Integral stocknorms in divergent systems with lot-sizes

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by K. van Donselaar

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Abstract

In this paper an overview is given of integral stocknorm formulas for several periodic review production/inventory systems. To derive these formulas, all systems up to the most complex system (a 2-stage divergent system with lot-sizing) are translated in terms of the most basic system: a 1-stage linear system without lot-sizing. The resulting formulas are simple and coherent.

In divergent systems imbalance may occur. Imbalance occurs if the inventory positions of the final products with a common component are not equivalent. In case of an integral re-order policy, imbalance may cause non-optimal re-order suggestions. For systems without lot-sizing the impact of this imbalance is negligible.

In this paper the role of imbalance due to lot-sizing is investigated. This role will appear to be small in many cases. In those cases where imbalance is no longer negligible the stocknorms can be adapted to take account of this effect. A method to do so is suggested and tested.

Systems with depot as well as without depot will be considered and compared. It will appear that the following rule holds in general: The positive effect of decreased imbalance in case a depot is present will appear to be small compared with the negative effect of decreased ability to satisfy customers' demand. The only exception to this rule are systems with a large lot-size for the common component combined with large coefficients of variation for the final products' demand.

Attention will be restricted to the identical products case.

1. Introduction.

* The goal.

The goal of this paper is to gain insight into the effect of integral stocknorms on service level in divergent systems. An example of a divergent system is given in Figure 1.


```
common          final
part            products
```
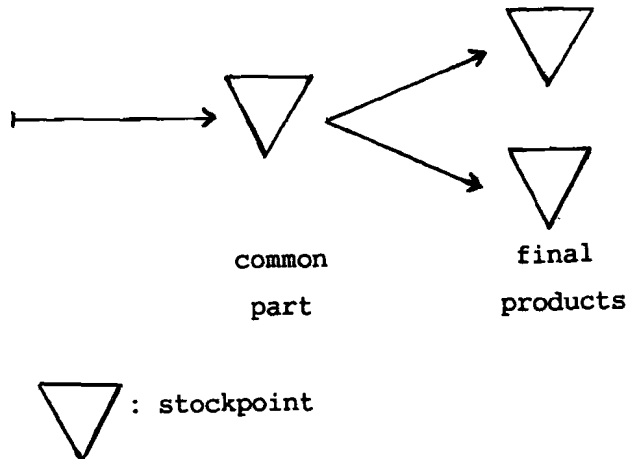
▽ : stockpoint

Figure 1. A divergent system with depot consisting of
2 final products having one part in common.

In a divergent system a common part is produced which goes into several final products. In this paper the common part is assumed to be controlled by means of an integral re-order rule: If all inventory in the divergent system (including the final products' inventories expressed in equivalent amounts of the common part) is below an integral stocknorm, a new order will be placed.

If the inventory levels of the final products are very unbalanced, this integral (or aggregate) re-order rule might perform poorly. Take for example a divergent system with two final products: A and B. The demand for A and B equals 10 products/week. The inventories of A and B are resp. 200 and zero. Each unit of A and B needs one common part. Suppose there are no common parts on hand. All this implies that the actual integral inventory equals 200*1 (A) + 0*1 (B) + 0 (common part) = 200 common parts. If the integral stocknorm equals 120, then no common parts are ordered. However it is obvious that B needs extra supply. So extra common parts should have been ordered.

The error in this re-order suggestion results from the fact that only aggregate information is used ("there are 200 common parts in the system") combined with the fact that the system is out of balance ("all 200 common parts are used for A exclusively"). From this example it can be concluded that the assumption that inventories are balanced leads to an overestimation of the expected service level.

For divergent systems with identical final products the extend to which each of the final products' inventories (on hand plus on order) deviates from the average inventory at the final stage is called imbalance. In most literarture concerning integral re-order policies imbalance is assumed to have negligible impact on the system's service level.

Clark and Scarf [2] proved that integral or echelon re-order policies are optimal for linear multi-echelon inventory systems without setup-costs. From that they concluded that integral order policies are also optimal for divergent multi-echelon inventory systems if imbalance is assumed to be negligible. Eppen and Schrage [4] derived formulas for integral stocknorms in divergent systems, corresponding with a pre-determined service level, again assuming that imbalance is negligible.

In practice integral re-order policies are rarely implemented, despite the proven optimality. One of the reasons is the fear, that imbalance might appear to be large; e.g. due to lot-sizing. This paper investigates whether this fear is warranted or not and in which way large imbalance might be tackled.

Earlier research in this field has been performed by Zipkin [6]. He derived an aggregate (dynamic) program, which takes account of the actual state of imbalance. His approach yields a.o. guidelines for allocation policies in case the coefficient of variation of the demand is equal for all final products. It does not however provide a direct and simple relation between the integral stocknorm and the system's parameters. To come up with such a relation imbalance is tackled in a more structural way in this paper: Instead of continuously keeping track of the actual imbalance, the system stocknorm is increased structurally based on the average imbalance. This increase of the stocknorm compensates the negative impact on the service level caused by imbalance.

