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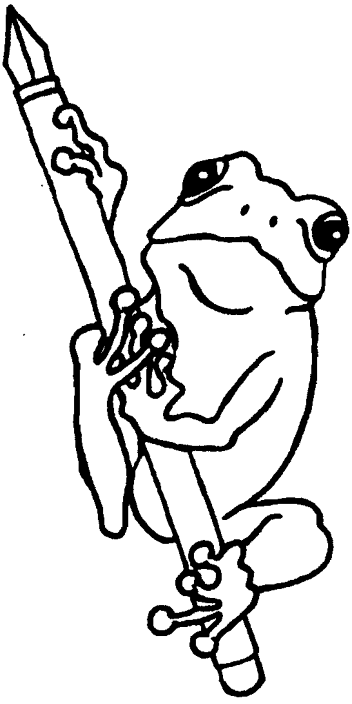

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Abstract: In the modelling of a very viscous drop that moves freely under the influence of surface tension it may be convenient to omit the inertial terms. Thus, a quasistatic model is obtained for which an exact statement is given using Lagrange coordinates. At any time the velocity and pressure field fulfill the Stokes equations with natural boundary conditions of Neumann type. It is shown that the solution of this problem for fixed time exists and is defined up to rigid body motions.

For the investigation of the time-dependent problem a result on global invertibility of small C^2-deformations is proved and the dependence of the solution on such deformations is investigated. A condition concerning this dependence is formulated under which the time-dependent problem has a unique solution on a short interval of time. Finally, a study of the asymptotic behaviour of globally existing solutions shows that (under reasonable regularity presumptions) our model resembles the fact that the drop approaches the state of a ball of resting liquid.

1 The quasistatic approximation

The motion of a free drop of incompressible viscous liquid under the influence of surface tension without external forces can be described as a free boundary value problem for the Navier-Stokes equations in the following way:

$$\left.\begin{array}{rcl} \varrho\left(\frac{\partial v}{\partial t}+(v\cdot\nabla)v\right)-\nu\Delta v+\nabla p &=& 0 \\ \operatorname{div} v &=& 0 \end{array}\right\} \text{ in } \Omega_t \qquad (1)$$
$$\mathcal{T}(v,p)n_t = \gamma\kappa_t n_t \quad \text{on } \Gamma_t = \partial\Omega_t$$

for any time $t \geq 0$, where $\Omega_t \subset \mathbb{R}^N$ is the (bounded) domain occupied by the fluid at time t and v and p are the time-dependent velocity and pressure fields on Ω_t. The density ϱ, the viscosity ν, and the surface tension coefficient γ are positive real material constants. $\mathcal{T}$ is a tensor given by

$$\mathcal{T}(v,p) := \nu\left(\nabla v + (\nabla v)^T\right) - p\mathcal{I}$$

where ∇v is the Jacobian of v and $\mathcal{I}$ denotes the identity tensor. By n_t the outer normal vector of Γ_t is denoted.

The scalar function $\kappa_t : \Gamma_t \longrightarrow \mathbb{R}$ expresses the local curvature behaviour of the surface Γ_t. If it is smooth enough, κ_t can be defined by

$$\kappa_t n_t = \Delta_{\Gamma_t} x$$

where Δ_{Γ_t} is the Laplace-Beltrami-operator on Γ_t. It has to be applied to every component of the vector x of the spatial coordinates which are to be considered as scalar functions on Γ_t. If $N = 2$ then κ_t is the usual curvature of the plane curve Γ_t, while for $N = 3$ it turns out to be the double mean curvature of the surface Γ_t. Furthermore, the initial velocity field

$$v(\cdot, 0) = v_0 \text{ in } \Omega \tag{2}$$

is given.

Existence and uniqueness statements for the solution of (1), (2) in Sobolev spaces of noninteger order have been established by Solonnikov ([13]-[15]).

For our further investigations we will introduce dimensionless variables. According to [9] and [8], a characteristic length x_k is chosen as a scaling factor resembling the spatial extent of the drop. Rewritten in the new dimensionless variables

$$\tilde{x} = \frac{x}{x_k}, \; \tilde{t} = \frac{\gamma t}{x_k \nu}, \; \tilde{v} = \frac{\nu v}{\gamma}, \; \tilde{p} = \frac{x_k p}{\gamma},$$

the equations (1) take the form

$$\begin{array}{rcl} \mathrm{Re}\left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + (\tilde{v} \cdot \nabla)\tilde{v}\right) - \Delta\tilde{v} + \nabla\tilde{p} & = & 0 \\ \operatorname{div}\tilde{v} & = & 0 \end{array} \Bigg\} \text{ in } \Omega_{\tilde{t}} \\ \tilde{\mathcal{T}}(\tilde{v}, \tilde{p})\tilde{n}_{\tilde{t}} = \tilde{\kappa}_{\tilde{t}}\tilde{n}_{\tilde{t}} \quad \text{on } \Gamma_{\tilde{t}} = \partial\Omega_{\tilde{t}} \tag{3}$$

where $\tilde{\kappa}_{\tilde{t}}$ and $\tilde{n}_{\tilde{t}}$ are the curvature (in the sense described above) and the outer normal vector of $\Gamma_{\tilde{t}}$ in the dimensionless system and

$$\tilde{\mathcal{T}}(\tilde{v}, \tilde{p}) := \left(\nabla\tilde{v} + (\nabla\tilde{v})^T\right) - \tilde{p}\mathcal{I}.$$

The dimensionless constant

$$\mathrm{Re} = \frac{\varrho x_k \gamma}{\nu^2}$$

is the well-known Reynolds number which is also called Suratman number if the scaling factors are chosen as described above.

In the case of highly viscous liquids (and moderate values of the other constants involved) this number is very small. This case is realized, for instance, in the modelling of the so-called viscous sintering of glasses. There it makes sense to replace the equations (3) by the "quasistatic approximation"

$$\begin{array}{rcl} -\Delta v + \nabla p & = & 0 \\ \operatorname{div} v & = & 0 \end{array} \Bigg\} \text{ in } \Omega_t \\ \mathcal{T}(v, p)n_t = \kappa_t n_t \quad \text{on } \Gamma_t = \partial\Omega_t \tag{4}$$

(here and in the following, the tildes are omitted).

This change is motivated mainly by the easier numerical treatment of this approximation. It has to be pointed out, however, that in spite of a growing number of numerical realisations based on (4) (e.g. [12], [16]-[18]) only one analytic approach to the occuring moving boundary problem has been found by the author ([8], [6], [11]). It essentially uses methods of complex function theory and is therefore restricted to the two-dimensional case. In the following another approach will be presented that employs Lagrange coordinates.

Note that the replacement of Re << 1 by zero is decisively changing the structure of the whole problem. Contrary to (3), the equations (4) are *linear* in v. Moreover, for fixed t they form an *elliptic* system in which no time derivative occurs. Hence, for any time $t \geq 0$ the solution $(v(\cdot, t), p(\cdot, t))$ only depends on

Ω_t but not on the time evolution of the domain or the velocity and pressure field. As a consequence one obtains that no initial condition comparable to (2) belongs to our model because the initial domain is determining the velocity field at $t = 0$ (excepted some degrees of freedom, see below).

