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# ON THE IDENTIFICATION OF CONTINUOUS LINEAR PROCESSES

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Abstract

The estimation of parameters of continuous, linear and time-invariant processes is studied. It is assumed that the signals entering these processes are band limited. Sample values of the input and the disturbed output of the process are available. Two approaches are discussed: the use of a discrete time model of the continuous process and the more direct methods based on the derivatives or the spectra of the signals.
The properties of this discrete time model have been emphasized. For the more direct approaches, estimation schemes are developed, based on the instrumental variable technique.
Experimental results of simulated and practical continuous processes are discussed.

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# 1. Introduction

As part of the research program of the group Measurement and Control, Department of Electrical Engineering, Eindhoven University of Technology, techniques are developed to identify unknown processes.

In this report the identification is studied of continuous linear and time-invariant processes, using band-limited input signals. Though in general not necessary, it is also assumed that these signals are periodical.

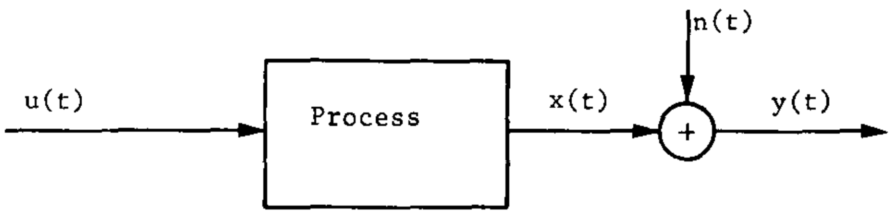



Fig. 1.1

It is assumed that the process of interest (fig. 1.1.) is completely described by a differential equation of known order. This differential equation is given by:

$$x(t) + a_1 \frac{dx(t)}{dt} + \ldots + a_q \frac{d^q x(t)}{dt^q} = b_0 u(t) + b_1 \frac{du(t)}{dt} + \ldots + b_p \frac{d^p u(t)}{dt^p} \quad (1.1)$$

with:

$u(t)$ = the input signal

$x(t)$ = the undisturbed output signal

$a_i$; $i = 1, \ldots, q$ and $b_j$; $j = 0, \ldots, p$ are the parameters of the process.

It will be assumed that the input signal and the disturbed output signal are available. The disturbed output signal is given by:

$$y(t) = x(t) + n(t) \quad (1.2)$$

with:

$n(t)$ = the additive noise.

As a result of our assumptions the identification problem can be reduced to the estimation of the coefficients of the differential equation (1.1), called the parameters of the process. Throughout the report this will be done using techniques which are linear in the parameters. However due to the additive noise at the output of the process, this will give in general biased results [Eykhoff, 1974] .
This bias is considered here of main concern and therefore techniques will be developed yielding bias-free parameters.

As in the discrete time case the bias problem is well understood and bias-free estimation techniques are available [Talmon, 1971] , [Smets, 1970] , we shall transform the differential equation (1.1) into a finite difference equation (chapter 2). In contrast to the known techniques using a data hold circuit [Smith, 1968] , [Sinha, 1972] , this will be carried out using the band-limited properties of the signals. It will be proven that the so-called bilinear z-transform will be very useful for this purpose. We shall give restrictions for its use and discuss the error involved by it.
On the other hand it is straightforward to develop least squares estimators for the assumed process, yielding directly the parameters of the continuous process. This can be done either by generating the derivatives from the signals [Vlek, 1973] or by spectral analysis of these signals [Shinbrot, 1954]
The additive noise, however, will give a bias to the estimated parameters. In contrast to the discrete time case it is yet not known which noise process will give a bias-free least squares estimator. To overcome this problem, in chapter 3 simple estimators, based on the instrumental variable technique [Wong, 1966] , [Smith, 1972] , are developed.

Finally some practical results are presented in chapter 4 concerning a simple model of a biological process.

## 2. On discrete time approach to the identification of continuous linear processes.

### 2.1. Discrete time model for the continuous process.

In this section we will establish relationships between the continuous process obeying the differential equation (1.1) and a discrete time model of this process working on the sample values of the input and output signals only (fig. 2.1).

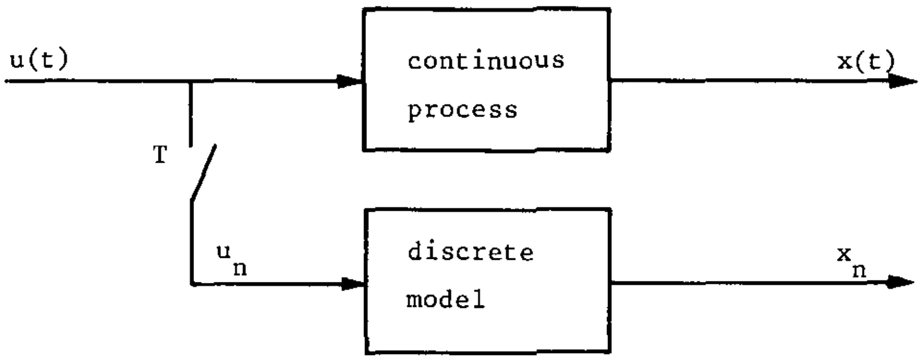



Fig. 2.1.

The discrete time model will be denoted by a difference equation of the following type:

$$x_n + \alpha_1 z x_n + \alpha_2 z^2 x_n + \dots + \alpha_g z^g x_n = \beta_o u_n + \beta_1 z u_n + \dots + \beta_f z^f u_n; \quad -\infty < n < \infty \tag{2.1}$$

with:

$u_n = u(nT)$, the sampled input sequence. T is the sampling time interval

$x_n$ = the sampled output sequence

$\alpha_i$; i =1, ..., g and $\beta_i$; j = 1, ..., f are the parameters of the discrete model

z = the shift operator; defined by: $z^k x_n = x_{n+k}$.

It is assumed here that the model given by this equation describes the process output completely, i.e. the output sequence of the discrete model is equal to the sample values of the process output.

A possible way to investigate the properties of equation (2.1) can be based on the interpolation polynomial through the sampling points of x(t) (fig. 2.2).

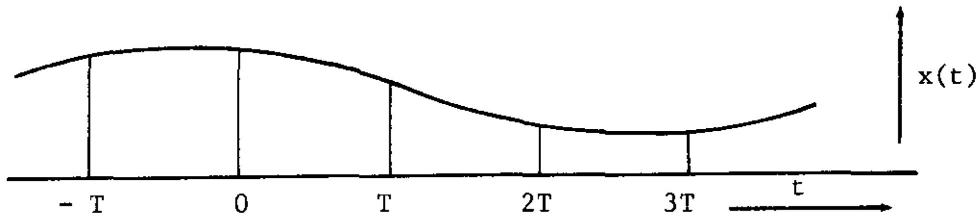



Fig. 2.2.

It will be assumed that this interpolation polynomial reconstructs the function x(t) without error, which will be the case for x(t) belonging to the class of polynomials but also for other functions which have a restricted variation between the sampling points. With this assumption x(t) can be written in a form originating from Newton (Appendix I):

$$x(t) = x_o + \binom{\tau}{1} (z-1)x_o + \binom{\tau}{2} (z-1)^2 x_o + \ldots + \binom{\tau}{k} (z-1)^k x_o + \ldots \qquad (2.2)$$

with:

$$x_o = x(t_o)$$

$$t = t_o + \tau T$$

Truncation of the righthand part of this equation after k terms yields an interpolation polynomial of order k, based on k + 1 sampling points.

The derivatives of eq. (2.2) are given by:

$$\frac{d^n x(t)}{dt^n} = \frac{d^n x(t)}{d\tau^n} \cdot \left(\frac{d\tau}{dt}\right)^n =$$

$$\left\{ \frac{d^n}{d\tau^n} \binom{\tau}{1} (z-1) + \ldots \frac{d^n}{d\tau^n} \binom{\tau}{k} (z-1)^k + \ldots \right\} \frac{x_o}{T^n} \qquad (2.3)$$

The derivatives of the binomial coefficients can be calculated from their generating function:

$$z^\tau = \sum_{k=o}^{\infty} \binom{\tau}{k} (z-1)^k \quad ; \quad |z-1| < 1.$$

from which it follows:

$$\frac{d^n z^\tau}{d\tau^n} = z^\tau (\log z)^n = \sum_{k=o}^{\infty} \frac{d^n}{d\tau^n} \binom{\tau}{k} (z-1)^k$$

Comparing this result with eq. (2.3) one finds for the point $\tau = 0$:

$$\frac{d^n x_o}{dt^n} = \frac{1}{T^n} (\log z)^n \{ x_o \} \tag{2.4}$$

It should be noted that this formula is only a shorthand notation of eq. (2.3) indicating that the coefficients of this expression, operating on the sequence $x_n$, are the same as those of the Taylor expansion of the $n^{th}$ power of log z around the point $z = 1$.

Also of interest is the inversion of eq. (2.4). This can be obtained by virtue of Taylor's theorem. Let x(t) be analytic at t, then:

$$x(t+nT) = x(t) + nTx^{(1)}(t) + \ldots + \frac{(nT)^k}{k!} x^{(k)}(t) + \ldots \tag{2.5}$$

which can be written as:

$$z^n x(t) = \{1 + nT\frac{d}{dt} + \ldots + \frac{(nT)^k}{k!} \frac{d^k}{dt^k} + \ldots \} x(t)$$

Comparing this with the Taylor expansion of the exponential function, it follows:

$$z^n x(t) = e^{nT \frac{d}{dt}} \{ x(t) \} \tag{2.6}$$

Which formula should be interpreted as a shorthand notation of eq. (2.5).

The results given by eq. (2.4) and (2.6) can be directly applied to the identification of the discrete time model by interpreting the relations as identities between operators. At this point, however, we first derive constraints for their validity. This can be easily done by transforming the problem to the s-plane. For this purpose we will assume in the sequel that the input signal of the process is band-limited and periodical, i.e. it can be represented by a Fourier series:

$$u(t) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k e^{jk\omega_o t} \tag{2.7}$$

with:

$N$ = the number of sampling points in one period

$\omega_o = \frac{2\pi}{NT}$

As u(t) is a real time function, it yields:

$$c_{-k} = c_k^*$$

The highest frequency of the signal u(t) is given by:

$$\omega_h = \frac{\pi}{T} = \frac{\omega_s}{2}$$

where $\omega_s$ stands for the sampling frequency. The signal obeys also Shannon's criterion and is uniquely determined by its values at the sampling points. Transforming the differential equation (1.1) to the s-plane, ignoring the initial conditions which is permitted for the considered signals, gives:

$$H(s) = \frac{X(s)}{U(s)} = \frac{b_o + b_1 s + \dots + b_p s^p}{1 + a_1 s + \dots + a_q s^q} \tag{2.8}$$

where U(s) and X(s) are the transforms of u(t) and x(t) respectively.

It is well known that the output of the process with the continuous signal of eq. (2.7) as input, is given by:

$$x(t) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} d_k e^{jk\omega_o t} \quad ; \text{ with } d_k = c_k H(jk\omega_o) \tag{2.9}$$

Relations of the same kind can be derived for the output of the discrete model (fig. 2.1). Therefore we shall calculate the stationary output of this discrete model. This can easily be done by making use of the so-called z-transform [Jury, 1964]. As the discrete model is a linear system, it suffices to calculate the stationary output for one of the components of the sampled input signal u(nT), also from eq. (2.7):

$$u_k(nT) = c_k e^{jk\omega_o nT}$$

where $u_k(nT)$ stands for the $k^{th}$ component of the input signal. In the application of the z-transform we shall truncate this sequence to $n \geq 0$, which will be allowed as only the output for large values of n is of interest. Now it follows, with $U_k(z)$ denoting the z-transform of $u_k(nT)$:

$$U_k(z) = Z\{u_k(nT)\} = \sum_{n=o}^{\infty} u_k(nT) z^{-n} = \sum_{n=o}^{\infty} c_k e^{jk\omega_o nT} z^{-n}$$

$$= c_k \{ 1 + e^{jk\omega_o T} . z^{-1} + e^{jk\omega_o 2T} . z^{-2} + \dots \}$$

So:

$$U_k(z) = \frac{c_k z}{z - e^{jk\omega_o T}} \qquad (2.10)$$

Let G(z) be the transfer function of the assumed discrete model, cf. eq. (2.1), its output is given by:

$$X_k(z) = G(z)U_k(z)$$

where $X_k(z)$ denotes the z-transform of the output sequence and G(z) is given by:

$$G(z) = \frac{\beta_o + \beta_1 z + \dots + \beta_f z^f}{1 + \alpha_1 z + \dots + \alpha_g z^g}$$

Assuming for simplicity only, $f < g$ and G(z) containing only simple poles, then $X_k(z)$ can be calculated from the partial fraction expansion:

$$X_k(z) = \frac{zr_1}{z - p_1} + \frac{zr_2}{z - p_2} + \dots + \frac{zr_g}{z - p_g} + \frac{ze_k}{z - e^{jk\omega_o T}}$$

with:

$p_j$ ; $j = 1, \ldots, g$ are the poles of $G(z)$

$$r_j = \lim_{z \to p_j} (z-p_j)G(z)U_k(z)z^{-1}$$

$$e_k = \lim_{z \to e^{jk\omega_o T}} (z-e^{jk\omega_o T})\, G(z)U_k(z)z^{-1}$$

Application of the inverse z-transform to the first g terms yields:

$$Z^{-1}\left[\frac{zr_1}{z-p_1} + \frac{zr_2}{z-p_2} + \ldots + \frac{zr_g}{z-p_g}\right] = r_1 p_1{}^n + r_2 p_2{}^n + \ldots + r_g p_g{}^n$$

The right hand part of this expression is the transient response of the discrete model. Assuming that $G(z)$ is a stable system, it holds:

$$|p_j| < 1$$

and consequently all terms of the transient response can be neglected by choosing n large enough. The steady state output then becomes:

$$X_k(z) = \frac{ze_k}{z - e^{jk\omega_o T}}$$

with:

$$e_k = \lim_{z \to e^{jk\omega_o T}} (z-e^{jk\omega_o T})G(z)U_k(z)z^{-1} = c_k G(e^{jk\omega_o T})$$

It will now be clear by comparison of this equation with the z-transform of the sample values of the output of the continuous process of which the $k^{th}$ component is given by (cf. eq.(2.9)):

$$X_k(z) = \frac{d_k z}{z - e^{jk\omega_o T}}$$

that necessary for both representing the same output, it should hold

$e_k = d_k$, for all k.

The z-transforms of the steady-state, sample values of the input and output signals, are therefore given by:

$$U(z) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{c_k z}{z - e^{jk\omega_o T}} \tag{2.11}$$

and:

$$X(z) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{d_k z}{z - e^{jk\omega_o T}} \; ; \quad \text{with } d_k = c_k G(z=e^{jk\omega_o T}) \tag{2.12}$$

Extensions of our results can be obtained by letting the number of samples N in the signal description (eq. (2.7)) go to infinity, which permits the treatment of the general class of band limited signals. The discrete variable $jk\omega_o$ can then be replaced by the continuous variable $j\omega$, $\frac{-\pi}{T} < \omega < \frac{\pi}{T}$. Then it follows from inspection of eq. (2.9) and eq. (2.12) that the transfer functions H(s) and G(z) are related to each other by the transform:

$$z = e^{sT} \; ; \quad \text{with } s = j\omega; \quad \frac{-\pi}{T} < \omega < \frac{\pi}{T} \tag{2.13}$$

(This tranform should be well distinguished from the z-transform used before. The z-transform is applicable toa continuous process of which the input consists of a sequence of delta functions; the spectrum of such input signals reaches to infinity. The main difference will then also be the restriction of the transform of eq. (2.13) to a finite interval of the frequency axis of the s-plane. The advantages of such a restriction are important: the transform of eq. (2.13) has an inverse, whereas the z-transform, applied to continuous systems, has no (unique)inverse).

The transform of eq. (2.13) respresents a mapping of the s-plane into the z-plane (fig. 2.3). This mapping has an unique inverse:

$$s = \frac{1}{T} \log z \tag{2.14}$$

where the principal value of the logarithm should be taken.

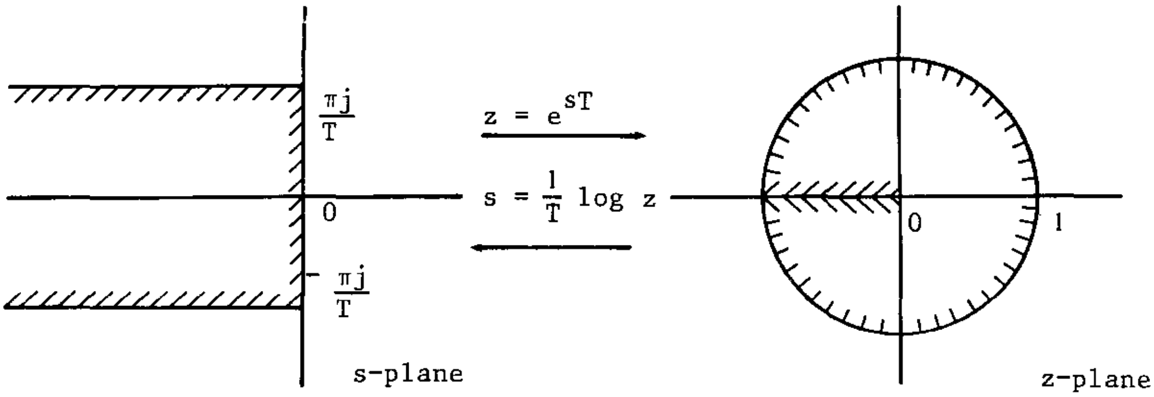



Fig. 2.3.

Eq. (2.13) and (2.14) are formally the same as eq. (2.4) and (2.6) respectively. The first ones however are somewhat easier to handle as they state a mapping of the s-plane into the z-plane, whereas the second ones yield relationships between operators.

In Appendix II a derivation of eq. (2.13) and eq. (2.14) is given, directly based on the definition integrals of the z-transform and the Laplace-transform respectively.

By making use of eq. (2.14), the discrete transferfunction can be directly obtained from its continuous counterpart (eq. (2.8)):

$$G(z) = \frac{b_o + b_1 \frac{1}{T} \log z + \ldots + b_p (\frac{1}{T} \log z)^p}{1 + a_1 \frac{1}{T} \log z + \ldots + a_q (\frac{1}{T} \log z)^q} \qquad (2.15)$$

## 2.2. The inversion problem

Regarding the transfer function of the discrete model, given by eq. (2.15), as a difference equation, it is clear, by series expansion of the logarithm, that this difference equation is of infinite order. The least squares estimators available for the identification of difference equations [Talmon, 1971] can not handle this case. Therefore the transfer function will be truncated to some finite number of terms, cf. the assumed form given by eq. (2.1). By this truncation an apprcximation of the discrete process as defined by eq. (2.15) is obtained:

$$G^*(z) = \frac{\beta_o + \beta_1 z + \ldots + \beta_f z^f}{1 + \alpha_1 z + \ldots + \alpha_g s^g} \qquad (2.16)$$

Transformation of this equation to the s-plane gives an approximation of the continuous process H(s):

$$H^*(s) = \frac{\beta_o + \beta_1 e^{sT} + \beta_2 e^{s2T} + \dots + \beta_f e^{sfT}}{1 + \alpha_1 e^{sT} + \dots + \alpha_g e^{sgT}} \qquad (2.17)$$

The introduction of the approximated discrete model gives rise to an error between its output and the sample values of the output of the continuous process (fig. 2.4), the model error.

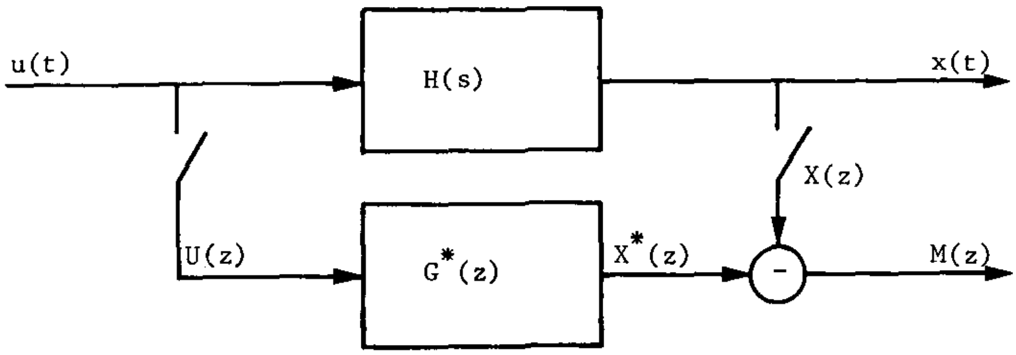



Fig. 2.4.

This model error is given by:

$$M(z) = X(z) - X^*(z) ,$$

or:

$$M(z) = (G(z) - G^*(z))U(z) \qquad (2.18)$$

For its power spectrum one finds:

$$\Phi_{MM}(z) = \left(G(z) - G^*(z))(G(z^{-1}) - G^*(z^{-1})\right).U(z).\overline{U(z^{-1})} \qquad (2.19)$$

where the bar indicates the complex conjugate.

At this point we will assume that the approximated discrete model $G^*(z)$ is derived from G(z) by minimizing the mean squared model error, given by eq. (2.19) when integrated around the unit circle in the z-plane.

As will be clear from inspection of eq. (2.19) the value of this mean squared error will be independent of the particular phase of the input signal U(z). With respect to the minimizing of eq. (2.18) the phase information of U(z) can be ignored. So, the approximated discrete model $G^*(z)$ is a fit to the function G(z) on the unit circle, in the classical least squares sense, with weighting function U(z), understood as stated above.

Now we can state the main problem of the identification of continuous processes using discrete models as: In which way can the original continuous process be reconstructed from its approximated discrete model?
In the sequel of this section we will discuss some approaches to this problem. One way of identification of the original process H(s) can be based on the approximated transfer function $H^*(s)$ (eq. (2.17)). Though nothing is said yet about the choice of the order of the discrete model given by eq. (2.16), it should be noted that at least will yield:

$$f + g \geq p + q$$

Otherwise the number of degrees of freedom will be insufficient to reconstruct the original process.
It can be tried to find a fit in the least squares sense, of the same form as H(s), to $H^*(s)$, with the input signal as weighting function. Calling this fit F(s), the error involved by it is:

$$M^*(j\omega) = (H^*(s) - F(s)).U(j\omega) \tag{2.20}$$

where $U(j\omega)$ stands for the spectrum of the input signal. Transformation to the z-plane gives:

$$M^*(z) = (G^*(z) - F(z))\, U(z)$$

Though formally the same as equation (2.18), it is not known if F(z) = G(z) holds. The minimization is obtained with respect to the parameters of F(z) (cf. eq. (2.15)), whereas the minimization of eq. (2.18) is obtained with respect to the parameters of $G^*(z)$.
The difficulties arise from the fact that the analytic solution of eq. (2.20) can not be found. Let:

$$F(s) = \frac{\sum_p b_p^* s^p}{\sum_q a_q^* s^q}$$

The mean squared error S, calculated from eq. (2.20), using the signals of eq. (2.7) is:

$$S = |M^*(j\omega)|^2 = \sum_k \left| \left( \frac{\sum_p b_p^* (jk\omega_o)^p}{\sum_q a_q^* (jk\omega_o)^q} - H^*(jk\omega_o) \right) |U(jk\omega_o)| \right|^2$$

Minimization of this expression with respect to its parameters gives:

$$\frac{\partial S}{\partial b_p^*} = 0 \quad ; \quad \frac{\partial S}{\partial a_q^*} = 0$$

This set of non-linear equations can not be solved analytically. As the approximation problem of eq. (2.20) for the assumed rational functions F(s) is basically non-linear , also no use can be made of the theory of orthogonal functions.

If the weighting function in eq. (2.20) has only significance in the neighbourhood of $\omega = 0$, then the reconstruction of H(s) can be based on the Taylor expansion of $H^*(s)$ round $s = 0$. For this purpose one can construct a rational function F(s) of the same form as H(s) whose Taylor expansion has $p + q + 1$ terms equal to the expansion of $H^*(s)$. Such a rational function will be called a (p,q) Taylor convergent. A simple method for their construction is given in Appendix III.
The same as has been said for the least squares fit to $H^*(s)$ applies here: it is not known to which degree of accuracy the function F(s) reconstructs the original process.

