

# Ray-optical analysis of a two dimensional aperture radiation problem

***Citation for published version (APA):***

Kalken, van, P. J. H., & Kooy, C. (1975). *Ray-optical analysis of a two dimensional aperture radiation problem*. (EUT report. E, Fac. of Electrical Engineering; Vol. 75-E-57). Technische Hogeschool Eindhoven.

***Document status and date:***

Published: 01/01/1975

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

***General rights***

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

***Take down policy***

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

th e

RAY-OPTICAL ANALYSIS OF A TWO DIMENSIONAL  
APERTURE RADIATION PROBLEM

by

P. J. H. van Kalken and C. Kooy

EINDHOVEN UNIVERSITY OF TECHNOLOGY  
EINDHOVEN THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING

Ray-optical analysis of a two dimensional  
aperture radiation problem

by

P.J.H. van Kalken and C. Kooy

T.H. Report 75-E-57

august 1975

ISBN 90 6144 057 2

Abstract

In this report the radiation from the open end of a parallel plane waveguide with slots at both edges is analysed, with the help of Keller's geometrical theory of diffraction. Comparison of the results with the known solution for the same problem without slots shows that the effect of the slots with respect to amplitude and phase of the radiation, is significant for the amplitude distribution only.

Subsequently special attention is given to the problem of two dimensional diffraction at an edge, when the latter is situated on the shadow boundary of the incident radiation. This situation presents itself as a sub-problem in the above mentioned parallel plane waveguide diffraction problem.

A modification of Sommerfelds classical method makes it possible to attack this problem. The first correction term in the half plane diffraction coefficient is found.

Contents

	page
I Radiation from the slotted open end of a parallel plane waveguide.	4
1. Description of the problem.	4
2. Primary diffraction at the edges A and C.	6
3. Reflection in a slot.	7
4. Survey of partial diffractions constituting the total radiation.	10
5. Numerical results.	15
II Diffraction at an edge, situated in a shadow boundary of the incident radiation.	21
1. Introduction.	21
2. Uniform asymptotic expression for secondary diffraction.	22
3. Application to slotted open end of parallel-plate waveguide.	23
4. The plane wave approximation.	27
5. Diffraction at the outer edge according to Sommerfeld.	29
6. A modification of Sommerfeld's problem.	33
7. Finding expressions for $F'(0)$ and $G'(0)$ .	39
8. Comparison of the effect of the first and the second term of the expansion of $\bar{u}$ around $z = 0$ .	44
Appendix A.	46
Appendix B.	47
References.	48

I. Radiation from the slotted open end of a parallel plane waveguide

1. Description of the problem

In 1968 Yee, Felsen and Keller applied Keller's geometrical theory of diffraction to the problem of reflection in an open-ended parallel-plane waveguide [1]. This paper is discussed and criticized in a report by Steures [2].

To analyse the problem of radiation from a slotted open-ended parallel-plane waveguide we use the same method. The configuration consists of (fig. 1) the perfectly conducting half planes  $y = a, z < 0$  (with edge  $y = a, z = 0$ ; indicated A) and  $y = -a, z < 0$  (edge  $y = -a, z = 0$ ; indicated C). The slots consist of the planes  $y = a + h, -d < z < 0$  (edge  $y = a+h, z = 0$ ; indic. B),  $a < y < a + h, z = -d$  and  $y = -a - h, -d < z < 0$  (edge  $y = -a - h, z = 0$ ; indic. D),  $-a - h < y < -a, z = -d$ .

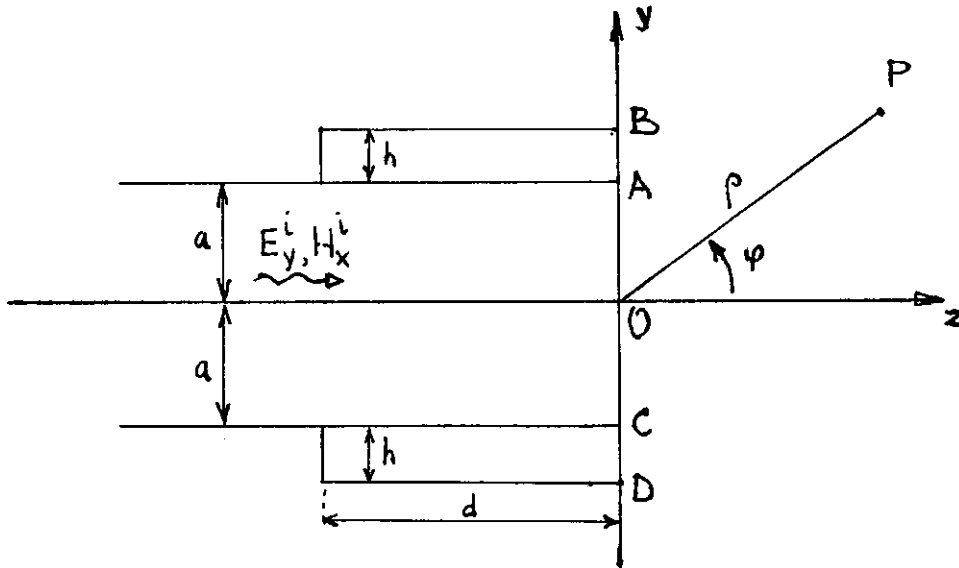


Fig. 1.

Assuming plane wave excitation  $E_y^i, H_x^i$  from the left we can write for the complex field  $H_x^i$  (adopting a time-factor according to  $\exp(j\omega t)$ )

$$H_x^i(z) = e^{-jkz} . \tag{1}$$

Diffraction at the point A and C thus arises as a consequence of incident primary rays in the direction  $\phi = 0$ , for which

$$H_x^i(0) = 1 . \quad (2)$$

In the notation of Yee, Felsen, Keller [1]:  $\theta_i = 0$ ;  $\mu_{pi} = 1$ . In the following we use the symbol  $H$  for  $H_x$  when it results from diffraction only; we use  $\hat{H}$  to indicate effects of diffraction of a reflected wave.

We consider the following partial effects as sufficient to characterize the total radiation, after summation.

For  $y > 0$ :

- a) primary diffraction at edge A as a consequence of the incident wave  $H_x^i$ , giving  $H_p^A(\rho, \phi)$ ,
- b) secondary diffraction at edge B as a consequence of reflection of  $H_p^A$  in the upper slot, giving  $\hat{H}_S^B(\rho, \phi)$ ,
- c) secondary diffraction at edge A as a consequence of reflection of  $H_p^A$  in the upper slot, giving  $\hat{H}_S^A(\rho, \phi)$ ,
- d) secondary diffraction at edge B as a consequence of  $H_p^A$  impinging directly on edge B, giving  $H_{AS}^B(\rho, \phi)$ .

The similar effects for  $y < 0$  are, respectively

$$e) H_p^C(\rho, \phi), \quad f) \hat{H}_S^D(\rho, \phi), \quad g) \hat{H}_S^C(\rho, \phi), \quad h) H_{CS}^D(\rho, \phi).$$

Further we have to add the contributions

- i) secondary diffraction at C as a consequence of  $H_p^A$  impinging directly on C, giving  $H_S^C(\rho, \phi)$ ,
- j) secondary diffraction at A as a consequence of  $H_p^C$  impinging directly on A, giving  $H_S^A(\rho, \phi)$ ,
- k) secondary diffraction at D as a consequence of  $H_p^A$  impinging directly on D, giving  $H_{AS}^D(\rho, \phi)$ ,
- l) secondary diffraction at B as a consequence of  $H_p^C$  impinging directly on B, giving  $H_{CS}^B(\rho, \phi)$ .

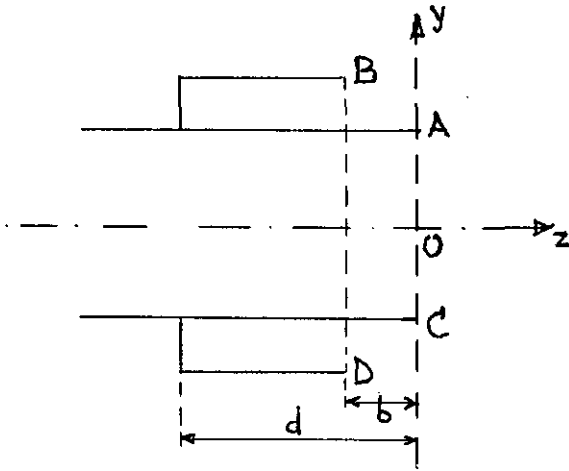
It is clear that the edges B and D are situated exactly on the shadow boundary of  $H_p^C$  and  $H_p^A$ , respectively. This situation constitutes a new basic diffraction problem, that is discussed in part II of this report.

The ray-optical method is an asymptotic method. This means that free space wavelength  $\lambda$  should be a fraction of transverse dimensions in order that significant results are to be expected. Yet, results of [1] show that even in case the smallest transverse dimension  $h = \frac{\lambda}{3}$  we have a fairly good agreement with the exact solution.

On the other hand we suppose that only dominant mode propagation is possible in waveguide and slots. These conditions together means that dimensions and frequency should be chosen such that

$$4a < \lambda < 3h . \quad (3)$$

We shall also investigate the radiation in case the outer edges of the slots, B and D, are not situated in the plane  $z = 0$ , so we put  $z = -b$  (fig. 2).



For  $b > 0$  both contributions  $H_{As}^D$  and  $H_{Cs}^B$  of course disappear.

Fig. 2.

## 2. Primary diffraction at the edges A and C

The primary diffraction at the edges A and C can be expressed with the well-known non-uniform asymptotic result

$$H_P^A(\rho_A, \theta) \sim D(\theta, 0) \frac{e^{-jk\rho_A}}{\sqrt{\rho_A}} , \quad (4)$$

where

$$D(\theta, \theta_i) = -\frac{e^{-j\pi/4}}{2\sqrt{2\pi k}} \left( \sec \frac{\theta - \theta_i}{2} + \sec \frac{\theta + \theta_i}{2} \right) \quad (5)$$

The non-uniformity in fact means that (4) and (5) are only significant away from the shadow boundaries.

The analogous result for edge C reads

$$H_P^C(\rho_C, \theta) \sim D(\theta, 2\pi) \frac{e^{-jk\rho_C}}{\sqrt{\rho_C}} \quad (6)$$



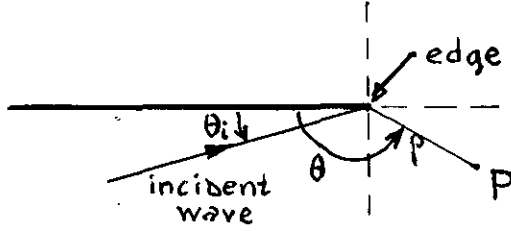


Fig. 3.

### 3. Reflection in a slot

From sufficiently large distance an edge can be seen as a magnetic current line source, whose magnetic current density distribution  $\bar{K}$  can be written (fig. 4):

$$\bar{K} = K\delta(y) \delta(z)\bar{x}_0 \quad (7)$$

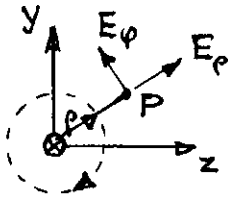


Fig. 4.

The electromagnetic field of  $\bar{K}$ , satisfying in every point  $(\rho, \phi, z)$ ,  $\rho \neq 0$

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} - \bar{K} \quad (8)$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E} \quad (9)$$

will be found by putting  $\bar{E} = -\nabla \times \bar{A}^*$ . (10)

Then it appears that

$$\nabla^2 \bar{A}^* + k^2 \bar{A}^* = -\bar{K} \quad (11)$$

on imposing

$$\nabla \cdot \bar{A}^* + j\omega\mu\chi = 0. \quad (12)$$

Condition (12) implies also

$$\nabla^2 \chi + k^2 \chi = 0. \quad (13)$$

In view of the behaviour of the fields for  $\rho \rightarrow \infty$  the only acceptable solution of (11) reads

$$\bar{A}^* = C H_0^{(2)}(k\rho) \bar{x}_0 \quad (14)$$

where C at the moment is unknown, but in the following is found as a function of the current strength K.

In the immediate vicinity of the line current ( $\rho \rightarrow 0$ ) we can write (fig. 4)

$$K = - \lim_{\rho \rightarrow 0} \oint \bar{E} \cdot d\bar{l} = - \lim_{\rho \rightarrow 0} \int_0^{2\pi} -E_\phi \rho d\phi. \quad (15)$$

Now 
$$E_{\phi} = -\frac{\partial \bar{A}^*}{\partial \rho} = -C \frac{d H_0^{(2)}(k\rho)}{d\rho} \quad (16)$$

with 
$$\frac{d H_0^{(2)}(k\rho)}{d\rho} = -k H_1^{(2)}(k\rho) = -k J_1(k\rho) + jk Y_1(k\rho) . \quad (17)$$

The leading term in the small argument expansion of  $J_1(x)$  and  $Y_1(x)$  is

$$\left. \begin{aligned} J_1(x) &= \frac{x}{2} + \dots\dots\dots \\ Y_1(x) &= -\frac{2}{\pi x} + \dots\dots \end{aligned} \right\} (x \rightarrow 0) \quad (18)$$

So we find

$$K = -kC \int_0^{2\pi} \lim_{\rho \rightarrow 0} [-\rho J_1(k\rho) + j\rho Y_1(k\rho)] d\rho$$

or

$$K = +C \int_0^{2\pi} j \frac{2}{\pi} d\phi = 4jC. \quad (19)$$

With (19) we arrive at

$$\bar{A}^* = \frac{K}{4j} H_0^{(2)}(k\rho) . \quad (20)$$

We observe

$$\nabla \cdot \bar{A}^* = \frac{\partial \bar{A}^*}{\partial x} = 0, \text{ so } \chi = 0 \quad (21)$$

For the field  $\bar{H}$  then follows

$$\bar{H} = -j\omega\epsilon\bar{A}^* - \nabla\chi = -j\omega\epsilon\bar{A}^*$$

or

$$\bar{H} = -\frac{\omega\epsilon}{4} K H_0^{(2)}(k\rho) \bar{x}_0$$

For large  $k\rho$  we have the approximation

$$\bar{H} \approx -\frac{\omega\epsilon}{4} K \sqrt{\frac{2}{\pi k\rho}} e^{-jk\rho + j\pi/4} \bar{x}_0 \quad (k\rho \rightarrow \infty) . \quad (23)$$

Comparison with (4) and (5) gives us the formal line current equivalence

$$K(\theta) = -\frac{4}{\omega\epsilon} D(\theta, 0) \sqrt{\frac{\pi k}{2}} e^{-j\pi/4} \quad (24)$$

or

$$K(\theta) = -j \frac{2}{\omega\epsilon} \cdot \frac{1}{\cos \frac{\theta}{2}} . \quad (25)$$

So we see that

$$K(2\pi) = \frac{2j}{\omega\epsilon} . \quad (26)$$

To find the complex amplitude of the reflected wave in the slot, caused by the diffraction source represented by (26), we imagine  $K$  situated on the wall of a half-infinite slot. The origin of the coordinate system is chosen on the line source for this occasion (fig. 5).

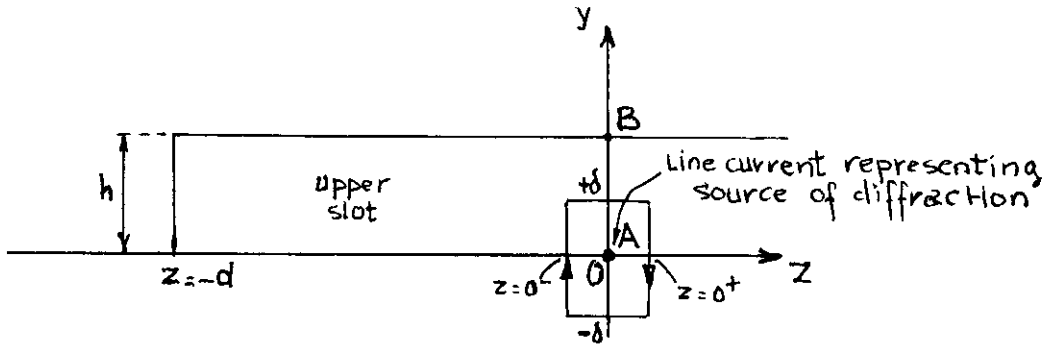


Fig. 5.

