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Summary

The continuous, translation invariant, linear operators from $L^p_{\mathrm{loc}}(\mathbb{R})$ into $L^p_{\mathrm{loc}}(\mathbb{R})$ and from $L^p_{\mathrm{comp}}(\mathbb{R})$ into $L^p_{\mathrm{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$ are characterized. This characterization is in terms of the convolution ring $\mathrm{ba}_c(\mathbb{R})$ consisting of all compactly varying, right continuous functions of bounded variation. It turns out that for $p = 1$ and $p = \infty$, each translation invariant operator on $L^p_{\mathrm{loc}}(\mathbb{R})$ leaves invariant the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$.

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1 Function spaces

For $1 \leq p < \infty$ by $L^p(I\!R)$ we denote the Banach space of (equivalence classes of) Lebesque measurable functions f on $I\!R$ for which $|f|^p$ is integrable, with associated norm

$$\|f\|_p = \left(\int_{I\!R} |f(t)|^p dt \right)^{1/p} .$$

By $L^\infty(I\!R)$ we denote the Banach space of essentially bounded measurable functions on $I\!R$ with norm

$$\|f\|_\infty = \text{essup}_{t \in I\!R} |f(t)| .$$

For $1 \leq p < \infty$ and $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the Banach space $L^q(I\!R)$ represents the dual of $L^p(I\!R)$ in the sense that each continuous linear functional F on $L^p(I\!R)$ is of the form

$$F(g) = \int_{I\!R} g(t)f(t)dt$$

where $f \in L^q(I\!R)$ with $\|F\|_{,p} = \|f\|_q$.

For $A \subset I\!R$ let 1_A denote the characteristic function of the set A. The space $L^p_{\text{loc}}(I\!R))$, $1 \leq p \leq \infty$, consists of all measurable functions f on $I\!R$ for which $f \cdot 1_A$ belongs to $L^p(I\!R)$ for all bounded Borel sets $A \subset I\!R$. The locally convex topology for $L^p_{\text{loc}}(I\!R)$ is brought about by the countable set of seminorms $\{s_{p,n} \mid n \in I\!N\}$ defined by

$$(1.1) \qquad s_{p,n}(f) = \|f 1_{[-n,n]}\|_p .$$

Thus $L^p_{\text{loc}}(I\!R)$ is a complete metrizable locally convex space, i.e. a Frechet space. A linear functional F on $L^p_{\text{loc}}(I\!R)$ is continuous if and only if there are $n \in I\!N$ and $C > 0$ (both depending on the choice of F) such that

$$(1.2) \qquad |F(g)| \leq C s_{p,n}(g) , \quad \forall g \in L^p_{\text{loc}}(I\!R) .$$

The space $L^p_{\text{comp}}(I\!R)$ is the subspace of $L^p(I\!R)$ consisting of all $f \in L^p(I\!R)$ for which $f = f \cdot 1_K$ for some compact set $K \subset I\!R$, i.e. for which the support $\text{supp}(f)$ is bounded. Introducing the Banach subspaces $L^p_{,n}(I\!R)$ of $L^p(I\!R)$ by

$$f \in L^p_{,n}(I\!R) : \Leftrightarrow f \in L^p(I\!R) \text{ with } \text{supp}(f) \subset [-n, n]$$

we have

$$L^p_{\text{comp}}(I\!R) = \bigcup_{n=1}^{\infty} L^p_{,n}(I\!R) .$$

So, most naturally, $L^p_{\mathrm{comp}}(\mathbb{R})$ carries the (strict) inductive limit topology generated by the strict inductive system of Banach spaces $\{L^p_{,n}(\mathbb{R}) \mid n \in \mathbb{N}\}$, i.e. $L^p_{\mathrm{comp}}(\mathbb{R})$ is a strict LB-space. (For a transparant introduction of strict inductive limits see [Co, Ch. IV].) Therefore, a linear functional F on $L^p_{\mathrm{comp}}(\mathbb{R})$ is continuous if and only if the restriction of F to each $L^p_{,n}(\mathbb{R})$ is continuous. Identifying $L^p_{,n}(\mathbb{R})$ and $L^p([-n,n])$ and having in mind that for $1 \leq p < \infty$, $L^q([-n,n])$ represents the dual of $L^p([-n,n])$ it can be proved that each continuous linear function F on $L^p_{\mathrm{comp}}(\mathbb{R})$ is of the form

$$(1.3) \qquad F(g) = \int_{\mathbb{R}} f(t)g(t)dt \ , \quad g \in L^p_{\mathrm{comp}}(\mathbb{R})$$

for some $f \in L^q_{\mathrm{loc}}(\mathbb{R})$ where $\|F|_{L^p_{,n}(\mathbb{R})}\| = s_{q,n}(f)$.

