described in this report can be applied for modeling the spread of infectious diseases.

There are no data available which give some indication of the spatial spread. So our analysis will be mainly qualitative. In a separate section, we discuss some numerical simulations.

2. A mathematical model

In this section, a mathematical model is presented that may describe the interaction between a human population and a polluted environment.

Consider a region Ω and a subregion $G \subset \Omega$. Let $u_G(t)$ denote the number of infective agents in the region G , and $v_G(t)$ the number of infected people in G . We define the concentration of infective agents $u(\underline{x}, t)$ and the concentration of infected people $v(\underline{x}, t)$ as follows:

$$(2.1) \quad \begin{aligned} u_G(t) &= \int_G u(\underline{x}, t) d\underline{x} \\ v_G(t) &= \int_G v(\underline{x}, t) d\underline{x} \end{aligned}$$

where $\underline{x} \in \Omega$ and t represents time.

The mean life time of the agents in the environment will be denoted by $1/a_{11}$, the mean infectious period of the human infectives by $1/a_{22}$. Let a_{12} be the average multiplication parameter of the infectious agents due to the human population. Thus per unit of time, $a_{12} v$ denotes the increase of the concentration of infective agents. Finally, we introduce a function $g(u)$, describing the "rate of infection" due to the agents on the human population.

The increase or decrease of u and v in time depends on several factors.

- 1) Natural decline: the number of ill people will decline as they recover from their illness (or die), the number of the infective agents will decline when medical treatment destroys the agents. When more people get ill, or the infective multiply, there will be an increase of u and v .
- 2) Interaction between the human population and the polluted environment, or between the infected and the infectives.
- 3) Spread of the infected and the infectives. When for example the ill people all stay in a hospital, there is no spread of the infected. (However, it is possible that the infected people spread infective agents by their faeces.)

Consider the number of infective agents at a time level $t + \Delta t$ where Δt is a small time interval. The number of infective agents at $t + \Delta t$ in a subregion G of Ω is equal to the number of infective agents at t , minus the number of infective agents that die during the time interval Δt , plus the number of infective agents that enters or leaves the area G during Δt across the boundary, plus the number of infective agents produced by the human population. In terms of an equation, we can write

$$(2.2) \quad \int_G u(\underline{x}, t + \Delta t) d\underline{x} = \int_G u(\underline{x}, t) d\underline{x}$$

$$\begin{aligned}
 & - a_{11} \int_t^{t+\Delta t} \int_G u(\underline{x}, \tau) d\underline{x} d\tau \\
 & + d_1 \int_t^{t+\Delta t} \int_{\partial G} (\nabla u(\underline{x}, \tau), \underline{n}) d\sigma d\tau \\
 & + a_{12} \int_t^{t+\Delta t} \int_G v(\underline{x}, \tau) d\underline{x} d\tau
 \end{aligned}$$

where \underline{n} is the outward directed normal on the boundary ∂G . With $\tau \in [t, t+\Delta t]$ and $u(\underline{x}, \tau) = u(\underline{x}, t) + O(\Delta t)$ for $\Delta t \rightarrow 0$, and using Green's first identity, we have after dividing by Δt

$$\begin{aligned}
 (2.3) \quad & \frac{1}{\Delta t} \int_G [u(\underline{x}, t+\Delta t) - u(\underline{x}, t)] d\underline{x} + \\
 & + a_{11} \int_G u(\underline{x}, t) d\underline{x} - d_1 \int_G \Delta u(\underline{x}, t) d\underline{x} \\
 & - a_{12} \int_G v(\underline{x}, t) d\underline{x} + O(\Delta t) = 0.
 \end{aligned}$$

Taking the limit $\Delta t \rightarrow 0$, we obtain

$$\begin{aligned}
 (2.4) \quad & \int_G \left[\frac{\partial u(\underline{x}, t)}{\partial t} + a_{11} u(\underline{x}, t) - d_1 \Delta u(\underline{x}, t) \right. \\
 & \left. - a_{12} v(\underline{x}, t) \right] d\underline{x} = 0.
 \end{aligned}$$

Equation (2.4) must hold for any subregion G of region Ω . Therefore

$$(2.5) \quad \frac{\partial u}{\partial t} = d_1 \Delta u - a_{11} u + a_{12} v.$$

In the same way, we can derive a differential equation for v .

Our mathematical model consists of the following set of coupled partial differential equations

$$\begin{aligned}
 (2.6) \quad & \frac{\partial u}{\partial t} = d_1 \Delta u - a_{12} u + a_{12} v \\
 & \frac{\partial v}{\partial t} = d_2 \Delta v + g(u) - a_{22} v
 \end{aligned}$$

where

$$\begin{aligned}
 (2.7) \quad & u = u(\underline{x}, t), v = v(\underline{x}, t), \text{ with } u, v \in C^2(\Omega) \text{ with respect to } \underline{x} \\
 & (\underline{x} \in \mathbb{R}^2 \text{ or } \underline{x} \in \mathbb{R}^3 : \text{one or dimensional case}) \\
 & a_{11}, a_{12}, a_{21}, a_{22}, d_1, d_2 \geq 0.
 \end{aligned}$$

The term $a_{12} v$ represents the interaction between the human population and the polluted environment, just as $g(u)$ does. Of course, we could have taken a function $h(v)$ instead of $a_{12} v$, but the choice of a linear function is reasonable: a doubling of the concentration of infected people

means approximately a doubling of the concentrating of infective agents brought into the environment by the infected. The reverse is not the case (a doubling of the concentration of infectious agents will not double the number of infected people), as we shall see later on. So the function g cannot be taken linear in general.

Using (2.6) as a model, we also need to prescribe initial and boundary conditions.

Initial conditions fix the concentrations u and v at the beginning of the interaction process. (We can also measure the concentrations u and v at a certain time t_0 (or give an estimation), and use these measurements as the initial conditions of the process beginning at $t = t_0$.)

The initial conditions give the concentrations at the starting time $t = 0$. We can write the initial conditions as

$$(2.8) \quad \begin{aligned} u(\underline{x}, 0) &= u_0(\underline{x}) \\ v(\underline{x}, 0) &= v_0(\underline{x}) \end{aligned}$$

for integrable functions $u_0(\underline{x})$ and $v_0(\underline{x})$ where $\underline{x} \in \Omega$.

Interesting initial conditions are the cases with a high concentration (for example of infective agents) in a subregion of Ω , and a low concentration in the rest of the region (a peak in concentration). Besides, also stationary solutions are of interest but not discussed here.