For a small contour around the line source we can write

$$\lim_{\delta \rightarrow 0} \oint \vec{E} \cdot d\vec{l} = -K$$

Denoting  $E_Y|_{z=0^+} = E_Y^+ ; H_X|_{z=0^+} = H_X^+ ; E_Z|_{y=-\delta} = E_Z^-$

$$E_Y|_{z=0^-} = E_Y^- ; H_X|_{z=0^-} = H_X^- ; E_Z|_{y=+\delta} = E_Z^+$$

we obtain

$$\int_{0^-}^{0^+} (E_Z^- - E_Z^+) dz + \int_{-\delta}^{+\delta} (E_Y^+ - E_Y^-) dy = K$$

$$E_Y^+ - E_Y^- = \frac{K}{2} \delta(y) = \frac{j}{\omega \epsilon} \delta(y). \quad (27)$$

For this source an expansion in TM-modes as representation of the field in the slots is complete. For the upper slot we can write then

$$H(y, z) = \sum_{m=0}^{\infty} A_m \cos \frac{m\pi}{h} y \cdot \cos [k_z^{(m)} (z+d)] \quad (-d < z < 0) \quad (28)$$

$$H(y, z) = \sum_{m=0}^{\infty} B_m \cos \frac{m\pi}{h} y \cdot e^{-jk_z^{(m)} z} \quad (z > 0) \quad (29)$$

where

$$k_z^{(m)} = \sqrt{k^2 - \left(\frac{m\pi}{h}\right)^2}$$

As for  $z=0$  we have  $H_x^- = H_x^+$  it follows that

$$A_m \cos k_z^{(m)} d = B_m \quad m = 0, 1, 2, 3, \dots \quad (30)$$

Also

$$E_Y^+ = \frac{1}{j\omega\epsilon} \frac{\partial H^+}{\partial z} = - \frac{1}{j\omega\epsilon} \sum_{m=0}^{\infty} j k_z^{(m)} B_m \cos \frac{m\pi}{h} y \quad (31)$$

$$E_Y^- = - \frac{1}{j\omega\epsilon} \sum_{m=0}^{\infty} k_z^{(m)} A_m \cos \frac{m\pi}{h} y \cdot \sin k_z^{(m)} d. \quad (32)$$

With (27) and (30) we arrive at

$$\sum_{m=0}^{\infty} \left[ \left( \frac{k_z^{(m)}}{\omega\epsilon} A_m \cos \frac{m\pi}{h} y \right) \left( -\cos k_z^{(m)} d + \frac{1}{j} \sin k_z^{(m)} d \right) \right] = \frac{j}{\omega\epsilon} \delta(y). \quad (33)$$

Multiplying left- and righthand member through with  $\cos \frac{n\pi}{h} y$  and integrating from  $y = 0$  to  $y = h$  gives

$$\frac{1}{\omega\epsilon} \Delta_n k_z^{(n)} h A_n (-\cos k_z^{(n)} d + \frac{1}{j} \sin k_z^{(n)} d) = \frac{j}{2\omega\epsilon} \quad (34)$$

with

$$\Delta_n = \begin{cases} 1 & \text{for } n = 0 \\ \frac{1}{2} & \text{for } n > 0 \end{cases} .$$

So we find

$$A_0 = - \frac{j}{2kh} e^{-jkd} . \quad (35)$$

With (28) and (35) the complex amplitude of the fundamental wave reflected from the end  $z = -d$  is found to be

$$H(z) = \frac{1}{2} A_0 e^{-jk(z+d)} = - \frac{j}{4kh} e^{-jk(z+d)} . \quad (36)$$

#### 4. Survey of partial diffractions constituting the total radiation

The asymptotic expressions for the various partial diffractions now read as follows:

a)  $H_P^A \sim D(\theta, 0) \frac{\exp(-jk\rho_A)}{\sqrt{\rho_A}} \quad (37a)$

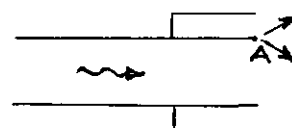


Fig. 6a.

$$b) \hat{H}_S^B \sim -\frac{j}{4kh} \exp(-2jkd) D(\theta, 0) \frac{\exp(-jk\rho_B)}{\sqrt{\rho_B}} \quad (37b)$$



Fig. 6b.

$$c) \hat{H}_S^A \sim -\frac{j}{4kh} \exp(-2jkd) D(\theta, 2\pi) \frac{\exp(-jk\rho_A)}{\sqrt{\rho_A}} \quad (37c)$$

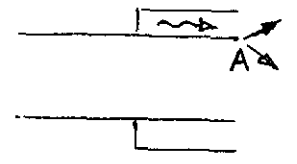


Fig. 6c.

$$d) H_{As}^B \sim H_p^A(h, \frac{3\pi}{2}) D(\theta, \frac{\pi}{2}) \frac{\exp(-jk\rho_B)}{\sqrt{\rho_B}} \quad (37d)$$

where

$$H_p^A(h, \frac{3\pi}{2}) \sim D(\frac{3\pi}{2}, 0) \frac{\exp(-jkh)}{\sqrt{h}}$$

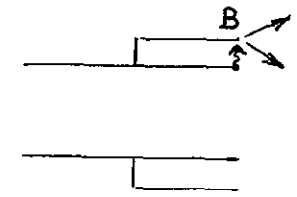


Fig. 6d.

$$e) H_p^C \sim D(\theta, 2\pi) \frac{\exp(-jk\rho_C)}{\sqrt{\rho_C}} \quad (37e)$$

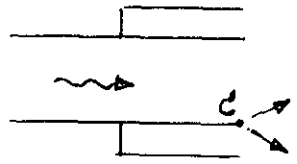


Fig. 6e.

$$f) \hat{H}_S^D \sim -\frac{j}{4kh} \exp(-2jkd) D(\theta, 2\pi) \frac{\exp(-jk\rho_D)}{\sqrt{\rho_D}} \quad (37f)$$

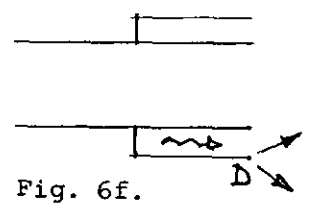


Fig. 6f.

$$g) \hat{H}_S^C \sim -\frac{j}{4kh} \exp(-2jkd) D(\theta, 0) \frac{\exp(-jk\rho_C)}{\sqrt{\rho_C}} \quad (37g)$$

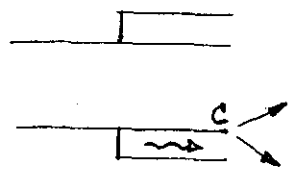


Fig. 6g.

$$h) H_{Cs}^D \sim H_p^C(h, \frac{\pi}{2}) D(\theta, \frac{3\pi}{2}) \frac{\exp(-jk\rho_D)}{\sqrt{\rho_D}} \quad (37h)$$

where

$$H_p^C(h, \frac{\pi}{2}) \sim D(\frac{\pi}{2}, 2\pi) \frac{\exp(-jkh)}{\sqrt{h}}$$

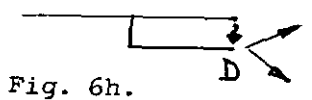


Fig. 6h.

$$i) H_S^C \sim H_P^A(2a, \frac{\pi}{2}) D(\theta, \frac{3\pi}{2}) \frac{\exp(-jk\rho_C)}{\sqrt{\rho_C}} \quad (37i)$$

with

$$H_P^A(2a, \frac{\pi}{2}) \sim D(\frac{\pi}{2}, 0) \frac{\exp(-2jka)}{\sqrt{2a}}$$

$$j) H_S^A \sim H_P^C(2a, \frac{3\pi}{2}) D(\theta, \frac{\pi}{2}) \frac{\exp(-jk\rho_A)}{\sqrt{\rho_A}} \quad (37j)$$

with

$$H_P^C(2a, \frac{3\pi}{2}) \sim D(\frac{3\pi}{2}, 2\pi) \frac{\exp(-j2ka)}{\sqrt{2a}}$$

$$k) H_{As}^D \sim \frac{1}{2} H_P^A(2a+h, \frac{\pi}{2}) D(\theta, \frac{3\pi}{2}) \frac{\exp(-jk\rho_D)}{\sqrt{\rho_D}} \quad (37k)$$

with

$$H_P^A(2a+h, \frac{\pi}{2}) \sim D(\frac{\pi}{2}, 0) \frac{\exp\{-jk(2a+h)\}}{\sqrt{2a+h}}$$

$$l) H_{Cs}^B \sim \frac{1}{2} H_P^C(2a+h, \frac{3\pi}{2}) D(\theta, \frac{\pi}{2}) \frac{\exp(-jk\rho_B)}{\sqrt{\rho_B}} \quad (37l)$$

with

$$H_P^C(2a+h, \frac{3\pi}{2}) \sim D(\frac{3\pi}{2}, 2\pi) \frac{\exp\{-jk(2a+h)\}}{\sqrt{2a+h}}$$

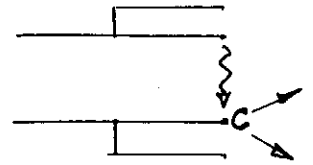


Fig. 6i.

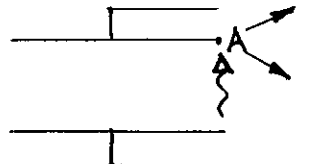


Fig. 6j.

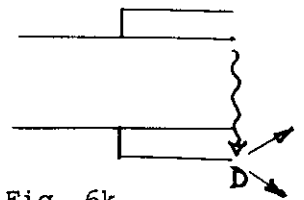


Fig. 6k.

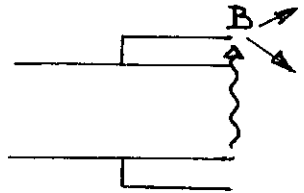


Fig. 6l.

It is a relative simple matter to consider a more general configuration with slot-edges B and D situated behind the mouth of the waveguide (fig. 7).

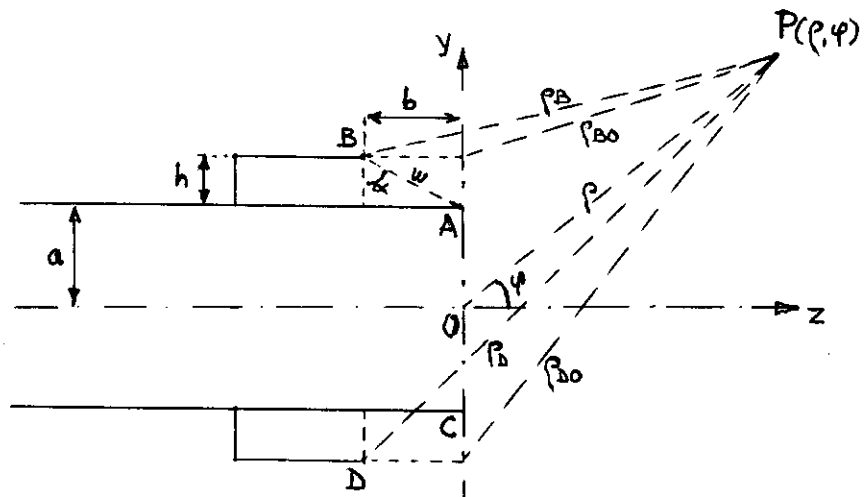


Fig. 7.

From fig. 7 can be seen that

$$\arctan \frac{b}{h} = \alpha \quad (38)$$

$$\sqrt{b^2 + h^2} = w \quad (39)$$

$$\left. \begin{aligned} \rho_B &\approx \rho_{Bo} + b \cos \phi \\ \rho_D &\approx \rho_{Do} + b \cos \phi \\ \rho_A &\approx \rho - a \sin \phi \\ \rho_{Bo} &\approx \rho - (a+h) \sin \phi \\ \rho_C &\approx \rho + a \sin \phi \\ \rho_{Do} &\approx \rho + (a+h) \sin \phi \\ \theta &\approx \phi + \pi \end{aligned} \right\} \begin{array}{l} \text{for P sufficiently} \\ \text{far from 0, so} \\ \rho \gg a \end{array} \quad (40)$$

We obtain then

$$\hat{H}_S^B \approx -\frac{j}{4kh} \exp \{-jk(2d-b)\} D(\theta, 0) \frac{\exp(-jk\rho_B)}{\sqrt{\rho_B}} \quad (41a)$$

$$\hat{H}_{As}^B \approx H_P^A(w, \frac{3\pi}{2} + \alpha) D(\theta, \frac{\pi}{2} + \alpha) \frac{\exp(-jk\rho_B)}{\sqrt{\rho_B}} \quad (41b)$$

$$\hat{H}_S^D \approx -\frac{j}{4kh} \exp \{-jk(2d-b)\} D(\theta, 2\pi) \frac{\exp(-jk\rho_D)}{\sqrt{\rho_D}} \quad (41c)$$

$$\hat{H}_{Cs}^D \approx H_P^C(w, \frac{\pi}{2} - \alpha) D(\theta, \frac{3\pi}{2} - \alpha) \frac{\exp(-jk\rho_D)}{\sqrt{\rho_D}} \quad (41d)$$

Defining

$$G = \frac{1}{\cos^{\frac{1}{2}}(\phi - \frac{\pi}{2} + \alpha)} - \frac{1}{\cos^{\frac{1}{2}}(\phi + \frac{\pi}{2} - \alpha)} \quad (42)$$

and

$$\Delta_b = \begin{cases} 0 & \text{for } b > 0 \\ 1 & \text{for } b = 0 \end{cases} \quad (43)$$

we arrive with (5) at

$$\hat{H}_P^A \approx \frac{\exp(-j\frac{\pi}{4})}{\sin^{\frac{1}{2}} \phi \cdot \sqrt{2\pi k}} \cdot \frac{\exp(-jk\rho + jka \sin \phi)}{\sqrt{\rho}} \quad (44a)$$

$$\hat{H}_S^B \approx - \frac{j}{4kh} \frac{\exp(-j \frac{\pi}{4})}{\sin \frac{\phi}{2} \cdot \sqrt{2\pi k}} \frac{1}{\sqrt{\rho}} \exp\{-2jk2d - jk\rho + jk(a+h)\sin \phi + jkb - jkbcos \phi\} \quad (44b)$$

$$\hat{H}_S^A \approx + \frac{j}{4kh} \frac{\exp(-j \frac{\pi}{4})}{\sin \frac{\phi}{2} \cdot \sqrt{2\pi k}} \cdot \frac{1}{\sqrt{\rho}} \exp\{-jk2d - jk\rho + jka \sin \phi\} \quad (44c)$$

$$\hat{H}_{As}^B \approx \frac{\exp(-j \frac{\pi}{2})}{\sin \frac{1}{2}(\alpha + \frac{\pi}{2}) \cdot \sqrt{2\pi k}} \frac{1}{\sqrt{w}} \frac{G}{\sqrt{\rho}} \frac{\exp\{-jkw - jk\rho + jk(a+h)\sin \phi - jkbcos \phi\}}{2 \sqrt{2\pi k}} \quad (44d)$$

$$\hat{H}_P^C \approx - \frac{\exp(-j \frac{\pi}{4})}{\sin \frac{\phi}{2} \cdot \sqrt{2\pi k}} \frac{1}{\sqrt{\rho}} \exp\{-jk\rho - jka \sin \phi\} \quad (44e)$$