* Description of the divergent system.

The divergent production/inventory system consists of two levels: a part level and a final product level. All final products require the same (common) part and it takes one part to produce one final product. Results for these type of systems can be translated easy to more complex systems, where each final product requires several common parts. Hereto all demand and inventory data should be translated in an equivalent amount of parts. Suppose for example that the final product is a table. Then the inventory and demand for the table should first be multiplied by four before the results of this paper can be applied to the legs which go into the table.

The final products are assumed to be identical with respect to their leadtime, lot-size and demand characteristics.

The final products are always made to stock. Whether the common part can be stored for a number of periods or whether it has to be allocated to the final products immediately is determined by the presence resp. absence of a central depot. In this paper both system, with and without depot for the common part, will be considered.

All order policies are periodic review integral (or echelon) re-order-point policies. That means, that if the integral inventory on hand plus on order is below its norm at the review moment, an order is placed to bring the inventory above the norm again. For systems without lot-sizing every order is equal to the demand in the previous period. For systems with lot-sizing every order is equal to or a multiple of the lot-size. The lot-size for the common part is assumed to be equal to or a multiple of the lot-size for the final products. Otherwise dead stock would result.

Demand is assumed to be stationary in time and distributed independently with respect to consecutive periods as well as with respect to different final products. Demand which can not be satisfied due to a shortage of supply will be backordered.

As an indicator for the system's performance, service levels (instead of costs) are being used.

* Definition of variables.

The main variables used in this text are:
N - number of final products

j   - index of a final product, j=1,..,N

r_{comm} - the (integral) re-order point of the common part

Q_{comm} - the lot-size for the common part

ℓ_{comm} - the leadtime for the common part

α   - pre-determined service level, defined as the percentage of periods, that demand can be met from on hand inventory instantaneously.

Φ(.)   - standard normal distribution function

k   - a safety-factor

For each of the final products:

μ   - the average demand

σ   - the standard deviation of demand

r   - the re-order point

Q   - the lot-size

ℓ - the leadtime.

* <u>The content of the paper.</u>

As mentioned before imbalance in a divergent system will be tackled by increasing the system's stocknorm. To come up with an appropriate stocknorm, simpler systems are studied first in the next Section. This will lead to a stocknorm formula for the divergent system with fixed lot-sizes, which uses imbalance as an input-variable.

In Section 3 estimators are derived for imbalance, which are a function of system parameters. Each estimator can be used in the stocknorm formula derived in Section 2.

In Section 4 the quality of the estimators for imbalance is tested by means of simulation. The estimators will appear to be poor estimators for the imbalance itself, but fortunately they are sufficiently good for the determination of appropriate stocknorms.

With these estimators the impact of imbalance on the service level is quantified in Section 5. It will be shown that in many cases imbalance has negligible influence on the service level. So then imbalance may be ignored when stocknorms are determined. There are however systematic exceptions where imbalance is too large to be ignored. In those cases imbalance should be incorporated in the stocknorm. The estimators found in Section 3 may be used for this.

In Section 6 an evaluation will be made on the merits and demerits of having a depot in a divergent system. The paper ends with a summary of the major conclusions.

2. Integral stocknorms for several systems.

To come up with a stocknorm formula for the divergent system with fixed lot-sizes, five simpler systems will be analyzed first:

* System I: 1-stage linear system without lot-sizing.

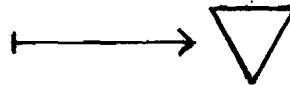Consider the system in Figure 2. This is the most basic production/inventory system.

Figure 2

The stocknorm proposed for this system is based on the assumption that demand is normally distributed (see e.g. [5]) :

$$r = (\ell+1)\mu + k\sqrt{(\ell+1)}\sigma \qquad (1)$$

with k such that $\quad \Phi(k) = \alpha \qquad (1')$

In words this formula reads: In order to be able to meet demand directly from stock a pre-determined percentage of periods, the content of the system after ordering should equal the average demand during 'leadtime plus review period' $(\ell+1)\mu$ plus some safety stock. This safety stock is equal to the safety factor times the standard deviation of demand during leadtime plus review period.

In case demand is (partially) known, $(\ell+1)\mu$ should be replaced by the demand forecast for the next $(\ell+1)$ periods and $\sqrt{(\ell+1)}\sigma$ should be replaced by the standard deviation of the forecast error. This observation applies to all systems studied in this paper.

* System II: 1-stage linear system with lot-sizing.

This system is the same as in Figure 2, except for the ordering policy. Instead of ordering exactly the demand of the previous period,

the system should order a multiple of the lot-size Q to bring the inventory level above the re-order point r. It is known, that the inventory level after possible ordering is uniformly distributed between r and r+Q (see [5]). Let f(.) be the corresponding uniform probability density function. Then r can be solved from the equation

$$\int_{-\infty}^{\infty} E(\text{service level}|\text{inventory}=x).f(x)\ dx =$$

$$\int_{r}^{r+Q} P((\ell+1)\text{ periods of demand}\leq x).\frac{1}{Q}\ dx = \frac{1}{Q}.\int_{r}^{r+Q} \Phi(\frac{x-(\ell+1)\mu}{\sqrt{(\ell+1)}\sigma})\ dx = \alpha$$

Note that the service level is measured each period and not per order.