Boundary conditions fix the concentrations u and v on the boundary $\partial\Omega$ of the region Ω . We take the following general boundary conditions

$$(2.9) \quad \begin{aligned} \rho_1 u + \sigma_1 \frac{\partial u}{\partial n} &= 0 \\ \rho_2 v + \sigma_2 \frac{\partial v}{\partial n} &= 0 \end{aligned} \quad \text{on } \partial\Omega, \quad t \geq 0.$$

Here, $\frac{\partial}{\partial n}$ denotes the normal derivative on the boundary $\partial\Omega$. For instance, $\frac{\partial u}{\partial n}$ can be interpreted as a flux of infective agents in (if < 0) or out (if > 0) the region Ω .

Consider the first equation in (2.9). In literature, three cases are distinguished.

- 1) $\sigma_1 = 0, \rho_1 = 1$, known as homogeneous Dirichlet boundary conditions. They are not interesting in this case because they would mean that the number of ill people is zero at the boundary.
- 2) $\sigma_1 = 1, \rho_1 = 0$, known as homogeneous Neumann boundary conditions. This condition says that all the infected people stay within Ω .

3) $\sigma_1 = 1$, $\rho_1 \neq 0$, known as homogeneous Robin boundary conditions.

We are mainly interested in the third case because the first two are included in the third one. We can interpret this condition as follows. The flux of infective agents will be linearly proportional to the concentration u at the boundary. So, for example, if the concentration of infectives at the boundary is high, there will be a flux of infectives out of the region Ω .

With the initial conditions (2.8) and the boundary conditions (2.9), the model is almost complete. So far, we have not mentioned a choice for the function g in (2.6). The simplest choice for g would be a function linear in u . This would mean that the concentration of infected people would double if the concentration of infected doubles. For a low concentration u , this seems reasonable, however for a high concentration of u this is not realistic. Still, it is worthwhile to examine. Another choice for the function g is a function that tends to a fixed value when u tends to infinity, e.g.,

$$(2.10) \quad g(u) = \alpha \frac{u}{1 + \beta u} \quad ; \quad \alpha, \beta \geq 0.$$

The graph of the function g in (2.10) is drawn in Figure 2.1.

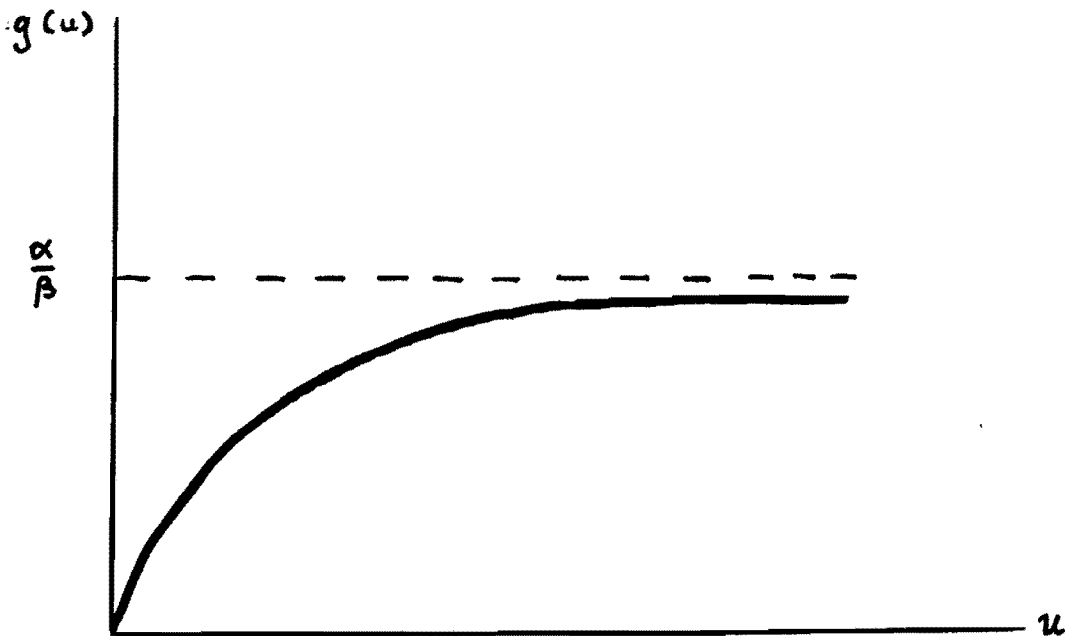


Figure 2.1. The function $g(u) = \alpha \frac{u}{1 + \beta u}$.

Even this form of g seems not so realistic. When the concentration of u is very low, hardly any people will be infected, so the slope of the function g will be flat for small u . The slope will become steeper when u increases. Again, the concentration of infected people will not rise much

when the concentration u is very high. These considerations inspire us to look at a function that has the following form

$$(2.11) \quad g(u) = \alpha \frac{u^2}{1 + \beta u^2} \quad ; \quad \alpha, \beta \geq 0.$$

The graph of this function is drawn in Figure 2.2.

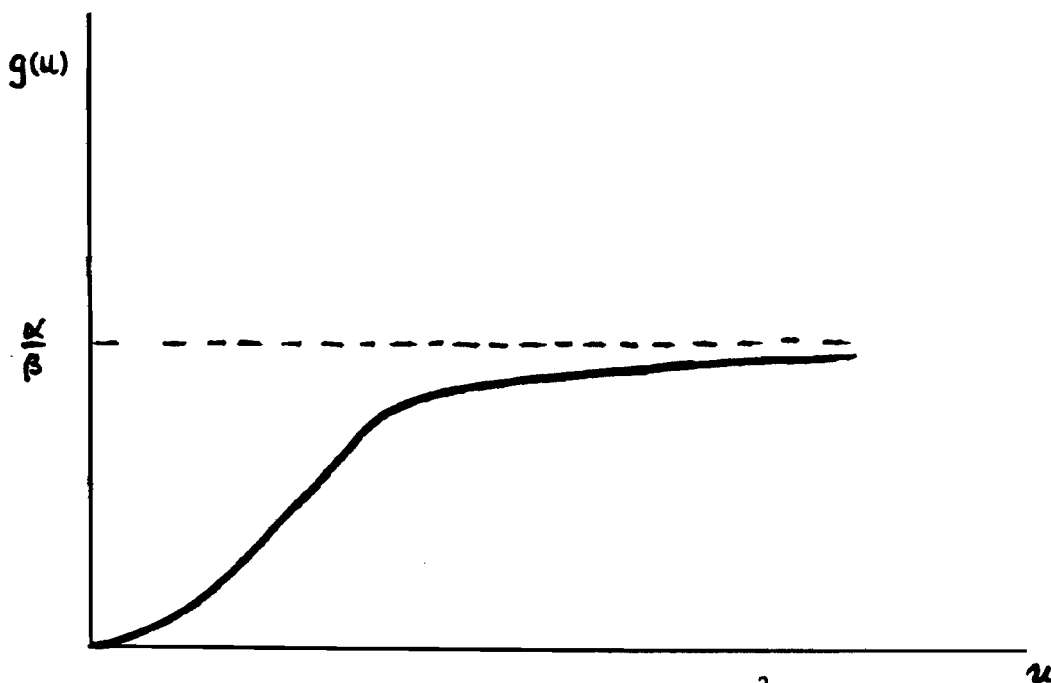


Figure 2.2. The function $g(u) = \alpha \frac{u^2}{1 + \beta u^2}$.