$$\hat{H}_S^D \approx + \frac{j}{4kh} \frac{\exp(-j \frac{\pi}{4})}{\sin \frac{\phi}{2} \cdot \sqrt{2\pi k}} \frac{1}{\sqrt{\rho}} \exp\{-jk2d - jk\rho - jk(a+h)\sin \phi + jkb - jkbcos \phi\} \quad (44f)$$

$$\hat{H}_S^C \approx - \frac{j}{4kh} \frac{\exp(-j \frac{\pi}{4})}{\sin \frac{\phi}{2} \cdot \sqrt{2\pi k}} \frac{1}{\sqrt{\rho}} \exp\{-jk2d - jk\rho - jka \sin \phi\} \quad (44g)$$

$$\hat{H}_{Cs}^D \approx - \frac{\exp(-j \frac{\pi}{2})}{\sin \frac{1}{2}(\alpha + \frac{\pi}{2}) \cdot \sqrt{2\pi k}} \frac{1}{\sqrt{w}} \frac{G}{\sqrt{\rho}} \frac{\exp\{-jkw - jk\rho - jk(a+h)\sin \phi - jkb \cos \phi\}}{2 \sqrt{2\pi k}} \quad (44h)$$

$$\hat{H}_S^C \approx - \frac{\exp(-\frac{j\pi}{2})}{\pi k} \cdot \frac{\sin \frac{\phi}{2}}{\cos \phi} \cdot \frac{1}{\sqrt{2a}} \cdot \frac{1}{\sqrt{\rho}} \exp\{-2jka - jk\rho - jka \sin \phi\} \quad (44i)$$

$$\hat{H}_S^A \approx + \frac{\exp(-j\frac{\pi}{2})}{\pi k} \cdot \frac{\sin \frac{\phi}{2}}{\cos \phi} \cdot \frac{1}{\sqrt{2a}} \cdot \frac{1}{\sqrt{\rho}} \exp\{-2jka - jk\rho + jka \sin \phi\} \quad (44j)$$

$$\hat{H}_{As}^D \approx - \Delta_b \frac{\exp(-j\frac{\pi}{2})}{\pi k} \frac{\sin \frac{\phi}{2}}{\cos \phi} \frac{1}{\sqrt{2a+h}} \frac{1}{\sqrt{\rho}} \exp\{-2jka - jkh - jk\rho - jk(a+h)\sin \phi\} \quad (44k)$$

$$\hat{H}_{Cs}^B \approx + \Delta_b \frac{\exp(-j\frac{\pi}{2})}{\pi k} \cdot \frac{\sin \frac{\phi}{2}}{\cos \phi} \frac{1}{\sqrt{2a+h}} \cdot \frac{1}{\sqrt{\rho}} \exp\{-2jka - jkh - jk\rho + jk(a+h)\sin \phi\} \quad (44l)$$

We find the resulting field  $H(\rho, \phi)$  in a distant point by summing these contributions. Writing this sum as

$$H(\rho, \phi) = (U + jV) \frac{\exp(-jk\rho)}{\sqrt{\pi k\rho}} \quad (45)$$



where

$$\begin{aligned}
 U = & \frac{\sin(ka \sin \phi)}{\sin \frac{\phi}{2}} + \frac{1}{4kh} \frac{\sin\{k(a+h)\sin \phi\}}{\sin \frac{\phi}{2}} \cos(kb - kbcos \phi - 2kd) \\
 & + \frac{1}{4kh} \frac{\sin\{k(a+h)\sin \phi\}}{\sin \frac{\phi}{2}} \sin(kb - kbcos \phi - 2kd) \\
 & - \frac{1}{4kh} \frac{\sin(ka \sin \phi)}{\sin \frac{\phi}{2}} \cos 2kd + \frac{1}{4kh} \frac{\sin(ka \sin \phi)}{\sin \frac{\phi}{2}} \sin 2kd \\
 & + \frac{1}{2\sqrt{\pi kw}} \frac{G}{\sin \frac{1}{2}(\alpha + \frac{\pi}{2})} \sin\{k(a+h)\sin \phi\} \cos(kw + kbcos \phi) \\
 & + \sqrt{\frac{2}{\pi ka}} \cdot \frac{\sin \frac{\phi}{2}}{\cos \phi} \sin(ka \sin \phi) \cos 2ka + \frac{2\Delta_b}{\sqrt{\pi k(2a+h)}} \cdot \frac{\sin \frac{\phi}{2}}{\cos \phi} \cdot \sin\{k(a+h)\sin \phi\} \cos(kh+2ka)
 \end{aligned} \tag{46}$$

and

$$\begin{aligned}
 V = & \frac{\sin(ka \sin \phi)}{\sin \frac{\phi}{2}} - \frac{\sin\{k(a+h)\sin \phi\}}{4kh \sin \frac{\phi}{2}} \cos(kb - kbcos \phi - 2kd) \\
 & + \frac{\sin\{k(a+h)\sin \phi\}}{4kh \sin \frac{\phi}{2}} \sin(kb - kbcos \phi - 2kd) \\
 & + \frac{\sin(ka \sin \phi)}{4kh \sin \frac{\phi}{2}} \cos 2kd + \frac{\sin(ka \sin \phi)}{4kh \sin \frac{\phi}{2}} \sin 2kd \\
 & - \frac{1}{2\sqrt{\pi kw}} \cdot \frac{G}{\sin \frac{1}{2}(\alpha + \frac{\pi}{2})} \sin\{k(a+h)\sin \phi\} \sin(kw + kbcos \phi) \\
 & - \sqrt{\frac{2}{\pi ka}} \cdot \frac{\sin \frac{\phi}{2}}{\cos \phi} \sin(ka \sin \phi) \sin 2ka \\
 & - \frac{2\Delta_b}{\sqrt{\pi k(2a+h)}} \cdot \frac{\sin \frac{\phi}{2}}{\cos \phi} \cdot \sin\{k(a+h)\sin \phi\} \sin(kh + 2ka) .
 \end{aligned} \tag{47}$$

### 5. Numerical results

Let us define  $U + jV \stackrel{\text{def}}{=} A' \exp(jF)$  (48)

and denote  $A' = A'_0$  when  $b = 0$  and  $\phi = 0$ . From (46) and (47) it is seen that

$$U_{b=0} = 2ka + \frac{1}{2}\cos 2kd - \frac{1}{2}\sin 2kd \quad (49)$$

$$\phi=0$$

$$V_{b=0} = 2ka - \frac{1}{2}\cos 2kd - \frac{1}{2}\sin 2kd \quad (50)$$

$$\phi=0$$

$$\text{So } (U^2 + V^2)_{b=0} = 8k^2 a^2 + \frac{1}{2} - 4ka \sin 2kd = \{A'_0(k)\}^2 \quad (51)$$

$$\phi=0$$

Now we normalize  $A'$ , as defined in (48), with respect to  $A'_0$ , computed for  $k = k_t = \frac{\pi}{2d}$  (corresponding with slot-depth  $d = \frac{\lambda}{4}$ ).

So our numerical calculations concern the quantities  $A$  and  $F$  according to

$$\left. \begin{aligned} A &= \sqrt{\frac{U^2 + V^2}{\frac{1}{2} + 8k_t^2 a^2}} \\ F &= \arctan \frac{V}{U} \end{aligned} \right\} \quad (52)$$

as a function of  $\phi$ , for different values of  $b$ ,  $k$  and  $d$ . We choose

$$a = 18,7 \text{ mm}$$

$$h = 3/2 a$$

and compute the  $\phi$ -dependence of  $A$  and  $F$ , keeping  $d$  at 19,6 mm and  $b = 0$ , for  $k = 75$ ;  $k = k_t = 80$  and  $k = 85$ .

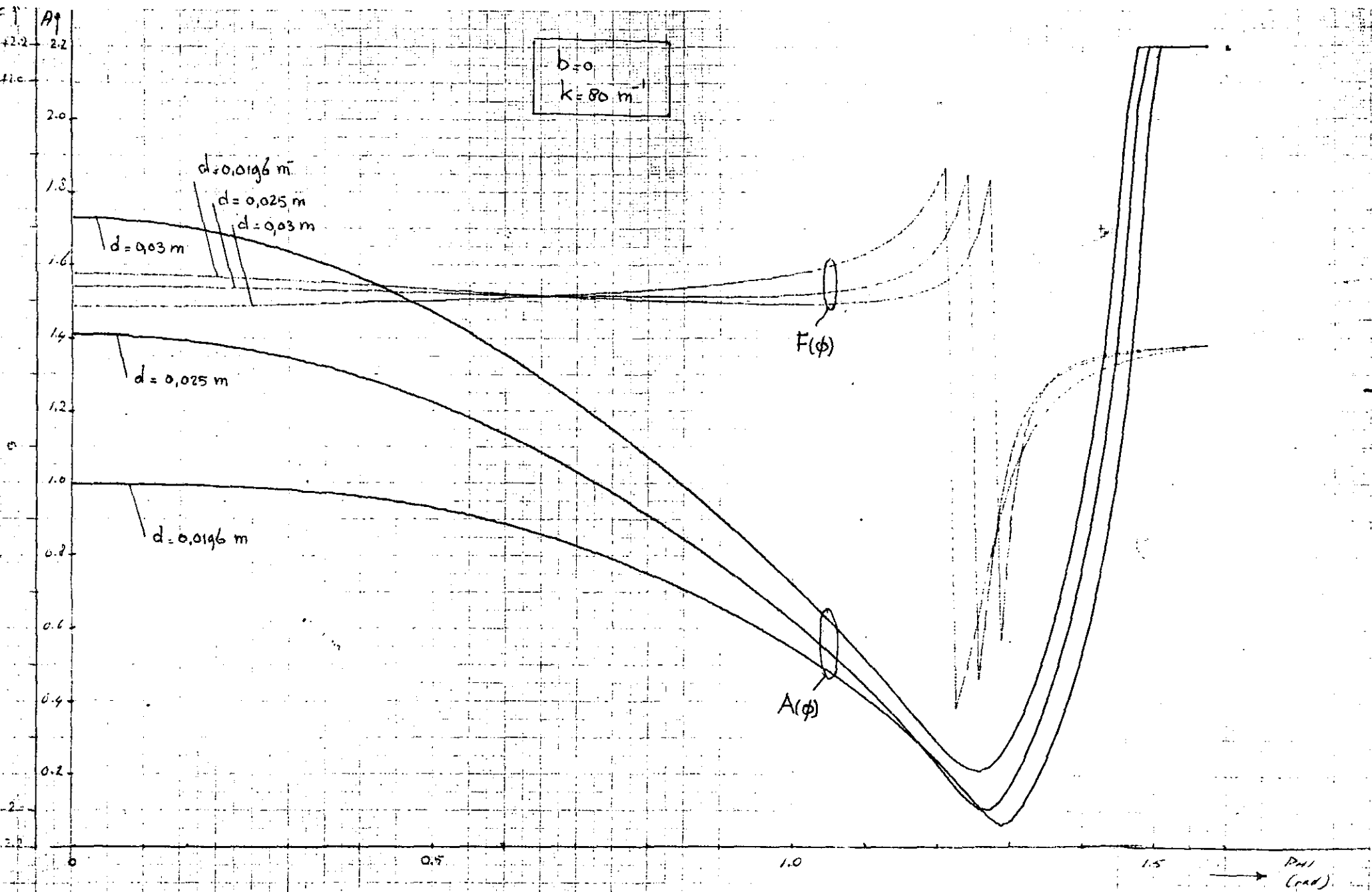
Subsequently we investigate the effect of slot-depth by keeping the frequency fixed at  $k = k_t = 80$  and calculating  $A$  and  $F$  for  $d = 19,6$ ; 25 and 30 mm, while still  $b = 0$ .

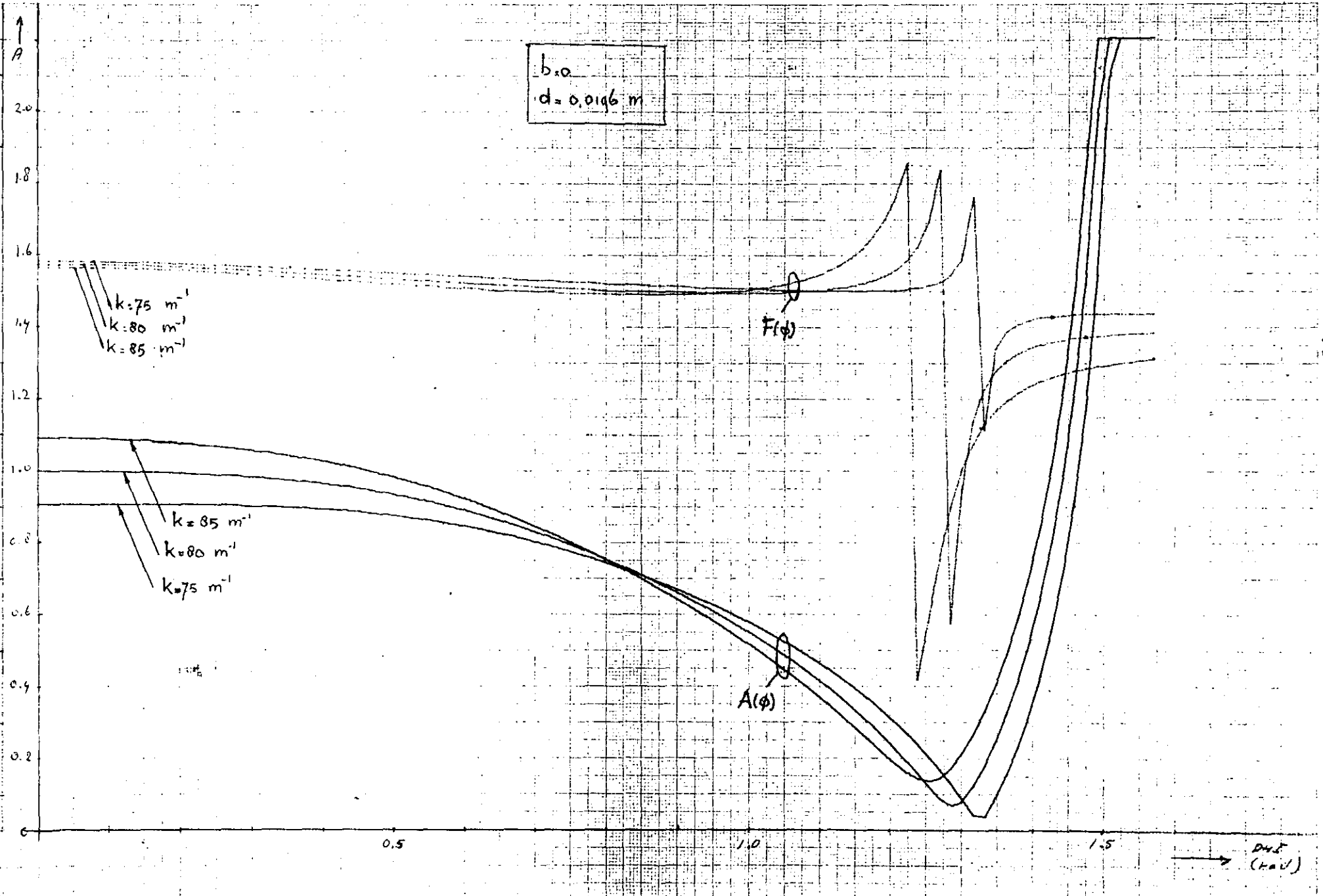
Finally the effect of  $b$  follows from calculation of  $A$  and  $F$  for  $k = k_t = 80$ ;  $d = 19,6$  mm and choosing successively  $b = 0$ ;  $\frac{1}{2}h$ ;  $h$ ;  $2h$ .

The numerical results show that the position  $b$  of the outer edges of the slots has some influence on the directivity. With the slots becoming more effective (smaller  $b$ ) it is seen that the radiation far from the main direction weakens. A similar, but more significant, effect on the directivity is found by increasing the slot depth  $d$ . As the overall level of the radiation is now rising with increasing slot depth, the conclusion seems justified that with increasing  $d$  better adaption of the waveguide to free space is obtained.