In practice the above equation is difficult to handle. Fortunately in literature there are several heuristics to determine the re-order point r. In this paper a simple approximative formula is proposed to determine r, which only takes account of the mean and standard deviation of the inventory position after possible ordering.
The reason for coming up with this formula is twofold:
-1. The aim is to have coherent formulas for all inventory systems.
     That implies that an expression of the same type as formula (1) has to be found for the system with lot-sizing.
-2. The above equation for the system with lot-sizing does not explicitly express the stocknorm in terms of the system parameters.

To come up with a formula which satisfies both these conditions, the following observation is used: The inventory position is uniformly distributed between r and r+Q. As a consequence its mean and variance are resp. r+Q/2 and $Q^2/12$. The service level α equals P(IAD≥0), where IAD is the inventory in the stockpoint just after demand took place. Note that IAD equals the inventory (on hand plus on order) after possible ordering minus (ℓ+1) periods of demand.
So E(IAD) = r+Q/2-(ℓ+1)μ and var(IAD) = $Q^2/12+(\ell+1)\sigma^2$.
Assuming (in concordance with the derivation of formula (1)) that IAD is normally distributed yields:

$$r = (\ell+1)\mu - Q/2 + k\sqrt{[(\ell+1)\sigma^2+Q^2/12]} \qquad (2)$$

with k such that $\Phi(k) = \alpha$ \qquad\qquad (2').

This formula performs well, except when the variance term due to lot-sizing far exceeds the variance term due to demand uncertainty.

Experiments reported below show, that if Q becomes relatively large, then it is no longer advisable to assume a normal distribution for IAD. Since in this case the variance term of the uniform distribution $(Q^2/12)$ clearly dominates over the variance term of the demand $((\ell+1)\sigma^2)$, it is natural to assume then that IAD has a uniform distribution function.

If IAD is supposed to be uniformly distributed between the lowerbound LB ($\leq$0) and the upperbound UB ($\geq$0) then IAD has a mean M and variance V with:

$$M = (LB+UB)/2, \qquad V = (UB-LB)^2/12.$$

So UB and LB can be written as folllows:

$$LB = M - 0.5\sqrt{(12V)}, \quad UB = M + 0.5\sqrt{(12V)}.$$

It can be readily seen that P(IAD$\geq$0) equals (UB-0)/(UB-LB) since IAD is a uniform distributed variable with LB$\leq$0 and UB$\geq$0.
This probability can be expressed in terms of M and V as follows:

$$P(IAD\geq0) = (M + 0.5\sqrt{(12V)}) / \sqrt{(12V)}.$$

Since this probability should equal α, M should equal

$$M = (\alpha-0.5)\cdot\sqrt{(12V)}$$

Since it is known for IAD that M and V equal

$$M = r+Q/2-(\ell+1)\mu \; ; \qquad V = Q^2/12+(\ell+1)\sigma^2,$$

the stocknorm r should be chosen such, that

$$r+Q/2-(\ell+1)\mu = (\alpha-0.5)\cdot\sqrt{(12V)}.$$

Therefore:

$$r = (\ell+1)\mu-Q/2 + (\alpha-0.5)\sqrt{12}\cdot\sqrt{(Q^2/12+(\ell+1)\sigma^2)}$$

This corresponds to formula (2) with

$$k = \sqrt{12} \cdot (\alpha-0.5) \qquad\qquad (2'')$$

In all simulations reported in this paper demand was gamma distributed. The arguments for this are listed in Burgin [1].
The fact that demand was gamma distributed was not taken into account in the determination of the stocknorms, since

1. the gamma distribution function is more difficult to handle than the normal or uniform distribution function. For example no explicit standardized formula for the stocknorm can be derived.

2. As soon as Q>0, IAD is no longer purely gamma distributed, but the sum of a gamma and a uniform distributed variable, which is even more difficult to handle.

Rather the stocknorms were determined by using the approximative formulas (2), (2') and (2"). Calculations were performed to find out

when formula (2") should be preferred over formula (2'). These
calculations determined the expected service level for a system with
gamma distributed demand and stocknorms based on (2) and (2') resp.
(2"), that is: stocknorms based on the assumption that IAD is normally
resp. uniformly distributed. The parameters of the system were:
$\mu=10$, $\sigma=10$, $(\ell+1)=6$, target service level $=95\%$. The lot-size Q was
varied (see Figure 3). It appears that as long as $Q^2/12 \geq 4\cdot(\ell+1)\sigma^2$,
formula (2") should be preferred over formula (2'), since it yields a
service level which is closest to the target service level.