In (2.10) and (2.11), g is nonlinear and so will be the partial differential equations.

In Sections 3.1 and 3.2, the linear case will be considered, in Section 3.3 a nonlinear case with Neumann boundary conditions will be studied.

3. Mathematical analysis of the model

This section is devoted to a qualitative analysis of the model derived in Section 2. First we will solve the model using homogeneous Robin boundary conditions while we assume that the function g is linear. After that, we will look more carefully at the case $d_2 = 0$. Next, we will look at the model using homogeneous Neumann boundary conditions while the function g is taken to be non-linear.

3.1. General solution of the linear model

In this section we will solve the following system of parabolic differential equations

$$(3.1) \quad \begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u - a_{11} u + a_{12} v ; t > 0, x \in \Omega \\ \frac{\partial v}{\partial t} = d_2 \Delta v + a_{21} u - a_{22} v ; t > 0, x \in \Omega \\ u + \sigma \frac{\partial u}{\partial n} \Big|_{\underline{x} \in \partial \Omega} = 0, \quad v + \sigma \frac{\partial v}{\partial n} \Big|_{\underline{x} \in \partial \Omega} = 0 \\ u(\underline{x}, 0) = u_0(\underline{x}), \quad v(\underline{x}, 0) = v_0(\underline{x}) ; \underline{x} \in \Omega. \end{cases}$$

Let the function space X_σ be defined by

$$(3.2) \quad X_\sigma = \{ \phi \in C^2(\Omega) \cap C^1(\bar{\Omega}) \mid \phi + \sigma \frac{\partial \phi}{\partial n} \Big|_{\underline{x} \in \partial \Omega} = 0 \}.$$

Observe that for all $\phi, \psi \in X_\sigma$

$$(\Delta \phi, \psi)_{L_2(\Omega)} = (\phi, \Delta \psi)_{L_2(\Omega)}.$$

The differential operator A acting on the function space $X_\sigma \times X_\sigma$ is defined by

$$(3.3) \quad A(u, v)^T = (d_1 \Delta u - a_{11} u + a_{12} v, d_2 \Delta v + a_{21} u - a_{22} v)^T.$$

In order to solve (3.1), we first solve the eigenvalue problem (separation of variables)

$$(3.4) \quad A(u, v)^T = \lambda(u, v)^T ; u, v \in X_\sigma.$$

First consider the eigenvalue problem

$$(3.5) \quad -\Delta \phi = \omega \phi ; \phi \in X_\sigma.$$

There exists a real-valued discrete spectrum of eigenvalues such that

$$(3.6) \quad 0 < \omega_1 < \omega_2 \leq \omega_3 \leq \dots, \omega_n \rightarrow \infty (n \rightarrow \infty).$$

If Ω is one-dimensional, (3.5) belongs to the well-known class of Sturm-Liouville problems.

The corresponding eigenfunctions $\phi_1, \phi_2, \dots \in X_\sigma$, form an orthonormal basis in $L_2(\Omega)$, since Δ is

symmetric in X_σ .

The first equation in (3.4) leads to

$$(3.7) \quad v = \frac{1}{a_{12}} (\lambda + a_{11} - d_1 \Delta) u .$$

Substitution of (3.7) in the second equation of (3.4) yields

$$(3.8) \quad (d_1 d_2 \Delta^2 - (d_1(a_{22} + \lambda) + d_2(a_{11} + \lambda)) \Delta + (a_{11} + \lambda)(a_{22} + \lambda) - a_{12} a_{21}) u = 0 .$$

Equation (3.8) is equivalent with

$$(3.9) \quad (\Delta + \mu^{(1)}) u = 0, \quad v(\Delta + \mu^{(2)}) u = 0$$

for certain values of $\mu^{(1)}$ and $\mu^{(2)}$.

From this, we may conclude the following:

- i) u is an eigenfunction of the operator $-\Delta$, hence $u = \phi_n$ for a certain $n \in \mathbb{N}$
- ii) equation (3.7) now indicates

$$v = \frac{1}{a_{12}} (\lambda + a_{11} + d_1 \omega_n) \phi_n$$

- iii) (3.8) gives with $u = \phi_n$

$$(3.10) \quad d_1 d_2 \omega_n^2 + (d_1(a_{22} + \lambda) + d_2(a_{11} + \lambda)) \omega_n + (a_{11} + \lambda)(a_{22} + \lambda) - a_{12} a_{21} = 0$$

- iv) Every eigenvalue ω_n of problem (3.5) leads to two eigenvalues $\lambda_{1,n}$ and $\lambda_{2,n}$ of problem (3.4). These numbers $\lambda_{1,n}$ and $\lambda_{2,n}$ satisfy equation (3.10).

With i)-iv), we come to the following conclusions

$$(3.11) \quad \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} (u, \phi_n)_{L_2} \\ (v, \phi_n)_{L_2} \end{bmatrix} \phi_n$$

and

$$(3.12) \quad A(u, v)^T = \sum_{n=1}^{\infty} A(\omega_n) \begin{bmatrix} (u, \phi_n)_{L_2} \\ (v, \phi_n)_{L_2} \end{bmatrix} \phi_n .$$

where the matrix $A(\omega)$ is defined by

$$(3.13) \quad A(\omega) = \begin{bmatrix} d_1 \omega - a_{11} & a_{12} \\ a_{21} & d_2 \omega - a_{22} \end{bmatrix} .$$

Remark: the convergence of the series in (3.11) and (3.12) is in $L_2(\Omega) \times L_2(\Omega)$.

Finally, we have to solve the equation (see (3.1))

$$(3.14) \quad \begin{cases} \frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = A(u, v) \\ \text{initial conditions : } u(0) = u_0, \quad v(0) = v_0. \end{cases}$$

It is easy to check that the solution of (3.14) is given by

$$(3.15) \quad \begin{bmatrix} u \\ v \end{bmatrix} (\underline{x}, t) = \sum_{n=1}^{\infty} e^{tA(\omega_n)} \begin{bmatrix} (u_0, \phi_n)_{L_2} \\ (v_0, \phi_n)_{L_2} \end{bmatrix} \phi_n(\underline{x}).$$

Remarks.

1. Note that there is a similarity between the solution (3.15) and the solution of the differential equation $\frac{d}{dt} \underline{u} = A \underline{u}$ where A is a matrix.