The effect of changing frequency on the amplitude distribution is rather small but confirms the expectations.

To conclude we observe that the presence of the slots appears not to effect significantly the phase distribution  $F(\phi)$  of the radiation.

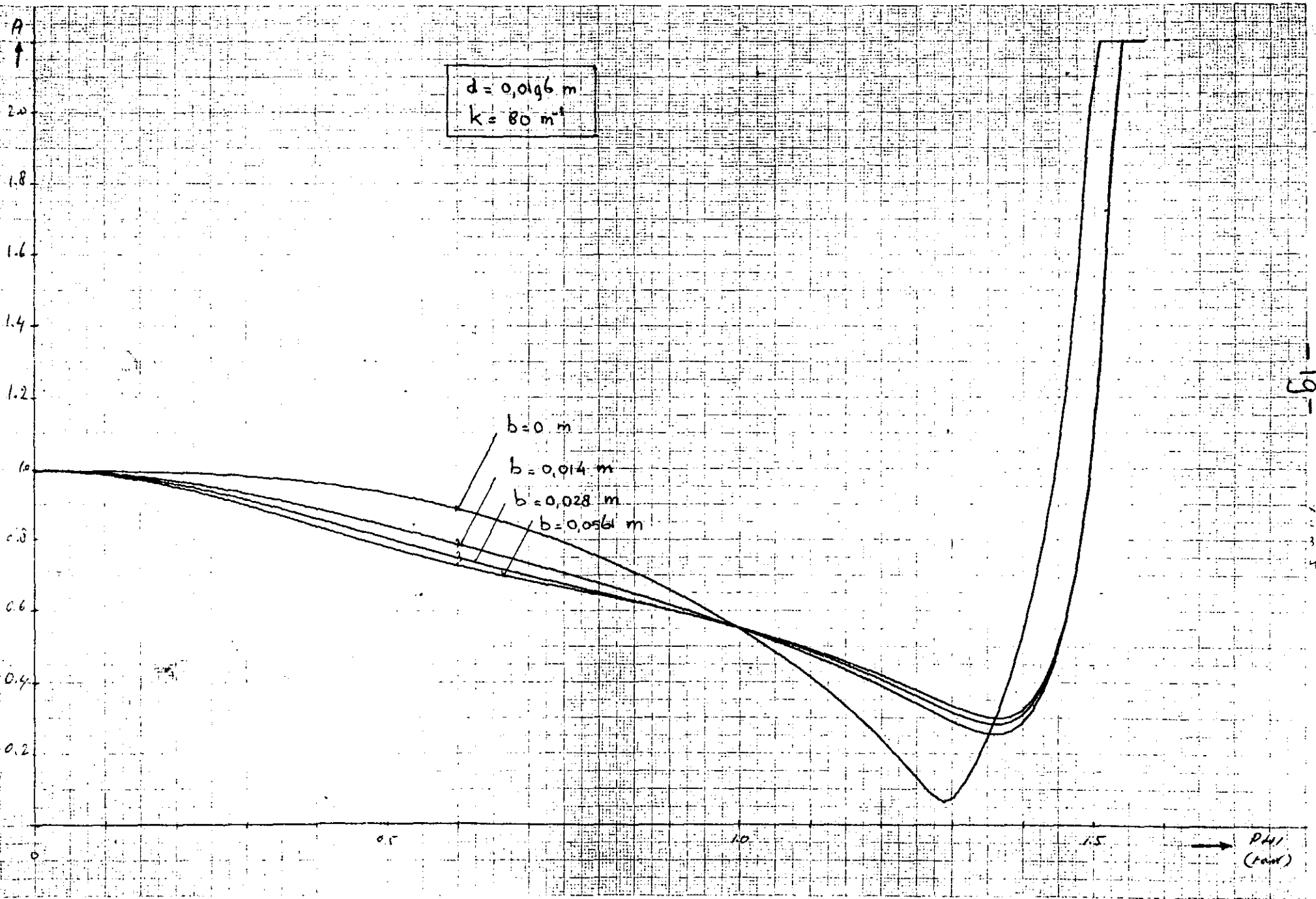




b = abkaykay

$d = 0,016 \text{ m}$   
 $k = 80 \text{ m}^2$

$b = 0 \text{ m}$   
 $b = 0,014 \text{ m}$   
 $b = 0,028 \text{ m}$   
 $b = 0,056 \text{ m}$

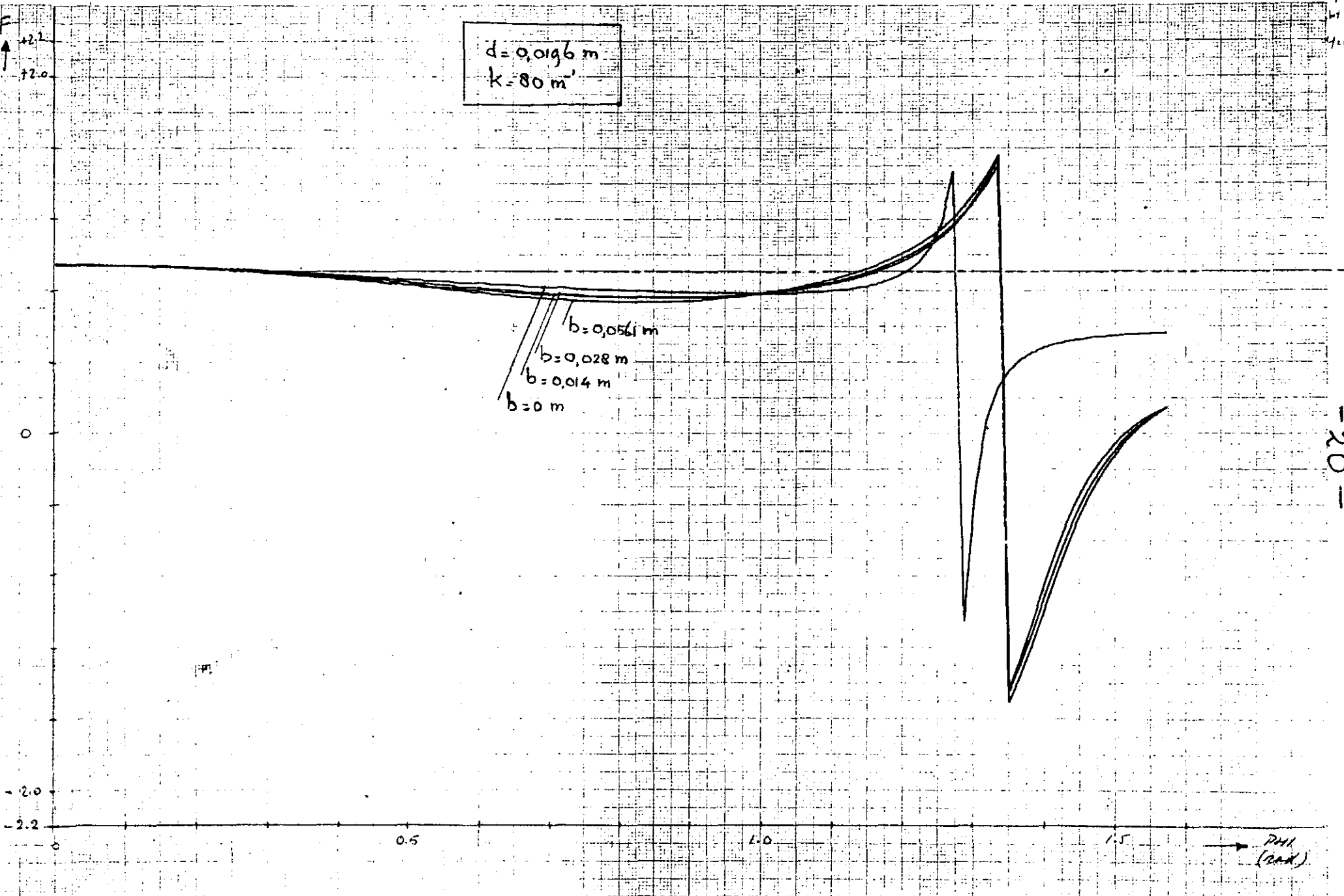


b - afhanalyt bereid

F=22  
Wt 2.2  
d=0.1

$d = 0,0196 \text{ m}$   
 $k = 80 \text{ m}^{-1}$

$b = 0,0561 \text{ m}$   
 $b = 0,028 \text{ m}$   
 $b = 0,014 \text{ m}$   
 $b = 0 \text{ m}$



-20-

r (rad)

II. Diffraction at an edge, situated in a shadow-boundary of the incident radiation

1. Introduction

In this part II we investigate more closely the diffraction process pictured in fig. i. P, Q and R are edges,  $u_1$  is the primary, incident, wave.  $u_2$  is the wave resulting from diffraction at P,  $u_3$  is the result of  $u_2$  after passing the edge Q. Finally  $u_4$  is the secondary diffracted wave as a result of  $u_3$  impinging on R. It is clear that R is situated in the shadow-boundary of  $u_3$ . The qualitative behaviour of  $|u_3|$  as a function of  $z$  will be as shown in fig. ii.

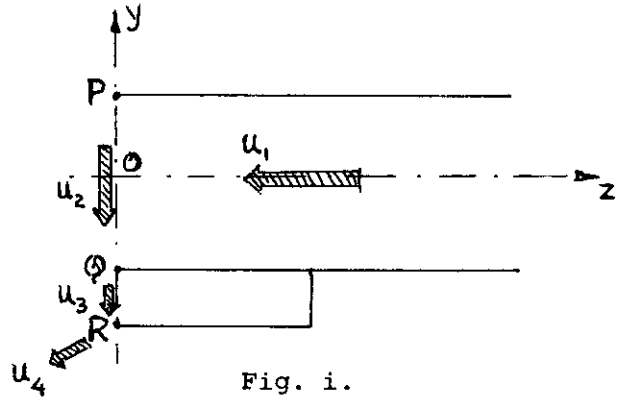


Fig. i.

This peculiar position of R defines a special diffraction problem which is the subject of this part of the report. We will try to find a first correction to the contributions  $H_{Cs}^B$ , (expression 37l), and  $H_{As}^D$ , (expression 37k) of part I where we simply accounted for the special position of R with a factor  $\frac{1}{2}$ .

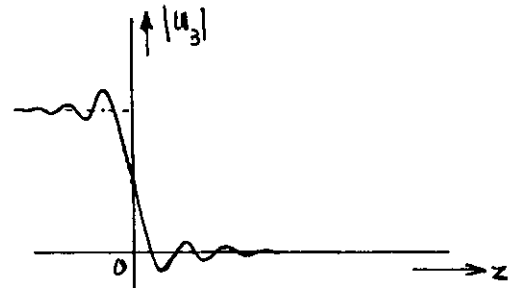


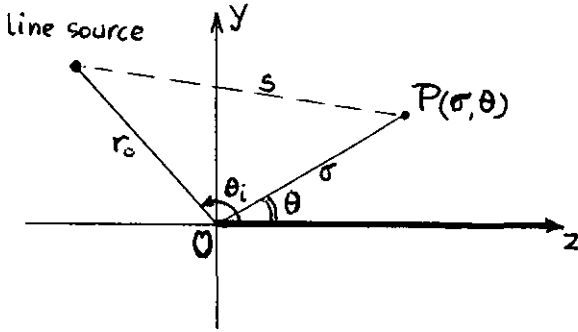
Fig. ii.

To do this we use an approach presented by Lewis and Boersma [5] and by Ahluwalia, Lewis and Boersma [6] to find a uniform asymptotic solution of the basic problem of diffraction by a plane screen. Subsequently a modification of Sommerfeld's classical function-theoretic method is set up leading to the expression for the first correction term in the diffraction field.

In this part II a time-factor  $\exp(-j\omega t)$  is understood.

2. Uniform asymptotic expression for secondary diffraction

We consider a perfectly conducting half-plane  $y = 0, z > 0$  (with edge  $O$  at  $y = 0, z = 0$ ) and a line source parallel with the edge, at  $\sigma = r_0, \theta = \theta_i$



(fig. iii). This line source can be the edge of another perfectly conducting half-plane, at which for instance primary diffraction takes place.

The distance between line source and a point of observation  $P(\sigma, \theta)$  we denote by  $s$ .

Fig. iii.

The wave  $u_0$  emanating from the line source, where [5], [6]:

$$u_0 \sim \ell^{jks} \sum_{m=0}^{\infty} (jk)^{-m} Z_m(\sigma, \theta) \tag{1}$$

causes a field  $u$  by diffraction at  $O$ . We can write

$$u = U(\sigma, \theta) + U(\sigma, 2\pi - \theta) \quad (-\pi \leq \theta \leq \pi) \tag{2}$$

$$U \sim \ell^{jk\hat{s}} \left[ f(\xi\sqrt{k}) \sum_{m=0}^{\infty} (jk)^{-m} Z_m + \frac{c}{\sqrt{k}} \sum_{m=0}^{\infty} (jk)^{-m} v_m \right] \quad (k \rightarrow \infty) \tag{3}$$

with:

$$\left. \begin{aligned} \hat{s} &= r_0 + \sigma \\ c &= \frac{1}{\sqrt{\pi}} \exp(j \frac{\pi}{4}) \end{aligned} \right\} \tag{4}$$

$$f(x) = -j c \exp(-jx^2) \int_{-\infty}^x \exp(jt^2) dt \tag{5}$$

$$\xi^2 = \hat{s} - s \quad \text{sgn } \xi = \text{sgn } \cos [\frac{1}{2}(\theta_i - \theta)] \tag{6}$$

while

$$\frac{\partial u}{\partial y} = 0 \quad \text{on the half-plane.} \tag{7}$$

$\theta_i$  is the angle denoting the direction of incidence of the source-wave (fig. iii). We derive an expression for  $v_0$  in the same way as done in [7].



Defining  $\theta_i - \theta = \nu$ , we get

$$\xi^2 = r_o + \sigma - s \tag{8}$$

and

$$s = \sqrt{\sigma^2 + r_o^2 - 2\sigma r_o \cos \nu} \tag{9}$$

following (1.19) and (2.2.12) of [7] we can write

$$\lambda_o(\psi) = - \frac{c}{2\sqrt{2} \cos \nu/2} Z_o(0,0) \tag{10}$$

For the source-wave we write (see also [1])

$$U_o \sim \frac{\exp(j(ks + \frac{\pi}{4}))}{2\sqrt{2\pi ks}} = \exp(jks)Z_o = C(ks) \tag{11}$$

So

$$Z_o(0,0) = \frac{\exp(j \frac{\pi}{4})}{2\sqrt{2\pi k r_o}} \tag{12}$$

Following (1.17) and (1.18) of [7] we get

$$\hat{v}_o(\sigma) = \frac{\lambda_o}{Y} = - \frac{c}{2\sqrt{2\sigma} \cos \nu/2} Z_o(0,0) \tag{13}$$

and

$$\hat{v}_o = c(v_o - \frac{1}{2} \xi^{-1} Z_o) \tag{14}$$

Then follows

$$v_o = \frac{\hat{v}_o}{c} + \frac{Z_o}{2\xi} \tag{15}$$

and

$$v_o = \frac{\exp(j \frac{\pi}{4})}{8 \sqrt{\pi k}} \left( \frac{\sqrt{2}}{\xi \sqrt{s}} - \frac{\sec \nu/2}{\sqrt{r_o \sigma}} \right) \tag{16}$$

Higher order terms ( $m \neq 0$ ) will not be considered.

### 3. Application to slotted open end of parallel-plate waveguide

Consider a two-dimensional configuration (fig. iv), consisting of the perfect conducting half-planes  $y = r_o, z > 0$  (with edge B at  $y = r_o, z = 0$ );  $y = 0, z > 0$  (with edge O at  $y = 0, z = 0$ ) and  $y = -b, z > 0$  (with edge A at  $y = -b, z = 0$ ).