Also, from the characterization of the continuous linear functionals on $L^p_{\mathrm{loc}}(\mathbb{R})$ as presented, we conclude that $L^q_{\mathrm{comp}}(\mathbb{R})$ represents its dual for $1 \leq p < \infty$. Indeed, let F be a linear functional on $L^p_{\mathrm{loc}}(\mathbb{R})$ satisfying (1.2) for some $n \in \mathbb{N}$. Then for all $g \in L^p_{\mathrm{loc}}(\mathbb{R})$, $F(g) = F(g \cdot 1_{[-n,n]})$ and $F|_{L^p_{,n}(\mathbb{R})}$ is continuous. So there exists $f \in L^q_{,n}(\mathbb{R})$ such that

$$(1.4) \qquad F(g) = F(g \cdot 1_{[-n,n]}) = \int_{\mathbb{R}} f(t)g(t)dt \ .$$

For notational convenience we introduce the bilinear form $\langle , \rangle_p$ on $L^p_{\mathrm{loc}}(\mathbb{R}) \times L^q_{\mathrm{comp}}(\mathbb{R})$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ by

$$(1.5) \qquad \langle g, f \rangle_p = \int_{\mathbb{R}} g(t)f(t)dt \ .$$

We conclude that each continuous linear functional on $L^p_{\mathrm{comp}}(\mathbb{R})$, $1 \leq p < \infty$, is given by

$$g \mapsto \langle f, g \rangle_q$$

and each continuous linear functional on $L^p_{\mathrm{loc}}(\mathbb{R})$

$$g \mapsto \langle g, f \rangle_p \ .$$

In the sequel we use the spaces $C(\mathbb{R})$, $C_c(\mathbb{R})$ and $\mathrm{ba}_c(\mathbb{R})$. Here $C(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$; it is a closed subspace of $L^\infty_{\mathrm{loc}}(\mathbb{R})$. So $C(\mathbb{R})$ is a Frechet space with respect to the seminorms $s_{\infty,n}$, $n \in \mathbb{N}$. The space $C_c(\mathbb{R})$ is the subspace of $C(\mathbb{R})$ consisting of all $f \in C(\mathbb{R})$ with bounded support. Define

$$C_{,n}(\mathbb{R}) = \{f \in C_c(\mathbb{R}) \mid \mathrm{supp}(f) \subset [-n,n]\} \ .$$

Then $C_{,n}(\mathbb{R})$ is a closed subspace of $L^\infty_{,n}(\mathbb{R})$ and

$$(1.6) \qquad C_c(\mathbb{R}) = \bigcup_{n \in \mathbb{N}} C_{,n}(\mathbb{R}) \ .$$

We see that $C_c(\mathbb{R})$ is a strict LB-space. The space $\mathrm{ba}_c(\mathbb{R})$ consists of all right–continuous functions of bounded variation on $\mathbb{R}$, i.e. a right–continuous function μ belongs to $\mathrm{ba}_c(\mathbb{R})$ if there exists $C > 0$ such that for any ordered tuple $t_1 < t_2 < ... < t_{N+1}$, $N \in \mathbb{N}$,

$$\sum_{j=1}^{N} |\mu(t_{j+1}) - \mu(t_j)| \leq C$$

and with the additional property that there exists $T > 0$ such that

$$\mu(t) = 0 \quad \text{for } t < -T \ ,$$

$$\mu(t) = \mu(T) \quad \text{for } t > T \ .$$

The space $\mathrm{ba}_c(\mathbb{R})$ represents (isomorphically) the dual of $C(\mathbb{R})$ in the sense that each continuous linear functional F on $C(\mathbb{R})$ is of the form

$$(1.7) \qquad F(g) = \int_{\mathbb{R}} g(t)d\mu(t) \ , \quad g \in C(\mathbb{R}) \ ,$$

where the integral is interpreted as a Riemann-Stieltjes integral. Moreover, $\mathrm{ba}_c(\mathbb{R})$ is a convolution ring without zero divisors, where the convolution is defined by

$$(1.8) \qquad (\mu_1 * \mu_2)(t) = \int_{\mathbb{R}} \mu_1(t-\tau)d\mu_2(\tau) \ .$$

For an extensive discussion of the convolution ring $\mathrm{ba}_c(\mathbb{R})$ we refer to [So] and [ES].
The dual of $C_c(\mathbb{R})$ can be represented by right continuous functions μ on $\mathbb{R}$ which are locally of bounded variation. We sketch the proof. First observe that if μ is a right continuous function such that for each $n \in \mathbb{N}$, μ has bounded variation on $[-n, n]$, the integral

$$F_\mu(g) = \int_{\mathbb{R}} g(t)d\mu(t)$$

is well–defined for each $g \in C_c(\mathbb{R})$ as a Riemann–Stieltjes integral, and for $g \in C_{,n}(\mathbb{R})$, $n \in \mathbb{N}$,

$$|F_\mu(g)| \leq \mathrm{var}(\mu|_{[-n,n]})\|g\|_\infty \ .$$

So F_μ is a continuous linear functional on $C_c(\mathbb{R})$. For the converse we apply the classical Riesz representation theorem for the dual of the Banach space $C[a, b]$. Identifying $C_{,n}(\mathbb{R})$ and the closed subspace $C_0[-n, n]$

$$C_0[-n,n] = \{f \in C[-n,n] \mid f(n) = f(-n) = 0\}$$

of $C[-n, n]$ we see that for each $n \in \mathbb{N}$ there is a right continuous function μ_n of bounded variation on $[-n, n]$ with $\mu_n(0) = 0$ such that

$$F(g) = \int_{-n}^{n} g(t)d\mu_n(t)\ , \quad g \in C_{,n}(\mathbb{R})$$

where F is a given continuous linear functional on $C_c(\mathbb{R})$. Since for all $n \in \mathbb{N}$ and $g \in C_{,n}(\mathbb{R})$