2. When the matrices $A(\omega_n)$, $n = 1, 2, \dots$, are diagonalizable we have (\sim means "similar to", see also iii) and iv))

$$A(\omega_n) \sim \begin{bmatrix} \lambda_{n,1} & 0 \\ 0 & \lambda_{n,2} \end{bmatrix}, \quad e^{tA(\omega_n)} \sim \begin{bmatrix} e^{t\lambda_{n,1}} & 0 \\ 0 & e^{t\lambda_{n,2}} \end{bmatrix}$$

and we can write (3.15) as

$$(3.16) \quad \begin{bmatrix} u \\ v \end{bmatrix} (\underline{x}, t) = \sum_{n=1}^{\infty} e^{t\lambda_{1,n}} \underline{\xi}_{1,n}(u_0, v_0) \phi_n(\underline{x}) + \sum_{n=1}^{\infty} e^{t\lambda_{2,n}} \underline{\xi}_{2,n}(u_0, v_0) \phi_n(\underline{x})$$

where $\underline{\xi}_{i,n}$, $i = 1, 2$, $n = 1, 2, \dots$, are vectors in \mathbb{R}^2 , which depend on the initial functions u_0 and v_0 .

3. From (3.16), one immediately sees that the solution (0,0) is stable if $\lambda_{i,n} < 0$, $n = 1, 2$, $n = 1, 2, \dots$. After some elementary calculations, it follows from (3.10) that the solution (0,0) is stable if the following threshold is satisfied

$$(3.17a) \quad d_1 d_2 \omega_n^2 + (d_1 a_{22} + d_2 a_{11}) \omega_n + a_{11} a_{22} - a_{12} a_{21} > 0, \quad n \in \mathbb{N}$$

or equivalently (solve equation (3.10))

$$(3.17b) \quad \theta := \frac{(a_{11} + d_1 \omega_1) \cdot (a_{22} + d_2 \omega_1)}{a_{12} a_{21}} > 1.$$

3.2. The special case $d_2 = 0$

In this section, we will look more carefully at the linear model in the case the parameter $d_2 = 0$. From (3.6) and (3.13), we see that in the case $d_1, d_2 > 0$ both eigenvalues $\lambda_{n,1}, \lambda_{n,2}$ of $A(\omega_n)$ tend to $-\infty$ for $n \rightarrow \infty$. In the case $d_2 = 0$, this no longer holds and the eigenvalues $\lambda_{2,n}$ of $A(\omega_n)$ will tend to zero for $n \rightarrow \infty$. This is the reason why $d_2 = 0$ is a special case.

First, we will take $\Omega := [0, 1]$ (one-dimensional model). This model is used for numerical simulation in Chapter 4. Without loss of generality, we put $d_1 = 1$. The model then becomes

$$(3.18) \quad \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - a_{11} u + a_{12} v \\ \frac{\partial v}{\partial t} = a_{12} u - a_{22} v \\ u + \sigma \frac{\partial u}{\partial n} \Big|_{x=0, x=1} = 0 ; t > 0 \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) ; x \in \Omega. \end{cases} \quad x \in \Omega, \quad t > 0$$

Note that for the function v no boundary conditions are prescribed. As in Section 3.1 we first solve the problem

$$(3.19) \quad \begin{cases} \frac{d^2 u}{dx^2} - a_{11} u + a_{12} v = \lambda u \\ a_{21} u - a_{22} v = \lambda v \\ u + \sigma \frac{du}{dx} \Big|_{x=0, x=1} = 0. \end{cases}$$

Substitution of the second equation in the first one in (3.19) gives

$$(3.20) \quad -\frac{d^2 u}{dx^2} = \mu^2 u$$

where

$$(3.21) \quad \mu^2 = \frac{a_{12} a_{21} - (a_{11} + \lambda)(a_{22} + \lambda)}{a_{22} + \lambda}, \quad (\lambda \neq -a_{22}).$$

Equation (3.20) together with the boundary condition in (3.19) is a Sturm-Liouville problem, hence μ^2 is real and eigenfunctions related to different eigenvalues are orthonormal

The general solution of (3.20) is

$$(3.22) \quad u_\mu(x) = \alpha_{1,\mu} \sin(\mu x) + \alpha_{2,\mu} \cos(\mu x), \quad \mu \neq 0.$$

Furthermore, we assume that infectious agents have the region when the concentration in Ω is higher than the concentration outside Ω and enter the region in the reverse situation when the concentration in Ω is lower than outside Ω . This implies that $\sigma > 0$. So, therefore a linear or exponential solution of (3.20) cannot satisfy the boundary condition and hence all possible

solutions are given by (3.22) with μ real. The values of μ now follow from (3.22) and the boundary condition; a short calculation shows that the numbers μ must satisfy the equation

$$(3.23) \quad h(\mu) := \tan \mu + \frac{2\sigma\mu}{1-\sigma\mu^2} = 0.$$

Because $h(\mu) = -h(-\mu)$ for all $\mu \in \mathbb{R}$ and $h(\mu) = 0 \Rightarrow \mu \in \mathbb{R}$ we only have to look for positive solutions of (3.23). We find a non-decreasing sequence $(\mu_n)_{n \in \mathbb{N}}$ with $\mu_n \rightarrow \infty$ ($n \rightarrow \infty$). Using these numbers μ_n , $n \in \mathbb{N}$, we can calculate the corresponding numbers $\lambda_{n,1}$, $\lambda_{n,2}$ $n \in \mathbb{N}$ from (3.21). In this way we obtain

$$(3.24) \quad \lambda_{n,i} = \frac{-(a_{11} + a_{22} + \mu_n)^2 \pm \sqrt{(a_{11} - a_{22} + \mu_n^2)^2 + 4a_{12}a_{21}}}{2}, \quad i = 1, 2, \quad n \in \mathbb{N}.$$

It is now easy to check that the solution of (3.18) is given by

$$(3.25) \quad \begin{cases} u(x, t) = \sum_{n=1}^{\infty} (\alpha_{n,1} e^{\lambda_{n,1}t} + \alpha_{n,2} e^{\lambda_{n,2}t}) u_{\mu_n}(x) \\ v(x, t) = \sum_{n=1}^{\infty} (\alpha_{n,1} \frac{a_{21}}{\lambda_{n,1} + a_{22}} e^{\lambda_{n,1}t} + \alpha_{n,2} \frac{a_{21}}{\lambda_{n,2} + a_{22}} e^{\lambda_{n,2}t}) u_{\mu_n}(x). \end{cases}$$

In case of non-degenerate eigenvalues (which is a justifiable assumption because the numbers a_{ij} , $i, j = 1, 2$, must be obtained through measurements), the coefficients $\alpha_{n,i}$, $n \in \mathbb{N}$, $i = 1, 2$, can be calculated from the initial conditions.