Finally the reconstruction of H(s) by means of a data hold circuit should be mentioned [Smith, 1968] , [Sinha, 1972] . For this purpose one assumes that the $k^{th}$ derivative of the input signal of the continuous process (fig. 2.5) consists of a sequence of impulses (k-1 order hold).

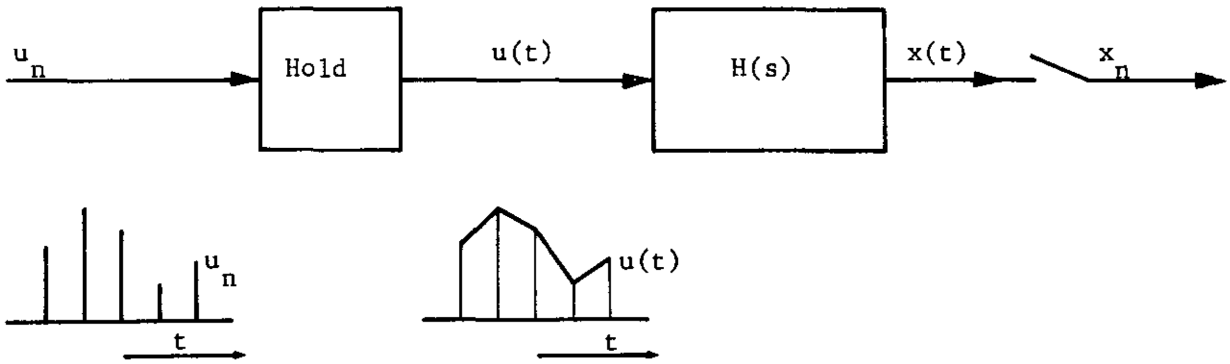



Fig. 2.5.

The signal u(t) can also completely be reconstructed from its sample values using the holdcircuit. This hold can be understood as a (simple) integrating rule. The discrete transfer function of the process of fig. 2.5 is:

$$G(z) = ZL^{-1}\{K(s)H(s)\} \tag{2.21}$$

where Z stands for the z-transform, $L^{-1}$ for the inverse Laplace transform and K(s) is the transfer function of the holdcircuit.

From eq. (2.21) H(s) can be calculated:

$$H(s) = \frac{LZ^{-1}\{G(z)\}}{K(s)} \tag{2.22}$$

For band limited signals the assumptions for the validity of these transformations are violated. The relation between the sampled inputs and outputs of the continuous process is given by eq. (2.15) which differs from eq. (2.21). However one can use eq. (2.22) for the reconstruction of H(s), by putting $G^*(z)$ into it, yielding an approximation $H^*(s)$ of the continuous process. As the transfer function of the hold circuit K(s) is a low pass process, the approximated transfer function $H^*(s)$ at large values of s will heavily depend on the used hold. (This differs from the case used by the derivation of eq. (2.21) and (2.22), where the hold has a special function, viz. the exact reconstruction of the input signal). Reasonable results of this reconstruction can also only be expected for simple low pass processes [ Wooldering, 1972 ] .

## 2.3. Approximation operators

Instead of reconstruction of the original continuous process from eq. (2.16) or (2.17), it can also be tried to estimate the parameters of H(s) directly from the discrete data.

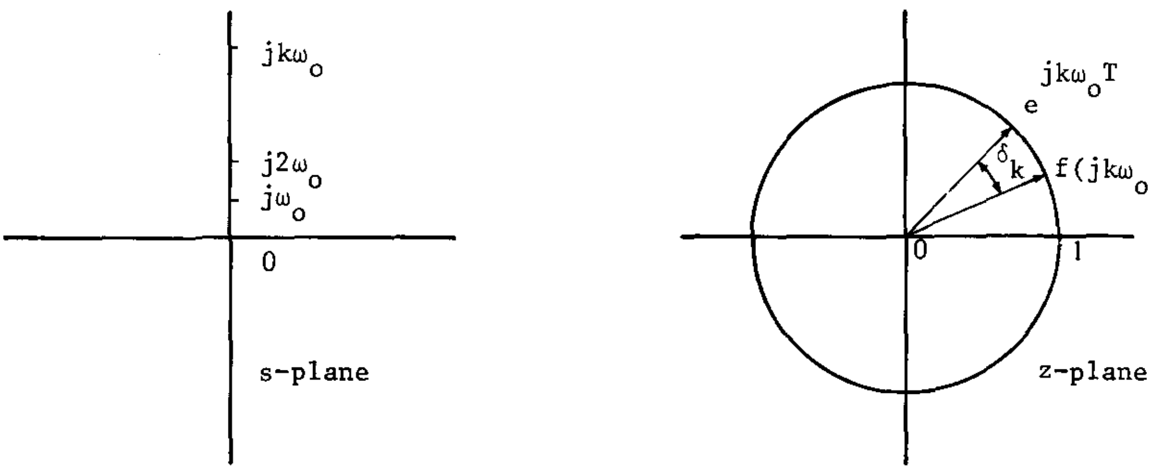
At this point we will recollect that the available information of the signals (eq. (2.7) and (2.9)) are the complex numbers $c_k$ and $d_k$, known at the frequenties $k\omega_o$. The transfer function H(s) is also known at these frequencies. The reconstruction of H(s) from these data will be the search for that interpolating rational function which fits the data in a least squares sense.
The discrete problem can be formulated in a similar way: The discrete transfer function is known at the points $z = e^{jk\omega_o T}$ in the z-plane, to them an interpolating function is sought.
The points $jk\omega_o$ in the s-plane can be projected on the unit circle in the z-plane (fig. 2.6) with some suitable conformal mapping:

$$z = f(s) \tag{2.23}$$

The transform of H(s) belonging to this mapping will be denoted as:

$$H^*(z) = H(f^{-1}(z))$$

where $f^{-1}(z)$ is the inverse of f(s).




As stated above the discrete transfer function G(z) is known at the points $z = e^{jk\omega_o T}$. Fitting of $H^*(z)$ in the least squares sense to G(z) will now give a difference between the interpolating points, viz. $f(jk\omega_o)$ instead of $e^{jk\omega_o T}$. The difference originates from the arguments:

$$\delta_k = \arg(f(jk\omega_o) - jk\omega_o T \tag{2.24}$$

We shall assume that the frequency distribution of the signal u(t) is sufficiently fine in such a way that the discrete variable $k\omega_o$ can be replaced by the continuous variable $\omega$. It will also be assumed that the frequency spectrum , $\Phi_{uu}(\omega)$ of u(t), is flat and has a highest frequency $\omega_h$, small with respect to the sampling frequency (fig. 2.7).

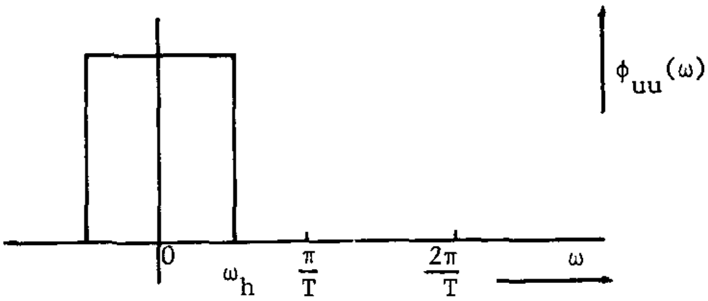



Fig. 2.7.

Now eq. (2.24) reads as:

$$\delta(\omega) = \arg\,(f(j\omega)) - \omega T \qquad (2.25)$$

By Taylor expansion of $\delta(\omega)$ and selecting f(s) in such a way that the first k terms of this expansion equate to zero, $\delta(\omega)$ behaves in the neighbourhood of $\omega = 0$ as:

$$\delta(\omega) \propto \omega^k \qquad (2.26)$$

Regarding the highest frequency of the signal as the variable $\omega$ then, with the assumptions made, $\delta(\omega)$ will go to zero faster than the highest frequency of the signal, provided $k \geq 2$. So for some $\omega > 0$, $\delta(\omega)$ can be made arbitrarily small.
The condition for the neglecting of the error in the interpolation points will also be:

$$\omega_h << \omega_s \qquad (2.27)$$

where $\omega_h$ represents the highest signal frequency and $\omega_s$ the sampling frequency.
This condition can be extended to other frequency spectra than those of fig. 2.7. As stated before (section 2.2), the fit of $H^*(z)$ is obtained by using the input signal as weighting function. With this knowledge the following criterion can be formulated (fig. 2.8):

"If for some $\omega_1$ eq. (2.27) holds and for $\omega > \omega_1$ the spectrum of the signal decreases fast enough, then the error in the interpolating points can be neglected". (2.28)

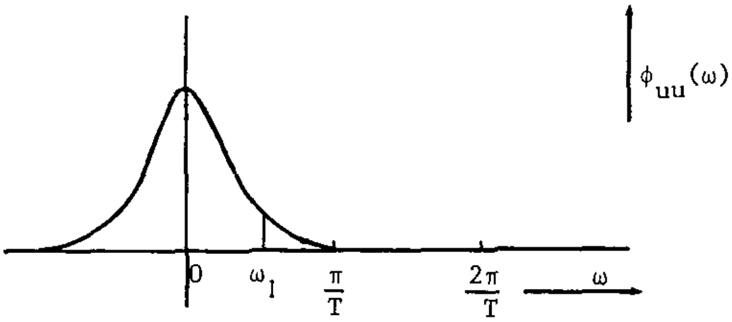



Fig. 2.8.

For the signals discussed above one can identify the transform $H^*(z)$ obtained from H(s) with the approximated discrete model $G^*(z)$, discussed in section 2.2 (cf. eq. (2.16)):

$$G^*(z) \approx H^*(z) \tag{2.29}$$

This relation holds exactly if the difference in interpolation points, given by eq. (2.24) may be ignored. The fit $H^*(z)$ will therefore be called almost least squares with respect to G(z), the exact discrete model. With respect to this discrete model (eq. 2.15) the constructed function f(s) acts as a linear approximation operator:

$$\frac{1}{T} \log z \rightarrow f^{-1}(z) \tag{2.30}$$

The actual choice of f(s) is restricted in our case: if eq. (2.29) is assumed to be valid, both $H^*(z)$ and H(s) are rational functions. Restricting the order of the approximated discrete model to the same order of the continuous process, one immediately obtains transforms of the type:

$$z = \frac{a + bs}{c + ds}$$

As all parameters of the continuous process and its discrete model will be real, also the coefficients of this transform are real. Furthermore the imaginary axis of the s-plane should be transformed onto the unitcircle of the z-plane, with the point s = 0 projected in z = 1. The left half s-plane should be mapped into the inner part of this circle. One finds:

$$z = \frac{1 + bs}{1 - bs} \tag{2.31}$$

with b a constant, still to be determined.

With this transform the imaginary s-axis is projected symmetrically on the unit circle in the z-plane, with respect to s = 0
The argument of eq. (2.31) is given by:

$$\arg(z) = \arctan \frac{2b\omega}{1 - b^2\omega^2} , \qquad j\omega = s$$

Now from the goniometric identity:

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

with tan u = bω, it follows:

$$\arg(z) = 2 \arctan b\omega \tag{2.32}$$

Taylor expansion gives:

$$\arg(z) = 2.(b\omega + \frac{(b\omega)^3}{3} + \frac{(b\omega)^5}{5} + \ldots), \quad |b\omega| < 1$$

Comparing this expansion with eq. (2.25), one finds:

$$b = \frac{T}{2}$$

and eq. (2.26) reads:

$$\delta(\omega) \simeq \frac{1}{12} T^3 \omega^3 \tag{2.33}$$

The constructed transform is also:

$$z = f(s) = \frac{2 + Ts}{2 - Ts} \tag{2.34}$$

and its inverse:

$$s = f^{-1}(z) = \frac{2}{T} \frac{z - 1}{z + 1} \tag{2.35}$$

This transform is identical to the so-called bilinear z-transform [ Sinha, 1972] , [Haberland, 1973] . It should be noted, however, that the derivation presented here differs from those of the authors cited. They consider the reconstruction of the continuous process with the aid of a hold circuit. (cf. eq. (2.22)). The transform is then based on the continued fraction expansion of $e^{p_k T}$, with $p_k$ a pole of the continuous process. From this continued fraction a(1,1) Taylor convergent is formed, yielding:

$$e^{p_k T} \simeq \frac{2 + Tp_k}{2 - Tp_k}$$

For its validity it is necessary to assume:

$$p_k \ll \omega_s \qquad (2.36)$$

for all poles of the considered process. Comparing this with eq. (2.27) shows the main difference: instead of restrictions on the continuous process, a restriction is made here on the signal entering the process. It is also shown that the transform will then be almost least squares with respect to the exact discrete model. In Appendix II another application of this tranform is given.

If eq. (2.27) does not hold, the treatment is complicated substantially. In view of the fact that both the continuous process and its approximated discrete model are described by rational functions, one still can assume that the transform will be given by eq. (2.31) (restricting the order of the discrete model to the same order of the continuous process). However it is not clear if this approximation operator is still linear . In particular the constant b of eq. (2.31) might yet be a function of the parameters of the continuous process.
As an approximation, we will neglect this dependency. Instead of the Taylor expansion of eq. (2.25) around $\omega = 0$, this function can be minimized along an interval of the $\omega$-axis. This can be done by choosing the mean difference in the interpolating points equals zero. For a flat spectrum of the input signal (fig. 2.7) with highest frequency $\omega_h$ one gets from eq. (2.25) and (2.32):

$$\int_0^{\omega_h} \delta(\omega)d\omega = \int_0^{\omega_h} 2 \arctan b\omega d\omega - \int_0^{\omega_h} \omega T d\omega = 0 \qquad (2.37)$$

Instead of solving this equation, the reversed problem will be treated: to the given interpolating points of eq. (2.34) a fictitious sampling frequency $T^*$ is sought, fulfilling the conditions stated above:

$$\int_0^{\omega_h} 2 \arctan \frac{T\omega}{2}\, d\omega = \int_0^{\omega_h} \omega T^* d\omega \qquad (2.38)$$

from which it follows:

$$T^* = \frac{8}{T\omega_h^2} \left( \frac{T\omega_h}{2} \arctan \frac{T\omega_h}{2} - \frac{1}{2} \log \left(1 + \frac{T^2 \omega_h^2}{4}\right)\right) \qquad (2.39)$$

For a highest signal frequency equal to half the sampling frequency:

$$\omega_h = \frac{\pi}{T}$$

this leads to (fig. 2.9):

$$T^* = \frac{8T}{\pi^2} \left( \frac{\pi}{2} \arctan \frac{\pi}{2} - \frac{1}{2} \log\left(1 + \frac{\pi^2}{4}\right)\right) = 0.7743T$$

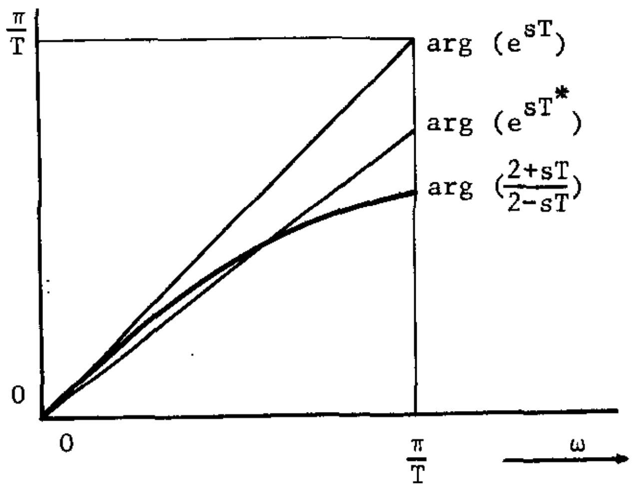



Fig. 2.9.

So the transform is:

$$e^{sT^*} \simeq \frac{2 + sT}{2 - sT} \quad ; \qquad T = 1.2916T^*$$

Replacement of $T^*$ by T gives:

$$z = \frac{2 + T_1 s}{2 - T_1 s} \quad ; \quad T_1 = 1.2916T \qquad (2.40)$$
$$\omega_h = \frac{1}{2}\omega_s \quad ;$$

and its inverse:

$$s = \frac{2}{T_1} \cdot \frac{z - 1}{z + 1} \qquad (2.41)$$

An other approach for signals violating eq. (2.27) is the following. The transform given by eq. (2.34) can be interpreted as an approximation of the series expansion of $e^{sT}$. Now, if more terms of this expansion are taken into account, the truncation error due to the finite number of terms will be smaller. So the parameters of the discrete model will be in a better agreement with the series expansion. This can be obtained by choosing in eq. (2.16), the approximated discrete model, the order $f > p$ and $g > q$. Transforming this model with eq. (2.34) to the s-plane gives an approximation of H(s), the order of which is (f, g). From this approximation a (p, q) convergent has to be formed.

Application of this kind of approximation can be found in the field of simplification of discrete transfer functions [Shih, 1973] .

## 2.4. Considerations about the errors involved in the bilinear z-transform.

The expected error in the parameters of the continuous process due to the difference in the interpolating points can be calculated along the same lines as eq. (2.41) was derived. For an arbitrary spectrum of the input signal, this spectrum should be introduced as a weighting function at both sides of eq. (2.38).
For a flat spectrum, this error has been calculated using eq. (2.39). The results, $T_1$, cf. eq. (2.40), are shown in fig. 2.10 as a function of the highest frequency of the signal. It should be noted that if it is known that the input signal fulfils the assumptions given, the constant $T_1$ can be taken from fig. 2.10 and the transform of eq. (2.40) can be carried out, assuming its linearity.

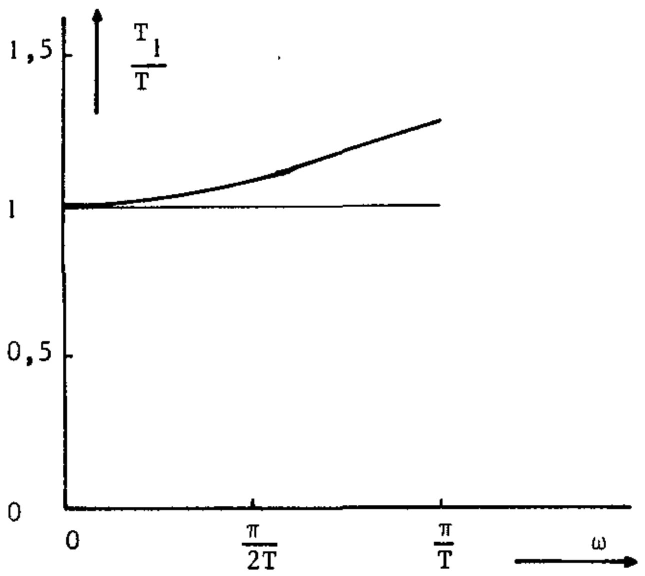



Fig. 2.10.

The relative error in the parameters of H(s) is also (obtained by comparing the results of substitution of eq. (2.40) respectively eq. (2.34) into $G^*(z)$, eq. (2.16)):

$$\frac{a_i}{a_i^*} = \left(\frac{T_1}{T}\right)^i \quad ; \quad i = 1, \ldots, q \tag{2.42}$$

where the asterisk denotes the estimated process parameter. A same formula holds for the coefficients $b_j$, $j = 1, \ldots, p$.

An other important criterion on which the relevancy of the bilinear z-transform can be based is the model error (fig. 2.4). To this aim we shall calculate the truncation error due to the replacement of the differential operator by this transform.

For some function $y = f(t)$ it follows:

$$\frac{dy(nT)}{dt} = y^{(1)}(nT) \simeq \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \cdot y(nT)$$

or:

$$y^{(1)}(nT) = \frac{2}{T}(y(nT) - y(nT-T)) - y^{(1)}(nT-T) + R(T).$$

where R(T) represents the truncation error.
R(T) can be calculated for polynomials, one finds R(T) = 0 for polynomials upto the second degree. So it follows, for sufficiently smooth functions:

$$R(T) = \frac{1}{6} T^2 f^{(3)}(t_o) \quad ; \quad nT < t_o < nT + T$$

As the operator (eq. (2.35)) is linear , the model error will satisfy a relation of the same form. For its power one finds:

$$\psi_{mm}(0) = CT^4 \qquad (2.44)$$

with C a constant, depending on both the signals and the process.
This expression shows that the model error can be made arbitrarily small by choosing the sampling frequency high enough.

Though until now no restrictions have been imposed on the process, it will be clear that with the finite frequency contents of the signal and with finite accuracy of the calculations, no reasonable estimate can be obtained for poles or zeros which become arbitrarily large.
This yields a discrepancy between the bilinear z-transform of the continuous process H(s) and the approximated discrete model $G^*(z)$, eq. (2.16). Suppose H(s) has a zero $s_o$, and the behaviour of the estimator for $s_o \to -\infty$ is of interest. The discrete model G(z), eq. (2.15), contains in this case a zero $z_o = e^{s_o T}$, so for $s_o \to -\infty$ , G(z) has a zero in z = 0.
It is reasonable to assume, the least squares fit $(G^*(z))$ of G(z) contains also a zero in z = 0. However with the transform of eq. (2.34) the zero $s_o$ becomes:

$$z_o^* = \frac{2 + Ts_o}{2 - Ts_o} \underset{s_o \to -\infty}{=} -1$$

Back transformation of $G^*(z)$ with eq. (2.35) gives for the zero in z = 0:

$$s_o^* = \frac{2}{T} \frac{z - 1}{z + 1} \underset{z \to o}{=} \frac{-2}{T} \qquad (2.45)$$

So instead of the zero $s_o = -\infty$ , a zero $s_o^* = \frac{-2}{T}$ is estimated.

For finite poles and zeros, large with respect to the frequency contents of the signal, an intermediate behaviour may be expected, also a bias with respect to the real value of the pole resp. zero.

If it is a priori known that H(s), eq. (2.8), contains a number of zeros in $s = -\infty$ $(p<q)$, then it is possible to transfer this knowledge to the discrete model $G^*(z)$ by demanding the same number of zeros in $z = 0$. Then for a number of $(q-p)$ zeros in $z = 0$ this gives in eq. (2.16): $\beta_o, \beta_1, \ldots, \beta_{q-p} = 0$. Rewriting this equation with the backward shift operator gives:

$$G^*(z) = \frac{\beta_o^* + \beta_1^* z^{-1} + \ldots + \beta_p^* z^{-p}}{1 + \alpha_1^* z^{-1} + \ldots + \alpha_q^* z^{-q}} \tag{2.46}$$

where the asterisk denotes the new parameters obtained. With this approach the approximated discrete model has also the same form as H(s), cf. eq. (2.8). Transformation of this discrete model with eq. (2.34) to the s-plane now, however, results in a multiple zero of order $q - p$ in $s = \frac{-2}{T}$ instead of in $s = -\infty$. These zeros should also still be removed from the transfer function (cf. section 2.2). The use of some other a-priori knowledge about the continuous process is treated in Appendix IV.

Finally some remarks should be made on the method by which the parameters of eq. (2.46) are obtained. If this is done with a method linear in the parameters, e.g. the so-called extended matrix method [Talmon, 1971] (see Appendix V for a short description), the estimator does not fulfil the conditions assumed in section 2.2 (cf. eq. (2.18)) as the output signal is also involved in the weighting function. So in general biased results have to be expected from this estimator with respect to the discrete model $G^*(z)$ fulfilling the conditions of eq. (2.18). Treating the model error (fig. 2.4) as an additive noise term, correlated with the output signal, shows that this bias will only be absent under the condition:.

$$\psi_{xe}(i) = 0 \;; \quad i = 1, 2, \ldots p \tag{2.47}$$

with:

- $x$ = the output signal
- $e$ = the equation error; defined by $e_n = (1 + \alpha_1^* z^{-1} + \ldots + \alpha_q^* z^{-q}) m_n$; $m_n$ is the sequence of model errors.