$$\int_{-n}^{n} g(t)d\mu_n(t) = \int_{-n-1}^{n+1} g(t)d\mu_{n+1}(t)$$

we have

$$\mu_{n+1}|_{(-n,n)} = \mu_n\ , \quad n \in \mathbb{N}\ .$$

So we can properly define μ on $\mathbb{R}$ by

$$\mu(t) = \mu_n(t)\ , \quad t \in (-n, n)$$

and we see that

$$F(g) = \int_{\mathbb{R}} g(t)d\mu(t)\ , \quad g \in C_c(\mathbb{R})\ .$$

Also, we shall employ the spaces $C^\infty(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$, which play a prominent role in classical distribution theory. The space $C^\infty(\mathbb{R})$ consists of all infinitely differentiable functions on $\mathbb{R}$. It is endowed with the Frechet topology brought about by the seminorms

$$w_n(f) = s_{\infty,n}(f^{(n)})\ , \quad n \in \mathbb{N}_0\ .$$

The space $C_c^\infty(\mathbb{R})$ consists of all functions in $C^\infty(\mathbb{R})$ with compact support and $C_c^\infty(\mathbb{R})$ is endowed most naturally with the strict inductive limit topology brought about by the closed subspaces $C_{,n}^\infty(\mathbb{R})$ of $C^\infty(\mathbb{R})$.

$$C_{,n}^\infty(\mathbb{R})\{f \in C^\infty(\mathbb{R}) \mid \mathrm{supp}(f) \subset [-n, n]\}\ .$$

So $C_c^\infty(\mathbb{R})$ is a strict LF-space, i.e. a strict countable inductive limit of Frechet spaces. In literature one often uses the notation $\mathcal{E}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ in stead of $C^\infty(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$, respectively. Part of the results mentioned here can be found in the monographs [DS] and [Sch].

2 Translation group, translation invariance

For a function f on $\mathbb{R}$ its translate $\sigma_t f$ is defined by

(2.1) $\quad (\sigma_t f)(\tau) = f(t + \tau)\ , \quad \tau \in \mathbb{R}\ .$

For measurable functions f_1 and f_2 on $\mathbb{R}$ with $f_1 = f_2$ almost everywhere, $\sigma_t f_1 = \sigma_t f_2$ almost everywhere. So the translation σ_t can be defined on all of the spaces $L^p_{\text{loc}}(\mathbb{R})$, $1 \le p \le \infty$. And for all $t \in \mathbb{R}$ the operator σ_t is continuous from $L^p_{\text{loc}}(\mathbb{R})$ into $L^p_{\text{loc}}(\mathbb{R})$, $L^p_{\text{comp}}(\mathbb{R})$ into $L^p_{\text{comp}}(\mathbb{R})$, $C(\mathbb{R})$ into $C(\mathbb{R})$ and $C_c(\mathbb{R})$ into $C_c(\mathbb{R})$. In fact, $(\sigma_t)_{t\in\mathbb{R}}$ is a group on each of these spaces. This translation group is strongly continuous for the spaces $C(\mathbb{R})$, $L^p_{\text{loc}}(\mathbb{R})$, $C_c(\mathbb{R})$ and $L^p_{\text{comp}}(\mathbb{R})$ whenever $1 \le p < \infty$. But not for the spaces $L^\infty_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{comp}}(\mathbb{R})$ which follows from the observation that

$$\|\sigma_t 1_{[0,1]} - 1_{[0,1]}\|_\infty = 1 \quad \forall t \in \mathbb{R} .$$

Being c_0-groups on Frechet spaces and strict inductive limits of Frechet spaces, respectively, we may apply the theory presented in [E1] and [E2]: In short, let V be a sequentially complete locally convex vector space and let $(\alpha_t)_{t\in\mathbb{R}}$ be a strongly continuous group of continuous linear operators on V. Then for each $\mu \in \text{ba}_c(\mathbb{R})$ the linear operator $\alpha[\mu]$ defined by the V-valued Riemann–Stieltjes integral

$$(2.2) \qquad \alpha[\mu]x = \int_{\mathbb{R}} \alpha_t x \; d\mu(t)$$

is continuous from V into V and for $\mu_1, \mu_2 \in \text{ba}_c(\mathbb{R})$

$$(2.3) \qquad \alpha[\mu_1 * \mu_2] = \alpha[\mu_1]\alpha[\mu_2]$$

where the convolution $*$ is defined in (1.8). Further it has been proved that for each $\mu \in \text{ba}_c(\mathbb{R})$ there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ in the linear span, $\text{span}(\{\sigma_t H \mid t \in \mathbb{R}\})$, such that for all $x \in V$

$$\lim_{n\to\infty} \alpha[\mu_n]x = \alpha[\mu]x .$$

Here H denotes the standard Heaviside function.

Let V denote any of the spaces $L^p_{\text{loc}}(\mathbb{R})$, $L^p_{\text{comp}}(\mathbb{R})$, $C(\mathbb{R})$, $C_c(\mathbb{R})$, $C^\infty(\mathbb{R})$, $C_c^\infty(\mathbb{R})$, where $1 \le p < \infty$, and let $\alpha_t = \sigma_t$ for all $t \in \mathbb{R}$. Then for $\mu \in \text{ba}_c(\mathbb{R})$, the operator $\sigma[\mu]$ is defined according to (2.2). So $\sigma[\mu]$ is a continuous translation invariant (i.e. $\sigma[\mu]\sigma_t = \sigma_t\sigma[\mu]$, $t \in \mathbb{R}$) linear operator from V into V. The question arises whether each continuous translation invariant linear operator from V into V is equal to $\sigma[\mu]$ for some $\mu \in \text{ba}_c(\mathbb{R})$. This question originates from the fact that for $V = C(\mathbb{R})$ it has been proven to be the case. But for $V = C^\infty(\mathbb{R})$ it is evidently not true; a continuous linear operator $\mathcal{L}$ from $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ is translation invariant if and only if $\mathcal{L} = p(d/dt)\sigma[\mu]$ for a polynomial p and $\mu \in \text{ba}_c(\mathbb{R})$. See [So].