As readily follows from (3.17a), the solution (3.25) is stable if ($d_1 = 1$, $d_2 = 0$, $\omega_n = \mu_n^2$)

$$(3.26) \quad a_{22}\mu_n^2 + a_{11}a_{22} - a_{12}a_{21} > 0 \quad \text{for all } n \in \mathbb{N}.$$

It is obvious that (3.26) holds true for all $n \in \mathbb{N}$ if it holds true for μ_1 . The value of μ_1 can be computed using (3.23). If we take $\sigma = 1/\alpha$, equation (3.23) becomes

$$(3.27) \quad \tan \mu = \frac{-2\mu}{\alpha - \mu^2}.$$

How the value of μ_1 depends on α is shown in Figure 3.1.

smallest eigenvalue
1-dimensional p.d.e.

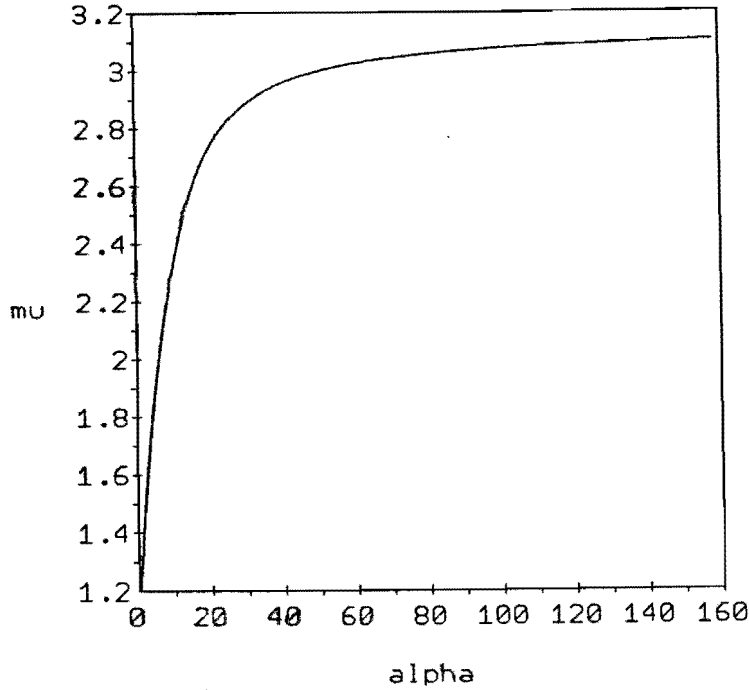


Figure 3.1. μ_1 as a function of α .

Next, we will look at a two-dimensional model. For Ω , we take a circular segment with radius $r = 1$. We assume the solution to be circular symmetric. Hence the model reduces to a one dimensional one. The model (in polar coordinates) now reads as

$$(3.28) \quad \begin{cases} \frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) u - a_{11} u + a_{12} v & 0 \leq r < 1, t > 0 \\ \frac{\partial v}{\partial t} = a_{21} u - a_{22} v \\ u + \sigma \frac{\partial u}{\partial r} \Big|_{r=1} = i, t > 0 \\ u(r, 0) = u_0(r), v(r, 0) = v_0(r), 0 \leq r < 1. \end{cases}$$

Because the derivation of the solution is similar to the previous case ($\Omega = [0, 1]$), we will omit it. The solution is given by

$$(3.29) \quad \begin{cases} u(r, t) = \sum_{n=1}^{\infty} (\alpha_{n,1} e^{\lambda_{n,1} t} + \alpha_{n,2} e^{\lambda_{n,2} t}) J_0(\mu_n r) \\ v(r, t) = \sum_{n=1}^{\infty} \left(\alpha_{n,1} \frac{a_{21}}{\lambda_{n,1} + a_{22}} e^{\lambda_{n,1} t} + \alpha_{n,2} \frac{a_{21}}{\lambda_{n,2} + a_{22}} e^{\lambda_{n,2} t} \right) J_0(\mu_n r) \end{cases}$$

where J_0 is a Bessel function of the first kind of order zero.

The condition for stability in the two-dimensional case is the same as for the one-dimensional

case. The numbers μ_n are solutions of the equation ($\alpha = 1/\sigma$)

$$(3.30) \quad \alpha J_0(\mu) - \mu J_1(\mu) = 0.$$

As for $\Omega = [0, 1]$, if the stability condition is satisfied for μ_1 , then it is satisfied for μ_n , $n \geq 1$. The relation between μ_1 and α is given in Figure 3.2.

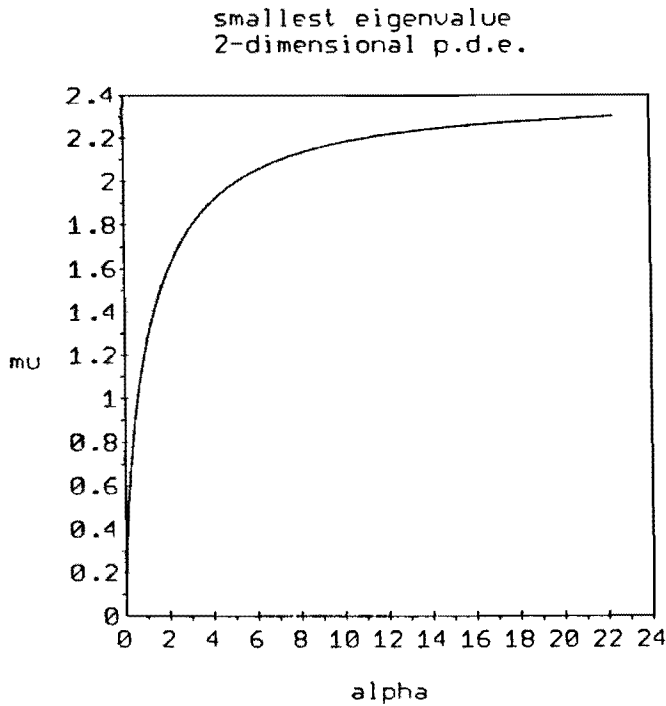


Figure 3.2. μ_1 as a function of α .

When the condition that the solution must be circular symmetric is dropped, we find a solution which is a series containing integer order Besselfunctions of the first kind.

3.3. Nonlinear systems

In this section we consider the model

$$(3.31) \quad \begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u - a_{11} u + a_{12} v \\ \frac{\partial v}{\partial t} = d_2 \Delta v + g(u) - a_{22} v \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad \underline{x} \in \partial\Omega \\ u(\underline{x}, 0) = u_0(\underline{x}), \quad v(\underline{x}, 0) = v_0(\underline{x}) \end{cases} \quad \underline{x} \in \Omega, \quad t > 0$$

where Ω is a bounded region in \mathbb{R}^1 or \mathbb{R}^2 . The function g will either be of the form

$$(3.32) \quad g(u) = \alpha \frac{u}{1 + \beta u} ; \alpha, \beta > 0$$

or

$$(3.33) \quad g(u) = \alpha \frac{u^2}{1 + \beta u^2} ; \alpha, \beta > 0 .$$

(See also equation (2.10) and (2.11).)