In general this equation does not hold.
For signals fulfilling the condition (2.28), where the spectra of x and m are almost disjunct, we assume that the correlation between x and m can be neglected for at least p points of its crosscorrelation function.
Putting (cf. fig. 2.4):

$$x = x^* + m$$

then it follows from eq. (2.47):

$$\psi_{me}(i) = 0 \quad ; \; i = 1, 2, \ldots, p.$$

This condition can be fulfilled by modelling the sequence $e_n$ as a white noise sequence (extended matrix method).
Apart from this case one can, however, always assume eq. (2.47) to be fulfilled, if the power of the equation error and so the model error is small enough. This can always be obtained by selecting the sampling frequency high enough (cf. eq. 2.44).

## 2.5. Experimental results of some simulated processes.

A number of continuous processes have been simulated using periodical input signals (see also chapter 3). One of the signals used was a recording of the heartpressure signal (cf. chapter 4), in fig. 2.11 the power spectrum of this signal has been sketched.

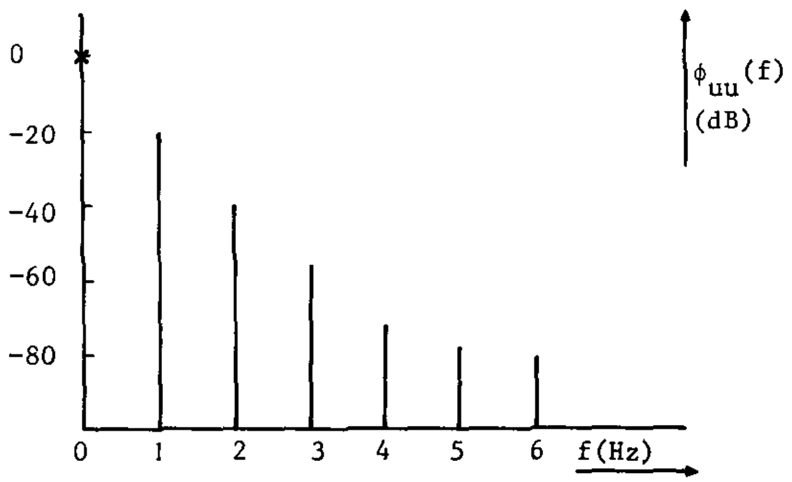



Fig. 2.11.

The power spectrum of the other signal used was flat, with an adjustable highest frequency (cf. fig. 2.7). As we are concerned about the error of the calculations random drawings are taken from the power spectrum of the used signals [ Papoulis, 1965].

From 5 such drawings the standarddeviations of the process parameters have been calculated, this standard deviation has a confidence interval of 90%.

Throughout the iterative version of the extended matrix method (Appendix V) was used here. The weighting factor in this algorithm was chosen to be:

$$\rho = 0.9913$$

and

$$\Delta\rho = 0.025$$

which choice gives results with almost the same standard deviations as when no weighting factor was applied, provided the number of samples is high enough. The use of a weighting factor results in estimated parameters which are slightly "better" than without a weighting factor.

The transfer function of the first process studied, is given by:

$$H(s) = \frac{b_o + b_1 s}{1 + a_1 s} \tag{2.48}$$

with:

$$b_o = 1$$

$$b_1 = 2.10^{-2}$$

$$a_1 = \text{variable}$$

The estimated discrete model will be:

$$G^*(z) = \frac{\beta_o + \beta_1 z^{-1}}{1 + \alpha_1 z^{-1}} \tag{2.49}$$

In addition to this model in most cases also two parameters of the noise process are estimated (cf. Appendix V). The number of samples taken into account was 500.

In fig. 2.12 (appendix) the estimated parameter $\hat{a}_1$ of the continuous process is shown, using the heart pressure signal as input. This estimate is obtained

from eq. (2.49) with the bilinear z-transform (eq. (2.34)). As can be taken from fig. 2.11 at every sampling frequency the criterion (2.28) is applicable.
At poles above the sampling frequency a bias is present in the estimates. It is thought that this bias is mainly caused by the correlation between the output signal and the model error, cf. section 2.4, it should be noted that the assumptions made there are more applicable at a higher sampling frequency. The influence of the estimated noise parameters is also shown.

In fig. 2.13 the power of the output signal compared to the power of the equation error (cf. eq. (2.47)) is shown. The latter power is almost equal to the power of the model error. As an increase of three times in the sampling frequency would give a decrease of 19dB in the model error, according to eq. (2.44), it can be seen that this equation is in good agreement with the calculated error. For high pass processes ($a_1 < 2.10^{-2}$) some deviations can be observed at the lowest sampling frequency. This is due to the necessary filtering of the signals at this aampling frequency, in order to satisfy Shannon's criterion.

The use of the signal with flat power spectrum in this case, is shown in fig. 2.14. Here the highest frequency of the input signal is varied from $\frac{1}{18}\omega_s$ till $\frac{1}{2}\omega_s$ where $\omega_s$, the sampling frequency, is fixed.
The poles above the sampling frequency are estimated badly, the calculated error grows fast. The bias in the estimated poles below the sampling frequency is in very good agreement with the calculations of section 2.4 (fig. 2.10). The results of the transform of eq. (2.40), applied to the case $\omega_h = \frac{1}{2}\omega_s$, are shown. Some deviations, however, remain: it is not known if this is due to the non-linearity of eq. (2.40) or due to the influence of the correlation between the output signal and the model error in the estimation scheme.
In fig. 2.15 the calculated power of the equation error is shown, the results are in good agreement with eq. (2.44).

As mentioned at the end of section 2.3, in the situation where eq. (2.17) is violated, it can also be tried to estimate a discrete model of higher order than given by eq. (2.49). This is applied to the previous case with $\omega_h = \frac{1}{2}\omega_s$. The discrete model used is:

$$G^*(z) = \frac{\beta_o + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} \qquad (2.50)$$

Transforming to the s-plane (eq. 2.34) gives:

$$H^*(s) = \frac{\hat{b}_o + \hat{b}_1 s + \hat{b}_2 s^2}{1 + \hat{a}_1 s + \hat{a}_2 s^2} \tag{2.51}$$

Now it is found in practice that the coefficients $\hat{b}_2$ and $\hat{a}_2$ are both very small when compared to the other parameters. At the same time the smallest coefficients of eq. (2.51) shall also be the most inaccurate, so it is not advisable to construct a (1, 1) Taylor convergent from $H^*(s)$. The best thing to do seems to be to neglect the coefficients $\hat{a}_2$ and $\hat{b}_2$ completely. In fig. 2.16 the estimated parameter $\hat{a}_1$ is shown (the parameter $\hat{b}_o$ compares always very good with $b_o$, whereas the parameter $\hat{b}_1$ shows a similar behaviour as $\hat{a}_1$ in the neighbourhood of $a_1 = 2.10^{-2}$). Comparison with fig. 2.14 shows that the results are remarkably better, especially for poles above the sampling frequency. In fig. 2.17 the calculated power of the equation error is shown, comparison of this error with that of fig. 2.15 shows that the first one is decreased with about 9dB.

Finally the influence of additive noise on the output signal (cf. fig. 1.1) is studied. The results are based on the use of the heart pressure signal as input to the processes. The sample values of the output signal are disturbed with a discrete white noise sequence. The discrete model used is given by eq. (2.49).

In fig. 2.18 and fig. 2.19 the estimated parameters obtained from this model, after transformation (eq. (2.34)) are recorded. The influence of the estimation of parameters of the noise process is shown.

An other simulated process is given by:

$$H(s) = \frac{b_o + b_1 s + b_2 s^2}{1 + a_1 s + a_2 s^2 + a_3 s^3} \tag{2.52}$$

with:

$b_o = 6.013\ 10^{-1}$ $\qquad a_1 = 8.473\ 10^{-2}$

$b_1 = 9.513\ 10^{-1}$ $\qquad a_2 = 5.793\ 10^{-3}$

$b_2 = 1.911\ 10^{-3}$ $\qquad a_3 = 4.105\ 10^{-5}$

This process has poles in: $-126.3$ rad/sec

$-7.41 \pm 11.75j$ rad/sec

and zeros in : $-0.633$ rad/sec

$-497.2$ rad/sec

$-\infty$ rad/sec

This process will also be met in chapter 3.

The input signal to this process will again be the heart pressure signal (fig. 2.11). Choosing the sample frequency high enough, the bilineair z-transform will apply, and the discrete model will be (cf. section 2.4):

$$G^*(z) = \frac{\beta_o + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \alpha_3 z^{-3}} \qquad (2.53)$$

The calculated power of the equation error of this model at a sample frequency of 99 Hz is -82dB compared to the power of the output signal. Transformation of $G^*(z)$ to the s-plane (eq. (2.34)) gives:

$$H^*(s) = \frac{\hat{b}_o + \hat{b}_1 s + \hat{b}_2 s^2 + \hat{b}_3 s^3}{1 + \hat{a}_1 s + \hat{a}_2 s^2 + \hat{a}_3 s^3} \qquad (2.54)$$

The estimated parameter $\hat{b}_3$ is small with respect to the other parameters and will be neglected. In fig. 2.20 the estimated parameters $\hat{a}_3$ and $\hat{b}_2$ are shown as a function of the signal to noise ratio at the output of the process, the remaining estimated parameters are shown in fig. 2.21. 1000 samples were used for these data.

# 3. Direct approaches to the identification of continuous processes.

## 3.1. Least squares estimators.

Let the output of the continuous process (fig. 1.1) be disturbed by an additive noise signal:

$$y(t) = x(t) + n(t)$$

Then eq. (1.1) can be written as:

$$y(t) = b_o u(t) + b_1 \frac{du(t)}{dt} + \ldots + b_p \frac{d^p u(t)}{dt^p} - a_1 \frac{dy(t)}{dt} - \ldots - a_q \frac{d^q y(t)}{dt^q} +$$

$$+ n(t) + a_1 \frac{dn(t)}{dt} + \ldots + a_q \frac{d^q n(t)}{dt^q} \qquad (3.1)$$

We shall first assume that a sequence of sample values of the signals and a sufficient number of their derivatives, are known. Denoting these sequences as $u_i, u_i^{(1)}, \ldots ; \frac{-N}{2} \leq i \leq \frac{N}{2}$, then eq. (3.1) gives:

$$y_i = b_o u_i + b_1 u_i^{(1)} + \ldots + b_p u_i^{(p)} - a_1 y_i^{(1)} - \ldots - a_q y_i^{(q)} +$$

$$+ n_i + a_1 n_i^{(1)} + \ldots + a_q n_i^{(q)} \quad ; \qquad (3.2)$$

$$-\frac{N}{2} \leq i \leq \frac{N}{2}$$

Let:

$$e_i = n_i + a_1 n_i^{(1)} + \ldots + a_q n_i^{(q)} \qquad (3.3)$$

where $e_i$ represents the sequence of equation errors. The N + 1 equations (3.2) can be written in a matrix notation:

$$\underline{y} = \Omega_1 \underline{b} + \underline{e} \tag{3.4}$$

with:

$$\underline{y}^T = [\, y_{-\ell}, \; \dots \; , y_i, \; \dots \; , y_\ell \,]$$

$$\underline{b}^T = [\, b_o, \, b_1, \, \dots, \, b_p, \, -a_1, \, \dots, \, -a_q \,]$$

$$\underline{e}^T = [\, e_{-\ell}, \, \dots, \, e_i, \, \dots, \, e_\ell \,]$$

$$\Omega_1 = \begin{bmatrix} u_{-\ell} & u_{-\ell}^{(1)} \dots u_{-\ell}^{(p)} & y_{-\ell}^{(1)} \dots y_{-\ell}^{(q)} \\ \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots \\ u_i & u_i^{(1)} \dots u_i^{(p)} & y_i^{(1)} \qquad y_i^{(q)} \\ \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots \\ u_\ell & u_\ell^{(1)} \dots u_\ell^{(p)} & y_\ell^{(1)} \dots y_\ell^{(q)} \end{bmatrix} ,$$

$$\ell = \frac{N}{2}$$

It is well known that the least squares solution of eq. (3.4) is given by:

$$\underline{\beta}_1 = \left[ \Omega_1^T \, \Omega_1 \right]^{-1} \Omega_1^T \underline{y} \, , \tag{3.5}$$

assuming that the matrix $\Omega_1^T \Omega_1$ is non-singular. If the equation error sequence $e_i$ is not identically zero, this solution has a bias with respect to the parameters $\underline{b}$, given by:

$$\underline{\beta}_1 - \underline{b} = \left[ \Omega_1^T \, \Omega_1 \right]^{-1} \Omega_1^T \underline{e} \tag{3.6}$$

The expected value of this bias can be calculated by using the limit in probability [ Goldberger, 1964 ] . So:

$$\lim_{N\to\infty} E(\underline{\beta}_1 - \underline{b}) = p\lim_{N\to\infty}\left(\left[\frac{\Omega_1^T\,\Omega_1}{N}\right]^{-1} \cdot \frac{\Omega_1^T\underline{e}}{N}\right)$$

$$= p\lim_{N\to\infty}\left[\frac{\Omega_1^T\,\Omega_1}{N}\right]^{-1} \cdot \; p\lim_{N\to\infty}\frac{\Omega_1^T\underline{e}}{N}$$

Assuming again that the matrix $\frac{1}{N}\,\Omega_1^T\Omega_1$ is nonsingular, this expectation will only be zero if the expected values of the elements of the vector $\frac{1}{N}\,\Omega_1^T\underline{e}$ are zero. Calculation of this vector gives:

$$\frac{1}{N}\Omega_1^T\underline{e} = \begin{bmatrix} \frac{1}{N}\sum\limits_{i=-\frac{N}{2}}^{\frac{N}{2}} u_i e_i \\ \frac{1}{N}\sum\limits_i u_i^{(1)} e_i \\ \vdots \\ \frac{1}{N}\sum\limits_i u_i^{(p)} e_i \\ \frac{1}{N}\sum\limits_i y_i^{(1)} e_i \\ \vdots \\ \frac{1}{N}\sum\limits_i y_i^{(q)} e_i \end{bmatrix} \qquad (3.7)$$

As the sequences $u_i$ and $e_i$ are assumed to be uncorrelated, the expected value of the first p + 1 elements will be zero. The other elements of the vector are only correlated with the additive noise $n_i$, so:

$$E\left(\frac{1}{N}\,\Omega_1^T\underline{e}\right)^T = \left[\,0,\ldots,\,0,\; E\left(\frac{1}{N}\sum_i n_i^{(1)}e_i\right),\;\ldots,\; E\left(\frac{1}{N}\sum_i n_i^{(q)}e_i\right)\right] \qquad (3.8)$$

If the noise process is stationary, the right hand members of this equation can be interpreted as crosscorrelation functions (fig. 3.1):

$$E\left(\frac{1}{N}\sum_i n_i^{(r)} e_i\right) \underset{N\to\infty}{=} \psi_{n^{(r)}e}(0) \quad ; \quad r = 1, \ldots, q \tag{3.9}$$

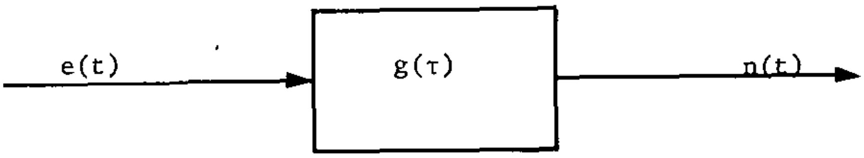



Fig. 3.1.

Now it follows:

$$\psi_{n^{(r)}e}(0) = \frac{d^r}{d\tau^r}\{g(\tau) * \psi_{ee}(\tau)\}_{\tau=0} \quad ; \quad r = 1, \ldots, q \tag{3.10}$$

where $\psi_{ee}(\tau)$ denotes the autocorrelation of the equation error and $g(\tau)$ is the impulse response of the process given by eq. (3.3).
No solutions of this equation, however, are known, yielding $\psi_{n^{(r)}e}(0) = 0$, $r = 1, \ldots, q$. If the equation error is a white noise sequence, i.e. $\psi_{ee}(\tau) = \delta(\tau)$, then it follows from eq. (3.10).

$$\psi_{n^{(r)}e}(0) = \left.\frac{d^r g(\tau)}{d\tau^r}\right|_{\tau=0} \quad ; \quad r = 1, \ldots, q \tag{3.11}$$

The Laplace transform of $g(\tau)$ is given by (cf. eq. 3.3):

$$L\{g(\tau)\} = G(s) = \frac{1}{1 + a_1 s + \ldots + a_q s^q} \tag{3.12}$$

So eq. (3.11) gives:

$$\psi_{n^{(r)}e}(0) = \lim_{s\to\infty} s.s^r G(s) \quad ; \; r = 1, \ldots, q$$

which limit does not exist for $r = q$.
In contrast to this stationary noise case it might , however, be possible to solve the bias problem with non-stationary noise sequences. This will be touched on at the end of this section.

If the process is assumed to behave stationary, i.e. if the initial conditions of the differential equation (1.1) can be neglected, the least squares estimator can also be based on the Laplace transform of eq. (3.1):

$$Y(s) = b_o U(s) + b_1 sU(s) + \dots + b_p s^p U(s) - a_1 sY(s) - \dots - a_q s^q Y(s) + \\ + N(s) + a_1 sN(s) + \dots + a_q s^q N(s) \qquad (3.13)$$

where Y(s), U(s) and N(s) represent transforms of the recorded signals y(t), u(t) and n(t) respectively.
For the identification of the parameters of this equation, we shall assume that the signals can be represented by Fourier series, of which N + 1 coefficients are available. Denoting these coefficients as $U_i$, $Y_i$ and $N_i$ respectively, then equation (3.13) becomes:

$$Y_i = b_o U_i + b_1 j\omega_i U_i + \dots + b_p (j\omega_i)^p U_i - a_1 j\omega_i Y_i - \dots - a_q (j\omega_i)^q Y_i + \\ + N_i + a_1 j\omega_i N_i + \dots + a_q (j\omega_i)^q N_i \; ; \qquad (3.14)$$

$$\frac{-N}{2} \leq i \leq \frac{N}{2} \; ;$$

where $\omega_i$ is the frequency at which the Fourier coefficients are given.
Let:

$$E_i = N_i + a_1 j\omega_i N_i + \dots + a_q (j\omega_i)^q N_i \qquad (3.15)$$

represent the sequence of equation errors in this case. The N + 1 equations (3.14) can be written in a matrix notation:

$$\underline{Y} = \Omega_2 \underline{b} + \underline{E} \qquad (3.16)$$

with:

$$\underline{Y}^T = [\, Y_{-\ell}, \ldots , Y_i, \ldots, Y_\ell \,]$$

$$\underline{b}^T = [\, b_o, b_1, \ldots, b_p, -a_1, \ldots, -a_q]$$

$$E^T = [\, E_{-\ell}, \ldots , E_i, \ldots, E_\ell \,]$$

$$\Omega_2 = \begin{bmatrix} U_{-\ell} & j\omega_{-\ell}\, U_{-\ell} & (j\omega_{-\ell})^p U_{-\ell} & j\omega_{-\ell}\, Y_{-\ell} & (j\omega_{-\ell})^q\, Y_{-\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_i & j\omega_i\, U_i & (j\omega_i)^p\, U_i & j\omega_i\, Y_i & (j\omega_i)^q\, Y_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_\ell & j\omega_\ell\, U_\ell & (j\omega_\ell)^p\, U_\ell & j\omega_\ell\, Y_\ell & (j\omega_\ell)^q\, Y_\ell \end{bmatrix}$$

$$\ell = \frac{N}{2}$$

The least squares solution of eq. (3.16) is given by:

$$\underline{\beta}_2 = \left[ \Omega_2^{T*}\, \Omega_2 \right]^{-1} \Omega_2^{T*}\, \underline{Y} \qquad (3.17)$$

where the asterisk denotes the complex conjugate. The bias of this estimator can be calculated along the same lines as in the previous case. The elements of $\frac{1}{N}\,\Omega_2^{T*}\underline{E}$ become:

$$\left(\frac{1}{N}\,\Omega_2^{T*}\,\underline{E}\right)^T = \left[\, 0, \ldots, 0, \frac{1}{N} \sum_{i=-\frac{N}{2}}^{\frac{N}{2}} (j\omega_i N_i)^* E_i, \ldots, \frac{1}{N} \sum_i ((j\omega_i)^q N_i)^* E_i \right] \qquad (3.18)$$

The sequence of equation errors in this case will in general not be stationary (the coefficients of the Fourier series expansion of a recorded stationary noise signal is only a stationary process if the noise has a flat power spectrum). The expected value of eq. (3.18) will now be interpreted as a cross-power:

$$\frac{1}{N}\sum_{i=-\frac{N}{2}}^{\frac{N}{2}} ((j\omega_i)^r N_i)^* E_i \underset{N\to\infty}{=} \int_{-j\infty}^{j\infty} (s^r N(s))^* E(s)\, ds; \tag{3.19}$$

$r = 1, \ldots, q \quad ; \; s = j\omega$

Upto a factor $2\pi j$, the right hand member of this equation is equal to that of eq. (3.9). So the bias problem of both estimators, eq. (3.5) and (3.17) is the same.

Now suppose E(s) is obtained by integrating a white noise process $\xi(t)$ (fig. 3.2); so:

$$E(s)\, E^*(s) = \frac{-P}{s^2} \tag{3.20}$$

where P is the power of $\xi(t)$.

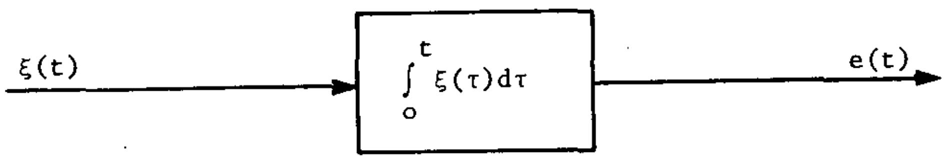



Fig. 3.2.

We now obtain from eq. (3.19):

$$\int_{-j\infty}^{j\infty} (s^r N(s))^* E(s)\, ds = - \int_{-j\infty}^{j\infty} s^r G(s) E(s) E(-s)\, ds$$

$$= P \int_{-j\infty}^{j\infty} s^{r-2} G(s)\, ds \quad ; \; r = 1, \ldots, q$$

where G(s) is given by eq. (3.12).

From this it follows:

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} s^{r-2} G(s)\, ds = \left. \frac{d^{(r-2)} g(\tau)}{d\tau^{(r-2)}} \right|_{\tau = 0} \qquad ; \quad r = 1, \ldots, q$$

for r - 2 < 0, the differentiation has to be replaced by an integration. As can be seen from:

$$\frac{d^{(r-2)} g(\tau)}{d\tau^{(r-2)}} = \lim_{s \to \infty} \frac{s . s^{r-2}}{1 + a_1 s + \ldots a_q s^q} \qquad ; \quad r = 1, \ldots, q$$

all limits exist and equate to zero.
The noise process of eq. (3.20) is the so-called Wiener-Lévy process [ Papoulis, 1965 ]. It should be noted that also a multiple integration of the white noise process $\xi(t)$, would yield bias free estimators. These processes are not stationary.
It will also be proposed that bias free estimators can be obtained (only in the limiting case $N \to \infty$) if the additive noise n(t) is derived from a white noise source $\xi(t)$ with a filter (fig. 3.3):

$$G^*(s) = \frac{1}{s^k} \cdot \frac{1}{1 + a_1 s + \ldots + a_q s^q} \qquad ; \quad k \geq 1 \qquad (3.21)$$

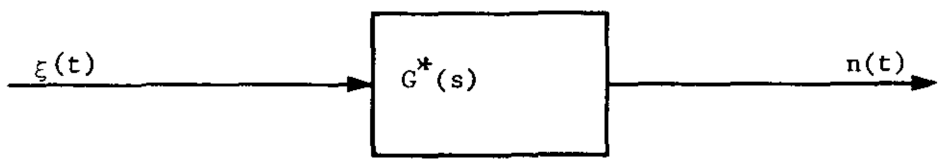



Fig. 3.3.