Next we discuss the spaces $C_c(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$. We are aware of the fact that the results derived here for these spaces can be found in literature, e.g. in [Sch]. However, they are not formulated in our terminology and we like to keep this paper as self-contained as possible introducing no more terminology as necessary.

Theorem 1. Let $\mathcal{L}$ from $C_c(\mathbb{R})$ into $C_c(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant, if and only if there exists $\mu \in \mathrm{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$.

Proof. Because of the previous observations we only have to prove necessity.
So assume that $\mathcal{L}$ is translation invariant. Then $(\mathcal{L}f)(t) = (\mathcal{L}\sigma_t f)(0)$ for all $t \in \mathbb{R}$ and $f \in C_c(\mathbb{R})$. The linear functional $f \mapsto (\mathcal{L}f)(0)$ is continuous on $C_c(\mathbb{R})$. So there exists a right continuous function $\tilde{\mu}$ on $\mathbb{R}$ with $\tilde{\mu}|_I$ of bounded variation for each bounded interval I such that

$$(\mathcal{L}f)(0) = \int f(\tau)d\tilde{\mu}(\tau) , \quad f \in C_c(\mathbb{R}) .$$

We conclude that

$$(\mathcal{L}f)(t) = \int f(t+\tau)d\tilde{\mu}(\tau) , \quad f \in C_c(\mathbb{R}),\ t \in \mathbb{R} .$$

Continuity of $\mathcal{L}$ means that there is $m \in \mathbb{N}$ such that

$$\mathcal{L}(C_{,1}(\mathbb{R})) \subset C_{,m}(\mathbb{R})$$

and

$$\max_{t\in[-m,m]} |(\mathcal{L}f)(t)| \leq C \max_{t\in[-1,1]} |f(t)|$$

for all $f \in C_{,1}(\mathbb{R})$. Hence for all $f \in C_{,1}(\mathbb{R})$ and all $t \in \mathbb{R}$ with $|t| \geq m$

$$\int f(t+\tau)d\tilde{\mu}(\tau) = 0 .$$

It follows that $\tilde{\mu}(t) = \tilde{\mu}(m)$ for $t > m$ and $\tilde{\mu}(t) = \tilde{\mu}(-m)$ for $t < -m$. Now put

$$\mu(t) = \tilde{\mu}(t) - \tilde{\mu}(-m) , \quad t \in \mathbb{R}.$$

Then $\mu \in \mathrm{ba}_c(\mathbb{R})$ and for all $f \in C_c(\mathbb{R})$ and $t \in \mathbb{R}$,

$$(\mathcal{L}f)(t) = \int f(t+\tau)d\mu(\tau) = (\sigma[\mu]f)(t) .$$

□

To derive a similar result for the space $C_c^\infty(\mathbb{R})$ we have to do some preparations. For $\psi \in \mathrm{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$, its derivative $\frac{d\psi}{dt}$ belongs to $C_c^\infty(\mathbb{R})$. Also, for $\varphi \in C_c^\infty(\mathbb{R})$, we have $J\varphi \in \mathrm{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$, where

$$(J\varphi)(t) = \int_{-\infty}^{t} \varphi(\tau)d\tau .$$

So we can reformulate a result of Dixmier and Malliavin, see [DM] and [E2], in our terminology:
(2.4) For all $g \in C^\infty(\mathbb{R})$ there are $\psi_1, \psi_2 \in \mathrm{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $g_1, g_2 \in C^\infty(\mathbb{R})$ such that

$$g = \sigma[\psi_1]g_1 + \sigma[\psi_2]g_2 .$$

Further, we observe that for $\varphi \in C_c^\infty(\mathbb{R})$ and $\psi \in \mathrm{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$

$$(2.5) \qquad \sigma[\psi]\varphi = \sigma[J\varphi]\frac{d\psi}{dt} .$$

We use the notation $\check{\mu}(t) = -\mu(-t)$ such that

$$\sigma[\check{\mu}]f = \int \sigma_{-t}f \, d\mu(t) .$$

Theorem 2. Let $\mathcal{L} : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if only if there are a polynomial p and $\mu \in \mathrm{ba}_c(\mathbb{R})$ such that $\mathcal{L} = p(\frac{d}{dt})\sigma[\mu]$.