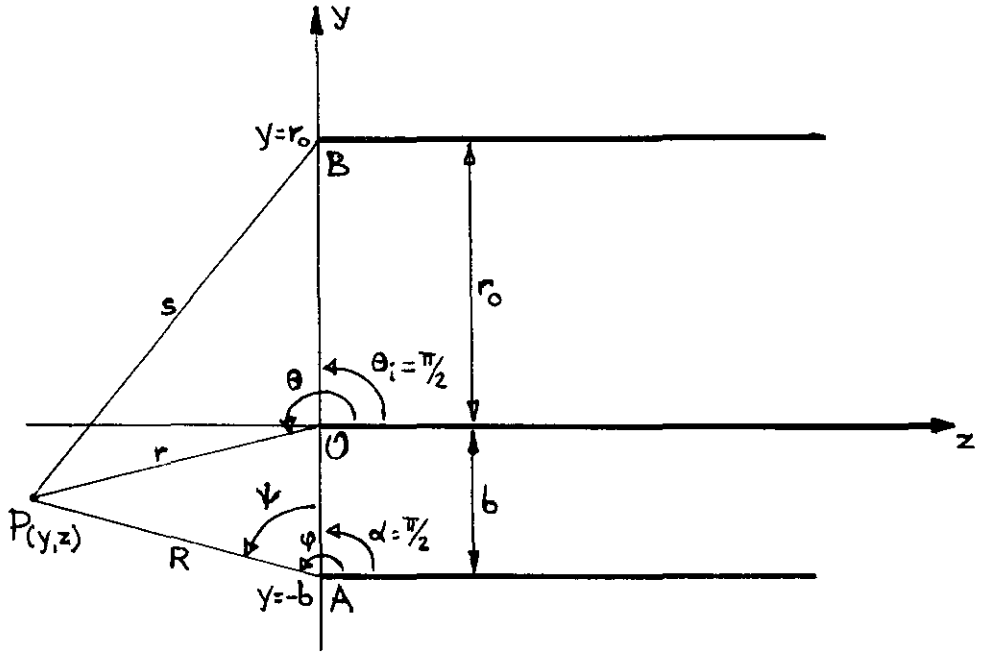


Fig. iv.

We assume that at the half-planes

$$\frac{\partial u}{\partial y} = 0, \quad \text{with } u \equiv H_x \quad (17)$$

$H_x$  is the x-component of the magnetic field-strength. In fig. iv we see that

$$\psi = \phi - \alpha = \phi - \frac{\pi}{2} \quad (18)$$

$$y = R \cos \psi - b \quad (19)$$

The edge at B ( $y = r_0, z = 0$ ) serves here as a source, generating a wave

$$u_0 \sim \frac{\exp j(ks + \frac{\pi}{4})}{2\sqrt{2\pi ks}} = C(ks)$$

Note that  $\theta_i = \frac{\pi}{2}$ .

The distance from a point of observation  $P(y,z)$  to the edge B we call  $s$ ; to the edge O we call  $r$ ; to the edge A we call  $R$ .

The angle between the positive  $z$ -axis and  $AP$  is called  $\phi$ ; that between  $AP$  and  $AO$  is called  $\psi$  (fig. iv).

In accordance with Sommerfeld [8] we call the angle of incidence of the wave from O at A:  $\alpha$ . Secondary diffraction of  $u_0$  occurs at O. The wave that arrives at A from O will be treated there as a plane wave with the amplitude-function of the real wave, for  $r = b$ .

From the considerations in the foregoing paragraph it follows that we can write

$$u(y, z) = U(r, \theta) + U(r, -\theta) \quad (20)$$

$$U(r, \theta) \sim \{ \exp jk(r+r_0) \} \left\{ \frac{\exp j\frac{\pi}{4}}{2\sqrt{2\pi ks}} f(\xi\sqrt{k}) + \frac{j}{8\pi k} \left[ \frac{\sqrt{2}}{\xi\sqrt{s}} - \frac{\sec[\frac{1}{2}(\theta_i - \theta)]}{\sqrt{r_0 r}} \right] \right\} \quad (21)$$

$$\xi = \sqrt{r + r_0 - s} \cdot \operatorname{sgn} \left[ \cos \frac{\theta_i - \theta}{2} \right] \quad (22)$$

$$f(\xi\sqrt{k}) = \frac{\exp(-j\frac{\pi}{4})}{\sqrt{\pi}} \{ \exp(-jk\xi^2) \} \int_{-\infty}^{\xi\sqrt{k}} \exp jt^2 dt \quad (23)$$

see also [2], p. 16.

Around  $(0, -b)$ , so for  $\theta \sim \frac{3\pi}{2}$ , we have

$$\begin{cases} \operatorname{sgn} \xi = - & \text{for } z > 0 & \text{at } y = -b \\ \operatorname{sgn} \xi = + & \text{for } z < 0 & \text{at } y = -b. \end{cases}$$

Denoting

$$\xi\sqrt{k} = x \quad (24)$$

$$f^*(x) = (\exp jx^2) f(x) \cdot \exp(j\frac{\pi}{4}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x \exp jt^2 dt \quad (25)$$

$$g = \frac{\sqrt{2}}{\xi\sqrt{s}} - \frac{\sec[\frac{1}{2}(\theta_i - \theta)]}{\sqrt{r_0 r}} \quad (26)$$

we can write

$$U(r, \theta) \sim \frac{f^*(x)}{2\sqrt{2\pi ks}} \exp jks + \frac{jg}{8\pi k} \exp jk(r+r_0) \quad (27)$$

We now first show that  $g$  exists for  $\theta = \frac{3\pi}{2}$  and  $\theta_i = \frac{\pi}{2}$ .

$$g = \sqrt{\frac{2}{(r_0 + r - s)s}} - \frac{1}{\sqrt{rr_0} \cos \beta/2} \quad (28)$$

with

$$\theta - \frac{\pi}{2} = \beta$$

when  $\theta \uparrow \frac{3\pi}{2}$ , then  $\operatorname{sgn} \xi = +$  and  $\beta \uparrow \pi$ , while  $\cos \beta/2$  can be approximated

$$\begin{aligned} \cos \beta/2 &= \cos \pi/2 + \left(\frac{\beta}{2} - \frac{\pi}{2}\right) \left(\frac{d \cos \beta/2}{d \beta/2}\right)_{\beta=\pi} + \frac{1}{2} \left(\frac{\beta}{2} - \frac{\pi}{2}\right)^2 \left(\frac{d^2 \cos \beta/2}{d \beta/2^2}\right)_{\beta=\pi} + \dots \\ &= \left(\frac{\pi}{2} - \frac{\beta}{2}\right) + 0 < \left(\frac{\beta}{2} - \frac{\pi}{2}\right)^3 > = \frac{1}{2}(\pi - \beta) [1 + 0 < (\pi - \beta)^2 >] \end{aligned}$$

Also

$$s = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \beta} = (r+r_0) \sqrt{1 - \frac{2rr_0}{(r+r_0)^2} (1+\cos \beta)}$$

$$\begin{aligned} \cos \beta &= \cos \pi + (\beta - \pi) \left(\frac{d \cos \beta}{d \beta}\right)_{\beta=\pi} + \frac{1}{2}(\beta - \pi)^2 \left(\frac{d^2 \cos \beta}{d \beta^2}\right)_{\beta=\pi} + \dots \\ &= -1 + \frac{1}{2}(\beta - \pi)^2 [1 + 0 < (\beta - \pi)^2 >] \end{aligned}$$

$$\begin{aligned} \text{So } s &= (r+r_0) \sqrt{1 - \frac{2rr_0}{(r+r_0)^2} \left\{ \frac{1}{2}(\beta - \pi)^2 [1 + 0 < (\beta - \pi)^2 >] \right\}} \\ &= (r+r_0) \sqrt{1 - \frac{rr_0}{(r+r_0)^2} (\beta - \pi)^2 + 0 < (\beta - \pi)^4 >} \\ &= (r+r_0) \left[ 1 - \frac{rr_0}{2(r+r_0)^2} (\beta - \pi)^2 + 0 < (\beta - \pi)^4 > \right] . \end{aligned}$$

Now we see that

$$\begin{aligned} (r+r_0 - s)s &= \frac{1}{2} r r_0 (\beta - \pi)^2 + 0 < (\beta - \pi)^4 > \\ \sqrt{(r+r_0 - s)s} &= \frac{\pi - \beta}{\sqrt{2}} \sqrt{rr_0 + 0 < (\beta - \pi)^2 >} \end{aligned}$$

we find that

$$g = \frac{2}{(\pi - \beta) \sqrt{rr_0 + 0 < (\beta - \pi)^2 >}} - \frac{2}{(\pi - \beta) \sqrt{rr_0} [1 + 0 < (\beta - \pi)^2 >]}$$

So

$$\begin{aligned} \lim g &= 0 \\ \theta \uparrow \frac{3\pi}{2} \end{aligned}$$

4. The plane wave approximation

The plane wave approximating  $U(r, \theta)$  in the neighbourhood of A we call  $\bar{U}_1$ .

Putting 
$$\left. \begin{aligned} s &= r + r_0 \\ r &= -y \quad (y < 0) \end{aligned} \right\} \quad (29)$$

then  $\bar{U}_1$  will be of the form

$$\bar{U}_1 \sim e^{jk(r_0 - y)} \quad (\dots\dots)$$

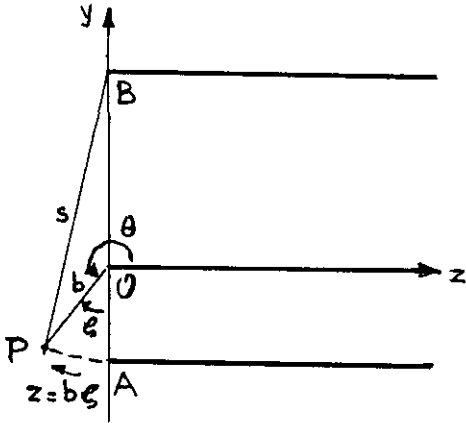


Fig. v.

In the amplitude function we can set

$$r = b.$$

We write for  $s, \xi, x$  and  $g$  resp.

$s_0, \xi_0, x_0$  and  $g_0$  when  $r = b$ . So (fig. v)

$$s_0 = \sqrt{b^2 + r_0^2 - 2br_0 \cos(\theta - \frac{\pi}{2})}. \quad (30)$$

With  $\zeta$  being the angle between OP and the negative y-axis (and  $\zeta$  positive for  $z > 0$  and negative for  $z < 0$ ) we obtain

$$\zeta = \theta - \frac{3\pi}{2} \quad (31)$$

and

$$\zeta b \sim z \quad (32)$$

$$\theta \sim \frac{z}{b} + \frac{3\pi}{2} \quad (33)$$

$$s_0 = \sqrt{b^2 + r_0^2 + 2br_0 \cos \frac{z}{b}} \quad (34)$$

So now we can write for  $\bar{U}_1$

$$\begin{aligned} \bar{U}_1 &\sim e^{jk(r_0 - y)} \left( \frac{f^*(x_0)}{2\sqrt{2\pi k s_0}} + \frac{jg_0}{8\pi k} \right) \\ &= e^{-jkR \cos \psi} e^{jk(r_0 + b)} \left( \frac{f^*(x_0)}{2\sqrt{2\pi k s_0}} + \frac{jg_0}{8\pi k} \right) \\ &= e^{-jkR \cos \psi} F_1(z) = e^{-jkR \cos \psi} F_1(-R \sin \psi) \end{aligned} \quad (35)$$

in which

$$F_1(z) = e^{jk(r_0 + b)} \left( \frac{f^*(x_0)}{2\sqrt{2\pi k s_0}} + \frac{jg_0}{8\pi k} \right) \quad (36)$$

Consider now  $U(r, -\theta)$ . We already have  $s = s(\theta)$ . When we define  $s' = s(-\theta)$  (37) then

$$s' = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \frac{z}{b}} \quad (38)$$

Constructing a plane wave, we can write for the phase

$$s' = r_0 - r = r_0 + y \quad (y < 0) \quad (39)$$

Starting from  $\xi = \xi(\theta)$  we get

$$\xi' = \xi(-\theta) = \sqrt{r + r_0 - s'} \operatorname{sgn} [\cos \frac{1}{2}(\theta_1 + \theta)] \quad (40)$$

In the neighbourhood of A  $\xi'$  is always negative. In the same way we get

$$x' = \xi' \sqrt{k} \quad (41)$$

and

$$g' = \frac{\sqrt{2}}{\xi' \sqrt{s'}} - \frac{\sec[\frac{1}{2}(\theta_1 + \theta)]}{\sqrt{r_0 r}} \quad (42)$$

Denoting  $\bar{U}_2$  the plane wave approximating  $U(r, -\theta)$  near A, then

$$\bar{U}_2 \sim e^{-jkR \cos \psi} e^{jk(r_0 + b)} \frac{jg_0'}{8\pi k} + e^{jkR \cos \psi} e^{jk(r_0 - b)} \frac{f^*(x_0')}{2\sqrt{2\pi k s_0'}} \quad (43)$$

$g_0'$ ,  $x_0'$  and  $s_0'$  mean  $g'$ ,  $x'$  and  $s'$  for  $r = b$ .

We define

$$F_2(z) = F_2(-R \sin \psi) = e^{jk(r_0 + b)} \frac{jg_0'}{8\pi k} \quad (44)$$

$$\chi = \pi - \psi \quad (45)$$

So  $\sin \psi = \sin \chi$

$$\cos \psi = -\cos \chi.$$

Next we define

$$G(z) = G(-R \sin \psi)$$

$$= e^{jk(r_0 - b)} \frac{f^*(x_0')}{2\sqrt{2\pi k s_0'}}$$

Then

$$\bar{U}_2 \sim e^{-jkR \cos \psi} F_2(-R \sin \psi) + e^{-jkR \cos \chi} G(-R \sin \chi) \quad (46)$$

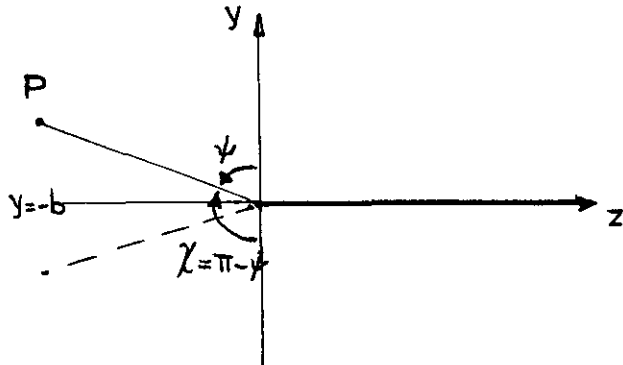


Fig. vi.

Near A we have

$$u(y, z) \sim \bar{u} = \bar{U}_1 + \bar{U}_2 \quad (47)$$

Splitting  $\bar{u}$  in a  $\psi$ -dependent and a  $\chi$ -dependent part

$$\left. \begin{aligned} \bar{u} &= e^{-jkR\cos\psi} F(-R\sin\psi) + e^{-jkR\cos\chi} G(-R\sin\chi) \\ &= \bar{u}_I + \bar{u}_{II} \end{aligned} \right\} \quad (48)$$

in which

$$F(-R\sin\psi) = F_1(-R\sin\psi) + F_2(-R\sin\psi) \quad (49)$$

and

$$\left. \begin{aligned} \bar{u}_I &= e^{-jkR\cos\psi} F(-R\sin\psi) \\ \bar{u}_{II} &= e^{-jkR\cos\chi} G(-R\sin\chi) \end{aligned} \right\} \quad (51)$$

### 5. Diffraction at the outer edge according to Sommerfeld

Following the classical procedure of Sommerfeld we suppose an incident wave  $\bar{u}_0$

$$\bar{u}_0 = A e^{-jkR\cos\psi} - A e^{-jkR\cos(\phi-\alpha)} \quad (52)$$

in which  $\phi$  and  $\alpha$  are the angles to be found in fig. iv.