2 Exact problem formulation

For the complete formulation of the time-dependent problem it is convenient to employ Lagrange coordinates ξ. Because of their property to remain fixed in time for any given particle of the liquid it is natural to use the initial domain Ω_0 as coordinate domain. The change between the usual spatial (Euler) coordinates and the Lagrange coordinates is described by the initial value problem

$$\begin{aligned} x(\xi, 0) &= \xi \\ \dot{x}(\xi, t) &= v(x(\xi, t), t) \end{aligned}$$

for all $\xi \in \Omega_0$, where the dot denotes differentiation with respect to t. It is clear that the evolution of the domain and the occuring velocity and pressure fields have to be described simultaneously. This can be done in the following way:

Given a simply connected, bounded C^2-domain $\Omega_0 \subset \mathbb{R}^N$, $N = 2, 3$, and a "final time" $t_0 > 0$. We are looking for (sufficiently smooth) functions

$$\begin{aligned} U : \Omega_0 \times [0, t_0] &\longrightarrow \mathbb{R}^N \\ P : \Omega_0 \times [0, t_0] &\longrightarrow \mathbb{R} \end{aligned}$$

and for a time-dependent C^2-diffeomorphism

$$x : \Omega_0 \times [0, t_0] \longrightarrow \mathbb{R}^N$$

such that $x(\cdot, t)$ is *globally invertible* for all $t \in [0, t_0]$. Furthermore, we demand x to be differentiable with respect to t and

$$\begin{aligned} \dot{x}(\xi, t) &= U(\xi, t) \quad \forall \xi \in \Omega_0 \, \forall t \geq 0 \\ x(\xi, 0) &= \xi \qquad\quad\; \forall \xi \in \Omega_0 \end{aligned} \tag{5}$$

The properties of x enable us to define $\Omega_t := x[\Omega_0, t]$ and the functions

$$\begin{aligned} u_t(x) &= U(\xi(x, t), t) \quad \forall x \in \Omega_t \\ p_t(x) &= P(\xi(x, t), t) \quad \forall x \in \Omega_t \end{aligned} \tag{6}$$

where $\xi(\cdot, t)$ denotes the inverse of $x(\cdot, t)$. For these functions we demand (cp. (4))

$$\begin{aligned} \left.\begin{aligned} -\Delta u_t + \nabla p_t &= 0 \\ \operatorname{div} u_t &= 0 \end{aligned}\right\} &\text{ in } \Omega_t \\ \mathcal{T}(u_t, p_t) n_t = \kappa_t n_t \quad &\text{on } \Gamma_t = \partial\Omega_t. \end{aligned} \tag{7}$$

Finally, the frame that is given by the above definitions and equations (5), (6) will be used to express a well-known result from elementary differential geometry that plays a crucial role in the following considerations. It is essentially equivalent to to the familiar statements on the "first variation of surfaces" (see e.g. [1]).

Lemma 1 *Let*

$$A(t) := \mathrm{mes}\,\Gamma_t$$

be the $(N-1)$*-dimensional surface measure of* Γ_t. *Under the above presumptions, the function* A *is differentiable with respect to* t *and for her time derivative*

$$\dot{A}(t) = -\int_{\Gamma_t} \kappa_t n_t \cdot u_t \, d\Gamma \tag{8}$$

holds.

3 The fixed time problem

The basis for the treatment of the quasistatic problem stated in the preceding section is the discussion of the boundary value problem (7) for *fixed time*, i.e. we are looking for (sufficiently smooth) functions $u:\ \Omega \longrightarrow \mathbb{R}^N$, $p:\ \Omega \longrightarrow \mathbb{R}$ that fulfill

$$\begin{array}{rcl} \left.\begin{array}{rcl} -\Delta u + \nabla p & = & 0 \\ \operatorname{div} u & = & 0 \end{array}\right\} & \text{in } \Omega \\ \mathcal{T}(u,p)n = \kappa n & \text{on } \Gamma = \partial\Omega. \end{array} \tag{9}$$

where Ω is a simply connected, bounded C^2-domain and the meaning of $\mathcal{T}$, κ and n is the same as before.

In order to establish a weak formulation of it we employ the following integral formula which is sometimes called *Green formula*[1] for the Stokes equations.

Lemma 2 *Consider two vector fields* $u, v \in (H^1(\Omega))^N$ *with* $\operatorname{div} u = 0$ *and a scalar field* $p \in L^2(\Omega)$. *Then the identity*

$$\begin{aligned} \int_\Omega (\Delta u - \nabla p) v \, dx \quad &+ \quad \frac{1}{2} \sum_{i,j=1}^{N} \int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dx \\ &- \quad \int_\Omega p \ \operatorname{div} v \, dx = \int_\Gamma \mathcal{T}(u,p) n \cdot v \, d\Gamma \end{aligned}$$

holds.

Clearly, all differentiations are to be understood in the generalized sense. The proof which is mainly an application of the Gauss identity can be found in [10].

Hence, introducing the continuous bilinear and linear forms

$$\begin{array}{ll} a: & (H^1(\Omega))^N \times (H^1(\Omega))^N \longrightarrow \mathbb{R} \\ b: & (H^1(\Omega))^N \times L^2(\Omega) \longrightarrow \mathbb{R} \\ f: & (H^1(\Omega))^N \longrightarrow \mathbb{R} \end{array}$$

by the definitions

$$\begin{aligned} a(u,v) &= \frac{1}{2} \sum_{i,j=1}^{N} \int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dx \\ b(v,p) &= -\int_\Omega p \ \operatorname{div} v \, dx \\ f(v) &= \int_\Gamma \kappa n \cdot v \, d\Gamma \end{aligned} \tag{10}$$

[1] The name is chosen in order to emphasize the strong similarities between the Laplace and the Stokes equations, cp. [10].

the following weak formulation for (9) can be established:

$$\begin{aligned} a(u,v)+b(v,p) &= f(v) \quad \forall v \in (H^1(\Omega))^N \\ b(u,q) &= 0 \quad\quad \forall q \in L^2(\Omega) \end{aligned} \tag{11}$$

This is a so-called *mixed variational problem.* For the theory and the numerical treatment of those problems the author refers e.g.to [2] or [7]. In this paper some of the theoretical results will be used.

The space $(H^1(\Omega))^N$is too large for reaching uniqueness of the solution because (as it will be shown later) adding an arbitrary rigid body motion to the velocity component of a solution (u,p) of (11) yields another solution. This is in accordance with intuition because it is to be expected that rigid body motions of the drop are not essential for the description of its deformation by surface tension. The space $V_0 \subset (H^1(\Omega))^N$ of the velocity fields belonging to these motions can be described by

$$V_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \right\}$$

for $N = 2$ and

$$V_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ 0 \\ x_1 \end{bmatrix}, \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \right\}$$

for $N = 3$. It is easily checked that

$$a(u_0,v) = a(v,u_0) = b(u_0,q) = 0 \ \ \forall u_0 \in V_0, \ v \in (H^1(\Omega))^N, \ q \in L^2(\Omega). \tag{12}$$