Proof. The sufficiency of the condition is readily established. We prove its necessity. From [E2] we conclude that

$$-- \ \forall \nu \in \mathrm{ba}_c(\mathbb{R}) : \ \sigma[\nu]\mathcal{L} = \mathcal{L}\sigma[\nu] ,$$

$$-- \ \forall j \in \mathbb{N} : \ (\frac{d}{dt})^j\mathcal{L} = \mathcal{L}(\frac{d}{dt})^j$$

and so for all $g \in C^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R})$ the function

$$t \mapsto \langle \sigma_t \mathcal{L}\varphi, g\rangle_1 , \quad t \in \mathbb{R}$$

belongs to $C^\infty(\mathbb{R})$. Moreover, for all $\psi \in \mathrm{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$

$$\sigma[\psi]\mathcal{L}\varphi = \sigma[J\varphi]\mathcal{L} \ \frac{d\psi}{dt} .$$

Let $g \in C^\infty(\mathbb{R})$. Then by (2.4) there are $\psi_1, \psi_2 \in \mathrm{ba}_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $g_1, g_2 \in C^\infty(\mathbb{R})$ such that

$$g = \sigma[\check{\psi}_1]g_1 + \sigma[\check{\psi}_2]g_2 .$$

Hence for all $\varphi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} \langle \mathcal{L}\varphi, g\rangle &= \langle \sigma[\psi_1]\mathcal{L}\varphi, g_1\rangle_1 + \langle \sigma[\psi_2]\mathcal{L}\varphi, g_2\rangle_1 \\ &= \langle \sigma[J\varphi]\mathcal{L} \ \frac{d\psi_1}{dt} , g_1\rangle_1 + \langle \sigma[J\varphi]\mathcal{L} \ \frac{d\psi_2}{dt} , g_2\rangle_1 \\ &= \int \varphi(t)(\langle \sigma_t\mathcal{L} \ \frac{d\psi_1}{dt} , g_1\rangle_1 + \langle \sigma_t\mathcal{L} \ \frac{d\psi_2}{dt} , g_2\rangle_1)dt . \end{aligned}$$

So the uniquely defined distribution $\mathcal{L}^*g$,

$$(\mathcal{L}^*g)(\varphi):\ \langle\mathcal{L}\varphi,g\rangle_1$$

is represented by the C^∞-function

$$t\mapsto\langle\sigma_t\mathcal{L}\ \frac{d\psi_1}{dt}\ ,g_1\rangle_1+\langle\sigma_t\mathcal{L}\ \frac{d\psi_2}{dt}\ ,g_2\rangle_1\ .$$

It follows that $\mathcal{L}^*$ maps $C^\infty(I\!R)$ into $C^\infty(I\!R)$ as a continuous, translation invariant linear mapping. We note that the continuity is a consequence of the Closed Graph Theorem. So as observed earlier, there are a polynomial p and $\mu\in\mathrm{ba}_c(I\!R)$ such that

$$\mathcal{L}^*=p(-\frac{d}{dt})\sigma[\check{\mu}]\ .$$

We conclude that $\mathcal{L}=p(\frac{d}{dt})\sigma[\mu]$ (and a forteriori that $\mathcal{L}$ extends to a continuous linear operator on $C^\infty(I\!R)$). □

Now let V be one of the Frechet spaces $L_{\mathrm{p,loc}}(I\!R)$, $1\le p<\infty$, and let $\mathcal{L}$ from V into V be continuous, translation invariant and linear. Then in [E1] we proved that $C^\infty(I\!R)$ is an invariant subspace of $\mathcal{L}$ and $\mathcal{L}|_{C^\infty(I\!R)}$ maps $C^\infty(I\!R)$ into $C^\infty(I\!R)$ continuously. (In fact, in the terminology of the mentioned paper, $L^p_{\mathrm{loc}}(I\!R)$ is a translatable Frechet space.) It follows from the observations at the beginning of this section that

$$\mathcal{L}f=p(\frac{d}{dt})\sigma[\mu]f\ ,\quad f\in C^\infty(I\!R)\ ,$$

for some $\mu\in\mathrm{ba}_c(I\!R)$ and polynomial p. This is something, but we can be a lot more precise. Denote by $W^{p,1}_{\mathrm{loc}}(I\!R)$ the subspace of $C(I\!R)$ consisting of all $f\in C(I\!R)$ for which there exists $g\in L^p_{\mathrm{loc}}(I\!R)$ such that

$$f(t)=f(0)+\int_0^t g(\tau)d\tau\ ,\quad t\in I\!R\ .$$

Then $W^{p,1}_{\mathrm{loc}}(I\!R)$ is the domain of the infinitesimal generator δ_σ $(=\frac{d}{dt})$ of the c_0-group $(\sigma_t)_{t\in I\!R}$. So equiped with the graph topology induced by δ_σ, i.e. the topology generated by the seminorms

$$s^1_{p,n}(f)=s_{p,n}(f)+s_{p,n}(\delta_\sigma f)\ ,$$

$W^{p,1}_{\mathrm{loc}}(I\!R)$ is a Frechet space. Observe that $\delta_\sigma f=g$ in the above definition. The inclusions $W^{p,1}_{\mathrm{loc}}(I\!R)\hookrightarrow C(I\!R)$ and $C^1(I\!R)\hookrightarrow W^{p,1}_{\mathrm{loc}}(I\!R)$ are continuous. Here $C^1(I\!R)$ is the space of all continuously differentiable functions on $I\!R$ with natural Frechet topology.
Now if $\mathcal{L}:\ L^p_{\mathrm{loc}}(I\!R)\to L^p_{\mathrm{loc}}(I\!R)$ is continuous, translation invariant and linear, $W^{p,1}_{\mathrm{loc}}(I\!R)=$