In this case, any point $(u^0, v^0) \in \mathbb{R}^2$ satisfying

$$(3.34) \quad \begin{cases} -a_{11} u^0 + a_{12} v^0 = 0 \\ g(u^0) - a_{22} v^0 = 0 \end{cases}$$

yields an equilibrium solution $u = u^0, v = v^0$ of (3.31).

For functions g as in (3.32), we find two equilibrium points and for functions g as in (3.33) three equilibrium points (see Figure 3.3.) for certain values of a_{11}, a_{12} and a_{22} .

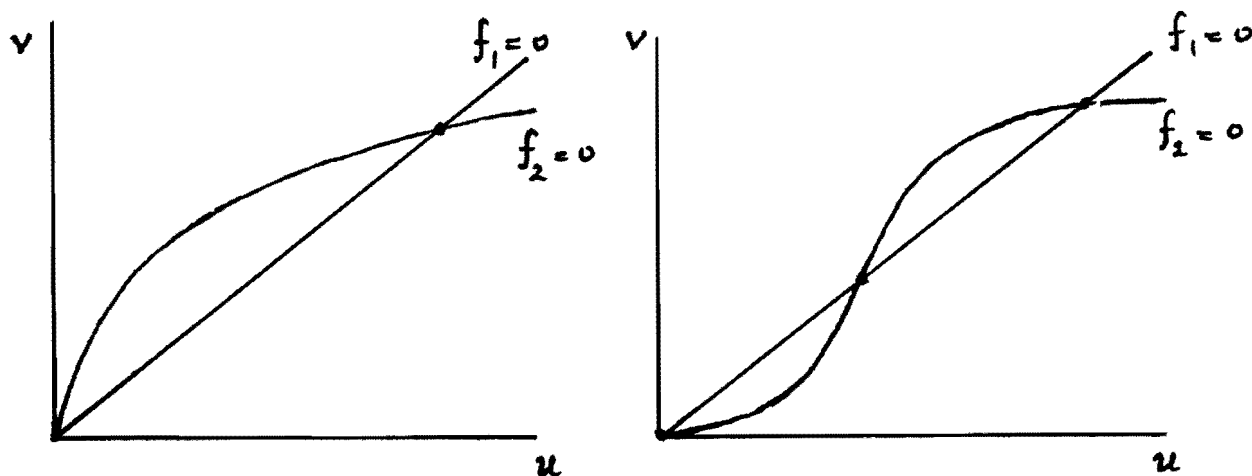


Figure 3.3. Equilibrium points.

In Figure 3.3., we used the definitions

$$(3.35) \quad \begin{cases} f_1 = -a_{11} u + a_{12} v \\ f_2 = g(u) - a_{22} v . \end{cases}$$

We will now investigate the local stability of these equilibrium points. This is done by linearizing the function g in the neighbourhood of an equilibrium point. We first take g of the form (3.32). The equilibrium solutions then are

$$(3.36) \quad \begin{cases} (u^0, v^0) = (0, 0) \\ (u^0, v^0) = \left(\frac{1}{\beta} (\gamma - 1), \frac{\alpha}{\beta \gamma} (\gamma - 1) \right) \end{cases}$$

where

$$(3.37) \quad \gamma = \frac{a_{12}}{a_{11} \cdot a_{22}} \cdot \alpha .$$

When $(u^0, v^0) = (0, 0)$ the linearized equations are

$$(3.38) \quad \begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u - a_{11} u + a_{12} v \\ \frac{\partial v}{\partial t} = d_2 \Delta v + g'(0) u - a_{22} v . \end{cases}$$

Using condition (3.17b), we find that $(u^0, v^0) = (0, 0)$ is locally stable if the following threshold is satisfied

$$(3.39) \quad (a_{11} + d_1 \omega_1) (a_{22} + d_2 \omega_1) > a_{12} g'(0) .$$

We will now investigate the local (in-) stability of the point $(u^0, v^0) = \left(\frac{1}{\beta} (\gamma - 1), \frac{\alpha}{\beta \gamma} (\gamma - 1) \right)$. For this purpose, we linearize system (3.31) in the neighbourhood of (u^0, v^0) . We write

$$(3.40) \quad \begin{cases} u = \xi + u^0 \\ v = \eta + v^0 . \end{cases}$$

The equations now become

$$(3.41) \quad \begin{cases} \frac{\partial \xi}{\partial t} = d_1 \Delta \xi - a_{11} \xi + a_{12} \eta \\ \frac{\partial \eta}{\partial t} = d_2 \Delta \eta + g'(u^0) \xi - a_{22} \eta . \end{cases}$$

Using (3.17b) we find:

$$(u^0, v^0) = \left[\frac{1}{\beta} \left[\frac{a_{12} \alpha}{a_{11} \cdot a_{22}} - 1 \right], \frac{\alpha}{\beta} - \frac{a_{11} \cdot a_{22}}{a_{12} \beta} \right]$$

is a local attractor if

$$(3.42) \quad (a_{11} + d_1 \omega_1) (a_{22} + d_2 \omega_1) > g'(u^0) a_{12} .$$

Next we take the function g of the form (3.33). The equilibrium solutions are

$$(3.43) \quad \begin{cases} (\mu^0, v^0) = (0,0) \\ (\mu^0, v^0) = \frac{1}{2\beta} (\gamma - \sqrt{\gamma^2 - 4\beta}) \left(1, \frac{a_{11}}{a_{12}}\right) \\ (\mu^0, v^0) = \frac{1}{2\beta} (\gamma + \sqrt{\gamma^2 - 4\beta}) \left(1, \frac{a_{11}}{a_{12}}\right). \end{cases}$$

For the equilibrium solution $(\mu^0, v^0) = (0,0)$, we find the same threshold as in (3.39). However, $g'(0) = 0$ so that $(\mu^0, v^0) = (0,0)$ is a locally stable equilibrium solution. For the other two equilibrium solutions, we also find a threshold as given in (3.42). Whether these points are stable solutions or not depends on the parameters a_{11} , a_{12} , a_{22} , d_1 and d_2 , the smallest eigenvalue ω_1 of the eigenvalue problem (3.5) and on the function g . Finally, the following remarks can be made

- i) When the second solution in (3.43) is stable then also the third one is stable.
When the third solution in (3.43) is unstable then also the second one is unstable.
- ii) When the parameters d_1 and d_2 are both equal to zero (ordinary differential equations) the second solution in (3.43) is always unstable and the third one is always stable. In the case of the PDE's, this no longer holds and the threshold becomes more complex then in the ODE-case.
- iii) When we take homogeneous Dirichlet or Robin boundary conditions in (3.31) we can also calculate whether $(0,0)$ is a locally stable solution. In fact we find the same threshold as in (3.17b). Only the numbers ω_n , $n \in \mathbb{N}$, will be different.