Furthermore, if we suppose $u_t \in V_0$ for a certain t in (8) we get

$$-f(u_t) = \dot{A}(t) = 0$$

because obviously the measure of a surface is not changed by any rigid body motion. (A strict proof for this can be given by employing invariant differential operators on surfaces.) Hence,

$$f(v) = 0 \ \ \forall v \in V_0 \tag{13}$$

In order to exclude rigid body motions from the space of variations in (11) we use an algebraic decomposition of $(H^1(\Omega))^N$which is constructed by help of the linear projection operator

$$\Pi_0 : \ (H^1(\Omega))^N \longrightarrow V_0$$

which is defined by

$$\Pi_0(v) = \Pi_T(v) + \Pi_R(v) - \Pi_T(\Pi_R(v)) \tag{14}$$

with

$$\Pi_T(v) = \frac{1}{|\Omega|} \int_\Omega v \, dx \tag{15}$$

and

$$\Pi_R(v) = \begin{cases} \frac{1}{2|\Omega|}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \int_\Omega \operatorname{rot} v\,dx & \text{for } N=2 \\ \frac{1}{2|\Omega|}\begin{bmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{bmatrix} \int_\Omega \operatorname{rot} v\,dx & \text{for } N=3 \end{cases} \tag{16}$$

for any $v \in (H^1(\Omega))^N$ where $|\Omega|$ denotes the measure of Ω. One directly checks

$$\Pi_T^2 - \Pi_T = \Pi_R^2 - \Pi_R = \Pi_R\Pi_T = 0.$$

Applying these properties it is easy to show that Π_0 is indeed a linear projector onto V_0 and furthermore

$$V := \ker \Pi_0 = \left\{ v \in (H^1(\Omega))^N \mid \int_\Omega v\,dx = 0, \int_\Omega \operatorname{rot} v\,dx = 0 \right\}.$$

Thus we get the direct algebraic decomposition

$$(H^1(\Omega))^N = \ker \Pi_0 + \operatorname{Im} \Pi_0 = V + V_0. \tag{17}$$

Using the subspace V we can reformulate the the problem (11) in the following way:

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) \quad \forall v \in V \\ b(u,q) &= 0 \quad\quad \forall q \in L^2(\Omega) \end{aligned} \tag{18}$$

Taking into account (17), (12), and (13) one directly obtains the following result on the relationship between the mixed variational problems (11) and (18):

Lemma 3 *The problem (11) has a solution $(u,p) \in (H^1(\Omega))^N \times L^2(\Omega)$ if and only if (18) has a solution $(u,p) \in (H^1(\Omega))^N \times L^2(\Omega)$. In this case, the set of all solutions of (11) is given by*

$$\{(u+u_0,p) \in (H^1(\Omega))^N \times L^2(\Omega) \mid (u,p) \in V \times L^2(\Omega) \text{ solves (18)}, u_0 \in V_0\}$$

Therefore it is sufficient to study the solvability of the restricted problem (18). This can be done by help of the following lemmas:

Lemma 4 *The bilinear form $a(\cdot,\cdot)$ is V-elliptic, i.e. there is a constant $\alpha = \alpha(\Omega) > 0$ depending only on the domain such that*

$$a(v,v) \geq \alpha \|v\|_1 \;\; \forall v \in V$$

where $\|\cdot\|_1$ denotes the usual (product space) norm of $(H^1(\Omega))^N$.

Proof: The above inequality is a direct consequence of two important inequalities of mathematical physics:

1. *Poincarés inequality:* There is a constant $\alpha_1 > 0$ depending only on Ω such that

$$\int_\Omega |\nabla w|^2\,dx + \left(\int_\Omega w\,dx\right)^2 \geq \alpha_1 \|w\|^2_{H^1(\Omega)} \;\; \forall w \in H^1(\Omega) \tag{19}$$

For a proof see e.g. [3].

2. *Korns second inequality:* There is a constant $\alpha_2 > 0$ depending only on Ω such that

$$a(v,v) \geq \alpha_2 \sum_{i,j=1}^{N} \int_\Omega \left(\frac{\partial v_i}{\partial x_j}\right)^2 dx \tag{20}$$

holds for all $v \in (H^1(\Omega))^N$ that fulfill

$$\int_\Omega \operatorname{rot} v \, dx = 0.$$

A proof can be found e.g. in [4].

Applying (19) to every component of an arbitrary $v \in V$ yields

$$\sum_{i,j=1}^{N} \int_\Omega \left(\frac{\partial v_i}{\partial x_j}\right)^2 dx \geq \alpha_1,$$

the statement of the lemma follows from this together with (20). ∎

Before investigating the bilinear form $b(\cdot,\cdot)$ it is necessary to introduce the following regularity result from the theory of elliptic boundary value problems (BVP):

Lemma 5 *For any $q \in L^2(\Omega)$ the (uniquely existing) solution of the BVP*

$$\begin{aligned} \Delta\Phi &= q \quad \text{in } \Omega \\ \Phi &= 0 \quad \text{on } \Gamma \end{aligned} \tag{21}$$

belongs to $H^2(\Omega) \cap H_0^1$ and there is a constant C depending only on Ω such that $\|\Phi\|_2 \leq \|q\|_0$, where $\|\cdot\|_2$ and $\|\cdot\|_0$ denote the usual norms of $H^2(\Omega)$ and $L^2(\Omega)$, respectively.

For the proof (of a much more general result) see e.g. [5].

Lemma 6 *The bilinear form $b(\cdot,\cdot)$ defined in (10) satisfies a so-called LBB-condition (BB-condition, inf-sup-condition), i.e. there is a constant $\beta > 0$ depending only on Ω such that*

$$\sup_{v \in V \setminus \{0\}} \frac{b(v,q)}{\|v\|_1} \geq \beta \|q\|_0 \; \forall q \in L^2(\Omega).$$

Proof: The statement of the lemma is equivalent to the surjectivity of the divergence operator from V to $L^2(\Omega)$, i.e. we have to show that

$$\forall q \in L^2(\Omega) \, \exists v \in V \; : \; \operatorname{div} v = q$$

(see [2]). This can be done by considering (21) for arbitrary $q \in L^2(\Omega)$ and setting $v = \nabla\Phi$. Indeed, we find $\operatorname{div} v = q$ and $v \in V$ because of $v \in (H^1(\Omega))^N$, $\operatorname{rot} v = \operatorname{rot} \nabla\Phi = 0$ and

$$\int_\Omega v \, dx = \int_\Omega \nabla\Phi \, dx = \int_\Gamma \Phi n \, d\Gamma = 0. \quad ∎$$

The lemmas 4 and 6 ensure the crucial presumptions of the existence and uniqueness theorem for solutions of mixed variational problems. Hence we get:

Proposition 1 *The problem (18) has exactly one solution* $(u,p) \in V \times L^2(\Omega)$.

The set of all solutions of (11) is given by lemma 3.

For the following considerations it is convenient to use a modified formulation of (18) in which the pressure does not occur.

Lemma 7 *The velocity component u in the solution of (18) is the only solution of the variational problem*

$$a(u,v) = f(v) \ \forall v \in W \tag{22}$$

where W is a (closed) subspace of V defined by

$$W = \{v \in V \mid \operatorname{div} v = 0 \ \textit{in} \ \Omega.\} \tag{23}$$

For the proof we again refer to [2] or [7].

Finally, it has to be pointed out that similar results can be obtained using the method of hydrodynamic potentials in strict analogy to the treatment of the second BVP for the Laplace equation in potential theory. For the first BVP this can be found in [10].

4 Local aspects of the quasistatic problem

We turn back now to the investigation of the quasistatic problem as stated in section 2. The results of the previous chapter enable us to give the following supplement:

In order to exclude the rigid body motions from solutions of the quasistatic problem we demand additionally to (7)

$$\int_{\Omega_t} u_t \, dx = 0, \ \int_{\Omega_t} \operatorname{rot} u_t \, dx = 0 \tag{24}$$

in the same way as in the fixed time problem.