$\text{dom}(\delta_\sigma)$ is an invariant subspace of $\mathcal{L}$ and $\mathcal{L}|_{W^{p,1}_{\text{loc}}(\mathbb{R})}$ is continuous on $W^{p,1}_{\text{loc}}(\mathbb{R})$, cf. [E1]. Consequently, the restriction $\mathcal{L}|_{C^1(\mathbb{R})}$ can be regarded as a translation invariant linear operator which maps $C^1(\mathbb{R})$ into $C(\mathbb{R})$ continuously. From the characterization proved in [So] we obtain that there exist constants a and b, and $\mu \in \text{ba}_c \in \mathbb{R}$ such that

$$\mathcal{L}f = \sigma[\mu](a\,\frac{d}{dt} + b)f\ , \quad f \in C^1(\mathbb{R})\ .$$

Theorem 3. Let $1 \le p < \infty$ and let $\mathcal{L}:\ L^p_{\text{loc}}(\mathbb{R}) \to L^p_{\text{loc}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there exist constants a and b, and $\mu \in \text{ba}_c(\mathbb{R})$ such that

$$\mathcal{L} = (a\delta_\sigma + b)\sigma[\mu]$$

where, in case $a \neq 0$, μ satisfies the additional condition

$$\sigma[\mu](L^p_{\text{loc}}(\mathbb{R})) \subset W^{p,1}_{\text{loc}}(\mathbb{R})\ .$$

Proof. Under the condition on μ be given the operator

$$(*) \qquad (a\delta_\sigma + b)\sigma[\mu]$$

is everywhere defined on $L^p_{\text{loc}}(\mathbb{R})$ and closed, whence continuity of $(*)$ follows from the Closed Graph Theorem. Translation invariance can be checked straightforwardly. The considerations which led to this theorem, show that any continuous translation invariant operator $\mathcal{L}$ on $L^p_{\text{loc}}(\mathbb{R})$ agrees with an operator of the form $(*)$ on the dense subspace $C^1(\mathbb{R})$. □

Remark: In the next section we prove that for $p = 1$ in Theorem 3, the constant a can be taken equal to zero. So the convolution ring $\text{ba}_c(\mathbb{R})$ and the collection of all translation invariant operators on $L^1_{\text{loc}}(\mathbb{R})$ are ring isomorphic. For $1 < p < \infty$ the question whether $a = 0$ may be taken, is still open.

For $1 < q \le \infty$, $L^q_{\text{loc}}(\mathbb{R})$ represents the dual of $L^p_{\text{comp}}(\mathbb{R})$ where $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. So if $\mathcal{K} : L^p_{\text{comp}}(\mathbb{R}) \to L^q_{\text{comp}}(\mathbb{R})$ is a continuous linear operator, then its dual $\mathcal{K}'$ is an everywhere defined closed linear operator on $L^q_{\text{loc}}(\mathbb{R})$ whence $\mathcal{K}'$ is continuous by the Closed Graph Theorem. If $\mathcal{K}$ is translation invariant, then $\mathcal{K}'$ also. Using these observations in combination with Theorem 3 we have

Theorem 4. Let $1 < p < \infty$ and let $\mathcal{L}$: $L^p_{\text{comp}}(\mathbb{R}) \to L^p_{\text{comp}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there exist constants a and b, and $\mu \in \text{ba}_c(\mathbb{R})$ such that

$$\mathcal{L} = (a\delta_\sigma + b)\sigma[\mu]$$

where, in case $a \neq 0$, μ satisfies the additional condition

$$\sigma[\mu](^p_{\text{comp}}(\mathbb{R})) \subset W^{p,1}_{\text{comp}}(\mathbb{R})\ .$$ □

Remark. $W^{p,1}_{\mathrm{comp}}(I\!R)$ is the subspace of $C_c(I\!R)$ consisting of all $f \in C_c(I\!R)$ for which there exists $g \in L^p_{\mathrm{comp}}(I\!R)$ such that

$$f(t) = \int_{-\infty}^{t} g(\tau)d\tau\ , \quad t \in I\!R\ .$$

3 Special cases: $L^1_{\mathrm{loc}}(I\!R)$ and $L^1_{\mathrm{comp}}(I\!R)$

In this section we shall prove that the operators $\sigma[\mu]$ for $\mu \in \mathrm{ba}_c(I\!R)$ establish all continuous translation invariant operators on $L^1_{\mathrm{loc}}(I\!R)$ and $L^1_{\mathrm{comp}}(I\!R)$, respectively. Therefore some auxilliary results are required.

We observed already that the translation group $(\sigma_t)_{t\in I\!R}$ is not a c_0-group on $L^\infty_{\mathrm{loc}}(I\!R)$ nor on $L^\infty_{\mathrm{comp}}(I\!R)$. So we cannot apply the theory developed in [E2] and we cannot introduce the operators $\sigma[\mu]$, $\mu \in \mathrm{ba}_c(I\!R)$, by the Riemann–Stieltjes integral

$$\int_{I\!R} \sigma_t f\ d\mu(t)$$

at least according to this theory. Instead we define the operators $\sigma[\mu]$ on $L^\infty_{\mathrm{loc}}(I\!R)$ and $L^\infty_{\mathrm{comp}}(I\!R)$ by duality: So

$$(3.1) \qquad \sigma[\mu] = (\sigma[\check{\mu}])'$$

in the sense of the duality between $L^1_{\mathrm{comp}}(I\!R)$ and $L^\infty_{\mathrm{loc}}(I\!R)$ and $L^1_{\mathrm{loc}}(I\!R)$ and $L^\infty_{\mathrm{comp}}(I\!R)$. The Closed Graph Theorem for Frechet spaces and for strict LB- spaces quarantees that $\sigma[\mu]$ on $L^\infty_{\mathrm{loc}}(I\!R)$ and $L^\infty_{\mathrm{comp}}(I\!R)$, thus defined, is continuous.