It is interesting to note that calculations for a system with normal
distributed demand (not reported here) lead to the same preference
rule.

Figure 3 also shows, that in case demand is actually gamma
distributed (which holds if $Q=0$), a stocknorm based on the assumption
of normal distributed demand yields a service level below the target.
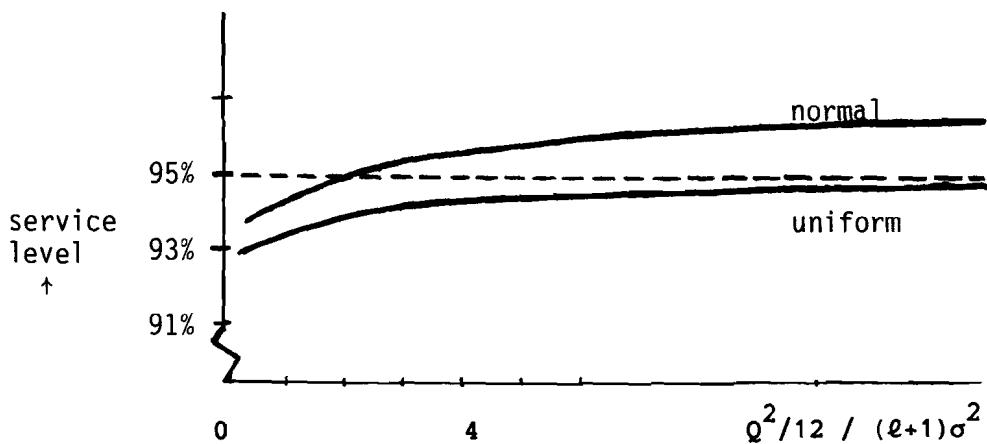
Expected service level for various values of Q
if the stocknorms are based on formulas (2') and (2").

Figure 3.

* <u>System III: 2-stage linear system without lot-sizing.</u>

System III is depicted in Figure 4. It consists of two consecutive
inventory systems, both without lot-sizing, like System I. For the
production of one unit of final product, one unit of the part is
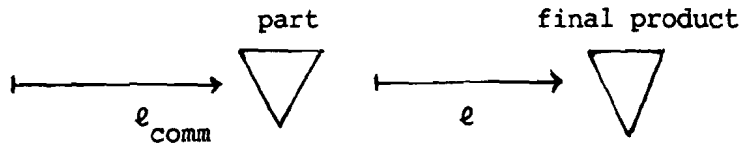needed. The leadtime for the final product and the part are resp. ℓ and
ℓ_{comm}.

Figure 4.

The re-order point for the final product (r) can be determined by means of equation (1):

$$r = (\ell+1)\mu + k\sqrt{(\ell+1)}\sigma^2 \qquad (3)$$

The re-order point for the entire system can be determined in an analogous way, that is: by assuming that system III is a special case of system I with leadtime $\ell_{comm}+\ell+1$:

$$r_{comm} = (\ell_{comm}+\ell+1)\mu + k\sqrt{(\ell_{comm}+\ell+1)}\sigma^2 \qquad (3')$$

Notice, that in formulas (3) and (3') k can no longer be solved using $\Phi(k)=\alpha$, since this would yield a service-level of α only in case of a 1-stage system. System III however is a 2-stage system, which implies, that the resulting service level is lower due to retaining inventory in an intermediate depot. This effect is thoroughly discussed in [3] and can be handled by adapting the safety-factor k.

An approximation (even simpler than the one proposed in [3]) to take account of this effect follows from the following observation:
Assume that the 2-stage system is treated like two 1-stage systems, resulting in formulas (3) and (3') with k simply solved from $\Phi(k)=\beta$. The actual service level in this 2-stage system will be less than β and larger than β^2. So the stockout probability is larger than $1-\beta$ and less than $1-\beta^2$. Note that

$$1-\beta^2 = 2(1-\beta) - (1-\beta)^2 \approx 2(1-\beta) \quad \text{if } \beta \to 1.$$

If it is known that the actual stock-out probability is somewhere between $1-\beta$ and $2(1-\beta)$, the most straightforward estimation of the stock-out probability is $1.5(1-\beta)$. This corresponds with a service level of $1-1.5(1-\beta) = 1.5\beta-0.5$. So if the target service level is α, β should be chosen such that $1.5\beta-0.5=\alpha$. That implies $\beta=1/3 + (2/3)\alpha$. In other words: In order to reach a service level α for System III, the safety factor in formulas (3) and (3') should be chosen such that:

$$\Phi(k) = 1/3 + (2/3)\alpha \qquad (3'').$$

From the derivation above it will be clear that formula (3") only yields a rough approximation of the service level.

* <u>System IV: 2-stage linear system with lot-sizing.</u>

System IV differs from the system in Figure 3 by having a lot-size for the final product (Q) as well as for the common part (Q_{comm}). Determination of the re-order points results from a combination of the concepts behind systems II and III.