The question of existence and uniqueness of solutions (x, U, P) for the quasistatic problem (5), (6), (7), (24) for a given initial domain Ω_0 seems to be not solved at the moment. The work of Hopper [8] shows that (for $N = 2$) there are solutions that are global in time (i.e. $(x(\cdot,t), U(\cdot,t), P(\cdot,t))$ exists for any $t \geq 0$) for certain classes of initial domains. It is a very challenging but apparently highly complicated problem to find conditions on Ω_0 that ensure the existence of global solutions. Numerical calculations as well as intuition indicate that for sufficiently "ill-shaped domains" the diffeomorphism x loses its global injectivity at a certain time t. This has to be interpreted as "collision" of one part of the drop with another one.

In this section we deal only with the less complicated problem of the *local* existence of the solution of the quasistatic problem, i.e. our attention is restricted to a sufficiently small interval of time starting at 0.

4.1 Global invertibility of small deformations

As it was pointed out, it is essential that the diffeomorphism $x(\cdot,t)$ is globally invertible. The first aim of this section is to provide a result that ensures global injectivity for *small deformations*, i.e. for diffeomorphisms that are close to the identity in a suitable norm. For this we have to demand the following weak regularity presumption on the domain which is (as well as the following lemmas) for the sake of simplicity and generality formulated with respect to normed spaces:

Condition 1 *There is a constant* $\varepsilon_0 > 0$ *such that for all* ε *with* $0 < \varepsilon \leq \varepsilon_0$ *and all* $x_0 \in \Omega$, $x \in B(x_0, \varepsilon) \cap \Omega$ *there is a finite set of points*

$$\{z_1, ..., z_n\} \subset B(x_0, \varepsilon) \cap \Omega$$

for which with $z_0 := x_0$ *and* $z_{n+1} := x$ *the following statements hold:*

1. $\forall \lambda \in [0,1] \ \forall k \in \{0, ..., n\}: \ \lambda z_k + (1-\lambda) z_{k+1} \in \Omega$
2. *There is a constant* $\omega \in \mathbb{R}$ *depending only on* ε_0 *such that*

$$\sum_{k=0}^{n} \|z_{k+1} - z_k\| \leq \omega \|x - x_0\|$$

This condition is obviously fulfilled for domains that are not too "ill-shaped". It can be used to prove the following generalization of the mean value theorem:

Lemma 8 *Let* E,F *be real normed linear spaces and* $\Omega \subset E$ *a domain that fulfills condition 1,* $x_0 \in \Omega$ *arbitrary,* $\varepsilon \in (0, \varepsilon_0)$. *Consider a Frechet-differentiable mapping* $g: \ \Omega \longrightarrow E$ *with*

$$\|g'(x)\|_{L(E,F)} \leq G \ \ \forall x \in \Omega \cap B(x_0, \varepsilon)$$

Then

$$\|g(x_1) - g(x_0)\|_F \leq \omega G \|x_1 - x_0\|_E \ \ \forall x_1 \in \Omega \cap B(x_0, \varepsilon)$$

Proof: Applying the polygon draught given by condition 1 and the usual mean value theorem one obtains

$$\|g(z_{n+1}) - g(z_0)\| \leq \sum_{k=0}^{n} \|g(z_{k+1}) - g(z_k)\| \leq G \sum_{k=0}^{n} \|z_{k+1} - z_k\| \leq \omega G \|x_1 - x_0\|.$$

(The subscripts at the norms are omitted here and in the following because it is clear which norms have to be used.) ■

The next lemma can be understood as a sharpening of the well-known theorem on the existence of a local diffeomorphism. It gives a lower bound for the diameter of the neighbourhood in which this theorem ensures injectivity.

Lemma 9 *Let* E,F *be real normed linear spaces,* $\Omega \subset E$ *a domain that fulfills the regularity condition 1 and* $f: \ \Omega \longrightarrow F$ *be a Frechet-differentiable mapping into the normed space* F *such that*

1. $f'(x) \in L(E, F)$ *is invertible for all* $x \in \Omega$
2. *there is a* $\gamma \in \mathbb{R}$ *such that* $\|[f'(x)]^{-1}\|_{L(F,E)} \leq \gamma \ \ \forall x \in \Omega$
3. $f': \ \Omega \longrightarrow L(E, F)$ *is Lipschitz-continuous in* Ω *with the Lipschitz constant* M.

Then f is "uniformly locally injective", i.e. there is a positive constant ε_1 depending only on Ω, γ, and M such that

$$(f(x_1) = f(x_0) \;\wedge\; \|x_1 - x_0\| \leq \varepsilon_1) \;\Rightarrow\; x_1 = x_0 \;\; \forall x_0, x_1 \in \Omega$$

Proof: We will show that any $\varepsilon_1 < \min\{\varepsilon_0, \frac{1}{\gamma\omega M}\}$ fulfills the statement of the lemma. The idea of the proof is the same as in the usual proof of the theorem on local diffeomorphisms. Consider fixed $x_0, x_1 \in \Omega$ with $f(x_1) = f(x_0)$ and $\|x_1 - x_0\| \leq \varepsilon_1$. Because of the first presumption, for any $x \in \Omega$, the equation $f(x) = f(x_0) = y_0$ is equivalent to the fixed point equation

$$x = T(x) := x - f'(x_0)^{-1}[f(x) - y_0].$$

The operator $T: \;\Omega \longrightarrow E$ defined above is Frechet-differentiable in any $x \in \Omega$ and has the derivative

$$T'(x) = I - f'(x_0)^{-1} f'(x) = f'(x_0)^{-1}(f'(x_0) - f'(x)). \tag{25}$$

This yields

$$\|T'(x)\| \leq \gamma M \|x - x_0\| \leq \gamma M \varepsilon_1 \;\; \forall x \in \Omega \cap B(x_0, \varepsilon_1).$$

Because of $\varepsilon_1 < \varepsilon_0$ we can apply lemma 8 now and obtain

$$\|x_1 - x_0\| = \|T(x_1) - T(x_0)\| \leq q \|x_1 - x_0\|$$

with $q := \gamma M \varepsilon_1 \omega < 1$. Thus we find $x_1 = x_0$. ■

We are prepared now to prove the announced result on global injectivity of mappings $f \in (C^2(\bar{\Omega}))^N$ that are close to the identity in the C^2-norm.

Proposition 2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain that fulfills the regularity condition 1. Then there exists a constant $\varepsilon_2 > 0$ depending only on Ω and N such that for all $f \in (C^2(\bar{\Omega}))^N$ from*

$$\|f - \mathrm{id}_\Omega\|_{C^2} < \varepsilon_2 \tag{26}$$

follows the global injectivity of f on Ω. Here id_Ω denotes the identical mapping of Ω on itself.