In the next section we will give some numerical results concerning the models we have discussed.

4. Conclusions

From our qualitative analysis, we have seen some ways in which the authorities can influence the spatial spread of the disease. Indeed, we found the following threshold parameter θ

$$(4.1) \quad \theta = \frac{(a_{11} + d_1 \omega_1)(a_{22} + d_2 \omega_1)}{a_{12} a_{21}}$$

where ω_1 is a positive, real number. This threshold value θ determines whether an epidemic tends to an endemic state or to extinction. We have

$$(4.2) \quad \begin{array}{ll} \theta < 1 : & \text{no extinction} \\ \theta > 1 : & \text{extinction} \end{array}$$

The threshold value θ can be influenced as follows. We only describe which actions lead to extinction, because this is what the authorities are interested in.

- 1) Decrease a_{12} or a_{21} , or both.
This means that the interaction between the infective agents and the infected people decreases. We can do this for example by improving the hygiene or vaccinating the people.

- 2) Increase d_2 .
This means that the infected people spread faster, which implies that there will be no peaks in the concentration of the infected people.

- 3) Increase a_{11} or a_{22} , or both.
This means that the natural decline of the infective agents or infected people is fastened. This can be done by exterminating the infective agents or curing the infective people, respectively.

5. Numerical simulations

In this section, we give some numerical simulations. Solutions of the set of coupled partial differential equations were calculated in the following cases.

- 1) One dimension, using homogeneous Neumann boundary conditions ($\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ on the boundary $\partial\Omega$) and having

$$d_1 = d_2 = 1/10$$

$$a_{11} = 0,15 \quad , \quad a_{12} = 0,1 \quad , \quad a_{22} = 0,15$$

$$\alpha = 1/100 \quad , \quad \beta = 1/1000$$

where

$$g(u) = \alpha \frac{u^2}{1 + \beta u^2} .$$

- 2) One dimension, using homogeneous Robin boundary conditions ($u + \frac{\partial u}{\partial n} = 0$, $v + \frac{\partial v}{\partial n} = 0$ on the boundary $\partial\Omega$) and having

$$d_1 = d_2 = 1/10$$

$$a_{11} = 0,1 \quad , \quad a_{12} = 0,2 \quad , \quad a_{22} = 0,1$$

$$\alpha = 1/100 \quad , \quad \beta = 1/1000$$

where

$$g(u) = \alpha \frac{u^2}{1 + \beta u^2} .$$

In Fig. 5.1 and 5.2, we have plotted $u(=u_1)$ and $v(=u_2)$, in the case of Neumann boundary conditions, at several time levels, indicated in the figure.

In Fig. 5.3 and 5.4, we did the same for Robin boundary conditions.

We see in all the pictures that peaks in concentrations are flattened quickly. This is due to the diffusion.

Concentration u_1

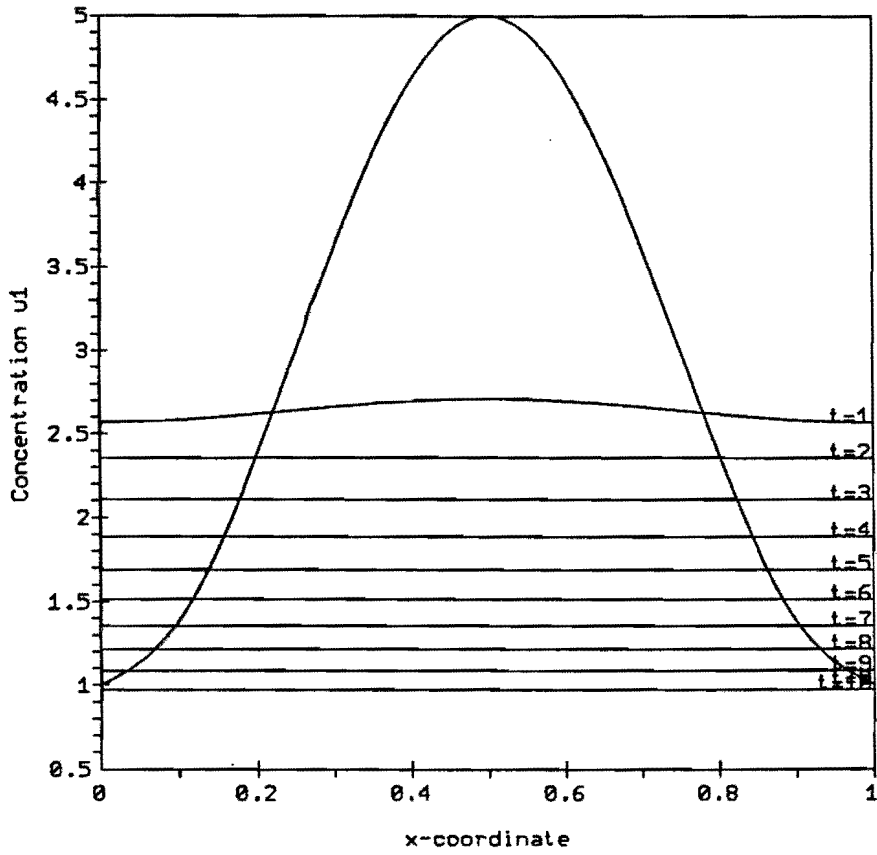


Figure 5.1.

Numerical results, homogeneous Neumann boundary conditions.