The final stage in the system can be viewed as an occurrence of system II. So formula (2) determines r, with k solved from (4)

$$\Phi(k) = 1/3 + (2/3)\alpha \qquad\qquad \text{if } Q^2/12 < 4(\ell+1)\sigma^2$$
$$k = \sqrt{12} \cdot (1/3 + (2/3)\alpha - 0.5) \qquad \text{elsewhere} \qquad\qquad (4)$$

Note that the service level used to determine k is set equal to $1/3 + (2/3)\alpha$ since System IV is a 2-stage system.

The stocknorm r_{comm} follows from interpreting System IV as a 1-stage linear system with leadtime $\ell_{comm} + \ell + 1$ and lot-size Q_{comm}:

$$r_{comm} = (\ell_{comm} + \ell + 1)\mu - \frac{Q_{comm}}{2} + k\sqrt{[(\ell_{comm} + \ell + 1)\sigma^2 + \frac{Q_{comm}^2}{12}]}, \qquad (4')$$

where k results from

$$\Phi(k) = 1/3 + (2/3)\alpha \qquad\qquad \text{if } Q_{comm}^2/12 < 4(\ell_{comm} + \ell + 1)\sigma^2$$
$$k = \sqrt{12} \cdot (1/3 + (2/3)\alpha - 0.5) \qquad \text{elsewhere}$$

* <u>System V: 2-Stage divergent system without lot-sizing.</u>

This system is the system with depot depicted in Figure 1. It differs from system III by having more than one final product requiring the same component. For each of the final products the re-order point r can be determined by means of equation (3). The re-order point for the common part can be calculated by collapsing the system into a linear 2-stage system as described in [3]. This collapsing is allowed since the imbalance of the system is negligible. The resulting formulas are:

$$r = (\ell+1)\mu + k\sqrt{(\ell+1)}\sigma \qquad\qquad\qquad (5)$$
$$r_{comm} = \Sigma(\ell_{comm} + \ell + 1)\mu + k\sqrt{[\ell_{comm}\Sigma\sigma^2 + \{\Sigma\sqrt{(\ell+1)}\sigma\}^2]}, \qquad (5')$$

where k results from (3"). Note that in case of no depot k has to be solved by means of equation (2').

* ## System VI: 2-Stage divergent system with lot-sizing.

System VI is analogous with System V. The only distinction is the introduction of lot-sizing in the products' ordering policies. This implies that the final stage consists of N final products having the same characteristics as the final product in System IV. Therefore the re-order points for the final products are determined in the same way as the re-order point for the final product in the 2-stage linear system with lot-sizing (System IV).

The re-order point for the entire system is, just like in System V, calculated by collapsing the system into a linear 2-stage system (and thus assuming that the imbalance is negligible). The mean and standard deviation of demand for this collapsed system are known from formula (5'). Substituting these in formula (4') yields:

$$r_{comm} = \Sigma(\ell_{comm} + \ell + 1)\mu - Q_{comm}/2 + k\sqrt{[\frac{Q_{comm}^2}{12} + \ell_{comm}\Sigma\sigma^2 + \{\Sigma\sqrt{[(\ell+1)\sigma^2]}\}^2]} \quad (6)$$

Formula (6) does not yet take into account the imbalance. In order to be able to incorporate imbalance in the stocknorm, imbalance has to be defined properly first. Recall that in this paper it is assumed that the final products have identically distributed requirements and equal lot-sizes. Imbalance of a final product j in such a divergent system is defined here as

$$imbalance_j = I_j - \bar{I},$$

where I_j is the economic inventory of product j after possible ordering and $\bar{I}$ is the average economic inventory of the N products ($\bar{I} = \Sigma I_j/N$). Note that $E[imbalance_j] = 0$. The variance of $imbalance_j$ will be denoted by σ_{imb}^2.

The term $(\ell+1)\sigma^2$ in formula (6) emerges from the fact that the service level is measured after $(\ell+1)$ periods demand for each final product, assuming balanced inventories. The fact that before demand takes place the system is out of balance, can be modelled as starting with a balanced system, which is first confronted with an extra artificial (possibly negative) demand with expectation zero and variance σ_{imb}^2 before the real demand takes place.

Thus in order to get a good estimation of the service level the variance for product j should be set at $(\ell+1)\sigma^2 + \sigma_{imb}^2$ instead of $(\ell+1)\sigma^2$. This yields formula (6'):

$$r_{comm} = \Sigma(\ell_{comm} + \ell + 1)\mu - Q_{comm}/2 +$$

$$+ k\sqrt{[\frac{Q_{comm}^2}{12} + \ell_{comm}\Sigma\sigma^2 + \{\Sigma\sqrt{[(\ell+1)\sigma^2 + \sigma_{imb}^2]}\}^2]} \qquad (6')$$

with k solved from

$$\Phi(k) = 1/3 + (2/3)\alpha \quad \text{if} \quad Q_{comm}^2/12 < 4(\ell_{comm}\Sigma\sigma^2 + \{\Sigma\sqrt{[(\ell+1)\sigma^2 + \sigma_{imb}^2]}\}^2)$$

$$k = \sqrt{12} \cdot (1/3 + (2/3)\alpha - 0.5) \qquad \text{elsewhere}$$

In Section 3 it is shown that σ_{imb}^2 depends on the ratio r_{comm}/r in case a depot is present. This effect is not taken into account here. In fact it is assumed that the safety factors for both echelons are always chosen equal. Recall that formula (6') is only an approximation of the stocknorm necessary to obtain a service-level α.