Proof: Because of (26) we have

$$\|f'(x) - I\|_\infty < N\varepsilon_2 \;\; \forall x \in \Omega$$

where $\|\cdot\|_\infty$ denotes the row sum norm of the space $\mathbb{R}^{N,N}$ of the quadratic (N, N)-matrices. We recall from linear algebra that the condition

$$\|f'(x) - I\|_\infty < \varepsilon_3 \;\; \forall x \in \Omega$$

where ε_3 is a certain positive constant depending only on N ensures the assumptions 1. and 2. of lemma 9 with a γ independent of f. Furthermore, from (26) follows

$$\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right| < \varepsilon_2 \; \forall i, j = 1, \ldots, N \; \forall x \in \Omega.$$

By applying lemma 8 to $f'(x)$ we find that also the third assumption of lemma 9 is fulfilled with a constant M independent of f. Assume now the existence of two points $x_0, x_1 \in \Omega$ such that $x_0 \neq x_1$ and $f(x_0) = f(x_1)$. For $\varepsilon_2 < \frac{\varepsilon_3}{N}$, lemma 9 yields $||x_0 - x_1|| > \varepsilon_1$. On the other side, we have

$$\begin{aligned} ||x_0 - x_1|| &= ||x_0 - f(x_0) + f(x_1) - x_1|| \\ &\leq ||x_0 - f(x_0)|| + ||x_1 - f(x_1)|| = \\ &= ||[f - \mathrm{id}_\Omega](x_0)|| + ||[f - \mathrm{id}_\Omega](x_1)|| \leq 2\varepsilon_2. \end{aligned}$$

This is a contradiction if we choose any $\varepsilon_2 < \min\{\frac{\varepsilon_3}{N}, \frac{\varepsilon_1}{2}\}$. Hence, for any such $\varepsilon_2 > 0$ the statement of the proposition is true. ■

4.2 On local existence and uniqueness of solutions

In the second part of this section we will sketch a possible approach to a local existence and uniqueness theorem. It is strongly oriented on the ideas of the proof of the Picard-Lindelöf theorem on ordinary differential equations. The crucial point is an inequality describing the dependence of the solution of the fixed time problem on deformations of the domain. Though the author was only able to prove a weaker version of it, the basic ideas of the approach could be useful for similar considerations.

In order to investigate the above mentioned dependence, consider an arbitrary fixed element z of the set

$$B_{C^2}(\mathrm{id}_{\Omega_0}, \varepsilon) = \{z \in (C^2(\bar{\Omega}_0))^N \mid ||z - \mathrm{id}_{\Omega_0}||_{(C^2(\bar{\Omega}_0))^N} \leq \varepsilon\}$$

with a sufficiently small ε.[2] Note that $\varepsilon < \varepsilon_2(\Omega_0)$ and condition 1 ensure by proposition 2 that z is a globally invertible C^2-diffeomorphism. Therefore it makes sense to consider the problem (18) or (22), respectively, on the bounded C^2-domain $\Omega := \tilde{\Omega} = z[\Omega_0]$. The latter has the unique solution $\hat{u} \in (H^1(\tilde{\Omega}))^N$. The composition $\hat{u} \circ z \in (H^1(\Omega_0))^N$ depends only on z, therefore it makes sense to introduce the operator

$$U : B_{C^2}(\mathrm{id}_{\Omega_0}, \varepsilon) \longrightarrow (H^1(\tilde{\Omega}))^N$$

by

$$U[z] = \hat{u} \circ z$$

which bears the necessary information. It will be investigated in the next lemmas.

Lemma 10 *For sufficiently small ε, there is a constant C such that*

$$||U[z] - U[\mathrm{id}_{\Omega_0}]||_1 \leq C||z - \mathrm{id}_{\Omega_0}||_{C^2} \quad \forall z \in B_{C^2}(\mathrm{id}_{\Omega_0}, \varepsilon) \qquad (27)$$

holds where $||\cdot||_1$ and $||\cdot||_{C^2}$ denote the norms in the spaces $(H^1(\Omega_0))^N$ and $(C^2(\bar{\Omega}_0))^N$, respectively.

Proof:

[2]This means we demand $||z - \mathrm{id}||_{C^2}$ to be small enough to ensure certain presumptions that will arise in the following (sometimes without explicit statement).

Step 1: The vector fields $u := U[\mathrm{id}_{\Omega_0}]$ and $\tilde{u} := U[z]$ are the solutions of the variational equalities

$$a(u,v) = f(v) \;\; \forall v \in W \tag{28}$$

and

$$\tilde{a}(\tilde{u},\tilde{v}) = \tilde{f}(\tilde{v}) \;\; \forall \tilde{v} \in \tilde{W}, \tag{29}$$

respectively, where the definitions of a, f, and W are given in (10) and (23) (where Ω has to be substituted by Ω_0). The variational equation for $\tilde{u}$ is obtained by introducing new coordinates $\xi \in \Omega_0$ in the equation (22) for the domain $\tilde{\Omega} = z[\Omega_0]$. The coordinate transformation is given by $x = z(\xi)$. Thus we get

$$\begin{aligned}
\tilde{W} &= \Big\{ v \in (H^1(\Omega_0))^N \mid \int_{\Omega_0} v \det A \, d\xi = 0, \\
&\qquad \int_{\Omega_0} \mathrm{rot}_z v \det A \, d\xi = 0, \ \mathrm{div}_z v = 0 \Big\}, \\
\tilde{a}(u,v) &= \frac{1}{2} \sum_{i,j,k=1}^{N} \int_{\Omega_0} \left(a^{jk} \frac{\partial u_i}{\partial \xi_k} + a^{ik} \frac{\partial u_j}{\partial \xi_k} \right) \left(a^{jk} \frac{\partial v_i}{\partial \xi_k} + a^{ik} \frac{\partial v_j}{\partial \xi_k} \right) \det A \, d\xi, \\
\tilde{f}(v) &= \int_{\Gamma_0} \tilde{\kappa}(z(\xi)) \tilde{n}(z(\xi)) \cdot v \gamma \, d\Gamma_0
\end{aligned}$$

where $A = \frac{\partial z}{\partial \xi}$ is the Jacobian of the transformation z, ${A^{-1}}^T = (a^{ij})$, and γ is a real factor arising from the change of the "surface element" from $d\tilde{\Gamma}$ to $d\Gamma_0$. The symbols $\tilde{\kappa}$ and $\tilde{n}$ denote the curvature (see section 1) and the outer normal vector of $\tilde{\Gamma}$, respectively, rot_z and div_z are the differential operators rot and div with respect to the new coordinates ξ.

Step 2: Taking into account that the inversion of regular matrices is a locally Lipschitz continuous operation we obtain by direct estimates

$$\begin{aligned}
|\tilde{a}(u,v) - a(u,v)| &\leq C\|z - id_{\Omega_0}\|_{C^2}\|u\|_1\|v\|_1 \\
|\tilde{f}(v) - f(v)| &\leq C\|z - id_{\Omega_0}\|_{C^2}\|v\|_1
\end{aligned} \tag{30}$$

for all $u, v \in (H^1(\Omega_0))^N$ where the constant C is independent of u and v. (Here an in the following, for the sake of brevity we write C for all occuring constants if we are not interested in their actual value.) Note that because of the curvature term second derivatives occur in the derivation of the second inequality.