The consequences of this filter scheme are not studied, it seems rather difficult to establish estimators fulfilling this scheme for an arbitrary noise signal n(t).

## 3.2. Instrumental variable methods.

As well known in the discrete time case [ Wong, 1966 ], [ Smets, 1970 ] the bias problem can be overcome by changing the least squares estimator somewhat. To this end the estimators (eq. (3.5) and (3.17)) are altered into:

$$\underline{\beta} = \left[Z^T\Omega\right]^{-1} Z^T\underline{y} \tag{3.22}$$

with a bias given by:

$$\underline{\beta} - \underline{b} = \left[Z^T\Omega\right]^{-1} Z^T\underline{e} \tag{3.23}$$

where Z represents the so-called instrumental variable matrix, of which the dimensions are the same as the matrix Ω. The elements of the matrix Z are chosen in such a way that they are uncorrelated with the additive noise n(t) and consequently they are uncorrelated with the equation error $\underline{e}$. Furthermore they are selected to have as maximal as possible a correlation with the signals u(t) and x(t), the latter being the undisturbed process output. (It is however not tried to give a minimum variance estimator). Assuming again the matrix $Z^T\Omega$ to be non-singular, it is easy to see that with the chosen matrix Z, the expected value of the bias, eq. (3.23), equals zero.

To obtain the elements of the matrix Z, two methods will be given. The first one is based on an auxillary model of the process (fig. 3.4).

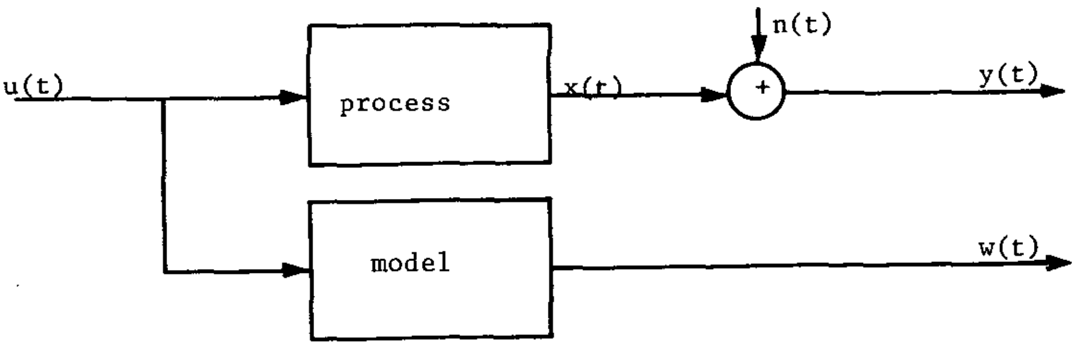



Fig. 3.4.

The output of this model w(t), can be used as an instrumental variable, fulfilling the stated assumptions if the model resembles the process rather well. Based on the least squares estimator (3.5), the i.v. estimator will be:

$$\underline{\beta}_{1i.v.} = \left[ Z_1^T \Omega_1 \right]^{-1} Z_1^T \underline{Y} \qquad (3.24)$$

with:

$$i^{th} \text{ row of } Z_1 = \left[ u_i, u_i^{(1)}, \ldots, u_i^{(p)}, w_i^{(1)}, \ldots, w_i^{(q)} \right]$$

The i.v. estimator based on the least squares estimator (3.17), using the Fourier series expansion of w(t), the coefficients of which are denoted by $W_i$, is:

$$\underline{\beta}_{2i.v.} = \left[ Z_2^{T*} \Omega_2 \right]^{-1} Z_2^{T*} \underline{Y} \qquad (3.25)$$

with:

$$i^{th} \text{ row of } Z_2 = \left[ U_i, j\omega_i U_i, \ldots, (j\omega_i)^p U_i, j\omega_i W_i, \ldots, (j\omega_i)^q W_i \right]$$

The model output can be calculated from:

$$w(t) = \hat{b}_o + \hat{b}_1 \frac{du(t)}{dt} + \ldots + \hat{b}_p \frac{d^p u(t)}{dt^p} - \hat{a}_1 \frac{dw(t)}{dt} - \ldots - \hat{a}_q \frac{d^q w(t)}{dt^q} \qquad (3.26)$$

where $\hat{a}_i$ and $\hat{b}_i$ denote the model parameters. In the second case the Laplace transform of this expression can be used.

The parameters entering these equations, however, are not known at all. Therefore, the estimation scheme can be started using a least squares estimator. The estimated process parameters obtained from this estimator, can be used in eq. (3.26). With the aid of this calculated model output, an i.v. estimation can be done. In order to obtain a maximal correlation between the model output w(t) and the undisturbed process output x(t), this procedure can be repeated, i.e. by calculating a new model output, using the estimated process parameters obtained from the i.v. estimator and so on.

The iterative version of the i.v. estimator [ Smets, 1970 ], (in Appendix V a short description is given) is also suited for this purpose.

Another i.v. technique can be based on the use of a delayed version of the disturbed output signal of the process as an instrumental variable. In the application of such a technique the statistical properties of the equation

error as well as those of the signals are important. In most cases the demands for the construction of the instrumental variable (uncorrelated with the equation error and a (maximum) correlation with the undisturbed process output) can hardly be satisfied. If the delay time is increased in order to obtain a smaller correlation between the delayed process output and the equation error sequence, the correlation between this delayed process output and the real-time process output will in general also decrease. This difficulty can be overcome completely if the signals entering the process are periodical, as their auto-correlation will then also be periodical. In that case the instrumental variable fulfils our assumptions, provided the delay time is chosen to be an integer number of the period time of the signal, assuming that with this delay time the correlation between the delayed process output and the equation error sequence can be neglected.
The matrix Z, entering the estimator (3.22) can be obtained from the matrix $\Omega_1$ given at eq. (3.4) as:

$$i^{th} \text{ row } Z_{1D} = \left[ u_i,\ u_i^{(1)}, \ldots\ldots, u_i^{(p)},\ y_{i+k}^{(1)}, \ldots\ldots\ldots, y_{i+k}^{(q)} \right] \qquad (3.27)$$

where k denotes the number of samples in one period.

Another possible choice can be based on the matrix $\Omega_2$ given at eq. (3.16):

$$i^{th} \text{ row } Z_{2D} = \left[ U_i,\ j\omega_i U_i,\ \ldots,\ (j\omega_i)^p U_i,\ (j\omega_i)\ \overline{Y}_i,\ \ldots,\ (j\omega_i)^q \overline{Y}_i \right] \qquad (3.28)$$

where the bar denotes the spectrum of the one period delayed output signal.

Finally it should be noted that the use of a delayed version of the input signal at the same time, should also result in a bias free estimator with respect to the input noise.

### 3.3. An estimator of mixed discrete and continuous type.

Though, in contrast to the discrete time case [Talmon, 1971], the condition of white residuals to obtain a bias free least squares estimator seems not to be met in the continuous time case (cf. section 3.1), it can be tried to obtain white residuals in the latter case too.
As in the estimator of eq. (3.5) only discrete time samples are involved, it should be possible to model the sequence of equation errors $e_i$ as being the output of some suitable discrete filter, the input of which is a white noise sequence. So:

$$e_i = \sum_{j=1}^{s} c_j \xi_{i-j} - \sum_{j=1}^{r} d_j e_{i-j} + \xi_i \qquad (3.29)$$

with:

$\xi_i$ = a white noise sequence

$c_j$ ; j = 1, ..., s and $d_j$ ; j = 1, ..., r are the parameters of the assumed noise process.

Equation (3.4) can now be rewritten, yielding:

$$\underline{y} = \Omega_3 \underline{b}_{CD} + \underline{\xi} \tag{3.30}$$

with:

$$\underline{y}^T = [\, y_{-\ell}, \ldots, y_i, \ldots, y_\ell \,]$$

$$\underline{b}_{CD}^T = [\, b_o, b_1, \ldots, b_p, -a_1, \ldots, -a_q, c_1, \ldots, c_s, -d_1, \ldots, -d_r \,]$$

$$\underline{\xi}^T = [\, \xi_{-\ell}, \ldots, \xi_i, \ldots, \xi_\ell \,]$$

$$\Omega_3 = \begin{bmatrix} u_{-\ell} & u_{-\ell}^{(1)} & \cdots & u_{-\ell}^{(p)} & y_{-\ell}^{(1)} & \cdots & y_{-\ell}^{(q)} & \xi_{-\ell-1} & \cdots & \xi_{-\ell-s} & e_{-\ell-1} & \cdots & e_{-\ell-r} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ u_i & u_i^{(1)} & \cdots & u_i^{(p)} & y_i^{(1)} & \cdots & y_i^{(q)} & \xi_{i-s} & \cdots & \xi_{i-s} & e_{i-1} & \cdots & e_{i-r} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ u_\ell & u_\ell^{(1)} & \cdots & u_\ell^{(p)} & y_\ell^{(1)} & \cdots & y_\ell^{(q)} & \xi_{\ell-s} & \cdots & \xi_{\ell-s} & e_{\ell-1} & \cdots & e_{\ell-r} \end{bmatrix}$$

$$\ell = \frac{N}{2}$$

The least squares solution is given by:

$$\underline{\beta}_{CD} = \left[ \Omega_3^T \Omega_3 \right]^{-1} \Omega_3^T \underline{y} \tag{3.31}$$

As the elements $\xi_i$ and $e_i$ of the matrix $\Omega_3$ are not available, they will be replaced by their estimates. To handle this an iterative version of the estimator (3.31) is required, yielding estimates of the process and noise parameters after each iteration (cf. Appendix V). The estimates of the elements $e_i$ and $\xi_i$ can then be calculated from:

$$\hat{e}_i = y_i + \sum_{j=1}^{q} \alpha_j y_i^{(j)} - \sum_{j=o}^{p} \beta_j u_i^{(j)}$$

and:

$$\hat{\xi}_i = \hat{e}_i + \sum_{j=1}^{r} \delta_j \hat{e}_{i-j} - \sum_{j=1}^{s} \gamma_j \hat{\xi}_{i-j}$$

where the Greek symbols denote the previously estimated parameters.

3.4. The calculation of derivatives and spectra.

A comprehensive study of the generation of the derivatives, necessary in a number of estimation schemes given, can be found in [ Vlek, 1973 ] .

We shall restrict ourselves completely to the case of band limited, periodical input signals. These signals can be represented by a Fourier series:

$$u(t) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k e^{jk\omega_o t} \tag{3.32}$$

with:

$\omega_o = \frac{2\pi}{NT}$ , T representing the sample time and N the number of samples in one period.

As the signal u(t) will be a real time signal, it holds:

$$c_k = c_k^* \tag{3.33}$$

where the asterisk denotes the complex conjugate. The signal of eq. (3.32) obeys Shannon's criterion; it is also completely determined by its values at the sample times nT. The coefficients $c_k$ are given by:

$$c_k = \frac{1}{NT} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} u(nT) e^{-jk\omega_o nT} \tag{3.34}$$

This expression is known as the discrete Fourier transform of the sequence u(nT). The inverse discrete Fourier transform of eq. (3.34) is given by:

$$u(nT) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k e^{jk\omega_o nT} \tag{3.35}$$

identical with eq. (3.32) for t = nT.
Eq. (3.34) and eq. (3.35) represent discrete variables, enabling the use of a digital computer for the necessary signal processing.
With respect to filters, described in the frequency domain, the initial conditions will be neglected as only the steady state response will be of interest. The output x(t) of some filter G(s) can then be calculated from:

$$x(nT) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} G(jk\omega_o)c_k e^{jk\omega_o nT} \tag{3.36}$$

In the same way the values of the $q^{th}$ derivative of the signal u(t) at the sample times nT can be calculated:

$$u^{(q)}(nT) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} (jk\omega_o)^q c_k e^{jk\omega_o nT} \tag{3.37}$$

Finally a remark should be made about the implementation of estimators based on the spectral components of the input and output signals (cf. eq. (3.16), eq. (3.25) and eq. (3.28)). The sets of complex equations given there can be made real by using eq. (3.33), which will be somewhat easier to handle.

## 3.5. Experimental results.

The performance of the estimators given in this chapter, are tested on a process, instrumented on an analogue computer. A description of this instrumentation and the meaning of the process can be found elsewhere [ Vlek, 1973 ]. The transfer function of this process is the same as the simulated process treated in section 2.5, eq. (2.52), which will be repeated here for convenience:

$$H(s) = \frac{b_o + b_1 s + b_2 s^2}{1 + a_1 s + a_2 s^2 + a_3 s^3} \tag{3.38}$$

with:

$b_o = 6.013 \; 10^{-1}$ $\qquad$ $a_1 = 8.473 \; 10^{-2}$

$b_1 = 9.513 \; 10^{-1}$ $\qquad$ $a_2 = 5.793 \; 10^{-3}$

$b_2 = 1.911 \; 10^{-3}$ $\qquad$ $a_3 = 4.105 \; 10^{-5}$

The input signal is the same one as used in section 2.5 (cf. fig. 2.11). From the input and output signals sample values are taken with a sampling frequency of 99.5 Hz. The results given in this section are based on three series of measurements, each containing 199 samples of the input and output signal. The calculated standard deviations of the estimated parameters, based on these three series will also have a confidence interval of about 80%.
If for some reasons more samples of the signals are necessary in the estimation scheme, these are obtained by periodical continuation of the available samples.
The noise on both input and output signal, amounted about -80dB with respect to the signals.
As it will appear, however, none of the presented estimators of this chapter can handle this noise level, and therefore the noise level will be reduced by the use of a low-pass filter. The transfer function K(s) of this filter was chosen to be a 3rd order polynomial filter with Paynter coefficients [ Kohr, 1967 ]:

$$K(s) = \frac{1}{1 + 0.2145\,\frac{s}{j\omega_A} + 0.1865\left(\frac{s}{j\omega_A}\right)^2 + \left(\frac{s}{j\omega_A}\right)^3} \qquad (3.39)$$

The output of the filter can be calculated with the aid of eq. (3.36).

In fig. 3.5 (Appendix, page 67) the results of the time-domain least squares estimator (3.5) and its i.v. extension (3.24) are shown as a function of the cutoff frequency of the Kohr filter. Only the estimated parameters $\hat{a}_3$ and $\hat{b}_2$ have been recorded, all others compare well with their nominal values. For these results the iterative i.v. algorithm (Appendix V) has been used; the weighting factor was selected to be 1. After a least squares estimation, applied to the first 200 samples (the results of which are shown), the i.v. estimation scheme was started. After every $50^{th}$ iteration this estimator has been interrupted in order to calculate a new model output. As it was observed that after the calculation of 3 such model outputs, no further improvement was

obtained, the estimator has been stopped at this point.
From the results (fig. 3.5) it is clear that with this i.v. technique, still a bias is present in the estimated parameters. The origin of this bias is not known, a possibility seems to be the effect of rounding-off errors in the iterative estimation scheme.

In fig. 3.6 the results of the frequency-domain least squares estimator (3.17) and its i.v. extension (3.25) are shown. As the signal entering the process consists only out of a few significant frequencies (cf. fig. 2.11), it appears not to be recommendable to use in this case an iterative estimation scheme. The results presented are therefore based on the definition of the estimators, i.e. obtained by matrix inversion.
After a least squares estimation, based on the 199 available frequency components of the signals (the results of which are shown), a model output is calculated and the i.v. estimator (eq. (3.25)) is applied. Repeating this procedure gave no further improvement.
From the results it can be seen that this i.v. technique is remarkably better than the previous one.

Comparison of the results of the least squares estimator in the time-domain (fig. 3.5) and the one in the frequency-domain (fig. 3.6) shows that they are the same, as should be (cf. section 3.1).

The use of the spectrum of a delayed version of the output signal as an instrumental variable is shown in fig. 3.7. These results are based on the estimator of eq. (3.22), with matrix Z given by eq. (3.28) (estimated parameters obtained by matrix inversion). The presented results are obtained using all available 199 frequency components of the signals; the least squares results of the previous estimator are also shown for reasons of comparison. The delay time was selected to be one period (99 samples, the available signals consist of two periods). The results show no bias, the calculated standard deviation, however, increases much faster than the previous i.v. estimator. The estimator is also somewhat less accurate at low cut off frequencies of the low-pass filter. The large standard deviations might be overcome by taking into account a larger number of samples. It should be noted also that the delay time could not be adjusted to exactly one period, as the number of available samples was odd.
In fig. 3.8 the remaining estimated parameters, obtained from this estimator, are shown.

Finally some results of the estimator of mixed discrete and continuous type (section 3.4) will be presented (table 3.1). These results are obtained *without additional filtering*.
The weighting factor in the iterative algorithm used (Appendix V) was selected to be: $\rho$ = 0.9913 and $\Delta\rho$ = 0.025. The number of iterations applied amounted to 1000; there are 3 backward parameters of the noise process estimated; the results of no noise parameters estimated (least squares) are also shown.
For reasons of comparison also the results of the complete discrete estimator (chapter 2) are shown (one forward and one backward parameter of the noise process estimated, further details are given in section 2.5). These results are also obtained without additional filtering. It should be noted however that the bilinear z-transform used in the construction of the discrete model itself acts like a low-pass filter.

| estimator | $\hat{a}_1$ | $\hat{a}_2$ | $\hat{a}_3$ | $\hat{b}_o$ | $\hat{b}_1$ | $\hat{b}_2$ |
|---|---|---|---|---|---|---|
| eq. (3.31); r = 0, s = 0 (least squares) | $7.17 \cdot 10^{-2}$<br>$\pm 0.12 \cdot 10^{-2}$ | $4.02 \cdot 10^{-3}$<br>$\pm 0.53 \cdot 10^{-3}$ | $7.82 \cdot 10^{-7}$<br>$\pm 3.29 \cdot 10^{-7}$ | $6.08 \cdot 10^{-1}$<br>$\pm 0.03 \cdot 10^{-1}$ | $8.69 \cdot 10^{-1}$<br>$\pm 0.16 \cdot 10^{-1}$ | $-6.68 \cdot 10^{-3}$<br>$\pm 1.38 \cdot 10^{-3}$ |
| eq. (3.31); r = 3, s = 0 (id. with noise parameters) | $7.71 \cdot 10^{-2}$<br>$\pm 0.21 \cdot 10^{-2}$ | $5.08 \cdot 10^{-3}$<br>$\pm 0.42 \cdot 10^{-3}$ | $8.59 \cdot 10^{-6}$<br>$\pm 1.69 \cdot 10^{-6}$ | $5.98 \cdot 10^{-1}$<br>$\pm 0.04 \cdot 10^{-1}$ | $9.20 \cdot 10^{-1}$<br>$\pm 0.20 \cdot 10^{-1}$ | $-3.72 \cdot 10^{-3}$<br>$\pm 1.23 \cdot 10^{-3}$ |
| eq. (2.34); r = 1, s = 1 (complete discrete scheme) | $8.43 \cdot 10^{-2}$<br>$\pm 0.05 \cdot 10^{-2}$ | $5.75 \cdot 10^{-3}$<br>$\pm 0.03 \cdot 10^{-3}$ | $3.98 \cdot 10^{-5}$<br>$\pm 0.17 \cdot 10^{-5}$ | $5.98 \cdot 10^{-1}$<br>$\pm 0.03 \cdot 10^{-1}$ | $9.41 \cdot 10^{-1}$<br>$\pm 0.03 \cdot 10^{-1}$ | $2.13 \cdot 10^{-3}$<br>$\pm 0.23 \cdot 10^{-3}$ |
| nominal values | $8.47 \cdot 10^{-2}$ | $5.79 \cdot 10^{-3}$ | $4.11 \cdot 10^{-5}$ | $6.01 \cdot 10^{-1}$ | $9.51 \cdot 10^{-1}$ | $1.91 \cdot 10^{-3}$ |

Table 3.1

When it is tried to obtain better results of this mixed estimator by filtering of the signals, the number of necessary noise parameters increases rapidly. This causes an increase in the standard deviations of the estimated process parameters, and the resulting estimates are not better than the least squares estimates.

## 4. Results of parameter estimation applied to a model of a biological process.

The process of interest represents a (part) of a hydraulic model of the human aorta[Leliveld, 1972],The results presented here concern the closing admittance of this model, the electrical analogue of which is shown in fig. 4.1.

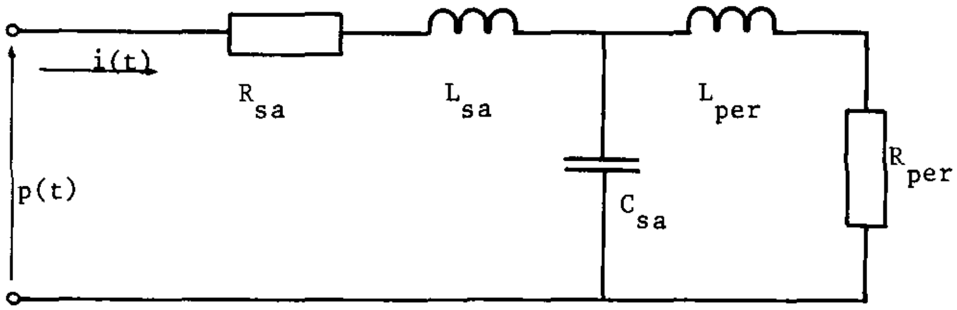



Fig. 4.1

The element values of this network as proposed by its designers are:

$$R_{sa} = 8.00_{10}{-2} \text{ mm } H_g \text{ ml}^{-1}\text{s},$$
$$R_{per} = 1.58 \text{ mm } H_g \text{ ml}^{-1}\text{s},$$
$$C_{sa} = 0.71 \text{ ml mm } H_g^{-1},$$
$$L_{sa} = \sim 2.16_{10}{-2} \text{ mm } H_g \text{ ml}^{-1}\text{s}^2,$$
$$L_{per} = \sim 6.22_{10}{-2} \text{ mm } H_g \text{ ml}^{-1}\text{s}^2.$$

The admittance of the network, with these element values will be:

$$H(s) = \frac{I(s)}{P(s)} = \frac{b_o+b_1s+b_2s^2}{1+a_1s+a_2s^2+a_3s^3} \tag{4.1}$$

with:

$b_o = 6.024_{10}{-1}$ $\qquad$ $a_1 = 1.045_{10}{-1}$

$b_1 = 6.758_{10}{-1}$ $\qquad$ $a_2 = 1.673_{10}{-2}$

$b_2 = 2.660_{10}{-2}$ $\qquad$ $a_3 = 5.746_{10}{-4}$

This process has poles in: $-$ 24.4 rad/sec,
$-$ 2.41 $\pm$ 8.1 j rad/sec,

and zeros in: - 24.8 rad/sec,
- 0.95 rad/sec,
- $\infty$ rad/sec.

The input p(t) of the process represents a simulated heartpressure (cf. fig. 2.11). The flow i(t) and the pressure p(t) (fig.4.1) are measured with an electrodynamical flow meter and a pressure transducer of the semiconductor type. The transfer functions of these transducers can be neglected with respect to this process and the used signals, as far as their frequency dependence is concerned. The absolute value of these transfer functions will be calculated from the estimated d.c. component and its claimed value ($b_o$).
The pressure and flowsignal are sampled with a sampling frequency of 100 Hz. The results presented concern 9 independent observations of the signals, each containing 300 samples. Also results are presented concerning the same process, of which the value of $C_{sa}$ is decreased with a factor 2. In this case 5 series of observations were available. The calculated standarddeviations concerning the estimated values will have a confidence interval of 98% in the first case and 90% in the second case. To the calculated standard deviations, however, not too much importance should be attached, as the main cause of the deviations in the estimated parameters is due to a rather deterministic error in the apparatus by which the samples are taken.

The parameters of eq. (4.1) are obtained by using the technique discussed in chapter 2. The discrete model given by eq. (2.53) applies here too. As the signals are contaminated with noise(the signal to noise ratio amounted to about 40dB), it is necessary to estimate parameters of the noise process as well, in order to obtain bias-free estimates of the processparameters. It appeared that the use of 2 such parameters (one forward and one backward) was sufficient.
It should be noted here that these noise parameters describe only the additive noise at the output of the process. Nothing has been said about the noise on the available input signal. This drawback is basic to all known estimators (cf. section 3.2 for a possible solution).