The field  $\bar{u}_1$  caused by diffraction at A must satisfy the following equations and conditions:

a)  $\Delta \bar{u}_1 + k^2 \bar{u}_1 = 0 \quad (53)$

b)  $\frac{\partial \bar{u}_1}{\partial y} = 0 \quad \text{for } \phi = \begin{cases} 0 \\ 2\pi \end{cases} \quad (54)$

c)  $\bar{u}_1$  everywhere finite and continuous outside the edge of the screen

d) the radiation condition.

e)  $\lim_{R \rightarrow 0} R \nabla \bar{u}_1 = 0.$

We suppose a function  $U^S$  with period  $4\pi$  in  $(\phi-\alpha)$ , satisfying conditions a) and

c) for

$$-2\pi \leq \phi - \alpha \leq 2\pi$$

and also conditions d) and e) (see [8] for a more detailed treatment).  
 The solution of our problem can then be expressed as

$$\bar{u}_1 = U^S(R, \phi - \alpha) + U^S(R, \phi + \alpha) \quad (55)$$

(see fig. vii and Appendix D).

When  $A \equiv 1$ , we can write  $\bar{u}_0$  as

$$\bar{u}_0 = \frac{1}{2\pi} \oint \frac{e^{j\beta}}{e^{j\beta} - e^{j\alpha}} e^{-jkR \cos(\phi - \beta)} d\beta \quad (56)$$

where the contour of integration encloses the pole  $\beta = \alpha$ .

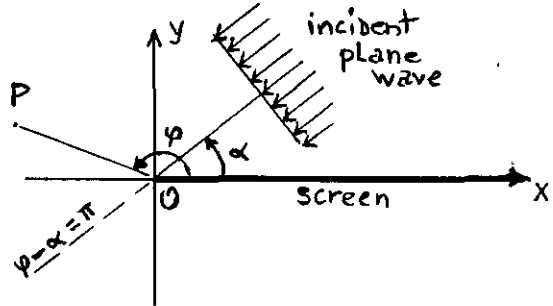


Fig. vii.

When we want that the path of integration extends to infinity then the integrand must go to zero there in the right way (fig. viii) L, M and N are situated on the real axis with  $L(\beta = \phi - \pi)$ ,  $M(\beta = \phi)$  and  $N(\beta = \phi + \pi)$ .

Denote

$$\beta = p' + jq$$

and

$$\phi - p' = p.$$

We have now

$$\begin{aligned} \cos(\phi - \beta) &= \cos(\phi - p' - jq) = \\ &= \cos(p - jq) = \end{aligned}$$

$$= \cos p \cosh q + j \sin p \sinh q.$$

When  $p = \phi - p' = k\pi$

i.e.  $\text{Re}(\beta) = \phi + k\pi$   $k=0, \pm 1, \pm 2, \pm 3, \dots$

or  $q = 0$

we have  $\text{Im} [\cos(\phi - \beta)] = 0$ .

This is true on the boundaries between the hatched and non-hatched domains (fig. viii)

$$\text{Im} [\cos(\phi - \beta)] = \sin p \sinh q.$$

$\sin p = \sin(\phi - p')$  as a function of  $p'$  and  $\sinh q$  as a function of  $q$  are scetched in fig. ix.

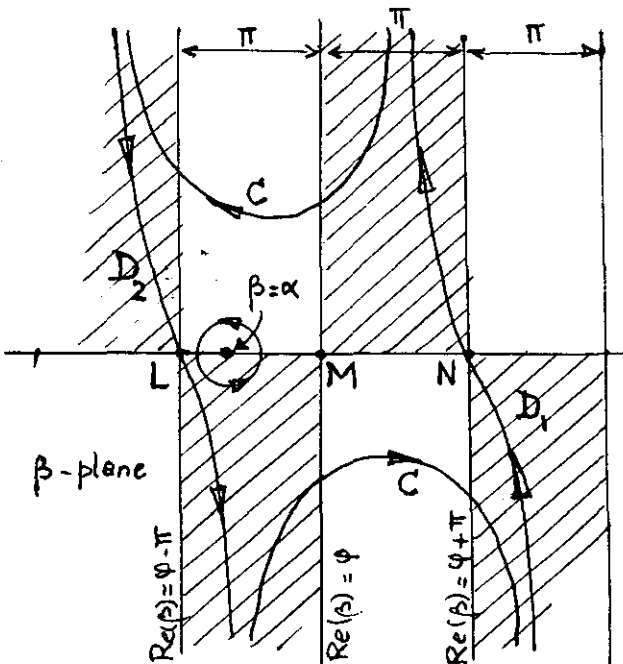


Fig. viii.



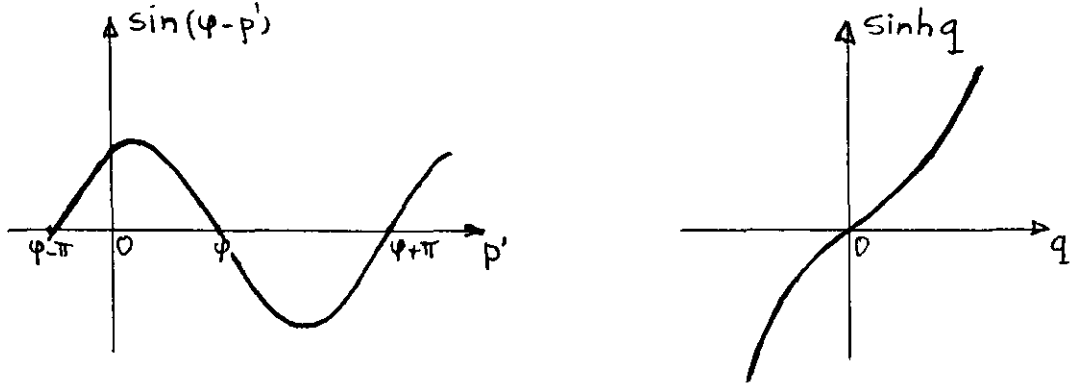


Fig. ix.

When  $q > 0$ , then  $\text{Im} [\cos(\phi-\beta)] < 0$  only when  $p'$  is situated between  $\phi + 2m\pi$  and  $\phi + (2m+1)\pi$  ( $m = \text{natural number}$ ).

When  $q < 0$ , then  $\text{Im} [\cos(\phi-\beta)] < 0$  only when  $p'$  is situated between  $\phi + (2m'+1)\pi$  and  $\phi + 2m'\pi$ . ( $m' = \text{natural number}$ ).

So in the hatched domains of fig. viii we have

$$\lim_{q \rightarrow \pm\infty} \exp [-jkR\cos(\phi-\beta)] = 0. \quad (59)$$

The contour of integration in fig. viii consists of the parts  $C$ ,  $D_1$  and  $D_2$ . A horizontal translation over  $2\pi$  maps the parts  $D_1$  and  $D_2$  on to each other. The integration over  $D_1$  and  $D_2$  thus cancel and only the part  $C$  remains. Can we find a function  $U^S$  with period  $4\pi$  in  $(\phi-\alpha)$ , satisfying Helmholtz' equation and the afore mentioned conditions then  $\bar{u}_1$  is known by (55).

We take

$$U^S = \frac{1}{4\pi} \int_C \frac{\exp(j\beta/2)}{\exp(j\beta/2) - \exp(j\alpha/2)} \exp\{-jkR\cos(\beta-\phi)\} d\beta \quad (60)$$

denoting

$$\begin{aligned} \beta - \phi &= \gamma \\ \phi - \alpha &= \psi \end{aligned} \quad (61)$$

we get

$$U^S = \frac{1}{4\pi} \int_C \frac{\exp(j\gamma/2)}{\exp(j\gamma/2) - \exp(-j\psi/2)} \exp(-jkR\cos\gamma) d\gamma \quad (62)$$

This  $U^S$  is nothing else than  $\exp[-jkR\cos(\phi-\alpha)]$ , so a solution of Helmholtz' equation.

In fig. x we find the path C in the  $\gamma$ -plane. The pole  $\beta = \alpha$  (fig. viii) is now

$$\gamma = \alpha - \phi = -\psi$$

For  $R \rightarrow \infty$  the integrand everywhere disappears in the right way in the hatched domains. The path C can now be modified so that the paths  $D_1$  and  $D_2$  remain, however, with reversed directions. The part

$$-\pi < \text{Re } \gamma < \pi,$$

on the real axis, is passed in opposite directions. As the integrand has period  $4\pi$  the integrals along  $D_1$  and  $D_2$  do not cancel.

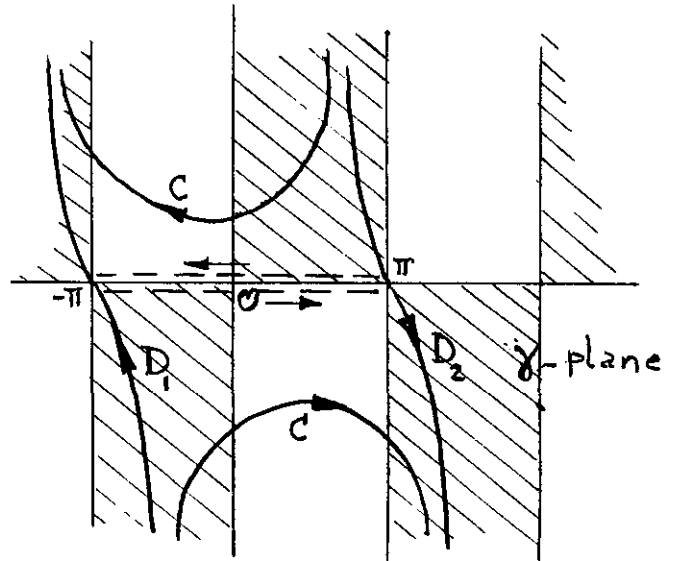


Fig. x.

Now

$$\lim_{R \rightarrow \infty} U^S = 0$$

when  $|\psi| > \pi$ , so in the "shadow" (fig. vii)

$$\lim_{R \rightarrow \infty} U^S = \exp[-jkR \cos \psi]$$

when  $|\psi| < \pi$ , so in the "illuminated" domain (fig. xi).

When  $|\psi| > \pi$  only the integrations along  $D_1$  and  $D_2$  remain. The latter is found from that along  $D_1$  by replacing

$$e^{j\gamma/2} \rightarrow e^{-j\gamma/2}$$

and by accounting for the reverse direction of travel.

So we find

$$U^S = \frac{1}{4\pi} \int_{D_2} e^{-jkR \cos \gamma} \Phi(\gamma) d\gamma \tag{63}$$

in which

$$\Phi(\gamma) = \frac{e^{j\gamma/2}}{e^{j\gamma/2} - e^{-j\psi/2}} - \frac{e^{j\gamma/2}}{e^{j\gamma/2} + e^{-j\psi/2}} = \frac{2 e^{j(\gamma-\psi)/2}}{e^{j\gamma} - e^{-j\psi}} \tag{64}$$

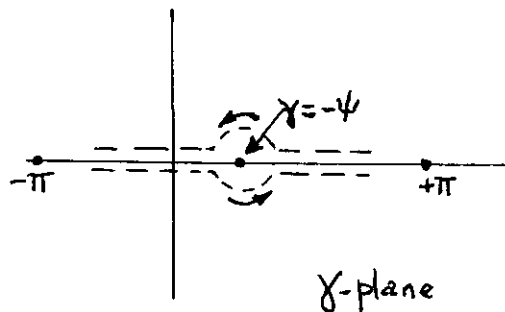


Fig. xi

6. A modification of Sommerfeld's problem

We now suppose that in stead of the wave  $\exp\{-jkR\cos(\phi-\alpha)\}$  we have the wave  $\bar{u}_I$  at A:

$$\bar{u}_I = \exp\{-jkR\cos\psi\} F(-R\sin\psi).$$

By diffraction  $u_1^S$  is generated. We derive  $u_1^S$  with the help of the function  $U_1^S$

$$U_1^S = \frac{1}{4\pi} \int_{D_2} e^{-jkR\cos\gamma} F(R\sin\gamma) \phi(\gamma) d\gamma \quad (65)$$

in which again

$$\phi(\gamma) = \frac{2 e^{j(\gamma-\psi)/2}}{e^{j\gamma} - e^{-j\psi}}$$

but now with  $\psi = \phi - \frac{\pi}{2}$ .

Introducing

$$R\sin\gamma = \bar{z} \quad (66)$$

we can write for sufficiently small  $\bar{z}$

$$F(\bar{z}) \sim F(0) + F'(0)\bar{z} \quad (67)$$

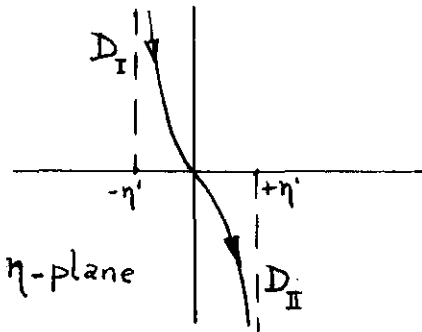


Fig. xii.

$$\text{with } \eta = \gamma - \pi \text{ (fig. xii)} \quad (68)$$

we can write

$$R\sin\gamma = R\sin(\eta+\pi) = -R\sin\eta \quad (69)$$

and

$$\int_{D_2} R\sin\gamma e^{-jkR\cos\psi} \phi(\gamma) d\gamma =$$

$$= \int_{-\eta'+j\infty}^0 e^{-jkR\cos(\pi+\eta)} R\sin(\pi+\eta) \phi(\pi+\eta) d\eta +$$

$$+ \int_0^{\eta'-j\infty} e^{-jkR\cos(\pi+\eta)} R\sin(\pi+\eta) \phi(\pi+\eta) d\eta$$

$$= - \int_0^{-\eta'+j\infty} -e^{jkR\cos\eta} R\sin\eta \phi(\pi+\eta) d\eta$$

$$- \int_0^{-\eta'+j\infty} e^{jkR\cos\eta} R\sin\eta \cdot \phi(\pi-\eta) d\eta \quad (\eta' \rightarrow 0) \quad (70)$$

$$= R \int_0^{-\eta'+j\infty} e^{jkR\cos\eta} \sin\eta \Psi(\eta) d\eta \quad (71)$$

in which

$$\Psi(\eta) = \Phi(\pi+\eta) - \Phi(\pi-\eta) = -4j \frac{\sin \eta/2 \sin \psi/2}{\cos\eta + \cos\psi} \quad (72)$$

This leads to the following expression for  $U_*$ , the part of  $U_1^S$  arising from the second term on the right of (67):

$$\pi U_* = -jF'(0) R \sin \psi/2 \int_0^{-\eta'+j\infty} e^{jkR\cos\eta} \frac{\sin \eta/2 \sin\eta}{\cos\eta + \cos\psi} d\eta \quad (73)$$

or

$$\pi U_* e^{jkR\cos\psi} = -jF'(0) R \sin \psi/2 \cdot P \quad (74)$$

in which

$$P = \int_0^{-\eta'+j\infty} e^{jkR(\cos\eta + \cos\psi)} \cdot \frac{\sin \eta/2 \sin\eta}{\cos\eta + \cos\psi} d\eta \quad (75)$$

Now we have

$$\frac{\partial P}{\partial R} = jk \int_0^{-\eta'+j\infty} e^{jkR(\cos\psi + \cos\eta)} \sin \eta/2 \sin\eta d\eta \quad (76)$$

As  $\cos\psi + \cos\eta = 2\cos^2(\psi/2) - 1 + 1 - 2\sin^2(\eta/2)$

we find

$$\frac{\partial P}{\partial R} = 2jk e^{2jkR\cos^2(\psi/2)} \int_0^{-\eta'+j\infty} e^{-2jkR\sin^2(\eta/2)} \sin^2(\eta/2) \cos \eta/2 d\eta \quad (77)$$

Partial integration leads to

$$\frac{\partial P}{\partial R} = - \frac{e^{2jkR\cos^2(\psi/2)}}{R} \left[ \sin \frac{\eta}{2} e^{-2jkR\sin^2(\eta/2)} \Big|_0^{-\eta'+j\infty} - \frac{1}{2} \int_0^{-\eta'+j\infty} e^{-2jkR\sin^2(\eta/2)} \cos \eta/2 d\eta \right]$$

with

$$\sin \eta/2 = \sqrt{\frac{\pi}{4kR}} s \quad (78)$$

we obtain

$$\frac{\partial P}{\partial R} = \frac{e^{2jkR\cos^2\psi/2}}{2R} \sqrt{\frac{\pi}{kR}} \int_0^{j\infty} e^{-j\pi \frac{s^2}{2}} ds$$

$$\frac{\partial P}{\partial R} = \frac{j(1+j)}{4} \left(\frac{\pi}{k}\right)^{1/2} \frac{e^{2jkR\cos^2(\psi/2)}}{R\sqrt{R}} \quad (79)$$

Let us now denote

$$\Lambda = \frac{e^{2jkR\cos^2(\psi/2)}}{R\sqrt{R}} \quad (80)$$

For  $\cos \psi/2 < 0$ , so in the shadow, we can set up the following reasoning.