Step 3: In order to investigate the relationship between the spaces W and $\tilde{W}$ we choose an arbitrary element $\tilde{w} \in \tilde{W}$ and construct an approximating element $w \in W$ by the ansatz

$$w = \tilde{w} + \nabla\Phi + v_0 \tag{31}$$

where we demand $\Phi = 0$ on Γ_0 and $v_0 \in V_0$. Applying the divergence operator to (31) yields

$$-\Delta\Phi = \mathrm{div}\, \tilde{w} = \mathrm{div}\, \tilde{w} - \mathrm{div}_z \tilde{w}.$$

The function on the right side obviously belongs to $L^2(\Omega_0)$, and in a similar way as in step 2 we can show

$$\|\mathrm{div}\, \tilde{w} - \mathrm{div}_z \tilde{w}\|_0 \leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2}\|\tilde{w}\|_1.$$

Hence, from lemma 5 we get

$$\|\nabla\Phi\|_1 \leq \|\Phi\|_2 \leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2}\|\tilde{w}\|_1.$$

Furthermore, applying the projector Π_0 as defined in (14) –(16) (with Ω_0 instead of Ω) to (31) and taking into account that $\Pi_0(\nabla\Phi) = 0$ we find

$$\Pi_0 v_0 = v_0 = \Pi_0 \tilde{w}.$$

In order to estimate the norm of this expression we obtain from $\tilde{w} \in \tilde{W}$

$$\begin{aligned} \|\Pi_T \tilde{w}\|_1 &= \frac{1}{|\Omega_0|} \left\| \int_{\Omega_0} \tilde{w}\, d\xi \right\|_1 = \frac{1}{|\Omega_0|} \left\| \int_{\Omega_0} \tilde{w}(\det A - 1)\, d\xi \right\|_1 \\ &\leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2} \|\tilde{w}\|_1. \end{aligned}$$

After an analogous consideration for Π_R we find

$$\|v_0\|_1 \leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2}\|\tilde{w}\|_1.$$

Thus we obtained

$$\forall \tilde{w} \in \tilde{W}\ \exists w \in W: \ \|\tilde{w} - w\|_1 \leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2}\|\tilde{w}\|_1. \tag{32}$$

If the roles of Ω_0 and $\tilde{\Omega}$ are interchanged and z is replaced by z^{-1} we prove in the same way

$$\forall w \in W\ \exists \tilde{w} \in \tilde{W}: \ \|\tilde{w} - w\|_1 \leq C\|z^{-1} - \mathrm{id}_{\tilde{\Omega}}\|_{C^2}\|w\|_1. \tag{33}$$

(The norms refer here to the analogous spaces on $\tilde{\Omega}$.) Provided that $\|z - \mathrm{id}_{\Omega_0}\|_{C^2}$ is small it is elementary to show

$$\|z^{-1} - \mathrm{id}_{\tilde{\Omega}}\|_{C^2(\bar{\tilde{\Omega}})} \leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2(\bar{\Omega}_0)} \tag{34}$$

where C is independent of z. The crucial fact is again the local Lipschitz continuity of matrix inversion. Hence, from (33) and (34) we can conclude that

$$\forall \tilde{w} \in \tilde{W}\ \exists w \in W: \ \|\tilde{w} - w\|_1 \leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2}\|w\|_1. \tag{35}$$

Step 4: For the final estimate the *second lemma of Strang* will be used. It gives the following inequality on the solutions u and $\tilde{u}$ of (28) and (29), respectively:

$$\|u - \tilde{u}\|_1 \leq C\left(\inf_{v\in W} \|\tilde{u} - v\|_1 + \sup_{w\in W\setminus\{0\}} \frac{|a(\tilde{u}, w) - f(w)|}{\|w\|_1}\right) \tag{36}$$

where the constant C essentially depends on the ellipticity constant of the bilinear form a. For the first summand on the right side an estimate is given by (32). In order to obtain an analogous bound for the second one, for any $w \in W$ we choose a $\tilde{w} \in \tilde{W}$ according to (35) and estimate

$$\begin{aligned} |a(\tilde{u}, w) - f(w)| &\leq |a(\tilde{u}, w) - \tilde{a}(\tilde{u}, w)| + |\tilde{a}(\tilde{u}, w) - \tilde{a}(\tilde{u}, \tilde{w})| \\ &\quad + |\tilde{f}(\tilde{w}) - \tilde{f}(w)| + |\tilde{f}(w) - f(w)| \\ &\leq \|a - \tilde{a}\|\|\tilde{u}\|_1\|w\|_1 + \|\tilde{a}\|\|\tilde{u}\|_1\|w - \tilde{w}\|_1 \\ &\quad + \|\tilde{f}\|\|\tilde{w} - w\|_1 + \|\tilde{f} - f\|\|w\|_1 \\ &\leq C\|z - \mathrm{id}_{\Omega_0}\|_{C^2}\|\|w\|_1 \end{aligned}$$

where (30) and again (32) have been used. Application of (36) completes the proof. ■

Under a further presumption this lemma can be generalized. For this purpose we introduce the notation

$$\mathcal{U}_{C^2}(\Omega_0,\varepsilon) = \{\Omega \subset \mathbb{R}^N \mid \exists z \in B_{C^2}(\mathrm{id}_{\Omega_0},\varepsilon) \,:\, \Omega = z[\Omega_0]\}$$

for the set of all domains that can be obtained as results of "small deformations" of the initial domain Ω_0. If Ω_0 fulfills condition 1 and ε is small then proposition 2 ensures that all elements of this set are bounded, simply connected C^2-domains.

Lemma 11 *Suppose that there are constants $\varepsilon > 0$, $M \in \mathbb{R}$ such that for the constant $C = C(\Omega)$ in lemma 5 $C(\Omega) \leq M$ holds for all $\Omega \in \mathcal{U}_{C^2}(\Omega_0,\varepsilon)$. Then there is a $\delta > 0$ such that*

$$\|U[y] - U[x]\|_1 \leq C\|y - x\|_{C^2} \tag{37}$$

for all $x, y \in B_{C^2}(\mathrm{id}_{\Omega_0},\delta)$.

Proof: The inequality (37) is equivalent to (27) if Ω_0 is replaced by $x[\Omega_0]$ and z is replaced by $y \circ x^{-1}$. A similar reasoning as for (34) shows that if $x, y \in B_{C^2}(\mathrm{id}_{\Omega_0},\delta)$ then $y \circ x^{-1} \in B_{C^2}(\mathrm{id}_{x[\Omega_0]}, K\delta)$ with a certain constant K independent of x and y. Hence, we only have to verify that (27) holds *with the same constant* C for all domains $\Omega \in \mathcal{U}_{C^2}(\Omega_0,\delta)$ with a certain $\delta > 0$.

This can be done by making sure that that all occuring constants that depend on the domain have uniform upper bounds on $\mathcal{U}_{C^2}(\Omega_0,\delta)$. For the ellipticity constant of the bilinear form a this can be done using the first equation of (30), the existence of a uniform bound for the constant $C(\Omega)$ in lemma 5 is demanded as a presumption of the lemma. ■

In the following the inequality (37) will be replaced by a sharper one in order to obtain a local existence an uniqueness result for the solution of the quasistatic problem. This sharpened inequality that plays the same role as the Lipschitz condition in the Picard-Lindelöf theorem has to be demanded here without knowledge about its validity.

For the statement and the proof of the final proposition of this section the function space

$$C^2_{t_0}(\bar{\Omega}_0) = C([0,t_0] \to (C^2(\bar{\Omega}_0))^N) = \{x : \; [0,t_0] \longrightarrow (C^2(\bar{\Omega}_0))^N \text{ continuous}\}$$

is introduced which is a Banach space with respect to the norm

$$\|x\|_{C^2_{t_0}} = \max_{t\in[0,t_0]} \|x(t)\|_{C^2}.$$

Furthermore, we will use its closed subset

$$B_{C^2_{t_0}}(\mathrm{id}_{\Omega_0},\varepsilon) = \{x \in C^2_{t_0}(\bar{\Omega}_0) \mid x(\cdot,t) \in B_{C^2}(\mathrm{id}_{\Omega_0},\varepsilon) \; \forall t \in [0,t_0]\}.$$

Proposition 3 *Let $\Omega_0 \subset \mathbb{R}^N$ be a bounded, simply connected C^2-domain that fulfills condition 1 and suppose that there is a $L \in \mathbb{R}$ such that for all $x, y \in B_{C^2}(\mathrm{id}_{\Omega_0},\varepsilon^*)$ the statements*

$$U[x], U[y] \in (C^2(\bar{\Omega}_0))^N$$

and

$$\|U[x]-U[y]\|_{C^2} \le L\|x-y\|_{C^2} \tag{38}$$

hold. Then the problem (5), (6), (7), (24) has one and only one solution on a (sufficiently short) time interval $[0,t_0]$ *with* $x \in C^2_{t_0}(\bar{\Omega}_0)$.