In the next Section an estimator for the variance of imbalance for a system with identical final products will be given.

3. An estimator for the size of imbalance in case of identical products

In order to derive approximations for the variance of imbalance, the cases without depot (subsection 3.2) and with depot (subsection 3.3) are analyzed separately. It will appear that the variance of imbalance in both systems is a function of the minimal variance of imbalance, which is defined and explored below.

3.1 Minimal variance of imbalance.

Consider a divergent system with depot. Assume there is an infinite amount of inventory available in the depot. This ensures the ability to bring all final product inventories up to a level, which is between their stocknorm r and r+Q. The variance of imbalance corresponding to these levels is solely due to the lot-sizes and not enlarged by non-availibility of inventory in the depot. Therefore this is called the minimal variance of imbalance.

The variance of imbalance equals $var(I_j - \bar{I}) = (N-1) \cdot var(I_j)/N$ if the final products face independent demand. According to Hadley and Whitin [5] I_j is uniformly distributed between r and r+Q if there are always sufficient components available. So in that case the variance of I_j

equals $Q^2/12$, resulting in the fact that the minimal variance of

imbalance equals $\dfrac{Q^2}{12} \cdot \dfrac{N-1}{N}$.

3.2 Imbalance in a system without depot.

Suppose in a system <u>without depot</u> an order of size Q_{comm} has to be distributed among the final products. Then it is no longer possible to end up with the minimal variance of imbalance. It can be shown (see Appendix), that in case of no depot the variance of imbalance just after allocating an order equals

$$\sigma^2_{imb} = \frac{Q^2}{12} \cdot \frac{(N-1)}{N} \cdot \frac{(N+2)}{N} \qquad (7)$$

In the system with depot and with a large inventory in the depot, it is possible to keep the distance between the inventory levels less than the lot-size Q each period. In case no depot is available, however, this is no longer possible: The system has to wait for the next order for the common part in order to be able to reduce the imbalance again.

To get a rough indication of this effect, the time between two orders for the common part is assumed to be constant. Note that the larger the coefficient of variation of demand, the more this assumption is violated. After allocating an order it takes $Q_{comm}/N\mu$ periods on average before the next order will be allocated. In the meantime the imbalance of the products will increase due to the variance in demand. The increase in the variance of imbalance per period equals $(N-1)\sigma^2/N$. If the time between two orders is less than one period, this increase is assumed to be negligible. If the time between two orders is larger than or equal to one period, then the variance increases at a constant rate during $Q_{comm}/N\mu-1$ periods, whereas in the $Q_{comm}/N\mu$-th period the variance drops back again. This leads to an estimated average increase in the variance of imbalance equal to

$$\frac{1}{2} \cdot \left(\frac{Q_{comm}}{N\mu} - 1 \right) \cdot \frac{N-1}{N} \sigma^2.$$

This combined with formula (7) gives an estimator for the variance of imbalance if no depot is available:

$$\sigma^2_{imb} = \begin{cases} \dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \cdot \dfrac{N+2}{N} + \dfrac{1}{2}\left(\dfrac{Q_{comm}}{N\mu} - 1 \right) \dfrac{N-1}{N} \cdot \sigma^2 & \text{if } Q_{comm} \geq N\mu \\[4mm] \dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \cdot \dfrac{N+2}{N} & \text{elsewhere} \end{cases} \qquad (8)$$

3.3 Imbalance in a system with depot.

For any divergent system with depot the variance of imbalance equals
at least the minimal variance of imbalance: non-availability of
inventory in the depot can only increase the variance of imbalance.
The exact equality holds if there is always enough inventory in the
depot, that is if the stocknorm for the whole system is large compared
to the stocknorm for the final products.

Likewise it can be argued that the variance of imbalance for a
system with depot will always be smaller than or equal to the variance
of imbalance for the same system without a depot. The equality holds if
the stocknorm for the specific products is extremely large compared to
the stocknorm for the whole system. These observations combined with
formula (8) lead to:

$$\dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \leq \sigma^2_{imb} \leq \dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \cdot \dfrac{N+2}{N} + \dfrac{1}{2} \cdot \left(\dfrac{Q_{comm}}{N\mu} - 1 \right) \cdot \dfrac{N-1}{N} \sigma^2 \qquad \text{if } Q_{comm} \geq N\mu$$

$$\dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \leq \sigma^2_{imb} \leq \dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \cdot \dfrac{N+2}{N} \qquad \text{elsewhere.}$$