Denote

$$\rho = \frac{1}{2\cos\psi/2} \sqrt{\frac{\pi}{k}} \frac{1}{\sqrt{R}} = aR^{-1/2} \quad (81)$$

in which

$$a = \frac{1}{2\cos\psi/2} \sqrt{\frac{\pi}{k}} \quad (82)$$

So

$$\frac{\partial \rho}{\partial R} = -\frac{a}{2R\sqrt{R}} \quad (83)$$

and we can write

$$\frac{\partial}{\partial \rho} (\dots) = -\frac{2R\sqrt{R}}{a} \frac{\partial}{\partial R} (\dots)$$

As a consequence

$$\Lambda = \frac{e^{\frac{1}{2}j\pi\rho^{-2}}}{R\sqrt{R}} = \frac{1}{R\sqrt{R}} \frac{\partial}{\partial \rho} \int_0^\rho e^{\frac{1}{2}j\pi\tau^{-2}} d\tau \quad (84)$$

or

$$\Lambda = -\frac{2}{a} \frac{\partial}{\partial R} \int_0^\rho e^{\frac{1}{2}j\pi\tau^{-2}} d\tau \quad (85)$$

with  $\tau = t^{-1}$  we get

$$\Lambda = -\frac{2}{a} \frac{\partial}{\partial R} \int_{-\infty}^{1/\rho} -e^{\frac{1}{2}j\pi t^2} t^{-2} dt \quad (86)$$

with

$$1/\rho = 2\cos\psi/2 \sqrt{\frac{kR}{\pi}}.$$

We have now

$$\Lambda = \frac{2}{a} \frac{\partial J}{\partial R} \quad (87)$$

with

$$J = \int_{-\infty}^p e^{\frac{1}{2}j\pi t^2} t^{-2} dt, \quad \text{divergent for } p > 0 \quad (88)$$

and

$$p = 2\cos\psi/2 \sqrt{\frac{kR}{\pi}}. \quad (89)$$

Going back to (79) and substituting

$$\frac{\partial P}{\partial R} = jk \frac{1+j}{4\pi} \left(\frac{\pi}{k}\right)^{3/2} 2 \sqrt{\frac{k}{\pi}} 2 \cos \psi/2 \frac{\partial J}{\partial R} \quad (90)$$

So

$$P = -(1-j) \cos \psi/2 \cdot J \quad (91)$$

From (74) we then find, for  $p < 0$

$$U_* = \frac{1+j}{\pi} F'(0) R \sin \psi/2 \cos \psi/2 e^{-jkR \cos \psi} J \quad (92)$$

Does  $U_*$  satisfy condition e (see par. 5)?

This conditions demands

$$\lim_{R \rightarrow 0} R \nabla U_* = 0$$

We have

$$\nabla U_* = \left( \frac{\partial U_*}{\partial R}, \frac{1}{R} \frac{\partial U_*}{\partial \psi} \right)$$

Denote

$$U_* = R \sin \psi e^{-jkR \cos \psi} J.K. \quad (93)$$

with

$$K = \frac{1+j}{2\pi} F'(0) \quad (94)$$

Now

$$\begin{aligned} \int_a^b e^{js^2} s^{-2} ds &= \frac{e^{js^2}}{s} \Big|_a^b - \int_{s=a}^{s=b} s d(e^{js^2} s^{-2}) \\ &= \frac{e^{js^2}}{s} \Big|_a^b - 2j \int_a^b e^{js^2} ds + 2 \int_a^b e^{js^2} s^{-2} ds . \end{aligned}$$

So

$$\int_a^b e^{js^2} s^{-2} ds = 2j \int_a^b e^{js^2} ds - \frac{e^{js^2}}{s} \Big|_a^b$$

or, with  $s = t \sqrt{\frac{\pi}{2}}$

$$\int_{a\sqrt{\frac{2}{\pi}}}^{b\sqrt{\frac{2}{\pi}}} e^{\frac{1}{2}j\pi t^2} t^{-2} dt = j\pi \int_{a\sqrt{\frac{2}{\pi}}}^{b\sqrt{\frac{2}{\pi}}} e^{\frac{1}{2}j\pi t^2} dt - \frac{e^{\frac{1}{2}j\pi t^2}}{t} \Big|_{a\sqrt{\frac{2}{\pi}}}^{b\sqrt{\frac{2}{\pi}}}$$

So 
$$J = j\pi \int_{-\infty}^{\rho} e^{\frac{1}{2}j\pi t^2} dt - \frac{e^{\frac{1}{2}j\pi p^2}}{p^2}$$

$$\frac{\partial J}{\partial p} = j\pi e^{\frac{1}{2}j\pi p^2} - p^{-2} (j\pi p^2 e^{\frac{1}{2}j\pi p^2} - e^{\frac{1}{2}j\pi p^2})$$

As

$\frac{\partial J}{\partial R} = \frac{\partial J}{\partial p} \cdot \frac{\partial p}{\partial R}$ , the term in  $\frac{\partial U_*}{\partial R}$  which most probably does not go to zero for  $R \rightarrow 0$  is

$$- \cos\psi/2 \sqrt{\frac{k}{\pi}} \frac{e^{\frac{1}{2}j\pi p^2}}{p^2 \sqrt{R}}$$

This term is of the order  $R^{-3/2}$ . But even this term goes to zero, for accounting for  $R$  in  $R\sqrt{U_*}$  and with  $R$  in (93), we see that in fact  $R\sqrt{U_*}$  goes to zero for  $R \rightarrow 0$  according to  $R^{1/2}$ .

In (92) we found, for  $p < 0$

$$U_* = \frac{1+j}{\pi} F'(0) R \sin\psi/2 \cos\psi/2 e^{-jkR\cos\psi} J.$$

For  $p \ll -1$  we can expand

$$J = \frac{e^{\frac{1}{2}j\pi p^2}}{j\pi p^3} \left( 1 + \frac{3}{j\pi p^2} + \frac{5.3}{(j\pi p^2)^2} + \dots \right) \tag{95}$$

retaining only the first term we can write

$$J \sim \sqrt{\frac{\pi}{kR}} \frac{e^{2jkR\cos^2\psi/2}}{8jkR\cos^3\psi/2} \tag{96}$$

thus

$$U_* \sim \frac{1-j}{8k\cos^2\psi/2} F'(0) \sin\psi/2 \frac{e^{jkR}}{\sqrt{\pi kR}} \tag{97}$$

Of course this result is only significant for large  $R$  in the shadow, away from the shadow-boundary  $\psi = \pi$ .

Now

$$U_* = U_*(\psi) = U_*(\phi - \frac{\pi}{2}) \quad -\frac{\pi}{2} \leq \psi \leq \frac{3\pi}{2}.$$

The solution for  $\mu_*$  in the shadow is

$$u_* = U_*(\phi - \frac{\pi}{2}) + U_*(\phi + \frac{\pi}{2}) \tag{98}$$

From (97):

$$U_*(\phi - \frac{\pi}{2}) \sim \frac{(\sin\phi/2 - \cos\phi/2)\sqrt{2}}{(\sin\phi/2 + \cos\phi/2)^2} \frac{1-j}{8k} F'(0) \frac{e^{jkR}}{\sqrt{\pi kR}} \quad (99)$$

$$U_*(\phi + \frac{\pi}{2}) \sim \frac{(\sin\phi/2 + \cos\phi/2)\sqrt{2}}{(\sin\phi/2 - \cos\phi/2)^2} \frac{1-j}{8k} F'(0) \frac{e^{jkR}}{\sqrt{\pi kR}} \quad (100)$$

So we arrive at

$$u_* \sim \frac{e^{jkR}}{\sqrt{\pi kR}} \frac{1-j}{2k\sqrt{2}} F'(0) \frac{\sin\phi/2(2\cos^2(\phi/2)+1)}{\cos^2\phi} \quad (101)$$

It must be kept in mind that  $u_*$  that part of  $u_1^S$  is which is associated with the term  $F'(0)\bar{z}$  in the expansion of  $F(\bar{z})$  around  $\bar{z} = 0$ .

To find an expression which is representative for the illuminated domain we reform (92) by applying partial integration to (88)

$$J = j\pi \int_{-\infty}^p e^{\frac{1}{2}j\pi t^2} dt - \frac{e^{\frac{1}{2}j\pi p^2}}{p} = j\pi \int_{-\infty}^p e^{\frac{1}{2}j\pi t^2} dt - \frac{e^{2jkR\cos^2\psi/2}}{2\cos\psi/2 \sqrt{\frac{kR}{\pi}}}$$

Substituted in (92) gives

$$U_* = -\frac{1+j}{2\sqrt{\pi}} F'(0) \frac{\sqrt{R}\sin\psi/2}{\sqrt{k}} e^{jkR} + j(1+j) F'(0) R \sin\psi/2 \cos\psi/2 \cdot e^{-jkR\cos\psi} \int_{-\infty}^p e^{\frac{1}{2}j\pi t^2} dt \quad (102)$$

Expanding J for  $p = 2\sqrt{\frac{kR}{\pi}} \cos\psi/2 \gg 1$  gives

$$J = j\pi(1+j) + \frac{e^{\frac{1}{2}j\pi p^2}}{j\pi p^3} + O(\frac{1}{p^5})$$

So that we find for large R in the illuminated domain

$$U_* \sim \underbrace{F'(0)R \sin\psi e^{-jkR\cos\psi}}_{\text{incident wave}} + \frac{1-j}{8k\cos^2\psi/2} F'(0)\sin\psi/2 \frac{e^{jkR}}{\sqrt{\pi kR}} \quad (103)$$



7. Finding expressions for  $F'(0)$  and  $G'(0)$

In order to derive an expression for  $F'(0)$  it is convenient to have all relevant formulas together. First we have

$$F(\bar{z}) = F_1(\bar{z}) + F_2(\bar{z})$$

in which

$$\bar{z} = R \sin \gamma$$

more explicitly for  $\bar{z}$  real, see (36) and (44)

$$F(\bar{z}) = e^{jk(r_0+b)} \left[ \frac{f^*(x_0)}{2\sqrt{2\pi k s_0}} + \frac{j(g_0+g'_0)}{8\pi k} \right] \quad (104)$$

in which

$$x_0 = \xi_0 \sqrt{k}$$

$$f^*(x_0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x_0} e^{jt^2} dt$$

(see paragraph 4 for the meaning of  $s_0$ ,  $\xi_0$ ,  $g_0$  and  $g'_0$ ).

For  $\bar{z} = 0$  we can write

$$\theta = \frac{3\pi}{2}, \quad s_0 = b + r_0, \quad \xi_0 = 0 \quad (105)$$

$$\frac{df^*(x_0)}{dx_0} = \frac{e^{jx_0^2}}{\sqrt{\pi}} \quad (106)$$

or

$$\frac{df^*(\xi_0 \sqrt{k})}{d\xi_0} = \sqrt{\frac{k}{\pi}} e^{jk\xi_0^2} \quad (107)$$

For  $\bar{z}$  real =  $z < 0$  we have

$$\frac{d\xi_0}{ds_0} = -(b+r_0-s_0)^{-\frac{1}{2}} \quad (108)$$

$$\frac{ds_0}{dz} = -r_0 \sin \frac{z}{b} (b^2+r_0^2+2br_0 \cos \frac{z}{b})^{-\frac{1}{2}} \quad (109)$$

$$\frac{df^*(x_0)}{dz} = \frac{df^*(x)}{d\xi_0} \cdot \frac{d\xi_0}{ds_0} \cdot \frac{ds_0}{dz} \quad (110)$$

or

$$\frac{df^*(x_0)}{dz} = \sqrt{\frac{k}{\pi}} e^{jk\xi_0^2} \frac{r_0}{2} \cdot \sin \frac{z}{b} (b+r_0-s_0)^{-\frac{1}{2}} (b^2+r_0^2+2br_0 \cos \frac{z}{b})^{-\frac{1}{2}} \quad (111)$$

Also we have

$$\frac{d}{dz} \left( \frac{f^*}{\sqrt{s_0}} \right) = \frac{\frac{df^*}{dz} \sqrt{s_0} - \frac{1}{2} f^* s_0^{-1/2} \frac{ds_0}{dz}}{s_0} \quad (112)$$

The factor  $(b+r_0-s_0)^{-1/2}$  seems to give singular points, however when  $z \uparrow 0$  we can write

$$\theta \sim \frac{z}{b} + \frac{3\pi}{2}$$

so

$$\cos\left(\frac{\pi}{2} - \theta\right) = -\cos \frac{z}{b}.$$

We obtain for  $s_0$ , as  $\cos \frac{z}{b} \sim 1 - \frac{1}{2} \left(\frac{z}{b}\right)^2$

$$\begin{aligned} s_0 &= \sqrt{b^2 + r_0^2 - 2br_0 \cos\left(\frac{\pi}{2} - \theta\right)} \\ &= \sqrt{b^2 + r_0^2 + 2br_0 \cos \frac{z}{b}} \\ &= (b+r_0) \sqrt{1 - \frac{2br_0}{(b+r_0)^2} (1-\cos \frac{z}{b})} \\ &\sim (b+r_0) \sqrt{1 - \frac{br_0}{(b+r_0)^2} \left(\frac{z}{b}\right)^2} \\ &= (b+r_0) \left[ 1 - \frac{br_0}{2(b+r_0)^2} \left(\frac{z}{b}\right)^2 \right] \\ &= b + r_0 - \frac{br_0}{2(b+r_0)} \left(\frac{z}{b}\right)^2 \end{aligned} \quad (113)$$

$$\xi_0 = \sqrt{b + r_0 - s_0} \sim \sqrt{\frac{br_0}{2(b+r_0)} \left(\frac{z}{b}\right)^2}$$

$$\xi_0 \sim -\frac{z}{b} \sqrt{\frac{br_0}{2(b+r_0)}} \quad (114)$$

Using  $\sin \frac{z}{b} \sim \frac{z}{b}$  we find

$$\frac{df^*}{dz} = \sqrt{\frac{k}{\pi}} e^{jk\xi_0^2} \frac{r_0}{2} \frac{\sin \frac{z}{b}}{\sqrt{b+r_0-s_0}} \cdot \frac{1}{\sqrt{b^2+r_0^2+2br_0 \cos \frac{z}{b}}}$$

$$\text{so } \left( \frac{df^*}{dz} \right)_{z=0} = -\sqrt{\frac{kr_0}{2\pi b(b+r_0)}} \quad (115)$$

Now we consider  $\frac{dg_0}{dz}$  for  $z \uparrow 0$ .