Proof: It is sufficient to show the existence and uniqueness of the diffeomorphism $x(\cdot,\cdot)$ because it is clear from the quasistatic model and the discussion of the system (7) that $U(\cdot,\cdot)$ and $P(\cdot,\cdot)$ are existing and uniquely determined by a given $x(\cdot,\cdot)$.

In analogy to the proof of the Picard-Lindelöf theorem, the considered problem can be reformulated as Volterra integral equation for x in the following way:

$$x(\xi,t) = \xi + \int_0^t U[x(\cdot,\tau)](\xi)\,d\tau =: Tx(\xi,t) \tag{39}$$

with $\xi \in \Omega_0$, $t \in [0,t_0]$, $x \in B_{C^2_{t_0}}(\mathrm{id}_{\Omega_0},\varepsilon^*)$. From (38) follows by elementary integral estimates that T is a well-defined operator on the closed subset $x \in B_{C^2_{t_0}}(\mathrm{id}_{\Omega_0},\varepsilon^*)$ into itself which is contractive, i.e.

$$\|Tx-Ty\|_{C^2_{t_0}} \le q\|x-y\|_{C^2_{t_0}} \quad \forall x,y \in B_{C^2_{t_0}}(\mathrm{id}_{\Omega_0},\varepsilon^*)$$

with $q<1$. Thus, application of the Banach fixed point theorem to T on $B_{C^2_{t_0}}(\mathrm{id}_{\Omega_0},\varepsilon^*)$ yields the existence of one and only one solution $x \in B_{C^2_{t_0}}(\mathrm{id}_{\Omega_0},\varepsilon^*)$ of (39) and hence of the original quasistatic problem. ■

5 Asymptotic behaviour of global solutions

Though global existence of a solution (x,U,P) for the quasistatic problem (5), (6), (7), (24) cannot be expected for general domains, the question is of interest which properties of the solution can be proved under the presumption that the solution does exist on a given time interval for a certain Ω_0, especially with regard to limit properties for $t \to \infty$. The special structure of the problem enables us to find such properties under some weak presumptions. These properties are in accordance with the expectations from physical reasonings.

At first two inequalities on the solution of an abstract (standard) variational problem are proved.

Lemma 12 *Let the bilinear form* $a: W\times W \longrightarrow \mathbb{R}$ *be* W*-elliptic and symmetric and let* f *be a bounded linear functional on* W*. Then for the solution* u *of the variational problem*

$$a(u,v) = f(v) \;\; \forall v \in W \tag{40}$$

the inequalities

$$\begin{aligned} \|f\|^2 &\le \|a\|f(u), \\ \|u\|^2 &\le \frac{1}{\alpha^2}\|a\|f(u) \end{aligned} \tag{41}$$

hold, where α *is the ellipticity constant of* a *and the norms of* a *and* f *are the usual norms in the dual spaces* W' *and* $(W\times W)'$.

Proof: It is well-known in the theory of variational equations that if a is *symmetric* then u is the solution of the minimization problem

$$\frac{1}{2}a(v,v) - f(v) \longrightarrow \min, \quad v \in W.$$

This means

$$\frac{1}{2}a(u,u) - f(u) = -\frac{1}{2}f(u) \leq \frac{1}{2}a(v,v) - f(v) \quad \forall v \in W. \tag{42}$$

A well-known consequence of the Hahn-Banach theorem ensures the existence of a v^* such that $f(v^*) = ||f||$, $||v^*|| = 1$. Setting $v = \frac{||f||}{||a||}v^*$ in (42) we obtain

$$-\frac{1}{2}f(u) \leq \frac{||f||^2}{2||a||} - \frac{||f||^2}{||a||} = -\frac{1}{2}\frac{||f||^2}{||a||}.$$

The first inequality in (41) directly follows from this. The second one is an immediate consequence of the first one and the usual estimate $||u|| \leq \frac{1}{\alpha}||f||$. ■

For the following more specialized considerations let Ω be a bounded, simply connected C^2-domain with boundary Γ again. We will need the following regularity result comparable to lemma 5:

Lemma 13 *Consider the BVP*

$$\begin{aligned} \Delta\Phi &= 0 \quad \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} &= \varphi \quad \text{on } \Gamma \end{aligned}$$

with $\int_\Gamma \varphi \, d\Gamma = 0$, $\varphi \in H^{\frac{1}{2}}(\Gamma)$. *Let* L *denote the set of its solutions. Then* $L \neq \emptyset$, $L \subset H^2(\Omega)$ *and*

$$\inf_{\Phi \in L} ||\Phi||_2 \leq C_2 ||\varphi||_{\frac{1}{2}}$$

where the constant C_2 *depends only on* Ω *and* $||\cdot||_2$ *denotes the usual norm on* $H^2(\Omega)$.

For the proof as well as for the definition and properties of the space $H^{\frac{1}{2}}(\Gamma)$ the reader is referred to e.g. [5] again. It has only to be remarked that we will use the norm $||\cdot||_{\frac{1}{2}}$ defined by

$$||\varphi||_{\frac{1}{2}} = \inf_{\substack{v \in H^1(\Omega) \\ T_\Gamma v = \varphi}} ||v||_{H^1}$$

for any $\varphi \in H^{\frac{1}{2}}(\Gamma)$ where T_Γ denotes the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Gamma)$.

By help of this regularity result the next lemma can be proved. It will be used in the following to obtain information on the shape of the domain for $t \to \infty$.

Lemma 14 *Consider the linear subspace* $W \subset (H^1(\Omega))^N$ *as defined in (23) and the linear form* f *on* $(H^1(\Omega))^N$ *defined by*

$$f(v) = \int_\Gamma \mu n \cdot v \, d\Gamma$$

with n denoting the outer normal vector on Ω and $\mu \in H^{-\frac{1}{2}}(\Gamma)$ such that f vanishes on the subspace of all constant functions $c \in (H^1(\Omega))^N$. Then the inequality

$$||\mu - \bar{\mu}||_{-\frac{1}{2}} \le C||f||_{W'}$$

holds where

$$\bar{\mu} = \frac{\int_\Gamma \mu \, d\Gamma}{\int_\Gamma d\Gamma}$$

has to be interpreted as a constant function on Γ and the constant C depends only on Ω.