In table 4.1 the means and standarddeviations of the estimated parameters of the two versions of the process are given.

| Nominal value of $C_{sa}$ | $\hat{a}_1$ | $\hat{a}_2$ | $\hat{a}_3$ | $\hat{b}_0$ | $\hat{b}_1$ | $\hat{b}_2$ |
|---|---|---|---|---|---|---|
| 0.71 | $1.65_{10^{-1}}$ <br> $\pm 0.22_{10^{-1}}$ | $1.25_{10^{-2}}$ <br> $\pm 0.07_{10^{-2}}$ | $2.61_{10^{-5}}$ <br> $\pm 0.58_{10^{-5}}$ | $2.31_{10^{-1}}$ <br> $\pm 0.14_{10^{-1}}$ | $3.31_{10^{-1}}$ <br> $\pm 0.70_{10^{-1}}$ | $1.68_{10^{-3}}$ <br> $\pm 0.41_{10^{-3}}$ |
| 0.36 | $1.10_{10^{-1}}$ <br> $\pm 0.03_{10^{-1}}$ | $7.96_{10^{-3}}$ <br> $\pm 0.37_{10^{-3}}$ | $1.92_{10^{-5}}$ <br> $\pm 0.91_{10^{-5}}$ | $3.18_{10^{-1}}$ <br> $\pm 0.09_{10^{-1}}$ | $1.88_{10^{-1}}$ <br> $\pm 0.08_{10^{-1}}$ | $1.11_{10^{-3}}$ <br> $\pm 0.34_{10^{-3}}$ |

Table 4.1

From the estimated parameters the element values of the process (fig.4.1) are calculated (Appendix VI). As can be seen from eq. (4.1) there are more parameters of the process available than necessary for these calculations. As $\hat{a}_3$ is the smallest parameter (and consequently the most inaccurate estimate) this parameter has been omitted in the calculations. The available estimated b-parameters are determined up to a constant, as said before already. In the calculations of the element values this constant has been determined from the estimated parameter $\hat{b}_0$ with respect to its nominal value. In principle the calculation of the element values is a non-lineair function of the estimated parameters. This can introduce an extra bias in the element values. This bias is investigated by comparing the element values calculated from the mean estimates (table 4.1) with the mean of the element values, obtained by calculation from each set of estimated parameters. The results of these calculations appear in table 4.2. The standard deviations given are obtained from the last mentioned calculations.

| Values obtained from: | $R_{per}$ | $L_{per}$ | $R_{sa}$ | $L_{sa}$ | $C_{sa}$ |
|---|---|---|---|---|---|
| individual parameters | 1.45 | $7.6_{10}{-3}$ | $2.1_{10}{-1}$ | $1.7_{10}{-2}$ | $8.1_{10}{-1}$ |
| | ± 0.02 | $\pm 2.1_{10}{-3}$ | $\pm 0.2_{10}{-1}$ | $\pm 0.3_{10}{-2}$ | $\pm 1.4_{10}{-1}$ |
| mean parameters | 1.45 | $7.4_{10}{-3}$ | $2.1_{10}{-1}$ | $1.7_{10}{-2}$ | $8.1_{10}{-1}$ |
| nominal values | 1.58 | $62.2_{10}{-3}$ | $0.8_{10}{-1}$ | $2.2_{10}{-2}$ | $7.1_{10}{-1}$ |
| individual parameters | 1.41 | $8.3_{10}{-3}$ | $2.6_{10}{-1}$ | $2.1_{10}{-2}$ | $4.2_{10}{-1}$ |
| | ± 0.02 | $\pm 2.6_{10}{-3}$ | $\pm 0.2_{10}{-1}$ | $\pm 0.1_{10}{-2}$ | $\pm 0.2_{10}{-1}$ |
| mean parameters | 1.41 | $8.3_{10}{-3}$ | $2.6_{10}{-1}$ | $2.1_{10}{-2}$ | $4.2_{10}{-1}$ |
| nominal values | 1.58 | $62.2_{10}{-3}$ | $0.8_{10}{-1}$ | $2.2_{10}{-2}$ | $3.6_{10}{-1}$ |

Table 4.2

As can be seen from the results most element values are estimated rather well. It also appears that the non-linear transform does not cause bias. The estimated value of $L_{per}$ compares badly with its nominal value; this nominal value itself, however, is only a guess. The decrease of $C_{sa}$ with a factor 2 is well estimated.

## 5. Conclusions and suggestions.

For the identification of linear continuous processes two different approaches are given: the use of a discrete time model and the more direct techniques, based on the derivatives or the spectra of the signals. Comparison of the results presented shows that the first one can handle a higher noise level than the second one, if no additional filtering is applied.

The investigations of the discrete time model are in good agreement with the results obtained from simulated continuous processes. It is shown that the bilinear z-transform constitutes a discrete model, which has almost least squares properties with respect to the continuous process.

In the case of estimators of the continuous type emphasis has been given to the development of techniques yielding bias-free estimates. The results of using the techniques presented, all show an improvement with respect to simple least squares estimators.

A number of basic problems not treated here, are suggested for future work on this topic:

- The treatment of additive noise on the input signal of the process. This is of great concern in practical applications of the estimators.
- Study of the statistical lower bounds for the standard deviation of the estimates.
- Application of an optimal filter in the estimation scheme (in order to obtain a maximal signal to noise ratio in these schemes).

<u>List of often used symbols.</u>

u(t) = process input signal

x(t) = process output signal

y(t) = disturbed output signal

n(t) = additive noise signal at the output of the process

U(z), X(z),.......= Z-transforms of the signals

U(s), X(s),.......= Laplace transforms of the signals.

p = the number of forward process parameters minus one

q = the number of backward process parameters

s = the number of forward noise process parameters

r = the number of backward noise process parameters

Figure captions

Fig. 2.12

Estimated backward parameter of first order process is shown versus the instrumented parameter. Input signal obtained from random drawings from the power spectrum of a recorded heart pressure signal. No noise added to the output signal. The sampling frequency $f_s$ has been varied:
1: $f_s$ = 99 Hz , 2: $f_s$ = 33 Hz and 3: $f_s$ = 11 Hz.
The dashed curve (1') shows what happens when no parameters of the "noise" process are estimated.

Fig. 2.13

The equation error of the discrete model concerning the process of fig. 2.12 is shown (F = 10 log Pout/Peta).
1: $f_s$ = 99 Hz , 2: $f_s$ = 33 Hz and 3: $f_s$ = 11 Hz.

Fig. 2.14

Estimated backward parameter of first order process is shown versus the instrumented parameter. Input signal obtained from random drawings from a flat power spectrum with adjustable highest frequency $f_h$, sampling frequency $f_h$ = 99 Hz. No noise added to the output signal.
1: $f_h$ = 49 Hz , 2: $f_h$ = 16 Hz and 3: $f_h$ = 5 Hz.
The dashed curve (3') shows the transform with $T_1$ = 1.29 T.

Fig. 2.15

The equation error of the discrete model concerning the process of fig. 2.14 is shown (F = 10 log Pout/Peta).
1: $f_h$ = 49 Hz , 2: $f_h$ = 16 Hz and 3: $f_h$ = 5 Hz.

Fig. 2.16

Estimated backward parameter of first order process is shown versus the instrumented parameter. Input signal obtained from random drawings from a flat power spectrum with highest frequency $f_h$ = 49 Hz , sampling frequency $f_s$ = 99 Hz. No noise added to the output signal. Here a second order discrete model has been used.

Fig. 2.17

The equation error of the discrete model concerning the process of fig. 2.16 is shown (F = 10 log Pout/Peta).

Fig. 2.18

Estimated backward parameter of first order process is shown versus the instrumented parameter. Recorded heart pressure used as input signal, sampling frequency $f_s$ = 99 Hz. White noise added to the output signal:
1: $^S/N = \infty$, 2: $^S/N$ = 50 dB, 3: $^S/N$ = 40 dB and 4: $^S/N$ = 30 dB.
Signal to noise ratio's measured at the output of the process.
The dashed curve (4') shows what happens when no parameters of the noise process are estimated ($^S/N$ = 30 dB).

Fig. 2.19

Estimated forward parameter of the first order process of fig. 2.18 is shown versus the instrumented backward parameter.
1: $^S/N = \infty$, 2: $^S/N$ = 50 dB, 3: $^S/N$ = 40 dB, 4: $^S/N$ = 30 dB and 4': no noise parameters estimated ($^S/N$ = 30 dB).

Fig. 2.20

Estimated parameters ($\hat{a}_3$ and $\hat{b}_2$) of a third order process are shown versus the signal to noise ratio ($^S/N$) at the output of the process. Recorded heart pressure used as input signal, sampling frequency $f_s$ = 99 Hz.
1: two forward and two backward parameters of the noise process estimated,
2: no noise parameters estimated.

Fig. 2.21

The remaining estimated parameters ($\hat{b}_o$, $\hat{b}_1$, $\hat{a}_1$ and $\hat{a}_2$) of the process of fig. 2.20 are shown.
Solid line: two forward and two backward parameters of the noise process estimated, dashed line: no noise parameters estimated.

Fig. 3.5

Estimated parameters ($\hat{a}_3$ and $\hat{b}_2$) of a third order process are shown versus the cutoff frequency of the used low-pass filter. The time domain least squares estimator is compared with its i.v. extension. A model of the process has been used for this i.v. technique.

Fig. 3.6

Estimated parameters ($\hat{a}_3$ and $\hat{b}_2$) of a third order process are shown versus the cutoff frequency of the used low-pass filter. The frequency domain least squares estimator is compared with its i.v. extension. A model of the process has been used for this i.v. technique.

Fig. 3.7

Estimated parameters ($\hat{a}_3$ en $\hat{b}_2$) of a third order process are shown versus the cutoff frequency of the used low-pass filter. The frequency domain least squares estimator is compared with its i.v. extension. A delayed version of the output signal is used for this i.v. technique.

Fig. 3.8

The remaining estimated parameters ($\hat{b}_o$, $\hat{b}_1$, $\hat{a}_1$ and $\hat{a}_2$) of the process of fig. 3.7 are shown.

Solid line: i.v. estimator, dashed line: l.s. estimator.