For $f \in L^\infty_{\mathrm{loc}}(I\!R)$ we define its trace $\sigma f :\ I\!R \to L^\infty_{\mathrm{loc}}(I\!R)$ by

$$(\sigma f)(t) = \sigma_t f\ , \quad t \in I\!R\ .$$

Since $(\sigma_t)_{t\in I\!R}$ is a strongly continuous group on $C(I\!R)$ and $C(I\!R)$ is closed on $L^\infty_{\mathrm{loc}}(I\!R)$, for each $f \in C(I\!R)$ its trace σf is a continuous function from $I\!R$ into $L^\infty_{\mathrm{loc}}(I\!R)$. The reverse is true also.

Lemma 5. Let $f \in L^\infty_{\mathrm{loc}}(I\!R)$. Then its trace σf is continuous as a function from $I\!R$ into $L^\infty_{\mathrm{loc}}(I\!R)$ if and only if $f \in C(I\!R)$.

Proof. Sufficiency of the condition is clear, we prove its necessity. Let σf be continuous from into $L^\infty_{\mathrm{loc}}(I\!R)$. Then for each $\varphi \in C^\infty_c(I\!R)$, the $L^\infty_{\mathrm{loc}}(I\!R)$-valued Riemann–Stieltjes integral

$$\gamma[\varphi]f = \int \varphi(\tau)\sigma_\tau f\ d\tau$$

exists in $L^\infty_{\mathrm{loc}}(I\!R)$. Because of (3.1) we have

$$\gamma[\varphi]f = \sigma[J\varphi]f$$

and

$$(\gamma[\varphi]f)(t) = \int_{-\infty}^{\infty} \varphi(\tau - t)f(\tau)d\tau \ .$$

We conclude that $\gamma[\varphi]f \in C^{\infty}(I\!R)$. Now let $(\varphi_k)_{k\in I\!N}$ be an approximate identity in $C_c^{\infty}(I\!R)$. Then the continuity of σf quarantees that

$$\lim_{k\to\infty} \gamma[\varphi_k]f = f$$

where the limit is taken in $L^{\infty}_{\text{loc}}(I\!R)$. So f is the $L^{\infty}_{\text{loc}}(I\!R)$-limit of a sequence in $C^{\infty}(I\!R)$ and therefore $f \in C(I\!R)$. □

The next result can be proved similarly.

Lemma 6. Let $f \in L^{\infty}_{\text{comp}}(I\!R)$. Then its trace σf is continuous from $I\!R$ into $L^{\infty}_{\text{comp}}(I\!R)$ if and only if $f \in C_c(I\!R)$. □

Remark. Of course Lemma 5+6 can be proved in a number of different ways, but our proof fits in the framework of this paper.

Consider a translation invariant continuous linear operator $\mathcal{K}$ on $L^{\infty}_{\text{loc}}(I\!R)$. Then for $f \in C(I\!R)$ the function

$$t \mapsto \mathcal{K}\sigma_t f \ , \quad t \in I\!R$$

is continuous from $I\!R$ in $L^{\infty}_{\text{loc}}(I\!R)$, because $\mathcal{K}$ is continuous. Since $\mathcal{K}$ is translation invariant $(\sigma\mathcal{K}f)(t) = \sigma_t\mathcal{K}f = \mathcal{K}\sigma_t f$ and so the trace of $\mathcal{K}f$ is continuous. By Lemma 5 we obtain $\mathcal{K}f \in C(I\!R)$. So $C(I\!R)$ is an invariant subspace of $\mathcal{K}$. Due to the characterization of the translation invariant operators from $C(I\!R)$ into $C(I\!R)$, there is $\mu \in \text{ba}_c(I\!R)$ such that $\mathcal{K}f = \sigma[\mu]f$ for all $f \in C(I\!R)$. Further, for all $g \in L^1_{\text{comp}}(I\!R)$ and $f \in C(I\!R)$

$$\langle \mathcal{K}f, g\rangle_{\infty} = \langle f, \sigma[\check{\mu}]g\rangle_{\infty}$$

because of the strong convergence of the L^1_{comp}-valued integral

$$\int_{I\!R} \sigma_{-\tau}g \ d\mu(\tau) \ .$$

We summarize in the following theorem.

Theorem 7. Let $\mathcal{K} : L^{\infty}_{\text{loc}}(I\!R) \to L^{\infty}_{\text{loc}}(I\!R)$ be a continuous, translation invariant linear opeator. Then there is $\mu \in \text{ba}_c(I\!R)$ such that $\mathcal{K}|_{C(I\!R)} = \sigma[\mu]$. Moreover, if $\mathcal{K}'(L^1_{\text{comp}}(I\!R)) \subset$

$L^1_{\mathrm{comp}}(\mathbb{R})$, then $\mathcal{K} = \sigma[\mu]$ and $\mathcal{K}'|_{L^1_{\mathrm{comp}}(\mathbb{R})} = \sigma[\check{\mu}]$.
(Here $\mathcal{K}'$ is the dual of $\mathcal{K}$ and we identify $L^1_{\mathrm{comp}}(\mathbb{R})$ as a closed subspace of $(L^\infty_{\mathrm{loc}}(\mathbb{R}))'$.)