Proof: From the Hahn-Banach theorem we obtain the existence of a $\varphi \in H^{\frac{1}{2}}(\Gamma)$ such that $||\varphi||_{\frac{1}{2}} = 1$ and

$$\int_\Gamma (\mu - \bar{\mu})\varphi \, d\Gamma = ||\mu - \bar{\mu}||_{-\frac{1}{2}}.$$

Furthermore, we define

$$\bar{\varphi} = \frac{\int_\Gamma \varphi \, d\Gamma}{\int_\Gamma d\Gamma}.$$

From this definition by the Schwarz inequality and the continuous embedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Omega)$ follows

$$\bar{\varphi} \le ||\varphi||_0 \le C_1 ||\varphi||_{\frac{1}{2}} = C_1.$$

where C_1 is the embedding constant of the above mentioned embedding. Because $\bar{\varphi}$ is a constant we get the estimates [3]

$$||\bar{\varphi}||_{\frac{1}{2}} = \inf_{\substack{v \in H^1(\Omega) \\ T_\Gamma v = \varphi}} ||v||_{H^1} \le ||\bar{\varphi}||_{H^1} = ||\bar{\varphi}||_0 = \bar{\varphi}\sqrt{|\Omega|} \le C.$$

Hence we have $||\varphi - \bar{\varphi}||_{\frac{1}{2}} \le ||\varphi||_{\frac{1}{2}} + ||\bar{\varphi}||_{\frac{1}{2}} \le C$. Consider now the elliptic BVP

$$\begin{aligned} \Delta\Phi &= 0 && \text{in } \Omega \\ \frac{\partial \Phi}{\partial n} &= \varphi - \bar{\varphi} && \text{on } \Gamma. \end{aligned} \tag{43}$$

It is solvable because of

$$\int_\Gamma (\varphi - \bar{\varphi}) \, d\Gamma = 0,$$

and the solution Φ is defined up to a constant. Hence, the function $\tilde{v} = \nabla\Phi$ is well-defined. Taking into account lemma 13 we find $\tilde{v} \in (H^1(\Omega))^N$ and

$$||v||_1 \le \inf_{c \in \mathbb{R}} ||\Phi_0 + c||_2 \le C_1 ||\varphi - \bar{\varphi}||_{\frac{1}{2}} \le C$$

where Φ_0 is an arbitrary solution of (43).

[3] Note that $\bar{\varphi}$ is to be understood either as a real number or as a constant function on Γ or Ω. It is always obvious from the context which interpretation has to be used.

Define now $v = \tilde{v} - \Pi_T \tilde{v} = (I - \Pi_T)\tilde{v}$ (see eq. (15)). Because of

$$\begin{aligned} \operatorname{div} v &= \operatorname{div} \tilde{v} = \Delta \Phi = 0, \\ \operatorname{rot} v &= \operatorname{rot} \tilde{v} = \operatorname{rot} \nabla \Phi = 0, \\ \Pi_T v &= (\Pi_T - \Pi_T^2)\tilde{v} = 0 \end{aligned}$$

we obtain $v \in W$. Taking into account that $\ker \Pi_T \perp \operatorname{Im} \Pi_T$, i.e. Π_T is an orthogonal projector in $(H^1(\Omega))^N$, we can conclude

$$\|v\|_1 = \|(I - \Pi_T)\tilde{v}\|_1 \le \|\tilde{v}\|_1 \le C.$$

Finally, from the fact that the function $v - \tilde{v} = \Pi_T \tilde{v}$ is constant on Ω and therefore $f(v) = f(\tilde{v})$, one gets

$$\begin{aligned} \|\mu - \bar{\mu}\|_{-\frac{1}{2}} &= \int_\Gamma (\mu - \bar{\mu})\varphi \, d\Gamma = \int_\Gamma (\mu - \bar{\mu})(\varphi - \bar{\varphi}) \, d\Gamma \\ &= \int_\Gamma (\mu - \bar{\mu}) n \cdot \tilde{v} \, d\Gamma = f(\tilde{v}) - \bar{\mu} \int_\Gamma n \cdot \tilde{v} \, d\Gamma \\ &= f(v) - \bar{\mu} \int_\Gamma \operatorname{div} \tilde{v} \, d\Gamma = f(v) \\ &\le \|f\|_{W'} \|v\|_1 \le C \|f\|_{W'}. \ \blacksquare \end{aligned}$$

On the basis of these lemmas it is possible to obtain the following statements on the behaviour of global solutions of the quasistatic problem (5), (6), (7), (24). As it was pointed out in section 3, we can characterize the function $u_t \in (H^1(\Omega_t))^N$ as solution of the variational problem

$$a_t(u_t, v) = f_t(v) \quad \forall v \in W_t$$

where a_t, f_t, and W_t are defined according to (10) and (23) if Ω is replaced by Ω_t. Consider now the function A as defined in lemma 1. Because of this lemma and lemma 12 we have

$$\dot{A}(t) = -\int_{\Gamma_t} \kappa_t n_t \cdot u_t \, d\Gamma_t = -f_t(u_t) \le 0$$

for all $t \ge 0$, hence A is a monotonously decreasing function that is obviously bounded from below. Thus $\dot{A}(t) \to 0$ and therefore

$$f_t(u_t) \to 0 \text{ as } t \to \infty. \tag{44}$$

Moreover, from $\dot{A}(t_0) = 0$ for a certain $t_0 \ge 0$ and the last inequality in lemma 12 we can conclude $u_{t_0} = 0$ and hence Ω_{t_0} is a (N-dimensional) ball, thus we find $u_t = 0$ and $\Omega_t = \Omega_{t_0}$ for all $t \ge t_0$. From this consideration follows for example that no (nontrivial) "periodic" solutions of the quasistatic problem exist.

Finally, the inequalities in the lemmas 12 and 14 give the possibility to obtain from (44) statements on the asymptotic behaviour of the velocity field and the shape of the domain. For this purpose we define the time-dependent "integral mean value"

$$\bar{\kappa}(t) = \frac{\int_{\Gamma_t} \kappa_t \, d\Gamma_t}{\int_{\Gamma_t} d\Gamma_t}$$

of the (double mean) curvature at time t.

Proposition 4

1. *Suppose there is a constant* $\alpha^* > 0$ *such that for the ellipticity constants* $\alpha = \alpha(\Omega_t)$ *of* a_t *(see lemma 4) holds* $\alpha(\Omega_t) \geq \alpha^*$ *for all* $t \geq 0$*. Then*

$$||u_t||_{(H^1(\Omega_t))^N} \to 0 \text{ as } t \to \infty.$$

2. *Suppose there are constants* $M_1, M_2 \in I\!R$ *such that* $C_1(\Omega_t) \leq M_1$ *and* $C_2(\Omega_t) \leq M_2$ *for all* $t \geq 0$ *where* C_1 *is the embedding constant occuring in the proof of lemma 14 and* C_2 *is the domain-dependent constant in lemma 13. Then*

$$||\kappa_t - \bar{\kappa}(t)||_{H^{-\frac{1}{2}}(\Gamma_t)} \to 0 \text{ as } t \to \infty. \quad (45)$$

Proof: The limit relations immediately follow from (44) and the lemmas 12 for the first part and 14 for the second part of the theorem by taking into account that the estimates in these lemmas hold *uniformly* in time because of the given assumptions. Note that the applicability of lemma 14 to the linear form f_t is ensured by (12). ■

The statement in this proposition can also be expressed as follows: If the described limit relations do not hold, then at least one of the constants $1/\alpha$, C_1, and C_2 has to tend to infinity which would indicate that the domain in some sense becomes more and more "irregular".

For the interpretation of the limit relation (45) it has to be pointed out that the only bounded, simply connected C^2-domain for which the (mean) curvature is constant on the boundary is the (N-dimensional) ball (see e.g. [1]). This allows us to interprete (45) in the way that Ω_t approaches a ball in a weak sense.

It is quite satisfying that the limit relations given in the above proposition are in accordance with physical reasonings: From considerations on the energy that is contained in a drop of viscous liquid under the influence of surface tension it is to be expected that this drop will approach a ball in which the fluid is resting. Hence, the quasistatic model qualitatively resembles the asymptotic properties of the physical problem.

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