Concentration u_2

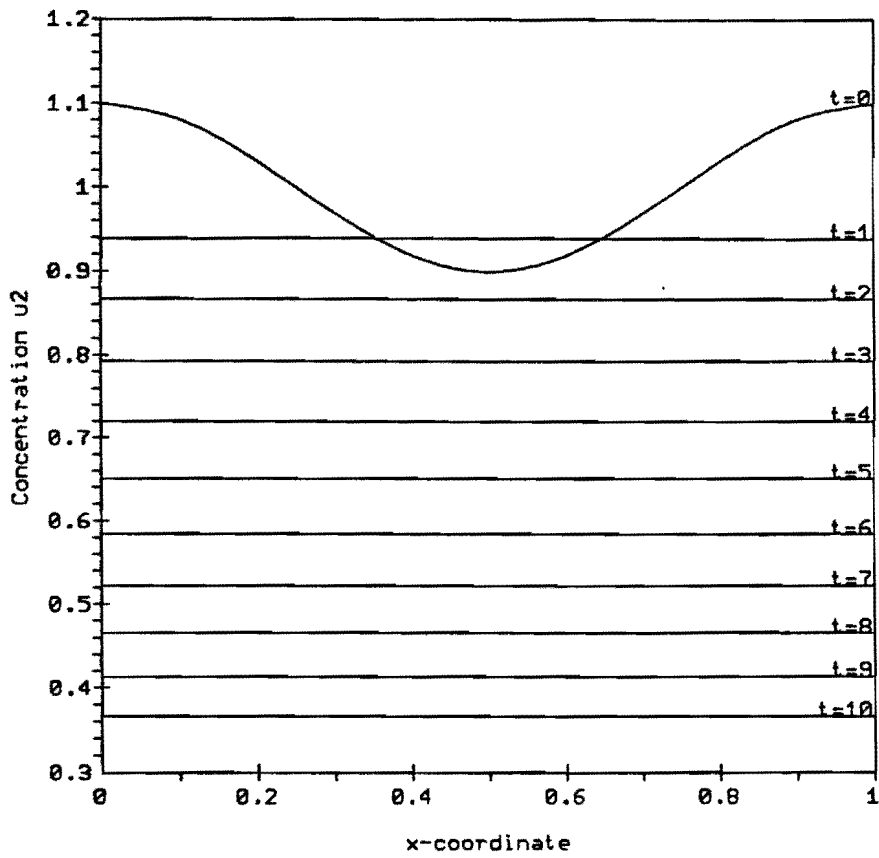


Figure 5.2.

Numerical results, homogeneous Neumann boundary conditions.

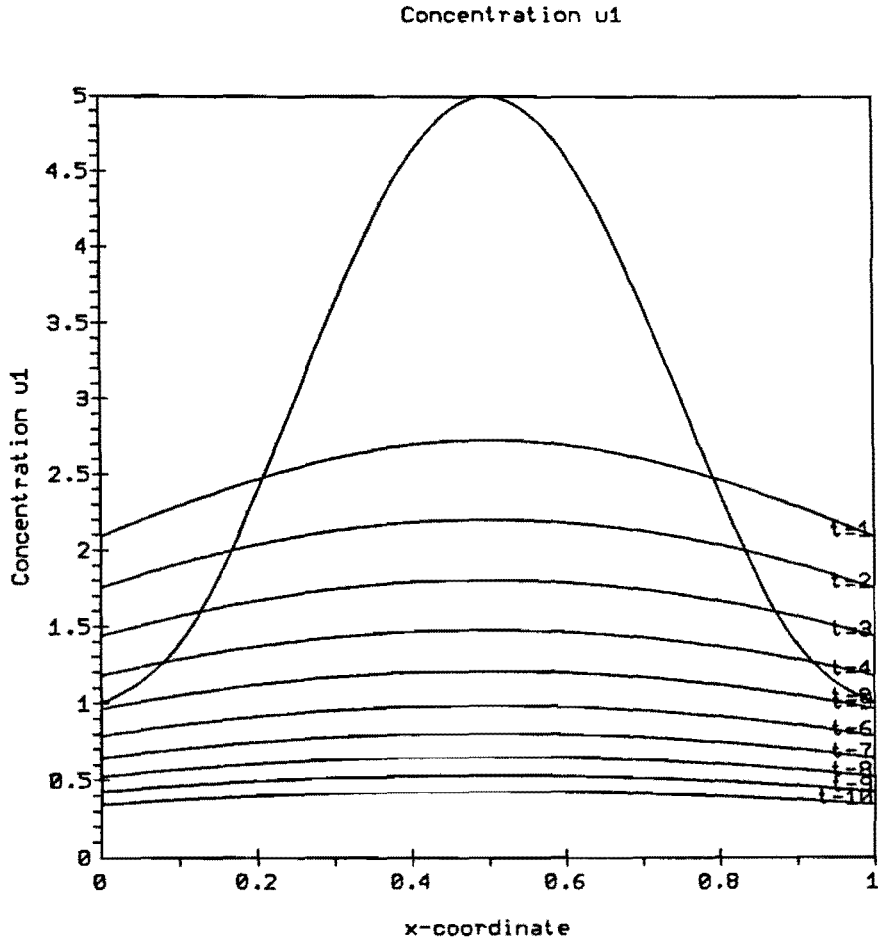


Figure 5.3.
Numerical results, homogeneous Robin boundary conditions.

Concentration u_2

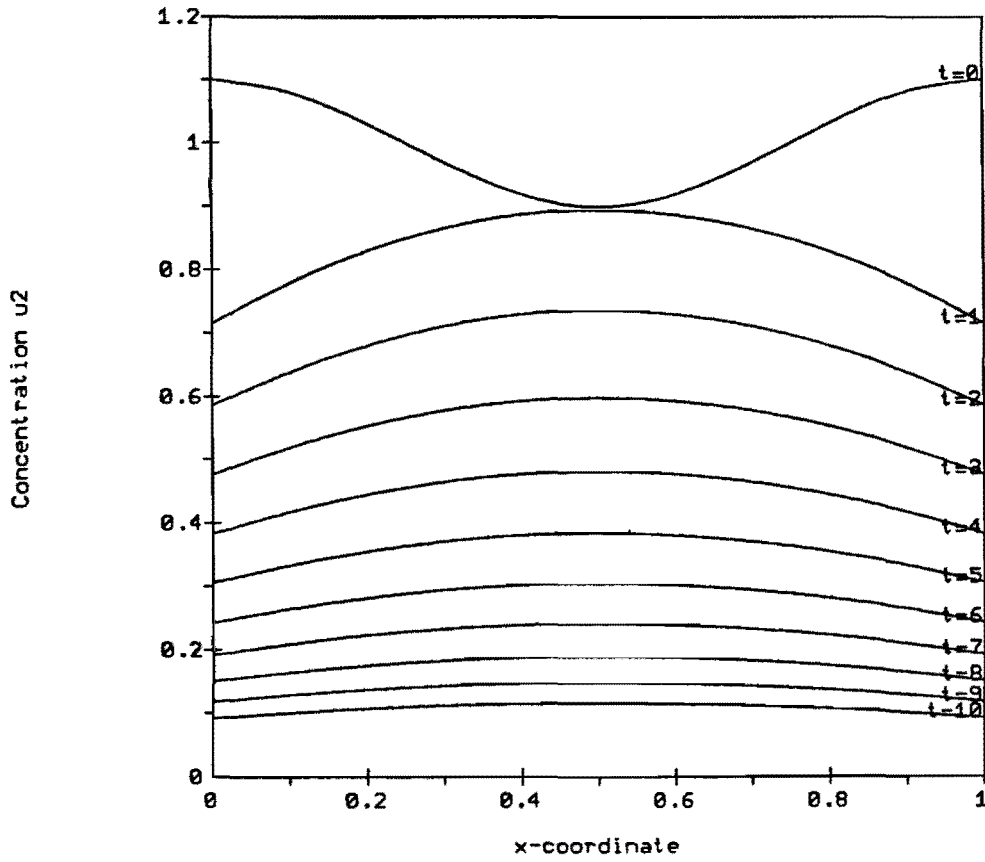


Figure 5.4.

Numerical results, homogeneous Robin boundary conditions.

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