Furthermore, if in a system with depot both the stocknorm for the
whole system as well as the stocknorms for the final products have
equivalent levels (that is: have the same safety factor), it is likely,
that the variance of imbalance is closer to

$$\dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \cdot \dfrac{N+2}{N} \qquad \text{than to} \qquad \dfrac{Q^2}{12} \cdot \dfrac{N-1}{N} \cdot \dfrac{N+2}{N} + \dfrac{1}{2} \cdot \left(\dfrac{Q_{comm}}{N\mu} - 1 \right) \cdot \dfrac{N-1}{N} \sigma^2 .$$

This is due to the fact that the last term is caused by non-
availability of components in the depot. As explained in the previous
paragraph this term is appropriate in case no depot is available. A
system with depot and a relatively low stocknorm for the whole system
compared to the stocknorms of the final products is comparable with a
system without depot. So the variance of imbalance for such a system is

close to the above right hand side of the inequality for the variance of imbalance in case $Q_{comm} \geq N\mu$.

This can be seen from Figure 5. There the variance of imbalance is shown for various divergent systems. Each simulated system there had a different integral stocknorm for the whole system (corresponding with a safety factor k_{comm}) whereas the stocknorm for the final products was kept constant with corresponding safety factor k equal to 1.83.

For small k_{comm} the variance of imbalance tends to equal

$$\frac{Q^2}{12} \cdot \frac{N-1}{N} \cdot \frac{N+2}{N} + \frac{1}{2} \cdot \left(\frac{Q_{comm}}{N\mu} - 1\right) \cdot \frac{N-1}{N} \sigma^2 = 258.3.$$

As k_{comm} increases, the variance of imbalance decreases, since the probability that the depot has no inventory decreases. If k_{comm} gets large, the variance of imbalance tends to equal $Q^2/12 \cdot (N-1)/N = 16.7$. When $k_{comm} = k = 1.83$, the variance of imbalance equals 34.3.

This supports the statement made earlier that as long as the safety factors are approximately equal, then the variance of imbalance (=34.3)

will be closer to
$$\frac{Q^2}{12} \cdot \frac{N-1}{N} \cdot \frac{N+2}{N}$$
(=33.3)

than to
$$\frac{Q^2}{12} \cdot \frac{N-1}{N} \cdot \frac{N+2}{N} + \frac{1}{2} \cdot \left(\frac{Q_{comm}}{N\mu} - 1\right) \cdot \frac{N-1}{N} \sigma^2$$
(=258.3).
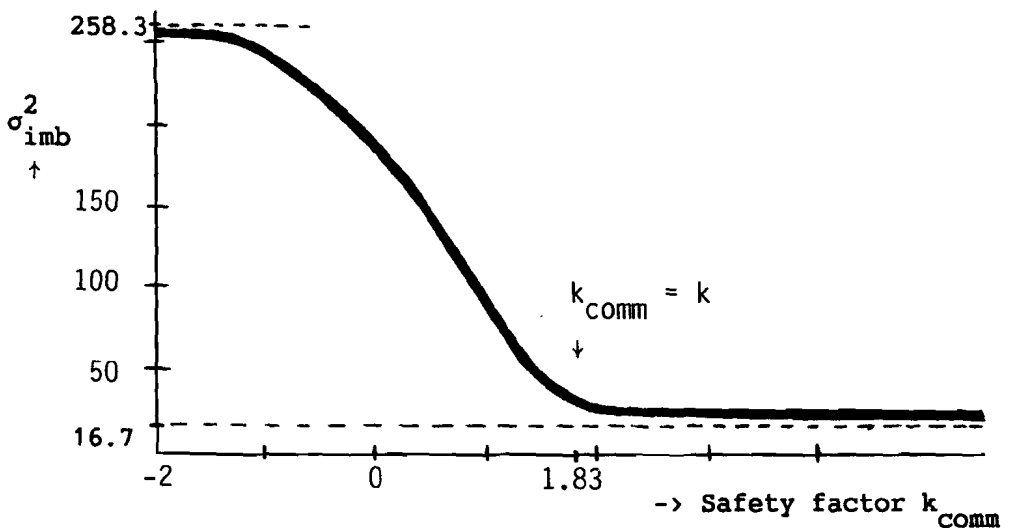
σ^2_{imb} for a system with depot and parameters Q_{comm}=200, Q=20, μ=10, σ=10, ℓ+1=3, N=2 and k=1.83.

Figure 5.

According to this statement the following approximation holds:

$$\frac{Q^2}{12} \cdot \frac{N-1}{N} \leq \sigma^2_{imb} \leq \frac{Q^2}{12} \cdot \frac{N-1}{N} \cdot \frac{N+2}{N}.$$

The limit of both the lower and upperbound is $Q^2/12$ as $N \to \infty$.