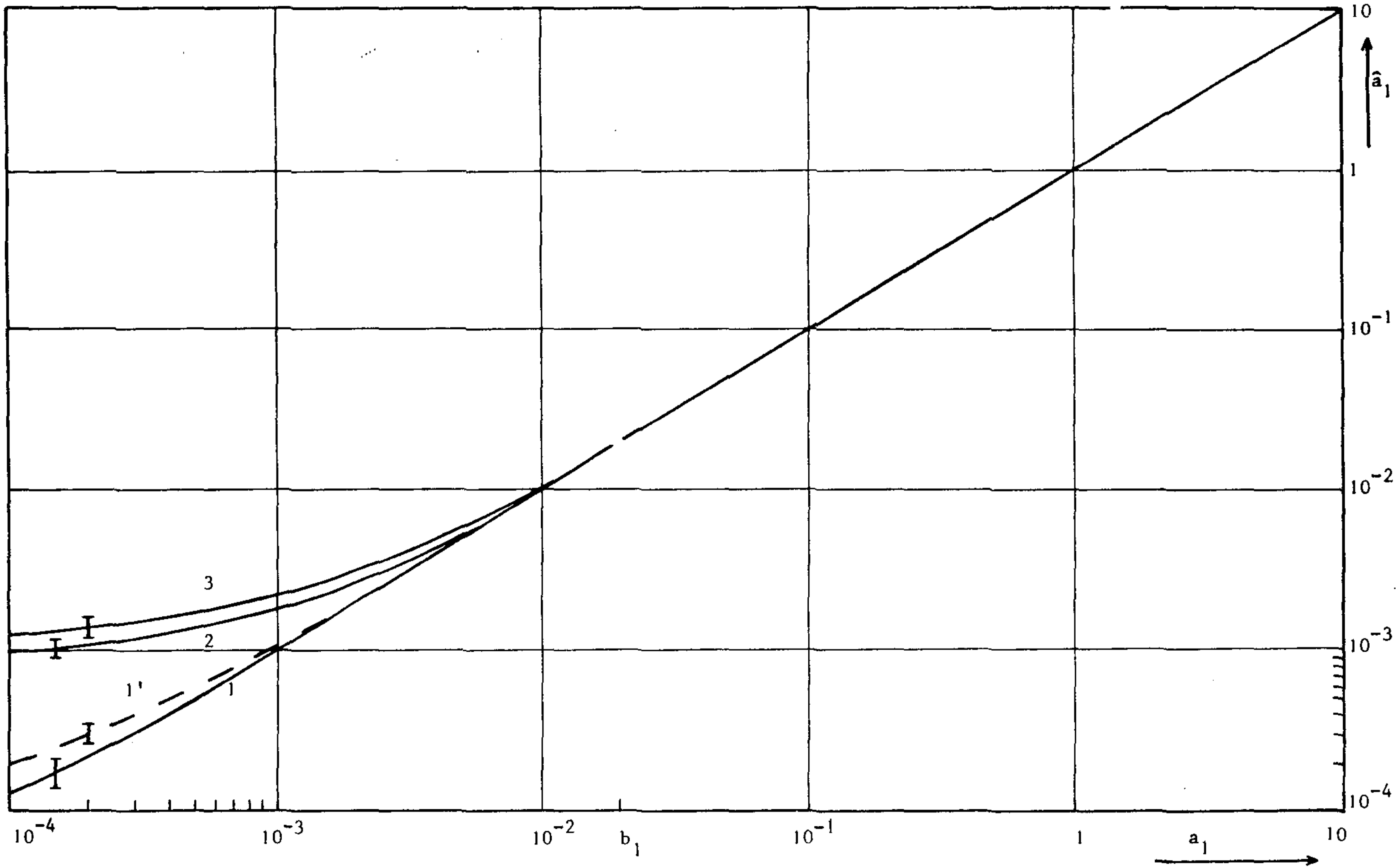



Figure 2.12

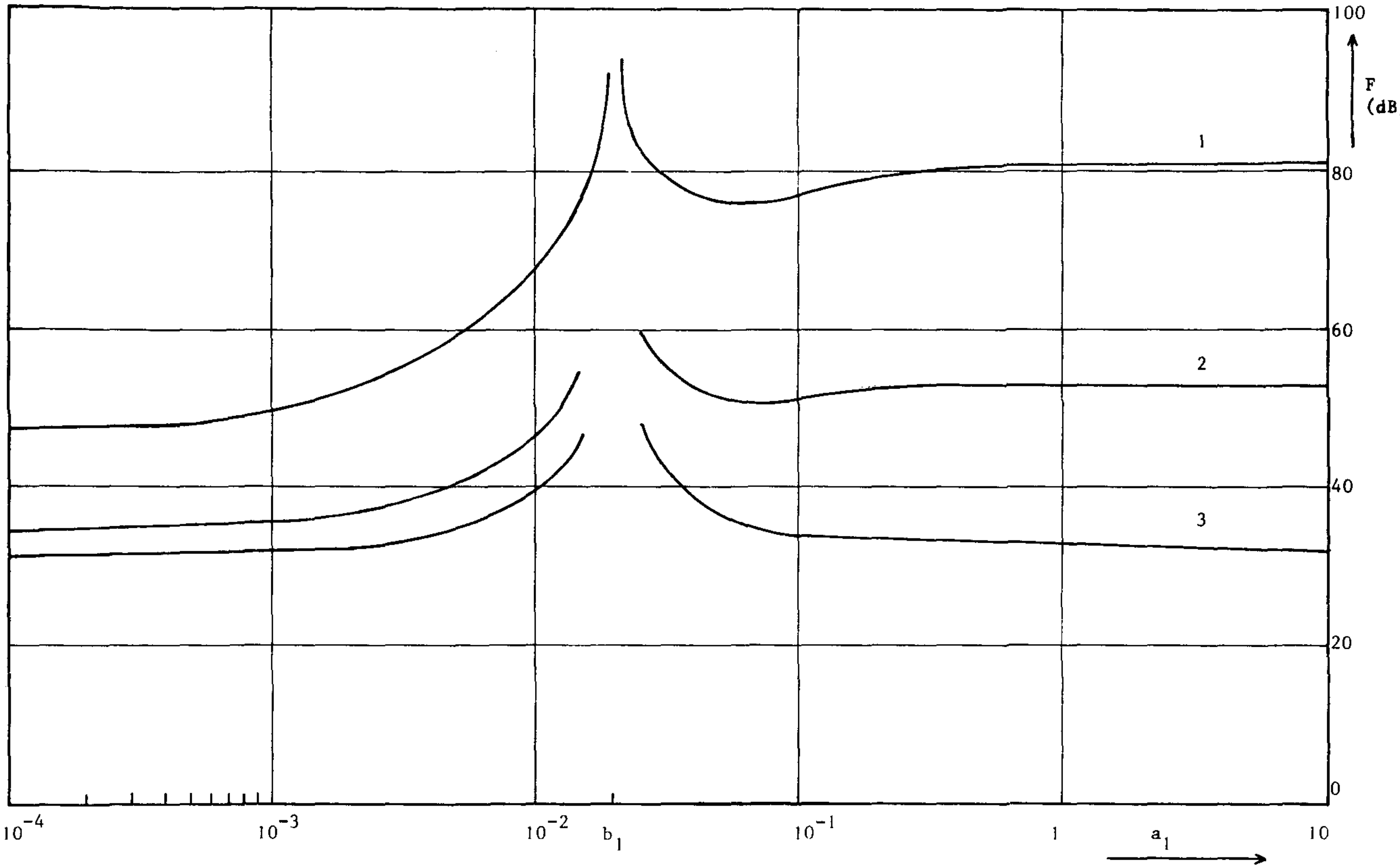



Figure 2.13

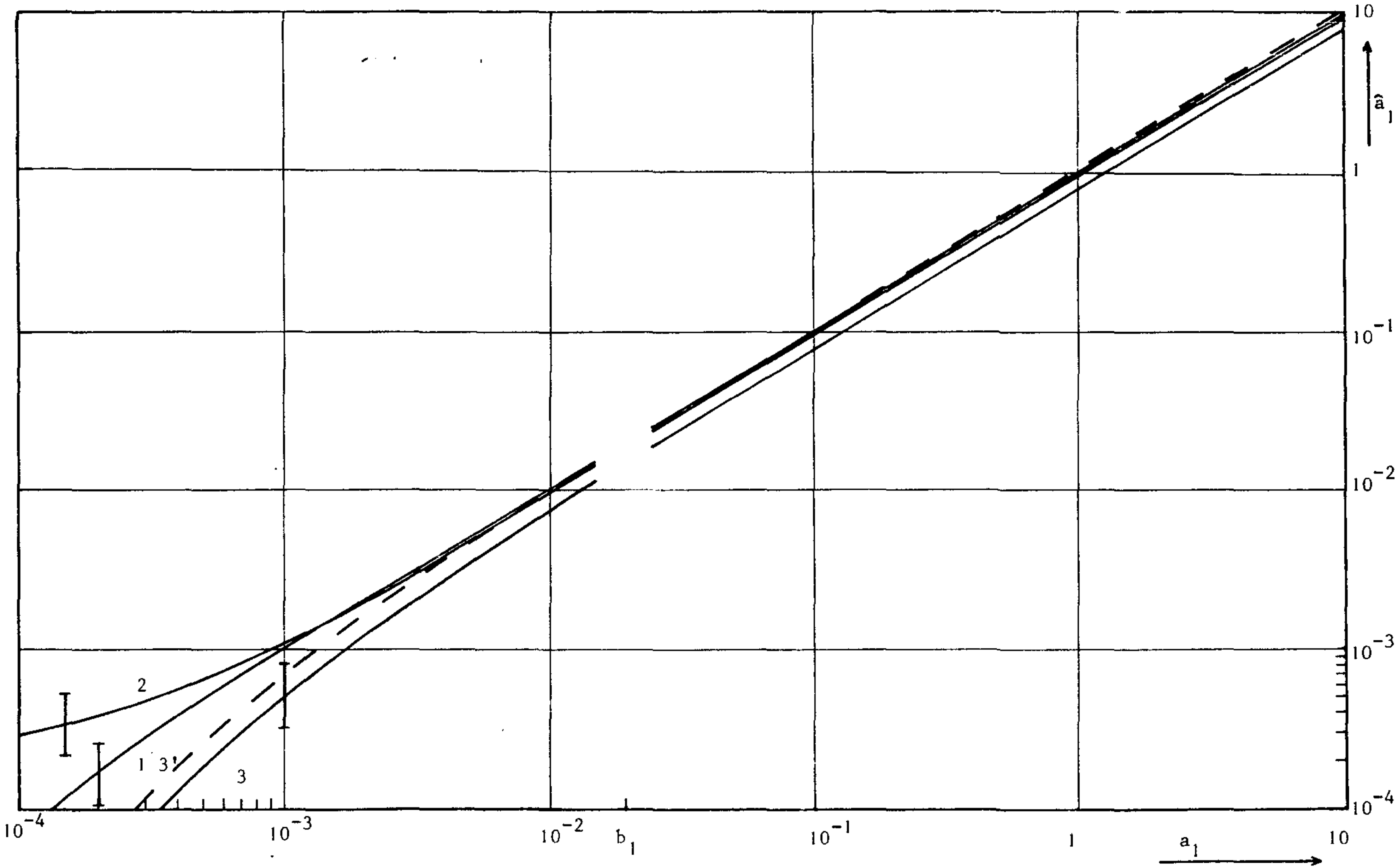



Figure 2.14

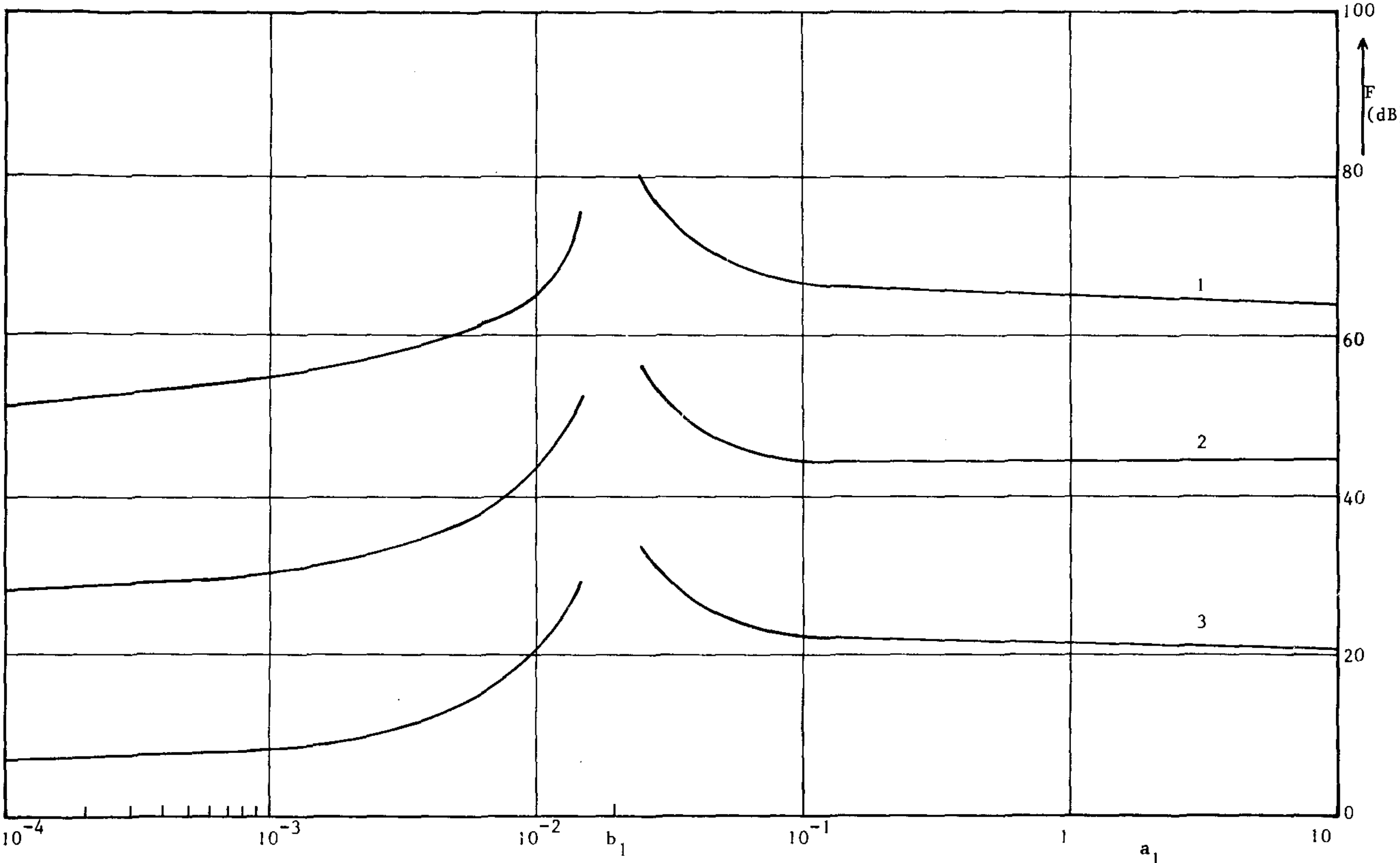



Figure 2.15

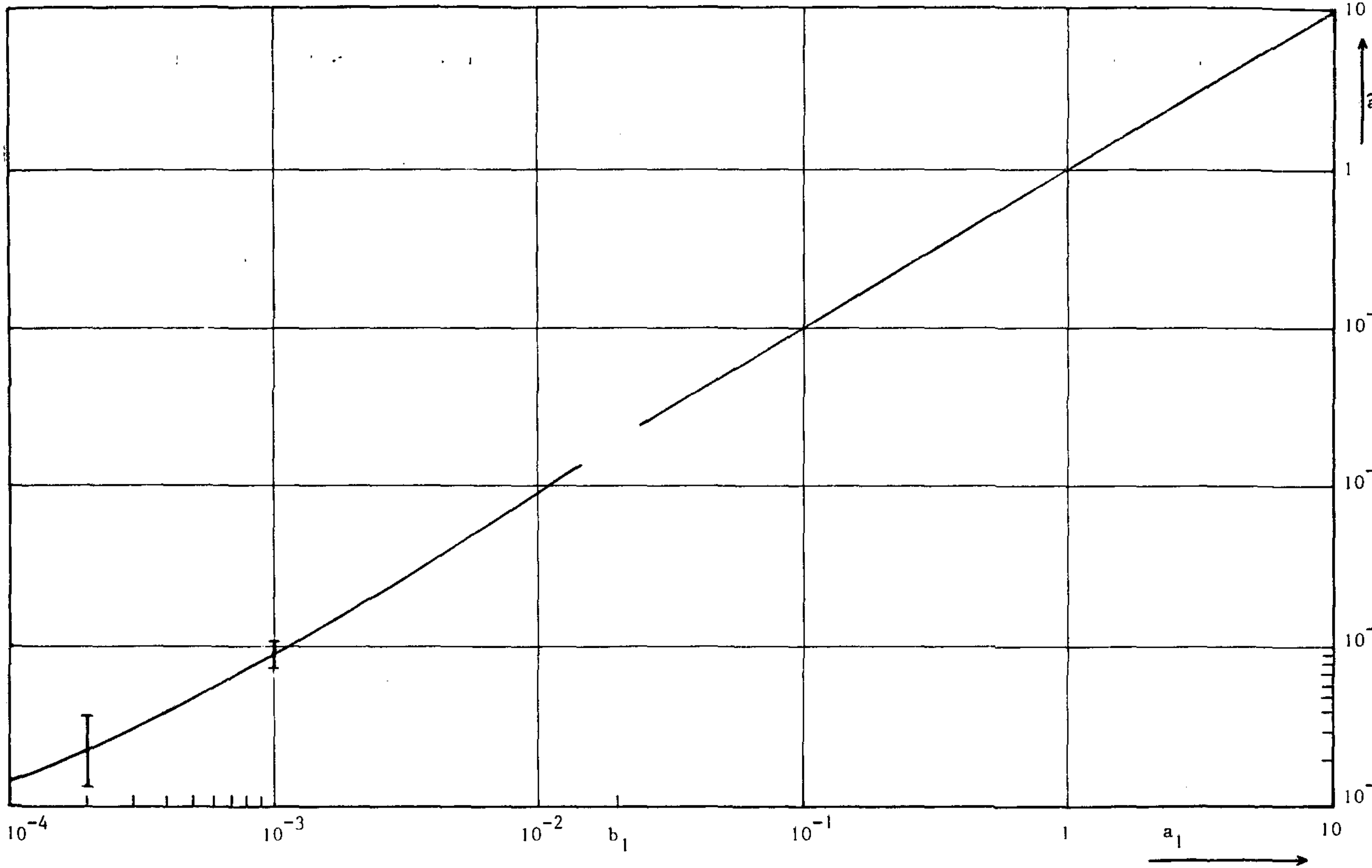



Figure 2.16

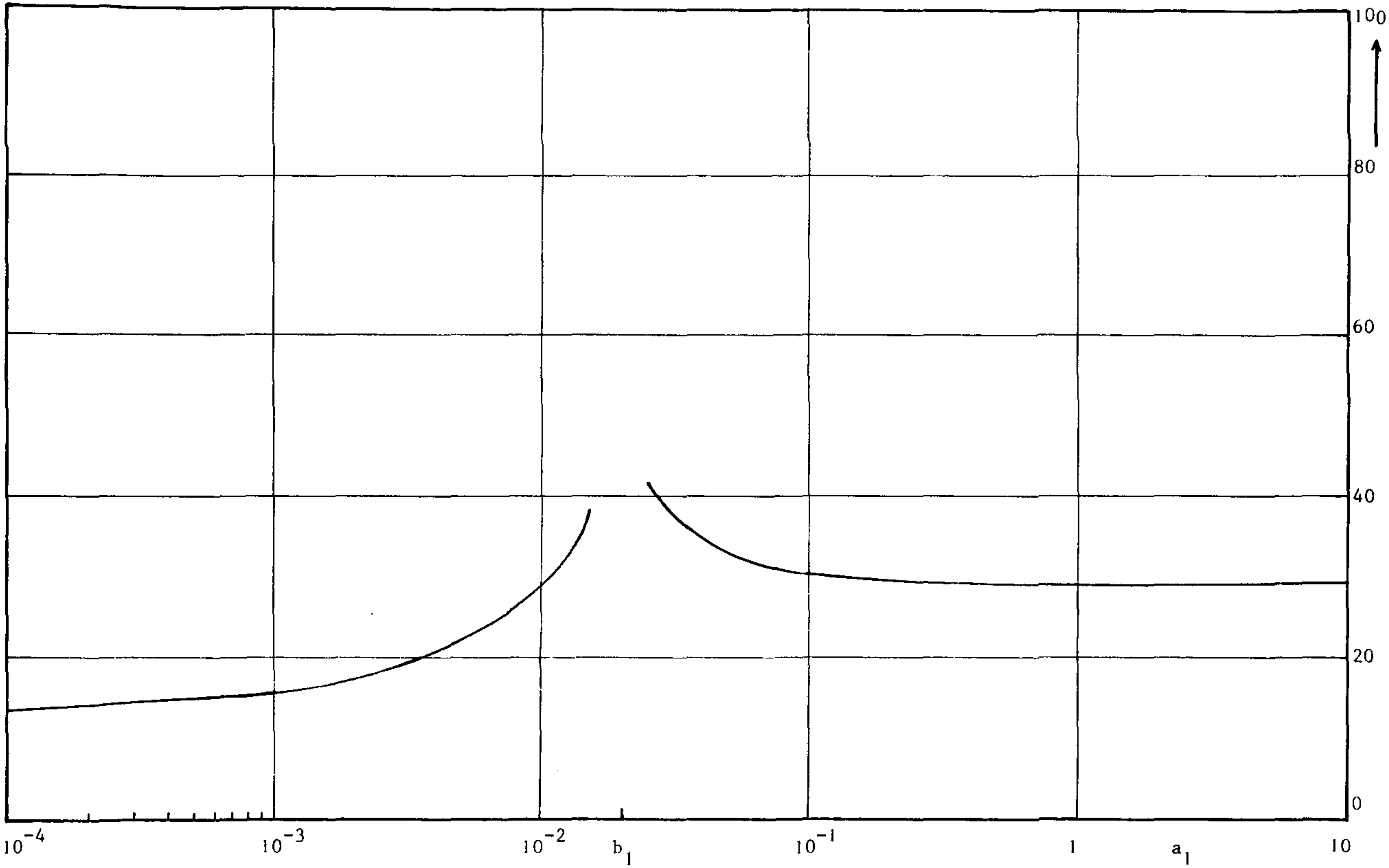



Figure 2.17

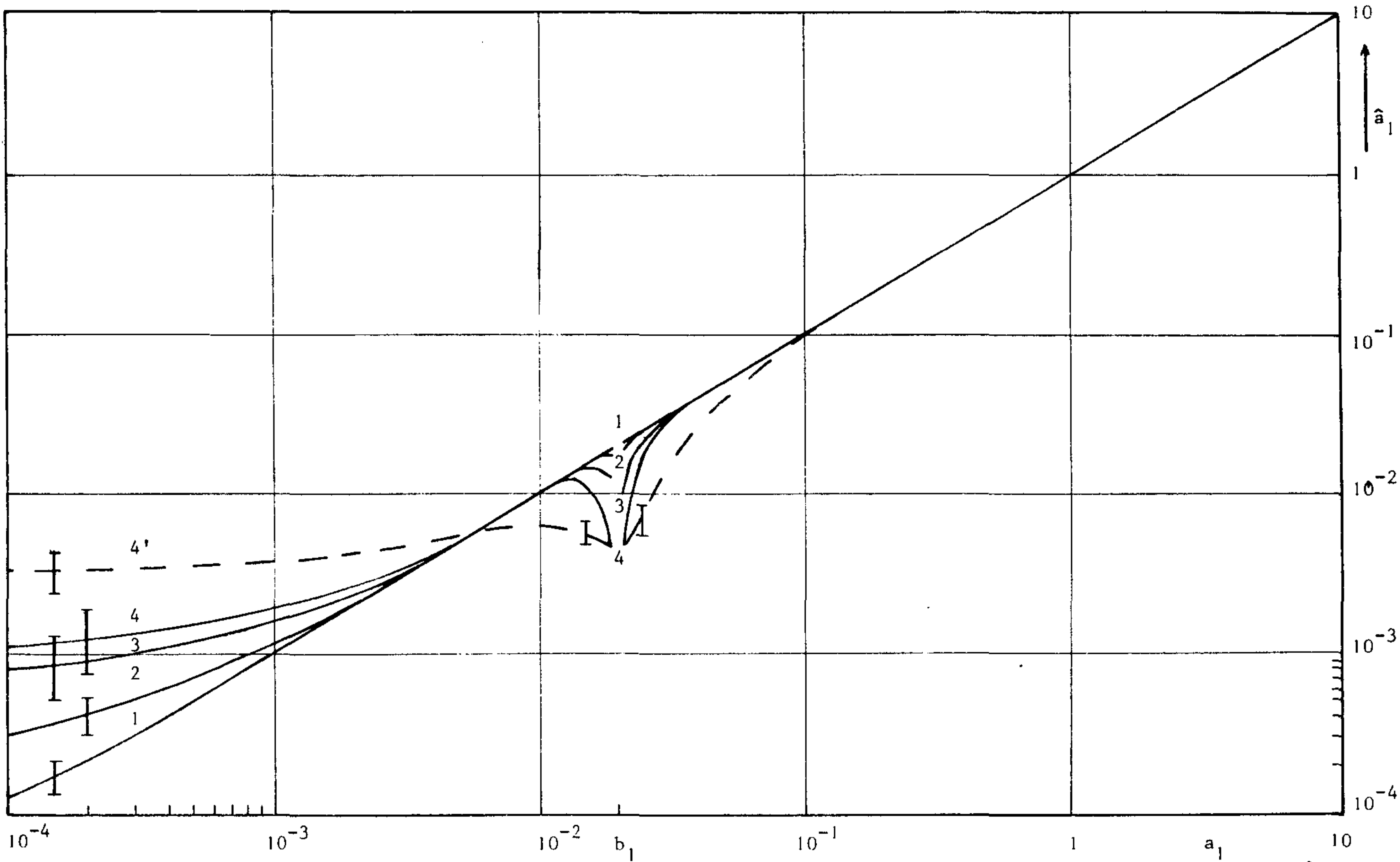



Figure 2.18

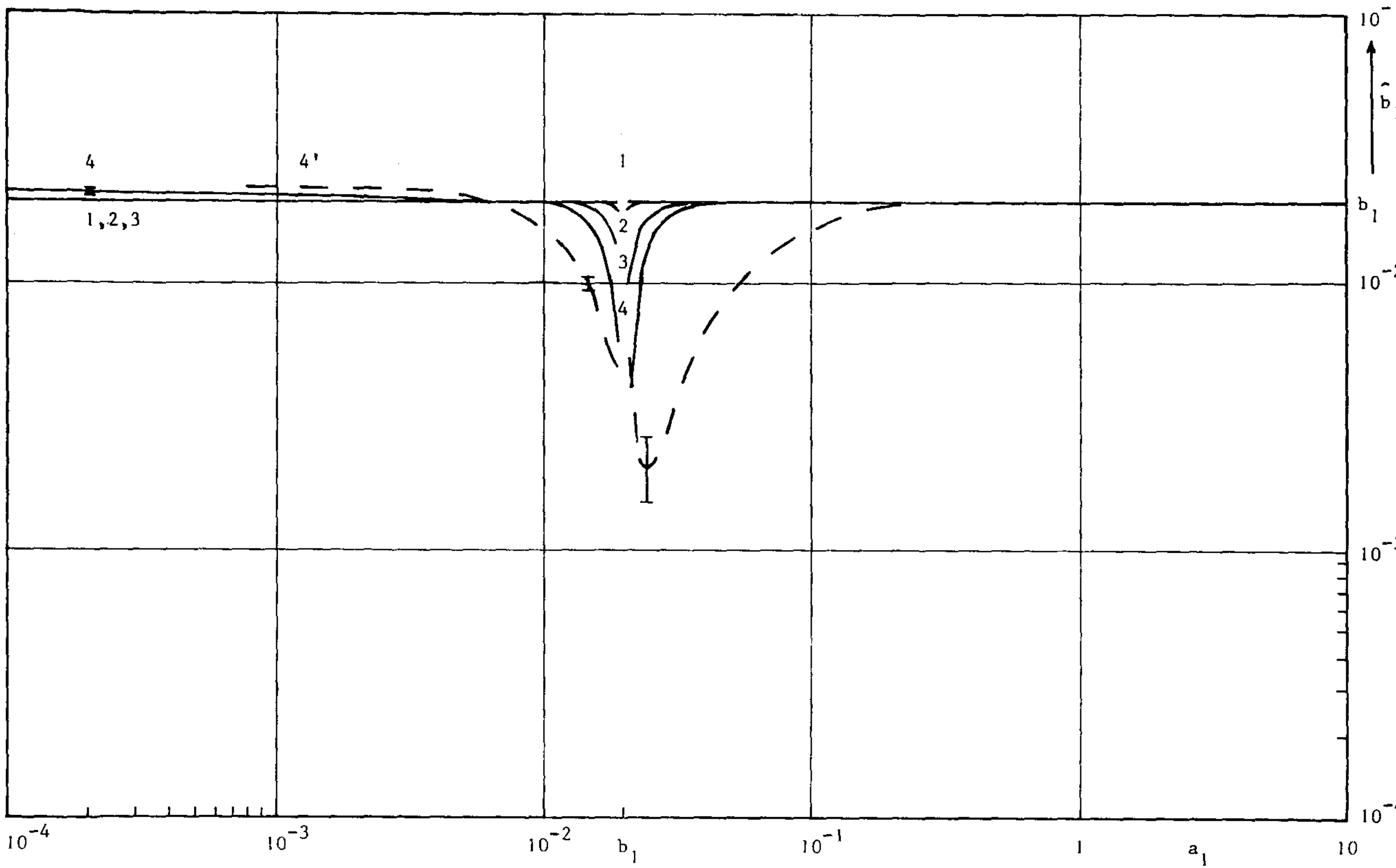
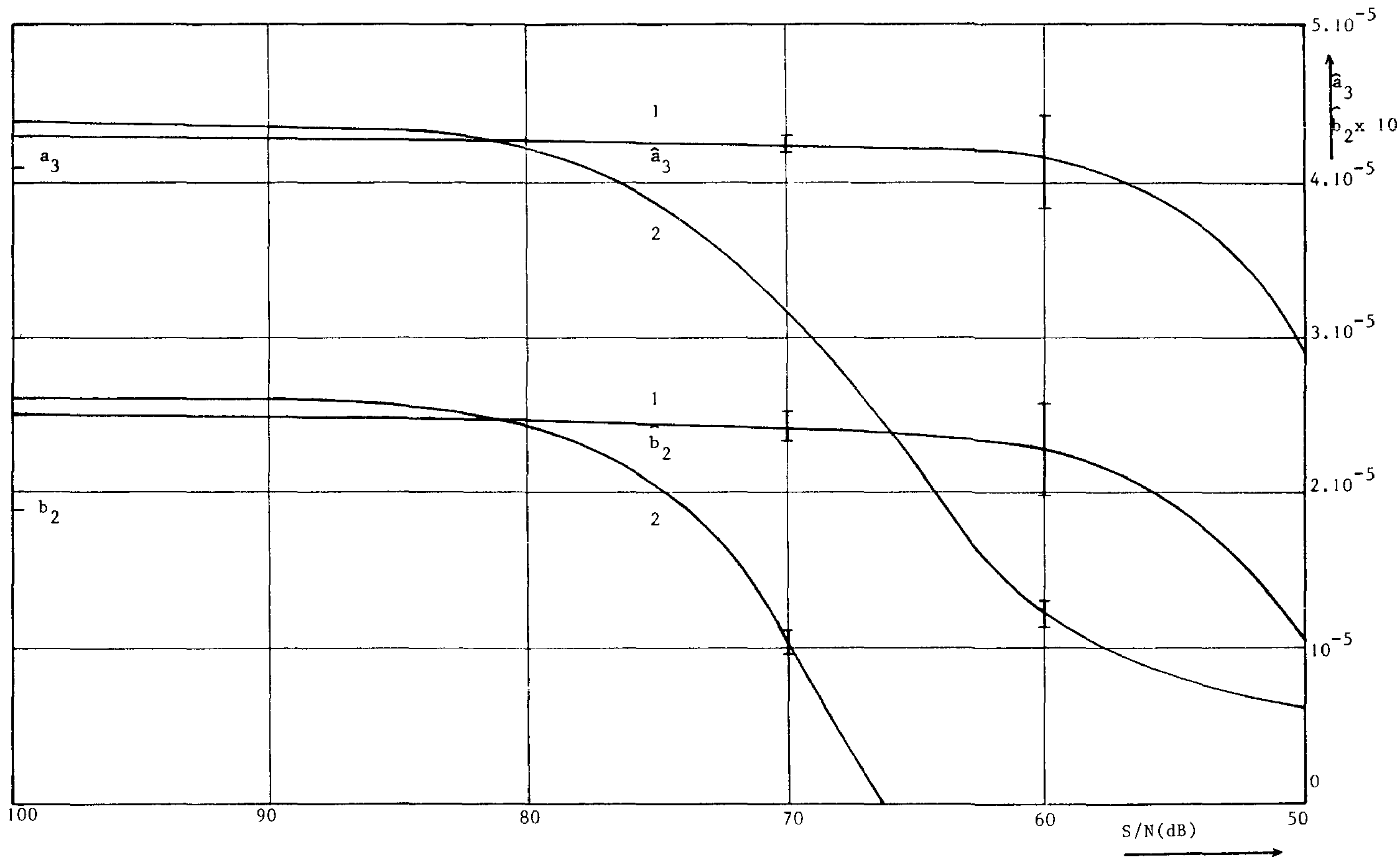
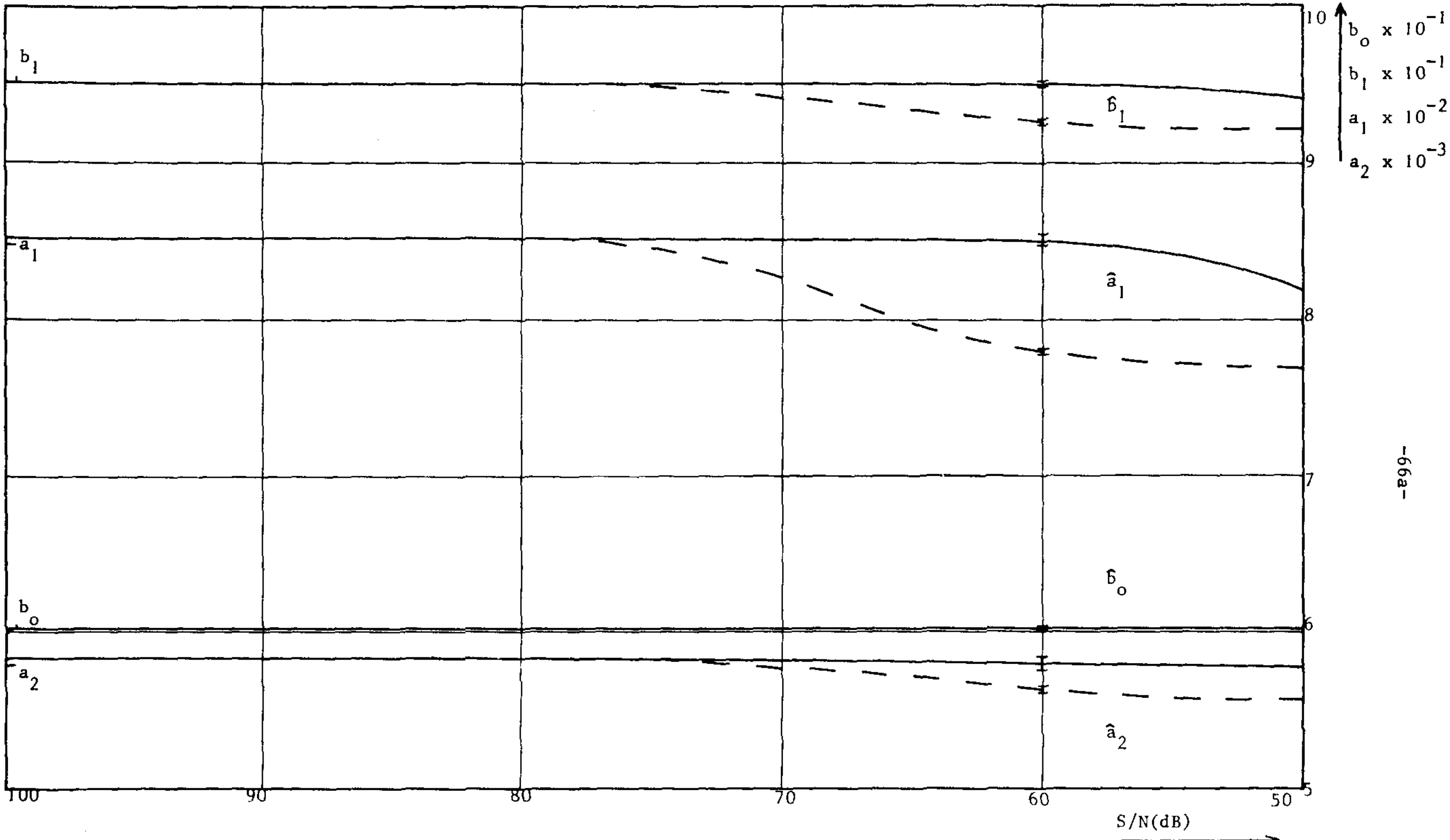