$$g_0 = \frac{\sqrt{2}}{\xi_0 \sqrt{s_0}} - \frac{1}{\sqrt{br_0} \cos[\frac{1}{2}(\frac{\pi}{2} - \theta)]}$$

$$\cos[\frac{1}{2}(\frac{\pi}{2} - \theta)] \sim -\sin \frac{z}{2b}.$$

Further

$$\frac{d(\xi_0^{-1})}{dz} = -\xi_0^2 \frac{d\xi_0}{ds_0} \cdot \frac{ds_0}{dz} = -\frac{r_0 \sin \frac{z}{b}}{2(b+r_0-s_0)\sqrt{b+r_0-s_0} \sqrt{b^2+r_0^2} + 2br_0 \cos \frac{z}{b}}$$

and

$$\frac{d(s_0^{-1})}{dz} = -\frac{1}{2} s_0^{-3/2} \frac{ds_0}{dz} = \frac{r_0 \sin \frac{z}{b}}{s_0 \sqrt{s_0} \sqrt{b^2+r_0^2} + 2br_0 \cos \frac{z}{b}}$$

When  $z \uparrow 0$  we can write for these quantities

$$\left( \frac{d(\xi_0^{-1})}{dz} \right)_{z \uparrow 0} \sim \frac{b}{z^2} \sqrt{\frac{2(b+r_0)}{br_0}} \quad (116)$$

$$\left( \frac{d(s_0^{-1})}{dz} \right)_{z \uparrow 0} \sim \frac{r_0 z}{b(b+r_0)^{5/2}} \quad (117)$$

$$\begin{aligned} \frac{d}{dz} \left( \frac{1}{\xi_0 \sqrt{s_0}} \right) &= \frac{d(\xi_0^{-1})}{dz} \cdot \frac{1}{\sqrt{s_0}} + \frac{1}{\xi_0} \frac{d(s_0^{-1/2})}{dz} \\ &\approx \frac{b}{z^2} \sqrt{\frac{2(b+r_0)}{br_0}} \cdot \frac{1}{\sqrt{b+r_0}} - \frac{r_0 z}{b \sqrt{br_0} (2b+r_0)^{5/2}} \\ &= \frac{\sqrt{2b}}{z^2 \sqrt{r_0}} - \frac{1}{(b+r_0)^2} \sqrt{\frac{2r_0}{b}} \end{aligned}$$

$$\frac{d}{dz} \left( \sin^{-1} \frac{z}{2b} \right) = -\frac{\cos \frac{z}{2b}}{2b \sin \frac{2z}{2b}} \sim -\frac{2b}{z^2}.$$

With these expressions we obtain the result

$$\frac{dg_o}{dz} \sim \frac{2}{z^2} \sqrt{\frac{b}{r_o}} - \frac{2\sqrt{\frac{r_o}{b}}}{(b+r_o)^2} - \frac{2}{z^2} \sqrt{\frac{b}{r_o}} = -\frac{2\sqrt{\frac{r_o}{b}}}{(b+r_o)^2} \quad (118)$$

Next we investigate  $\frac{dg_o'}{dz}$  for  $z \uparrow 0$ .

$$\frac{d}{dz} \left( \frac{1}{\xi_o' \sqrt{s_o'}} \right) = \frac{d\xi_o'^{-1}}{dz} \frac{1}{\sqrt{s_o'}} + \frac{1}{\xi_o'} \frac{ds_o'^{-1/2}}{dz}$$

$$\xi_o'^{-1} = -(b+r_o - s_o')^{-1/2}$$

$$\frac{d\xi_o'^{-1}}{dz} = -\frac{1}{2}(b+r_o - s_o')^{-3/2} \frac{ds_o'}{dz}$$

$$\frac{ds_o'^{-1/2}}{dz} = -\frac{1}{2} s_o'^{-3/2} \frac{ds_o'}{dz}$$

As we can see from (109) the last two expressions become zero when  $z \rightarrow 0$ .

$$\cos\left[\frac{1}{2}\left(\frac{\pi}{2} + \theta\right)\right] = -\cos \frac{z}{2b}$$

$$\frac{d}{dz} \left( \cos^{-1} \frac{z}{2b} \right) = \frac{1}{2b} \cos^{-2} \left( \frac{z}{2b} \sin \frac{z}{b} \right)$$

This term of  $\frac{dg_o'}{dz}$  also becomes zero for  $z \rightarrow 0$ .

We conclude that

$$\left( \frac{dg_o'}{dz} \right)_{z=0} = 0 \quad (119)$$

Returning to formula (112) we see that for  $z = 0$  we get

$$\frac{d}{dz} \frac{f_*}{\sqrt{s_o}} \sim -\frac{1}{\sqrt{s_o}} \sqrt{\frac{kr_o}{2\pi b(b+r_o)}} = -\frac{\sqrt{\frac{kr_o}{2\pi b}}}{b+r_o} \quad (120)$$

This gives

$$F'(0) = e^{jk(b+r_o)} \left\{ -\frac{1}{2\sqrt{2\pi}} \cdot \frac{1}{b+r_o} \sqrt{\frac{r_o}{2\pi b}} + \frac{j}{4\pi k} \left[ -\frac{\sqrt{\frac{r_o}{b}}}{(b+r_o)^2} \right] \right\}$$

$$= -e^{jk(r_o+b)} \frac{\sqrt{\frac{r_o}{b}}}{4\pi(r_o+b)} \left[ 1 + j \frac{1}{k(r_o+b)} \right] \quad (121)$$

When  $k(r_0+b) \gg 1$  we can approximate

$$F'(0) \sim -\frac{e^{jk(r_0+b)}}{4\pi(r_0+b)} \sqrt{\frac{r_0}{b}} \quad (122)$$

The incident wave at A contains also the term  $\bar{u}_{II}$ , for which we have according to (51)

$$\bar{u}_{II} = e^{-jkR\cos\chi} G(-R\sin\chi).$$

Diffraction generates a cylinder-wave  $u'_*$  emanating from A, which we can express in the well-known way with the help of a function  $U'_*$ .

We put again

$$G(z') \sim G(0) + G'(0)z' \quad (123)$$

and consider the effect of the second term only.

In (38) we defined

$$s'_0(z) = \sqrt{b^2 + r_0^2 - 2br_0 \cos \frac{z}{b}}.$$

while in (46) we used

$$G = e^{jk(r_0-b)} \frac{f^*(x'_0)}{2\sqrt{2\pi k s'_0}}$$

We have now

$$\frac{df^*(x'_0)}{d\xi'_0} = \sqrt{\frac{k}{\pi}} e^{jk\xi'^2_0}$$

$$\xi'_0 = -\sqrt{b+r_0-s'_0}$$

$$\frac{d\xi'_0}{ds'_0} = \frac{1}{2}(b+r_0-s'_0)^{-\frac{1}{2}}$$

$$\frac{ds'_0}{dz} = r_0 \sin \frac{z}{b} (b^2 + r_0^2 - 2br_0 \cos \frac{z}{b})^{-\frac{1}{2}}$$

$$\frac{df^*(x'_0)}{dz} = \frac{df(x'_0)}{d\xi'_0} \cdot \frac{d\xi'_0}{ds'_0} \cdot \frac{ds'_0}{dz}$$

$$= \sqrt{\frac{k}{\pi}} e^{jk\xi'^2_0} \frac{1}{2}(b+r_0-s'_0)^{-\frac{1}{2}} r_0 \sin \frac{z}{b} (b^2 + r_0^2 - 2br_0 \cos \frac{z}{b})^{-\frac{1}{2}} \quad (124)$$

For  $z = 0$  we obtain

$$s'_0(o) = r_0 - b \quad (125)$$

$$\xi'_0(o) = -\sqrt{2b} \quad (126)$$

So

$$\left( \frac{df^*(x'_0)}{dz} \right)_{z=0} = 0 \quad (127)$$

and

$$\left[ \frac{d}{dz} \left( \frac{f^*(x'_0)}{\sqrt{s'_0}} \right) \right]_{z=0} = 0 \quad (128)$$

The conclusion is that

$$G'(o) = 0 \quad (129)$$

$$u'_* = 0 \quad (130)$$

So the plane wave  $\bar{u}$  (see (48)) originates, as far as it's  $z$ -derivative is concerned, only a diffracted wave  $u_*$ .

We can thus write

$$u_*(R, \phi) \sim \frac{e^{jkR}}{\sqrt{\pi kR}} \frac{1-j}{2k\sqrt{2}} \frac{e^{jk(r_0+b)}}{4\pi(r_0+b)} \cdot \sqrt{\frac{r_0}{b}} \cdot \frac{\sin \phi/2}{\cos^2 \phi} (2\cos^2 \frac{\phi}{2} + 1) \quad (131)$$

under the conditions

$$R \rightarrow \infty \quad \text{so } p = 2\cos\left(\frac{\phi}{2} - \frac{\pi}{4}\right) \sqrt{\frac{kR}{\pi}} \ll -1 \quad (\text{see (89)})$$

$$3\pi/2 \ll \phi < 2\pi$$

$$k(r_0 + b) \gg 1.$$

8. Comparison of the effect of the first and the second term of the expansion of  $\bar{u}$  around  $z = 0$

According to [2], formulas (3.5) and (3.12) we find, when a wave  $u_0$  for which

$$u_0 \sim C(ks) \quad \text{see (11)}$$

is incident at  $O$  from  $y = r_0$ ,  $z = 0$ , that at the shadow boundary  $\theta = \frac{3\pi}{2}$  by

by diffraction at O a wave u is excited for which

$$u(r, \frac{3\pi}{2}) \sim \frac{1}{2} C[k(r+r_0)] + C(kr_0)C(kr) \quad (132)$$

At the edge A(r=b) this becomes

$$u(b, \frac{3\pi}{2}) \sim \frac{1}{2} C[k(r_0+b)] + C(kr_0)C(kb) \quad (133)$$

see appendix B.

The theory of [1] delivers the expression for the diffraction  $\hat{u}$  at A.

$$\hat{u}(R, \phi) \sim \{ \frac{1}{2} C[k(r_0+b)] + C(kr_0)C(kb) \} D(\phi, \frac{\pi}{2}) \frac{e^{jkR}}{\sqrt{R}} \quad (134)$$

with

$$D(\phi, \frac{\pi}{2}) = - \frac{1+j}{4\sqrt{\pi k}} \left( \sec \frac{\phi - \pi/2}{2} + \sec \frac{\phi + \pi/2}{2} \right) \quad (135)$$

( $\phi \neq \frac{3\pi}{2}$ )

This result we also find from Sommerfeld's formula (see [8], page 238).

Mind the adaption of the coordinate-axes (see Appendix A). We neglect the diffraction term in  $\hat{u}$  and obtain

$$\hat{u} \sim \frac{1}{2} C[k(r_0+b)] D(\phi, \frac{\pi}{2}) \frac{e^{jkR}}{\sqrt{R}} \quad (136)$$

So we find

$$\frac{u_*}{\hat{u}} \sim e^{j\pi/4} \cdot \frac{2\sqrt{r_0}}{\sqrt{\pi k b (r_0 + b)}} \cdot \frac{\sin \phi/2 (2\cos^2 \phi/2 + 1)}{\cos^2 \phi [\sec(\frac{\phi}{2} - \frac{\pi}{4}) + \sec(\frac{\phi}{2} + \frac{\pi}{4})]} \quad (137)$$

So it appears that the effect is a factor  $\sqrt{k}$  smaller than the "main effect" found with the theory of [1].

### Acknowledgment

The authors wish to thank prof. dr. J. Boersma for his remarks concerning the second part of this report.

Appendix A

Adaption of the coordinate axes

Sommerfeld's coordinate axes are those of fig. vii. In [8]  $\bar{u}_1$  is  $H_z$ . We work with the coordinate system pictured in fig. iv.

We see that for our  $H_x$  we have

$$H_x = - H_z^{\text{Sommerfeld}}$$

(see fig. 3)

As far as we follow Sommerfeld's method we use the xy-system.

So then  $u_x = - H_x$

Once arrived at the result (see formula 131) we adapt the sign to the yz-system, so that we subsequently have

$$u^* = + H_x$$

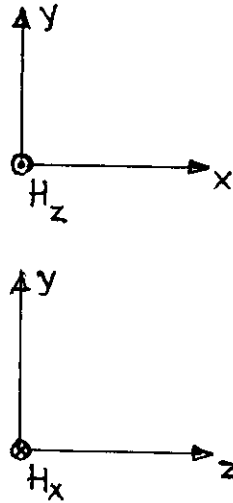


Fig. 3.



Appendix B

Reformulation of  $\bar{u}(b, \frac{3\pi}{2})$   
-----

We show that for  $r = b$  and  $\theta = \frac{3\pi}{2}$

$$\bar{u}(b, \frac{3\pi}{2}) \sim \frac{1}{2}C[k(b+r_0)] + C(kr_0)C(kb).$$

We have  $\phi = \frac{\pi}{2}$  and  $R \sim 0$ . According to (47) is

$$\bar{u} = \bar{U}_1 + \bar{U}_2.$$

We can put that

$$\bar{U}_1 \sim e^{jk(r_0+b)} \left[ \frac{e^{j\pi/4}}{4\sqrt{2\pi k(b+r_0)}} + 0 \right]$$

for around  $\theta = 3\pi/2$  the first term of  $g_0$  behaves like  $2(\frac{3\pi}{2} - \theta)^{-1}$  and the second term like  $-2(\frac{3\pi}{2} - \theta)^{-1}$ . According to (43) is

$$\bar{U}_2 \sim e^{jk(r_0+b)} \frac{jg'_0}{8\pi k} + e^{jk(r_0-b)} \frac{f^*(x'_0)}{2\sqrt{2\pi k s'_0}}$$

for  $\theta \sim 3\pi/2$  we have

$$g'_0 = \frac{\sqrt{2}}{\xi'_0 \sqrt{s'_0}} + \frac{1}{\sqrt{r_0 b}}$$

Now

$$f^*(x'_0) = e^{jx'^2_0} f(x) e^{j\pi/4} = e^{2jb} e^{j\pi/4} f(x)$$

with

$$f(x) \sim -\frac{e^{j\pi/4}}{2x\sqrt{\pi}} \quad \text{for } x \rightarrow -\infty \quad \text{see [2], formula (3.10)}$$

So

$$\bar{U}_2 \sim e^{jk(r_0+b)} \frac{j}{8\pi k \sqrt{r_0 b}}$$

With this result we find

$$\bar{U}_1 + \bar{U}_2 \sim \frac{1}{2}C[k(r_0+b)] + C(kr_0)C(kb).$$

References

- [1] H.Y. Yee, L.B. Felsen and J.B. Keller, "Ray theory of reflection from the open end of a waveguide", SIAM Journal on applied mathematics, Vol. 16 no. 2, March 1968, p. 268.
- [2] H.J. Steures, "Reflektie en transmissie in een golfpijp volgens een asymptotische methode van Yee, Felsen en Keller", internal report dept. of Mathematics, T.H. Eindhoven, May 1970.
- [3] R.F. Harrington, "Time-harmonic electromagnetic fields", McGraw-Hill, 1961.
- [4] E.V. Jull, "Reflection from the aperture of a long E-plane sectoral horn", IEEE transactions on antennas and propagation, Vol. AP-20, no. 1, January 1972, p. 62.
- [5] R.M. Lewis and J. Boersma, "Uniform asymptotic theory of edge diffraction", Journal of mathematical physics, Vol. 10 no. 12, December 1969, p. 2291.
- [6] D.S. Ahluwalia, R.M. Lewis and J. Boersma, "Uniform asymptotic theory of diffraction by a plane screen", SIAM Journal on applied mathematics, Vol. 16 no. 4, July 1968, p. 783.
- [7] M.J. van de Scheur, "Diffractie aan een spleet, bij een loodrecht invallende vlakke scalaire golf", internal report dept. of Mathematics, T.H. Eindhoven, September 1970.
- [8] A. Sommerfeld, "Optik. Vorlesungen über theoretische Physik, Band IV", Geest & Portig K-G, 1959, p. 224.
- [9] R.E. Collin and F.J. Zucker, "Antenna theory, part I", McGraw-Hill, 1969, p. 621.