Corollary 8. Let $\mathcal{K} : L^\infty_{\mathrm{loc}}(\mathbb{R}) \to L^\infty_{\mathrm{loc}}(\mathbb{R})$ be a continuous, translation invariant linear operator. Suppose $\mathcal{K}'(C^\infty_c(\mathbb{R})) \subset C^\infty_c(\mathbb{R})$. Then $\mathcal{K} = \sigma[\mu]$ for some $\mu \in \mathrm{ba}_c(\mathbb{R})$.

Proof. There is $\mu \in \mathrm{ba}_c(\mathbb{R})$ such that $\mathcal{K}|_{C(\mathbb{R})} = \sigma[\mu]$. So for all $\varphi \in C^\infty_c(\mathbb{R})$ and $f \in C(\mathbb{R})$

$$\langle f, \mathcal{K}'\varphi\rangle = \langle \mathcal{K}f, \varphi\rangle = \langle f, \sigma[\check{\mu}]\varphi\rangle \ .$$

Hence $\mathcal{K}' vphi = \sigma[\check{\mu}]\varphi$. Since $C^\infty_c(\mathbb{R})$ is dense in $L^1_{\mathrm{comp}}(\mathbb{R})$ it follows that

$$\mathcal{K}'|_{L^1_{\mathrm{comp}}(\mathbb{R})} = \sigma[\check{\mu}]$$

and so the result. □

The above theorem yields the characterization of the translation invariant operators on $L^1_{\mathrm{comp}}(\mathbb{R})$.

Theorem 9. let $\mathcal{L} : L^1_{\mathrm{comp}}(\mathbb{R}) \to L^1_{\mathrm{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Then there is $\mu \in \mathrm{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$.

Proof. Apply the preceding theorem to $\mathcal{K} = \mathcal{L}'$, the dual operator of $\mathcal{L}$. □

For each $\mu \in \mathrm{ba}_c(\mathbb{R})$ the operator $\sigma[\mu]$ on $L^\infty_{\mathrm{loc}}(\mathbb{R})$ has been defined using the duality of $L^1_{\mathrm{comp}}(\mathbb{R})$ and $L^\infty_{\mathrm{loc}}(\mathbb{R})$. From Theorem 5 we cannot conclude that the collection $\{\sigma[\mu] \mid \mu \in \mathrm{ba}_c(\mathbb{R})\}$ consists of precisely all continuous translation invariant linear operators on $L^\infty_{\mathrm{loc}}(\mathbb{R})$. Indeed the following question remains

- Does there exist a continuous translation invariant linear operator from $L^\infty_{\mathrm{loc}}(\mathbb{R})$ into $L^\infty_{\mathrm{loc}}(\mathbb{R})$ such that $\mathcal{K}f = 0$ for all $f \in C(\mathbb{R})$?

For the dual pair $L^\infty_{\mathrm{comp}}(\mathbb{R}) \times L^1_{\mathrm{loc}}(\mathbb{R})$ the discussion is similar. Indeed, for $f \in L^\infty_{\mathrm{comp}}(\mathbb{R})$ the trace σf is continuous if and only if $f \in C_c(\mathbb{R})$ according to Lemma 6. So if $\mathcal{K} : L^\infty_{\mathrm{comp}}(\mathbb{R}) \to L^\infty_{\mathrm{comp}}(\mathbb{R})$ is continuous from $\mathbb{R}$ into $L^\infty_{\mathrm{comp}}(\mathbb{R})$ for all $f \in C_c(\mathbb{R})$, whence $\mathcal{K}(C_c(\mathbb{R})) \subset C_c(\mathbb{R})$. Applying Theorem 1, this yields

Theorem 10. Let $\mathcal{K} : L^\infty_{\mathrm{comp}}(\mathbb{R}) \to L^\infty_{\mathrm{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Then there exists $\mu \in \mathrm{ba}_c(\mathbb{R})$ such that $\mathcal{K}|_{C_c(\mathbb{R})} = \sigma[\mu]$.
If $\mathcal{K}'(L^1_{\mathrm{loc}}(\mathbb{R})) \subset L^1_{\mathrm{loc}}(\mathbb{R})$, then $\mathcal{K} = \sigma[\mu]$ and $\mathcal{K}'|_{L^1_{\mathrm{loc}}(\mathbb{R})} = \sigma[\check{\mu}]$.

Corollary 11. Let $\mathcal{K} : L^\infty_{\mathrm{comp}}(\mathbb{R}) - L^\infty_{\mathrm{comp}}(\mathbb{R})$ be a continuous translation invariant linear operator. Suppose $\mathcal{K}'(C^\infty(\mathbb{R})) \subset C^\infty(\mathbb{R})$. Then $\mathcal{K} = \sigma[\mu]$ for some $\mu \in \mathrm{ba}_c(\mathbb{R})$.

Last but not least

Theorem 12. Let $\mathcal{L} : L^1_{\mathrm{loc}}(\mathbb{R}) \to L^1_{\mathrm{loc}}(\mathbb{R})$ be a continuous linear operator. Then $\mathcal{L}$ is translation invariant if and only if there is $\mu \in \mathrm{ba}_c(\mathbb{R})$ such that $\mathcal{L} = \sigma[\mu]$. □

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