Figure 2.19



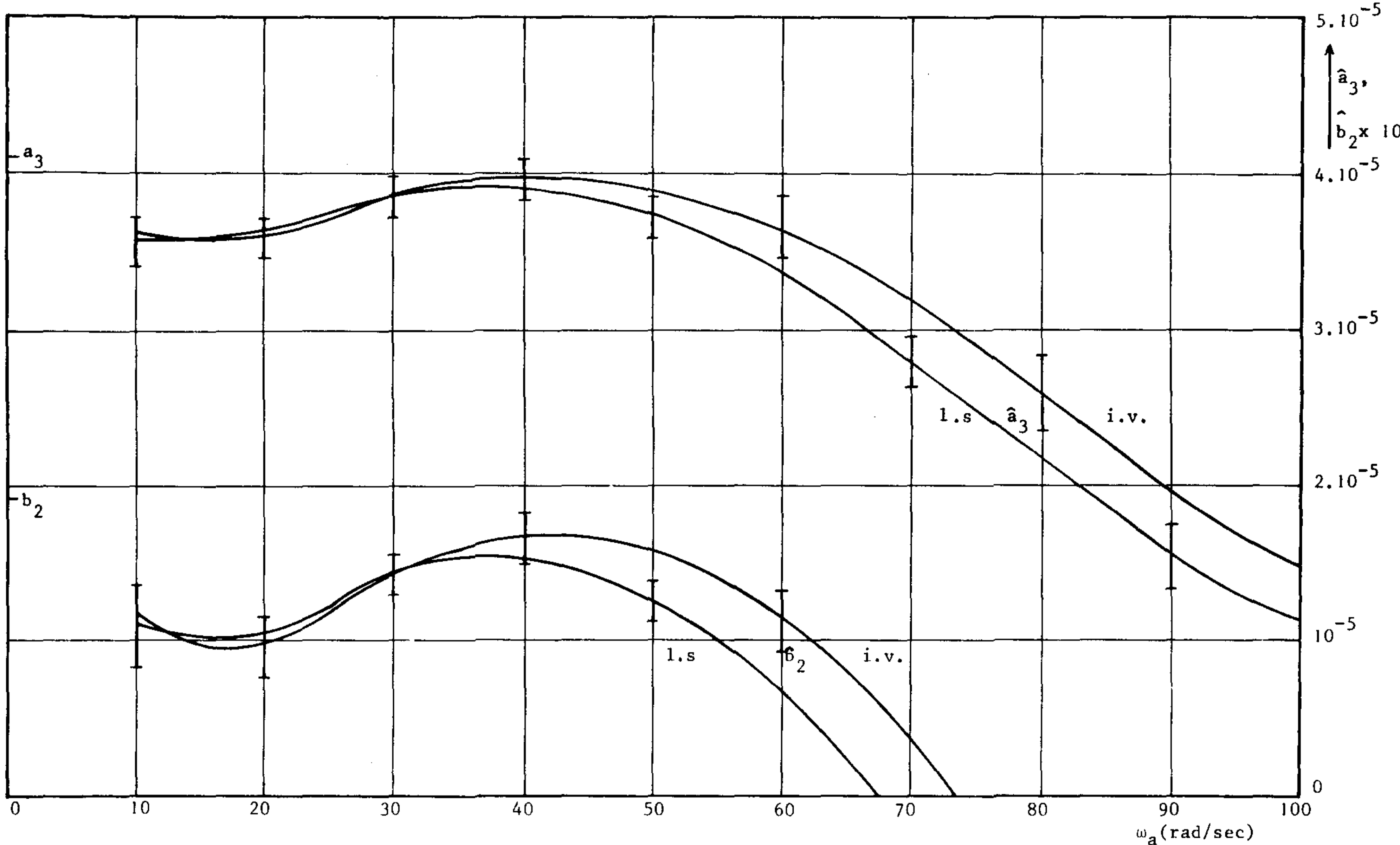
5.10$^{-5}$
4.10$^{-5}$
3.10$^{-5}$
2.10$^{-5}$
10$^{-5}$
0
$\hat{a}_3$, $\hat{b}_2$ x 10$^2$
$a_3$
$\hat{a}_3$
$b_2$
$\hat{b}_2$
1
2
1
2
100
90
80
70
60
50
S/N(dB)


Figure 2.20



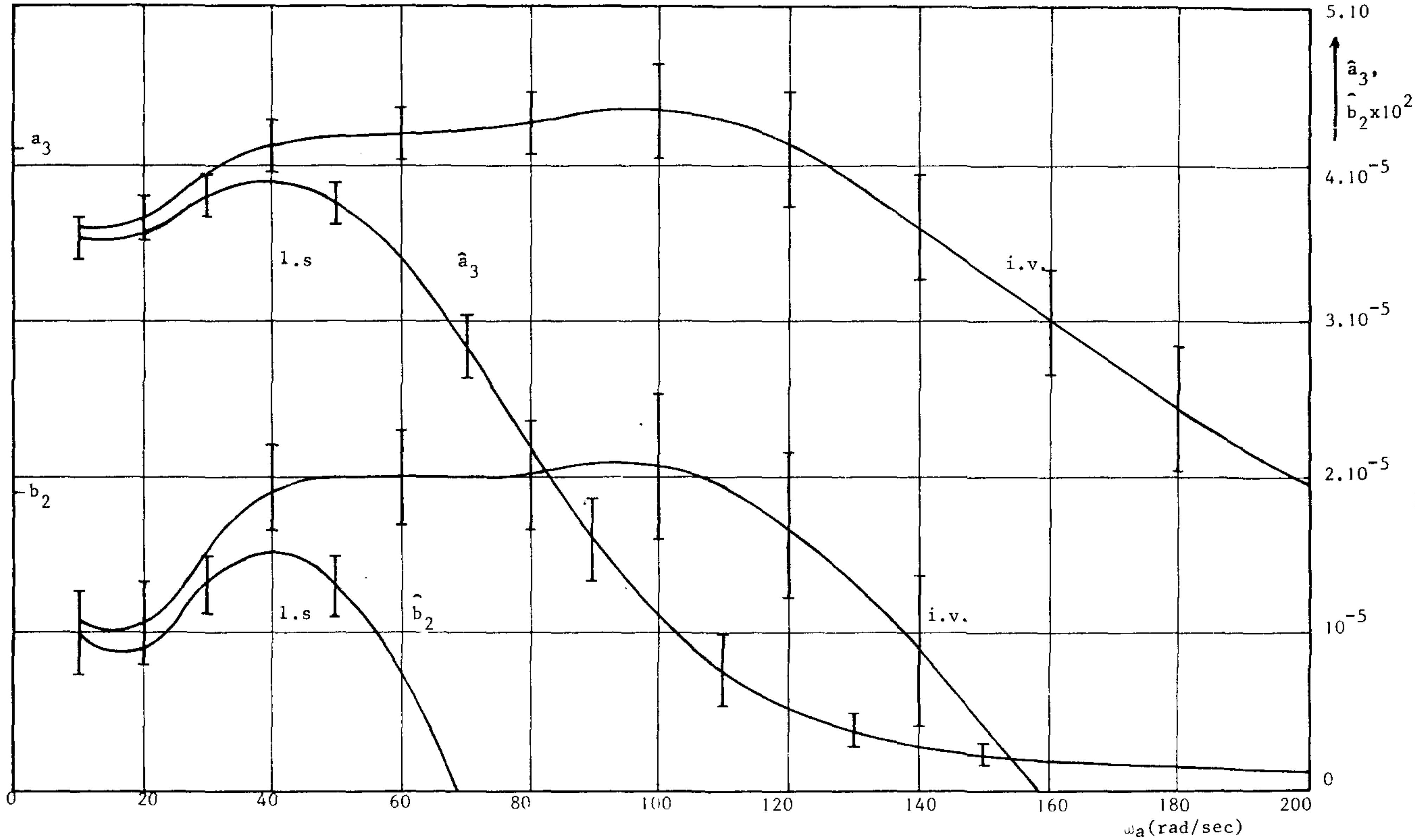
$b_1$
$a_1$
$b_o$
$a_2$
$\hat{b}_1$
$\hat{a}_1$
$\hat{b}_o$
$\hat{a}_2$
10
9
8
7
6
5
$b_o \times 10^{-1}$
$b_1 \times 10^{-1}$
$a_1 \times 10^{-2}$
$a_2 \times 10^{-3}$
100
90
80
70
60
50
S/N(dB)


Figure 2.21



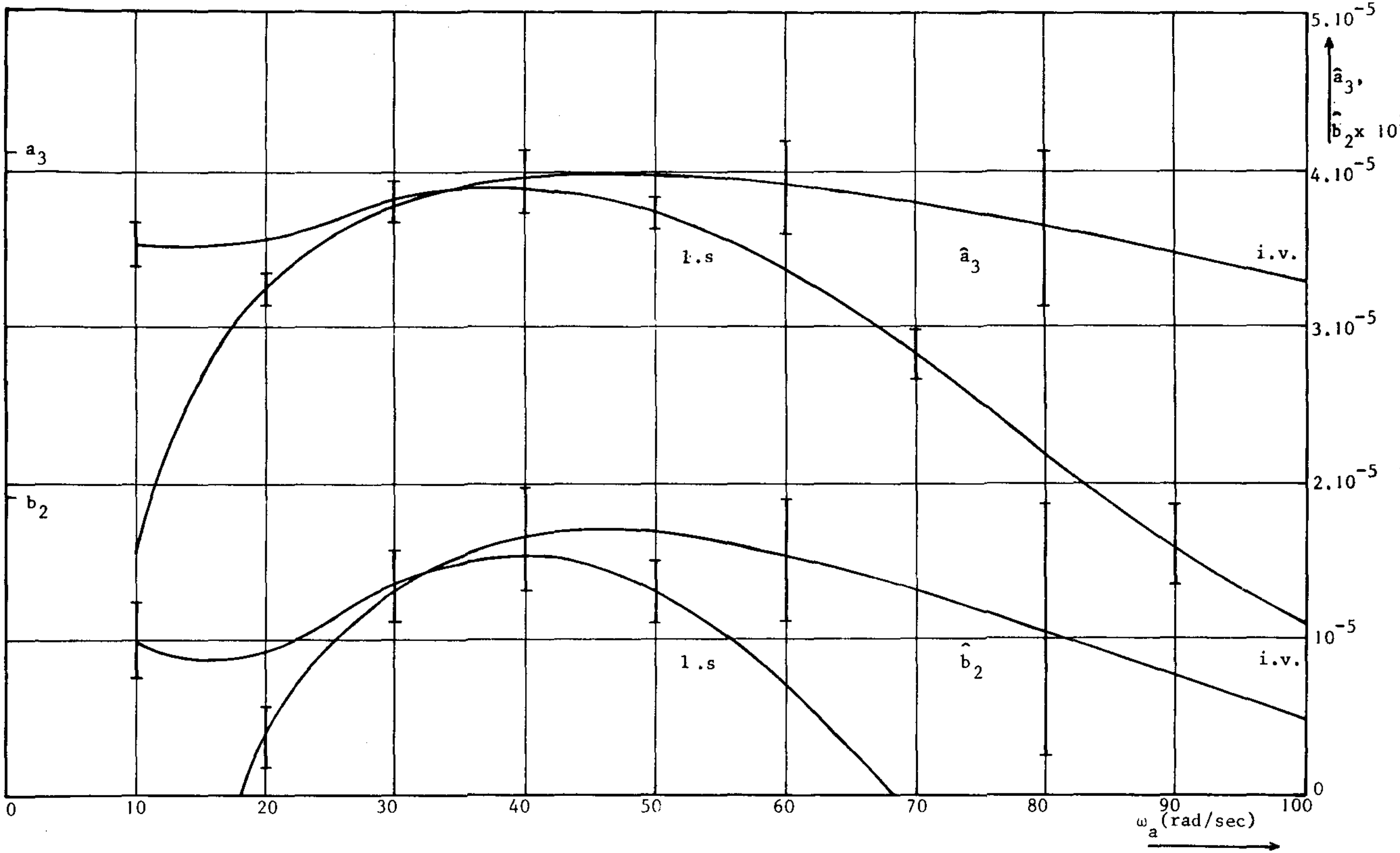
a3
b2
5.10^-5
â3, b̂2 x 10^2
4.10^-5
3.10^-5
2.10^-5
10^-5
0
l.s
â3
i.v.
l.s
b̂2
i.v.
0
10
20
30
40
50
60
70
80
90
100
ωa (rad/sec)


Figure 3.5



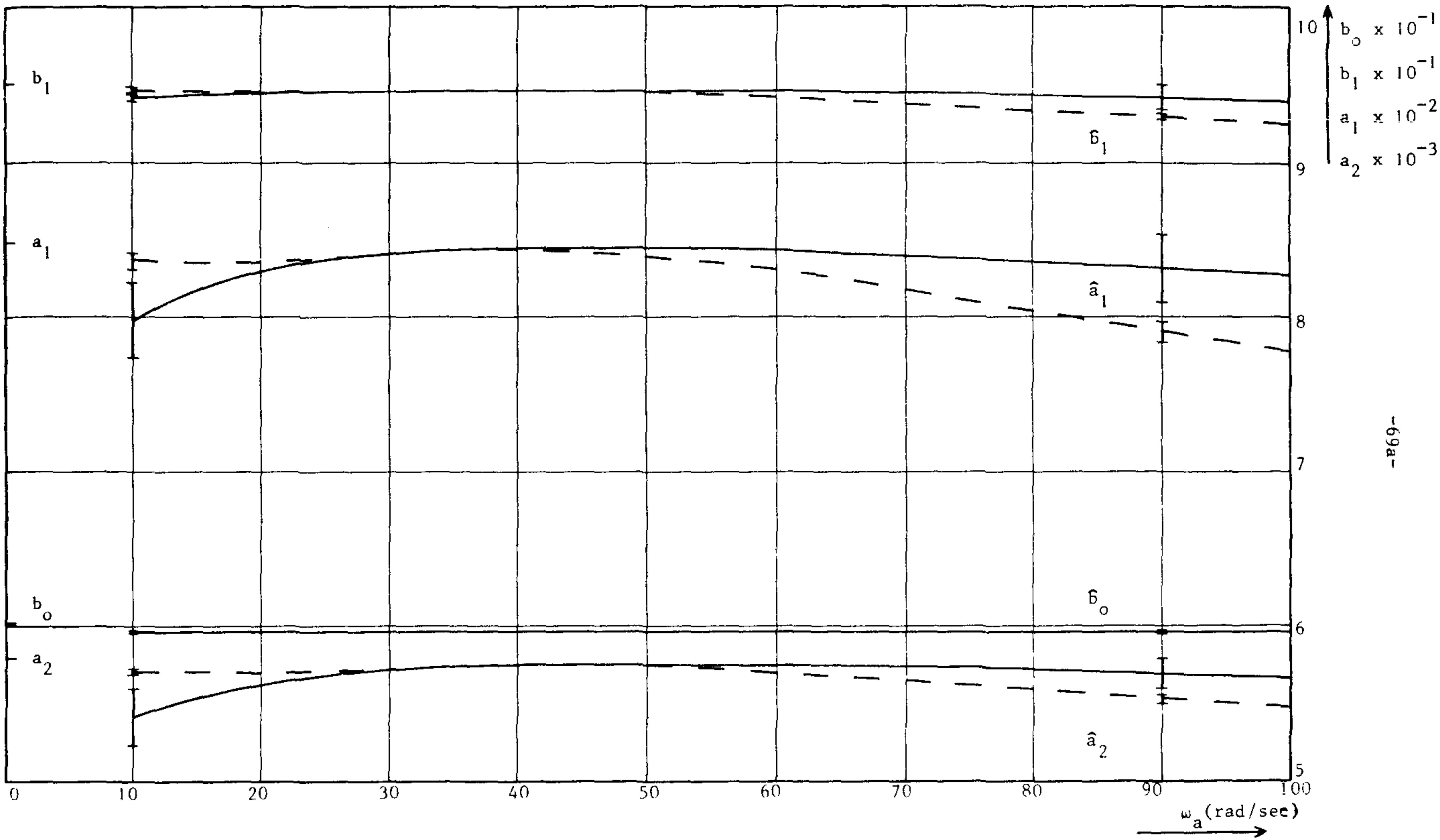
5.10
$\hat{a}_3$,
$\hat{b}_2 x10^2$
$a_3$
$4.10^{-5}$
l.s
$\hat{a}_3$
i.v.
$3.10^{-5}$
$2.10^{-5}$
$b_2$
l.s
$\hat{b}_2$
i.v.
$10^{-5}$
0
0
20
40
60
80
100
120
140
160
180
200
$\omega_a$(rad/sec)


Figure 3.6



a3
b2
5.10-5
4.10-5
3.10-5
2.10-5
10-5
0
â3, b̂2 x 10^2
l.s
â3
i.v.
l.s
b̂2
i.v.
0
10
20
30
40
50
60
70
80
90
100
ωa (rad/sec)


Figure 3.7



b1
a1
b0
a2
b̂1
â1
b̂0
â2
10
9
8
7
6
5
b0 x 10^-1
b1 x 10^-1
a1 x 10^-2
a2 x 10^-3
0
10
20
30
40
50
60
70
80
90
100
ωa(rad/sec)


Figure 3.8

## Appendix I: Newton's interpolation formula.

The interpolating polynomial of degree n in t, based on n+1 equidistant values of some function f(x), can be written as [ Sauer, 1968].

$$f(x) = f(x_o) + \binom{t}{1}\Delta f_o + \binom{t}{2}\Delta^2 f_o + \ldots\ldots + \binom{t}{n}\Delta^n f_o \qquad (I.1)$$

with:

$x = x_o + th$; h is the interval length,
$f_i$; $i = 0,1, \ldots, n+1$, the values of f(x) at $x = 0, h, 2h, \ldots, ih, \ldots$
$\Delta$ = the forward difference operator

This expression is known as Newton's interpolation formula.
The forward difference operator is defined by:

$$\Delta f_i = f_{i+1} - f_i \qquad (I.2)$$

Higher order operators can be obtained from the recurrent relationship:

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i \qquad (I.3)$$

The given interpolation polynomial can be rewritten, using the forward shift-operator z. The shiftoperator is defined by:

$$z^k f_i = f_{i+k} \qquad (I.4)$$

So it follows from eq. (I.2):

$$\Delta f_i = (z-1) f_i$$

Now assume the following relation to be valid for some k:

$$\Delta^k f_i = (z-1)^k f_i \qquad (I.5)$$

Then it follows from eq. (I.3) for k+1:

$$\Delta^{k+1} f_i = \Delta^k f_{i+1} - \Delta^k f_i = (z-1)^k f_{i+1} - (z-1)^k f_i$$

$$= z(z-1)^k f_i - (z-1)^k f_i = (z-1)^{k+1} f_i$$

So eq. (I.5) is also valid for k+1, when assuming its validity for k. Then from complete induction it follows that eq. (I.5) is valid for all k. Application of the derived relationship between the shiftoperator and the difference operator to eq. (I.1) gives:

$$f(x) = f(x_o) + \binom{t}{1}(z-1)f_o + \binom{t}{2}(z-1)^2 f_o + \ldots. + \binom{t}{n}(z-1)^n f_o \qquad (I.6)$$

This represents of course only another notation of the interpolating polynomial.

## Appendix II: Approximation to the inverse Laplace transform.

The inverse Laplace transform of some function F(s) is given by:

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{ts}\, ds \qquad \text{(II.1)}$$

where f(t) denotes the original time function.
It is assumed that the convergence region of the integral in this equation contains the imaginary axis of the s-plane, which allows the choice c=0. An approximation of eq.(II.1) can then be obtained for some functions F(s) of which the functional values for s=jω become arbitrary small, provided ω is choosen large enough.
Let $\omega = \frac{\pi}{T}$ denote a frequency above which the values of F(s), with s=jω, can be neglected. The integral of eq.(II.1) can then be splitted up into three parts:

$$f(t) = \frac{1}{2\pi j} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} F(s)e^{ts}\, ds + \frac{1}{2\pi j} \int_{-j\infty}^{-\frac{\pi}{T}} F(s)e^{ts}\, ds + \frac{1}{2\pi j} \int_{\frac{\pi}{T}}^{j\infty} F(s)e^{ts}\, ds$$

The approximated timefunction, $f^*(t)$, can be obtained by ignoring the last two integrals, yielding:

$$f^*(t) = \frac{1}{2\pi j} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} F(s)e^{ts}\, ds. \qquad \text{(II.2)}$$

For the calculation of this equation use can be made of the Z-transform [Jury, 1964]. By choosing a sampling frequency of twice the highest frequency, and substituting in eq.(II.2), the sampled values of $f^*(t)$ become:

$$f^*(nT) = \frac{1}{2\pi j} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} F(s)e^{snT}\, ds, \; n=0,1,2,\ldots\ldots \qquad \text{(II.3)}$$

Transformation of this equation to the z-plane, with

$$z = e^{Ts}; \; -\frac{\pi}{T} j < s < \frac{\pi}{T} j \qquad \text{(II.4)}$$

and

$$s = \frac{1}{T} \log z \; ; \text{ principal value of the logarithm} \qquad \text{(II.5)}$$

gives:

$$f^{*}(nT) = \frac{1}{2\pi j} \oint \frac{1}{T} F(s = \frac{1}{T} \log z) z^{n-1} \, dz \qquad \text{(II.6)}$$

where the contour integral has to be calculated on the unit circle of the z-plane.
The inverse z-transform of some function G(z) is given by:

$$g(nT) = \frac{1}{2\pi j} \oint G(z) z^{n-1} \, dz \qquad \text{(II.7)}$$

where g(nT) denotes the sequence of sample values of the original of G(z). Comparison of eq.(II.7) and (II.6) shows that, apart from a factor T, the function $F(s = \frac{1}{T} \log z)$ is the z-transform of the sequence $f^{*}(nT)$. In order to calculate the values of $f^{*}(nT)$ directly from $F(s = \frac{1}{T} \log z)$ a second approximation will be made. This is necessary as the function is transcendental in z, which makes it impossible to expand it into powers of $z^{-1}$ (long division). The coefficients of such an expansion are the values of the discrete time function.
For the approximation of $F(s = \frac{1}{T} \log z)$ one can use approximations of logz as substitutial approximation operators. A large number of such approximation operators are known, for a review and results see [Cuénod, 1969]. They can be constructed for example from the continued fraction expansion of logz, round z=1, and the use of its (1,1) Taylorconvergent as a linear approximation operator:

$$\frac{1}{T} \log z \approx \frac{2}{T} \frac{z-1}{z+1}$$

Or, when applied directly to F(s):

$$s \approx \frac{2}{T} \frac{z-1}{z+1} \qquad \text{(II.8)}$$

This equation is readily recognized as the bilinear z-transform. In the application discussed here the operator (II.8) is known as Tustin's operator [Tustin, 1947].

Using of this and other substitutial operators one is able to transform rational functions F(s) into rational functions F(z). The sequence of sampled values obtained by expansion of F(z) in powers of $z^{-1}$ yields also an approximation of the original time function f(t), the inverse Laplace transform of F(s).

## Appendix III: The construction of Taylor convergents.

Let $H^*(s)$ be a given rational function:

$$H^*(s) = \frac{c_o+c_1s+c_2s^2+\ldots+c_fs^f}{1+d_1s+d_2s^2+\ldots+d_gs^g} \qquad \text{(III.1)}$$

From this given function another rational function of lower order, H(s), has to be constructed:

$$H(s) = \frac{b_o+b_1s+b_2s^2+\ldots+b_ps^p}{1+a_1s+a_2s^2+\ldots+a_qs^q}\ ;\ p\leq f,\ q\leq g. \qquad \text{(III.2)}$$

When this construction is performed by demanding that a maximal number of coefficients, i.e. in general p+q+1, of the Taylor expansions of eq.(III.1) and eq.(III.2) are equal, such a construction is called a (p,q) Taylorconvergent of $H^*(s)$.

The coefficients of the Taylorconvergent H(s) can be obtained by multiplying the Taylor expansion of $H^*(s)$ with the denominator of $H^*(s)$ and the denominator of H(s), yielding:

$$(c_o+c_1s+c_2s^2+\ldots+c_fs^f)(1+a_1s+a_2s^2+\ldots+a_qs^q) \qquad \text{(III.3)}$$

Multiplying of the Taylor expansion of H(s) with the denominator of $H^*(s)$ and the denominator of H(s) gives:

$$(b_o+b_1s+b_2s^2+\ldots+b_ps^p)(1+d_1s+d_2s^2+\ldots+d_gs^g) \qquad \text{(III.4)}$$

By equating both polynomials (III.3) and (III.4) equal upto the degree p+q+1, a set of p+q+1 equations is obtained:

$$d_ib_o+d_{i-1}b_1+d_{i-2}b_2+\ldots+d_{i-p}b_p = \qquad \text{(III.5)}$$

$$c_i+c_{i-1}a_1+c_{i-2}a_2+\ldots+c_{i-q}a_q,$$

$$i = 0,1,2,\ldots,p+q;$$

with the convention: $a_o=1$ ; $a_i=0$ for $i > q$ or $i < 0$;

$b_i=0$ for $i > p$ or $i < 0$.

This set of equations can be written in a convenient matrix notation:

$$\underline{c} = Q\,\underline{b}, \tag{III.6}$$

with:

$$\underline{c}^T = [\,c_o, c_1, c_2, \ldots, c_f, 0, 0, \ldots, 0\,] \;;\; \text{dimension } p+q+1,$$

$$\underline{b}^T = [\,b_o, b_1, b_2, \ldots, b_p, -a_1, -a_2, \ldots, -a_q\,]\,,$$

$$Q = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
d_1 & 1 & 0 & 0 & & \vdots & c_o & 0 & 0 & & \vdots \\
d_2 & d_1 & 1 & 0 & & \vdots & c_1 & c_o & 0 & & \vdots \\
d_3 & d_2 & d_1 & 1 & & \vdots & c_2 & c_1 & c_o & & \vdots \\
\vdots & \vdots & \vdots & d_1 & & 0 & \vdots & \vdots & \vdots & \ddots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 1 & \vdots & \vdots & \vdots & & c_o \\
\vdots & \vdots & \vdots & \vdots & & d_1 & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & c_{f-1} & c_{f-2} & c_{f-3} & \cdots & c_{f-q} \\
d_{g-1} & d_{g-2} & d_{g-3} & d_{g-4} & \cdots & d_{g-p-1} & c_f & c_{f-1} & c_{f-2} & \cdots & c_{f-q+1} \\
d_g & d_{g-1} & d_{g-2} & d_{g-3} & \cdots & d_{g-p} & 0 & c_f & c_{f-1} & \cdots & c_{f-q+2} \\
0 & d_g & d_{g-1} & d_{g-2} & \cdots & d_{g-p+1} & 0 & 0 & c_f & \cdots & c_{f-q+3} \\
0 & 0 & d_g & d_{g-1} & \cdots & d_{g-p+2} & 0 & 0 & 0 & & \vdots \\
0 & 0 & 0 & d_g & \cdots & d_{g-p+3} & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0\,.. & d_g \,\ldots & d_q & 0 & 0 & 0\,.. & c_f\,\ldots & c_p
\end{bmatrix}$$

(with the same convention for the indices as in eq.(III.5)).

The coefficients (parameters) of H(s) can also be calculated from:

$$\underline{b} = Q^{-1}\underline{c} \tag{III.7}$$

assuming Q to be a non-singular matrix.

As can be seen from its construction only terms up to the degree p+q of $H^*(s)$ enter the calculations. It is therefore possible to apply this scheme also to functions of the type:

$$H^{**}(s) = \frac{c_o+c_1w(s)+c_2w^2(s)+\ldots+c_fw^f(s)}{1+d_1w(s)+d_2w^2(s)+\ldots+d_gw^g(s)} \qquad \text{(III.8)}$$

By Taylor expansion of w(s), $w^2(s)$, etc. up to the degree p+q, a rational function is constructed, of which the (p,q) Taylor convergent is the same as the one of $H^{**}(s)$.

## Appendix IV: Incorporation of some a-priori information of the continuous process in its discrete model.

It will be assumed here that the bilinear z-transform applies to the discrete model of the continuous process. As will be clear from substitution of this transform in the transfer function of the discrete model, the continuous parameters are lineair combinations of the discrete parameters. For example, the d.c. component $b_o$ is given by:

$$b_o = \sum_{i=o}^{p} \beta_i - b_o \sum_{i=1}^{q} \alpha_i \tag{IV.1}$$

where $\beta_i$, $\alpha_i$ are the parameters of the discrete model. Relations of the same kind apply to the other parameters of the continuous process.
If a number of parameters of the continuous process is known, or also, if linear relationships of these parameters are given, this information can be transferred to the discrete model, yielding a number, say k, of linear relations between the parameters of the discrete model. Let these relations be given by:

$$\begin{array}{l} \mu_{11}\,\beta_o + \mu_{12}\,\beta_1 + \ldots\ldots\ldots + \mu_{1z}\,\beta_z = \nu_1 \\ \mu_{21}\,\beta_o + \mu_{22}\,\beta_1 + \ldots\ldots\ldots + \mu_{2z}\,\beta_z = \nu_2 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \mu_{k1}\,\beta_o \quad \mu_{k2}\,\beta_1 + \ldots\ldots\ldots + \mu_{kz}\,\beta_z = \nu_k \end{array} \tag{IV.2}$$

With:

$\mu_{ij}$; $\nu_i$; i, j = 1,....,k are the given coefficients,

$\beta_i$ ; i = 0,....z are the parameters of the discrete model (for simplicity, no difference has been made in the notation of forward and backward parameters).

Using the elimination method of Gauss, this set can be written in a semi-diagonal form:

$$
\begin{array}{llll}
\mu^*_{11}\,\beta_o & & + \mu^*_{1,k+1}\,\beta_k \;\cdots\cdots + \mu^*_{1z}\,\beta_z = \nu^*_1 & \\
& \mu^*_{22}\,\beta_1 & + \mu^*_{2,k+1}\,\beta_k \;\cdots\cdots + \mu^*_{2z}\,\beta_z = \nu^*_2 & \\
& & \vdots \qquad\qquad\qquad \vdots \qquad \vdots & \quad (IV.3) \\
& \mu^*_{kk}\,\beta_{k+1} & + \mu^*_{k,k+1}\,\beta_k \;\cdots\cdots + \mu^*_{kz}\,\beta_z = \nu^*_k &
\end{array}
$$

The partial solution of these equations is:

$$
\begin{array}{ll}
\beta_o = \mu^{**}_{1,k+1}\,\beta_k + \cdots\cdots + \mu^{**}_{1z}\,\beta_z + \nu^{**}_1 & \\
\beta_1 = \mu^{**}_{2,k+1}\,\beta_k + \cdots\cdots + \mu^{**}_{2z}\,\beta_z + \nu^{**}_2 & \quad (IV.4) \\
\vdots & \\
\beta_{k-1} = \mu^{**}_{k,k+1}\,\beta_k + \cdots\cdots + \mu^{**}_{kz}\,\beta_z + \nu^{**}_k & \\
\beta_k = \qquad \beta_k & \\
\vdots & \\
\beta_z = \qquad\qquad\qquad \beta_z &
\end{array}
$$

where the parameters $\beta_k,\ldots,\beta_z$ have been added to make the following matrix notation possible:

$$\underline{\beta} = \mu\,\underline{\beta}^* + \underline{\nu}, \qquad (IV.5)$$

with:

$$\underline{\beta}^{*T} = [\,\beta_k, \beta_{k+1}, \ldots\ldots, \beta_z]$$

The elements of the matrix $\mu$ and the vector $\underline{\nu}$ can be obtained from eq. (IV.4).
Equation (IV.5) contains the available information of the continuous process, obtained by eliminating k discrete parameters.
Writing the output of the discrete model as:

$$\underline{y} = \Omega\,\underline{\beta}$$

It follows:

$$\underline{y} = \Omega \, \mu \, \underline{\beta}^{*} + \Omega \, \underline{\nu} \tag{IV.6}$$

The least squares solution of this equation is given by:

$$\hat{\underline{\beta}}^{*} = \left[ (\Omega \, \mu)^{T} \, \Omega \, \mu \right]^{-1} (\Omega \, \mu)^{T} (\underline{y} - \Omega \, \underline{\nu}) \tag{IV.7}$$

From these estimated parameters, estimates of the parameters of the discrete model can be obtained by using eq. (IV.5).

## Appendix V : Discrete estimation schemes.

### V.1 The extented matrix method.

Only a short description of this technique will be presented here. A detailed analysis of this method can be found in [Talmon, 1971].
Consider a discrete process (fig. V.1) whose input and output relationship is given by the difference equation:

$$x_k = \sum_{i=o}^{p} b_i\, u_{k-i} - \sum_{i=1}^{q} a_i x_{k-i}, \qquad q \geq p \tag{V.1}$$

with:

$x_k$ = the sequence of output samples,
$u_k$ = the sequence of input samples,
$b_i$, $a_i$ are the parameters of the discrete process.

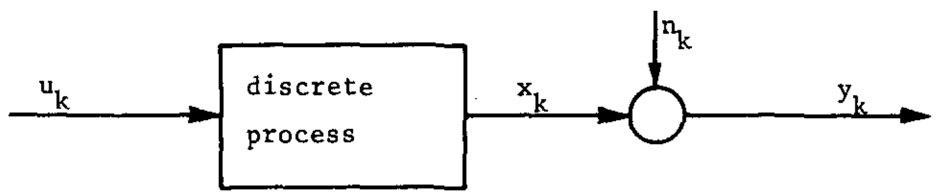



Fig. V.1

The available output is disturbed by an additive noise signal $n_k$:

$$y_k = x_k + n_k \tag{V.2}$$

Combining eq. (V.1) and eq. (V.2) gives:

$$y_k = \sum_{i=o}^{p} b_i\, u_{k-i} - \sum_{i=1}^{q} a_i\, y_{k-i} + n_k + \sum_{i=1}^{q} a_i\, n_{k-i}. \tag{V.3}$$

Define the equation error $e_k$ with:

$$e_k = n_k + \sum_{i=1}^{q} a_i n_{k-i} \tag{V.4}$$

Now suppose that the equation error sequence can be obtained from a white noise sequence $\xi_k$:

$$e_k = \xi_k + \sum_{i=1}^{s} c_i\, \xi_{k-i} + \sum_{i=1}^{r} d_i\, e_{k-i}, \tag{V.5}$$

with:

$c_i$, $d_i$ the forward, respectively backward parameters of the assumed noise process.

Together with eq. (V.4) and eq. (V.3) it follows:

$$y_k = \sum_{i=o}^{p} b_i u_{k-i} - \sum_{i=1}^{q} a_i y_{k-i} + \sum_{i=1}^{s} c_i \xi_{k-i} - \sum_{i=1}^{r} d_i e_{k-i} + \xi_k \quad (V.6)$$

The extented matrix method is based on this equation. The assumed white noise sequence $\xi_k$ guarantees that the application of a least squares estimator will be bias-free.

Let N samples of the input and output signals be available, yielding N-q equations of the type (V.6). These can be written in matrix notation:

$$\underline{y} = \Omega_1 \underline{b}_1 + \underline{\xi}, \quad (V.7)$$

With:

$$\underline{y}^T = [\, y_{q+1}, y_{q+2}, \ldots\ldots\ldots, y_N \,] \;,$$

$$\underline{\xi}^T = [\, \xi_{q+1}, \xi_{q+2}, \ldots\ldots\ldots, \xi_N \,] \;,$$

$$\underline{b}^T = [\, b_o, b_1, \ldots\ldots, b_p, -a_1, \ldots\ldots, -a_q, c_1, \ldots\ldots, c_s, -d_1, \ldots\ldots, -d_r \,]$$

$$\Omega_1 = \begin{bmatrix} u_{q+1} \cdots\cdots u_{q+1-p} & y_q \cdots\cdots y_1 & \xi_q \cdots\cdots \xi_{q+1-s} & e_q \cdots\cdots e_{q+1-r} \\ \vdots \qquad\qquad \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots \\ u_N \cdots\cdots u_{N-p} & y_{N-1} \cdots y_{N-q} & \xi_{N-1} \cdots \xi_{N-s} & e_{N-1} \cdots e_{N-r} \end{bmatrix}$$

A consistent estimator of $\underline{b}$ is given by:

$$\underline{\beta}_1 = \left[ \Omega_1^T \, \Omega_1 \right]^{-1} \Omega_1^T \, \underline{y} \quad (V.8)$$

The sequences $e_k$ and $\xi_k$ are not available. Therefore it will be necessary to use an iterative version of the estimator (V.8), yielding estimates of the parameters after every k samples, $k = 1,\ldots,N$. Estimates of the values of $e_k$ and $\xi_k$ can then be calculated from:

$$\hat{e}_k = y_k + \sum_{i=1}^{q} \alpha_i(k) y_{k-i} - \sum_{i=o}^{p} \beta_i(k) u_{k-i} \qquad \text{(V.9)}$$

and:

$$\hat{\xi}_k = \hat{e}_k + \sum_{i=1}^{r} \delta_i(k) \hat{e}_{k-i} - \sum_{i=1}^{s} \gamma_i(k) \hat{\xi}_{k-i}, \qquad \text{(V.10)}$$

where $\alpha_i(k)$, $\beta_i(k)$, $\delta_i(k)$ and $\gamma_i(k)$ denote estimates of the parameters after k iterations.

The iterative estimator of eq. (V.7) is given by:

$$\underline{\beta}_{k+1} = \underline{\beta}_k - P_k \underline{\omega}_{k+1} \{\rho + \underline{\omega}_{k+1}^T P_k \underline{\omega}_{k+1}\}^{-1} (\underline{\omega}_{k+1}^T \underline{\beta}_k - y_{k+1}),$$

with:

$$P_k = \frac{1}{\rho} (P_{k-1} - P_{k-1} \underline{\omega}_k \{\rho + \underline{\omega}_k^T P_{k-1} \underline{\omega}_k\}^{-1} \underline{\omega}_k P_{k-1}),$$

$\omega_{k+1}$ = $(k+1)^{th}$ row of the matrix $\Omega_I$; the elements $e_k$ and $\xi_k$ replaced by their estimates,

$\rho$ = a weighting-factor.

The weighting factor has been introduced as the estimates of the equation error and the white noise sample will be bad in the beginning. The use of a weighting factor however fixes a lower bound to the standard deviations of the estimated parameters. This can be precluded by increasing the weighting factor in an exponential way during the iterative scheme, viz.

$$\rho_{k+1} = \rho_k + (1 - \rho_k)\Delta\rho \qquad \text{(V.12)}$$

## V.2 The instrumental variable method.

A comprehensive study of this technique applied to discrete processes can be found in [Smets, 1970]. Only a short description is given here.

Based on eq. (V.3) and eq. (V.4) a set of equations can be obtained for N available samples of the input and output signals. These equations can be written as:

$$\underline{y} = \Omega_2 \, \underline{b}_2 + \underline{e}, \tag{V.13}$$

with:

$$\underline{y}^T = [\, y_{q+1}, \, y_{q+2}, \ldots\ldots\ldots, y_N \,] \,,$$

$$\underline{e}^T = [\, e_{q+1}, \, e_{q+2}, \ldots\ldots\ldots, e_N \,] \,,$$

$$\underline{b}_2^T = [\, b_o, \, b_1, \ldots\ldots\ldots, b_p, \, -a_1, \ldots\ldots\ldots, -a_q \,] \,,$$

$$\Omega_2 = \begin{bmatrix} u_{q+1} & \cdots & u_{q+1-p} & y_q & \cdots & y_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ u_N & \cdots & u_{N-p} & y_{N-1} & \cdots & y_{N-q} \end{bmatrix}$$

A consistent estimator of this equation is given by:

$$\underline{\beta}_2 = \left[ Z^T \, \Omega_2 \right]^{-1} Z^T \underline{y} \tag{V.14}$$

with:

$i^{th}$ row of $Z = [\, u_i, \ldots\ldots, u_{i-p}, \, w_{i-1}, \ldots\ldots\ldots, w_{i-q} \,]$

$i = q+1, \ldots\ldots, N$

$w_i$ = the output signal sequence of a model of the process

The consistency of this estimator is based on the fact that the model output sequence $w_k$ and the noise sequence $n_k$ will be uncorrelated.
The parameters of the model of the process are not available. An iterative version of the estimator (V.14) can therefore be used, yielding estimated parameters after every k samples, $k = 1, \ldots\ldots, N$. An estimate of the model output can then be calculated from:

$$\hat{w}_k = \sum_{i=o}^{p} \beta_i(k) u_{k-i} - \sum_{i=1}^{q} \alpha_i(k) \hat{w}_{k-i} \tag{V.15}$$

where $\beta_i(k)$ and $\alpha_i(k)$ denotes the estimated process parameters after k iterations.

The recursive form of the estimator (V.14) is:

$$\underline{\beta}_{k+1} = \underline{\beta}_k - p_k \underline{z}_{k+1} \{ \rho + \underline{\omega}_{k+1}^T p_k \underline{z}_{k+1} \}^{-1} (\underline{\omega}_{k+1}^T \underline{\beta}_k - y_{k+1}),$$

with:

$$p_k = \frac{1}{\rho} (p_{k-1} - p_{k-1} z_k \{ \rho + \underline{\omega}_k^T p_{k-1} \underline{z}_k \}^{-1} \omega_k^T p_{k-1},$$

$\underline{\omega}_{k+1}$ = $(k+1)^{th}$ row of the matrix $\Omega_2$,

$\underline{z}_{k+1}$ = $(k+1)^{th}$ row of the matrix Z; the elements $\omega_k$ replaced by their estimates,

$\rho$ = a weighting factor.

The weighting factor is used as in the beginning bad estimates of the model output are obtained (cf. section V.1)

## Appendix VI: The calculation of the elementvalues of a network.

Let H(s) be a given admittance function:

$$H(s) = \frac{b_o + b_1 s + b_2 s^2 + \ldots\ldots\ldots + b_p s^p}{1 + a_1 s + a_2 s^2 + \ldots\ldots\ldots + a_q s^q} \qquad \text{(VI.1)}$$

If the network represented by this transfer function is also specified, the element values of this network can be obtained by use of the results of network synthesis [Weinberg, 1962] . In general this will be a rather difficult task and in a number of cases no unique solution can be obtained (this occurs if the specified network contains more elements than the number of coefficients in eq. (VI.1).

In some cases, however, the element values of the network can easily be calculated. Consider for example the 'delay-line' of fig. VI.1:

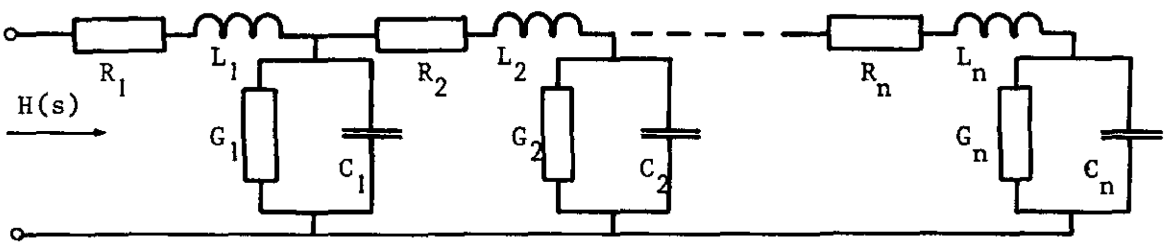



Fig. VI.1

If eq. (VI.1) is supposed to represent the input admittance of this network, it holds:

$$p = 2n-1, \quad q = 2n, \qquad \text{(VI.2)}$$

where n is the number of sections of the network.

The elementvalues of this network can now be calculated by first removing the first series branch from the impedance function $H^{-1}(s)$, observing that the order of the numerator of the remaining network decreases with 2:

$$H^{-1}(s) = \frac{1 + a_1 s + a_2 s^2 + \ldots\ldots + a_{2n} s^{2n}}{b_o + b_1 s + b_2 s^2 + \ldots\ldots + b_{2n-1} s^{2n-1}} =$$

$$R_1 + sL_1 + \frac{a_o^* + a_1^* s + \ldots\ldots + a_{2n-2}^* s^{2n-2}}{b_o + b_1 s + \ldots\ldots + b_{2n-1} s^{2n-1}}$$

The coefficients $R_1, L_1$ and $a_i^*$ can be obtained by long division, or solving 2n+1 linear equations.
Next the first parallel branch of the remaining admittance function is removed:

$$H^*(s) = \frac{b_o + b_1 s + b_2 s^2 + \ldots\ldots + b_{2n-1}\, s^{2n-1}}{a_o^* + a_1^* s + a_2^* s^2 + \ldots\ldots + a_{2n-2}^*\, s^{2n-2}} =$$

$$G_1 + sC_1 + \frac{b_o^* + b_1^* s + \ldots\ldots\ldots + b_{2n-3}^*\, s^{2n-3}}{a_o^* + a_1^* s + \ldots\ldots\ldots + a_{2n-2}^*\, s^{2n-2}}$$

These calculations can be repeated until the complete network has been realized.
The preceeding calculations yield the coefficients of the following continued fraction expansion of $H^{-1}(s)$:

$$H^{-1}(s) = R_1 + sL_1 + \cfrac{1}{G_1 + sC_1 + \cfrac{1}{R_2 + sL_2 + \ldots\ldots\ldots\ldots \ldots + \cfrac{1}{R_n + sL_n + \cfrac{1}{G_n + sC_n}}}}. \qquad \text{(VI.3)}$$

As an application we will finally treat the network discussed in chapter 4 of this report (fig. VI.2). Let H(s), with p = 2 and q = 3, again represent its input admittance.

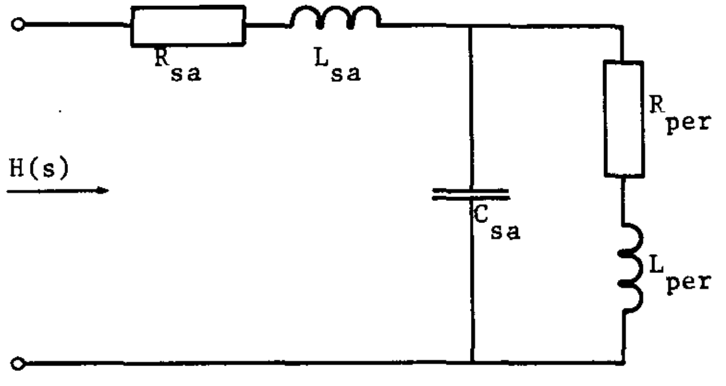



Fig. VI.2

First remove the series branch from the impedance function $H^{-1}(s)$:

$$H^{-1}(s) = \frac{1 + a_1 s + a_2 s^2 + a_3 s^3}{b_o + b_1 s + b_2 s^2} = \tag{VI.4}$$

$$R_{sa} + sL_{sa} + \frac{a_o^* + a_1^* s}{b_o + b_1 s + b_2 s^2} .$$

From the remaining admittance function, the first parallel branch is removed:

$$H^*(s) = \frac{b_o + b_1 s + b_2 s^2}{a_o^* + a_1^* s} = sC_{sa} + \frac{1}{R_{per} + sL_{per}} \tag{VI.5}$$

This removal, however, is only possible without error, if it holds:

$$b_1 a_1^* = b_2 a_o^* , \tag{VI.6}$$

the closure equation.

Together with the equations for $a_1^*$ and $a_o^*$ obtained by the removal of the series branch, this gives the following set of equations:

$$\begin{aligned}
1 &= a_o^* && + b_o R_{sa} \\
a_1 &= && a_1^* + b_1 R_{sa} + b_2 L_{sa} \\
a_2 &= && \quad b_2 R_{sa} + b_1 L_{sa} \\
a_3 &= && \quad\quad b_2 L_{sa} \\
0 &= -b_2 a_o^* + b_1 a_1^*
\end{aligned} \tag{VI.7}$$

As in practical application the coefficients entering these equations are only estimates, the set of equations given will be false.

This can be overcome by neglecting the $4^{th}$ equation, observing that the coefficient $a_3$ is very small, and will be the most inaccurate estimate. The equations can then be solved easily.

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