Besides that:

$$\frac{Q^2}{12} \cdot \frac{N-1}{N} \leq \frac{Q^2}{12} \quad \text{and} \quad \frac{Q^2}{12} \cdot \frac{N-1}{N} \cdot \frac{N+2}{N} \geq \frac{Q^2}{12} \quad \text{for all } N \geq 2.$$

So $Q^2/12$ is an appropriate approximation for σ^2_{imb} in case a depot is present and $N \geq 2$.

Summarizing: For a system with depot the variance of imbalance can be estimated as follows:

$$\sigma^2_{imb} = \begin{cases} Q^2/12 & \text{if } N \geq 2 \\ \\ 0 & \text{if } N = 1 \end{cases} \tag{9}$$

4. The quality of the estimators for σ^2_{imb}.

In this Section the estimators for the variance of imbalance are tested. For the reader not interested in detailed results the results can be summarized thus:

1. Although the estimators (8) and (9) appear to be poor estimators for the variance of imbalance under certain circumstances, they perform well enough to serve as estimators for imbalance in a stocknorm formula for any divergent system. This is due to the fact that whenever the estimator performs poorly, other elements in the stocknorm formula will dominate over the variance of imbalance.

2. In general $Q^2/12$ can also be used as an estimator for the system without depot, except in those cases where Q_{comm} and σ/μ are high.

Since the derivation of the estimators for σ^2_{imb} is heuristic, many simulations over a broad set of parameters have been performed. The purpose of these simulations is to test the quality of (8) and (9) as estimators for σ^2_{imb}. A total of 144 systems were simulated.

The parameters and their values were:

depot   present or not;

N   with values 2, 4 and 8;

σ   with values 5, 10 and 20 (μ was equal to 10);

ℓ   with values 1 and 3 (so ℓ+1 was equal to 2 or 4).

         The total leadtime was always equal to 6. So ℓ_{comm} = 6 - (ℓ+1).

f1   defined as Q/μ, with values 1 and 3.

f0   defined as Q_{comm}/NQ, with values 1 and 3.

The sytems with 8 final products performed in much the same way as the systems with 4 final products. Therefore it is expected that systems with a number of final products larger than tested here will behave similar to the systems with 8 final products. A coefficient of variation equal to 2 is rather large. It is merely used to gain insight in the effects of large uncertainty in the system. The average time between two orders is minimally one period and maximally three periods (f1) for the final products and nine periods (f0·f1) for the common part.

The estimators for the systems with and without depot will be investigated separately in subsections 4.1 and 4.2.

4.1 The system with depot.

As an indicator for the quality of (9) as an estimator for the variance of imbalance, the ratio σ^2_{imb}(sim) over σ^2_{imb}(est) will be used. The two variables mentioned last stand for the actual variance of imbalance, which was measured during the simulations, resp. for the estimated variance of imbalance using formula (9). Their ratio appeared to be 2.8 on average, the maximum ratio registered was 13.1 and the minimum ratio was 0.6; a poor result. Particularly if σ/μ and f0 were high and if f1 was low, the estimator performed badly.

Recalling that the primary purpose of estimating the variance of imbalance was to get good stocknorms, the quality of (9) as an estimator for total system standard deviation (totsd) is considered too. Total system standard deviation is defined as

$$totsd = \sqrt{\left[\frac{Q^2_{comm}}{12} + \ell_{comm}\Sigma\sigma^2 + \{\Sigma\sqrt{[(\ell+1)\sigma^2 + \sigma^2_{imb}]}\}^2\right]}.$$

Totsd(sim) and totsd(est) are defined as totsd with the variance of imbalance replaced by σ_{imb}^2(sim) resp. σ_{imb}^2(est).

The simulations showed an average ratio totsd(sim) over totsd(est) of 1.00, with a maximum of 1.04 and a minimum of 0.94. A good result, which can be rapidly explained by looking at the definition of totsd and recalling that the variance of imbalance was estimated badly for high σ and Q_{comm} (note that Q_{comm} equals f0f1Nμ): The factors which make up totsd, other than the variance of imbalance, are large if the latter is estimated badly.

Up till now stocknorms were based on σ_{imb}^2(est). The simulations for the 72 systems with depot using these stocknorms yielded a service level $\alpha(\sigma_{imb}^2$(est)) of 93.2% on average. To check the influence on the service level of mis-estimating the variance of imbalance by using (9), all 72 systems were simulated again, now using σ_{imb}^2(sim) to determine the stocknorms. The service level $\alpha(\sigma_{imb}^2$(sim)) turned out to be 93.2% on average again. So although formula (9) is a bad indicator for the variance of imbalance under certain circumstances, it can be used very well for determination of stocknorms for divergent systems with a depot.

In most simulations the service level appeared to be somewhat below the target level of 95%. The main cause for this is the fact that actual demand was gamma distributed (see Figure 3).

For the reader not only interested in aggregate and relative figures, Table 1 contains a representative subset of the simulation data. From Table 1 it is obvious, that σ_{imb}^2(sim) depends on σ. For example the simulated variance of imbalance for a system with f0=f1=1 and σ=5 resp. σ=20 equals 7.5 resp. 92.8. This observation is not incorporated in the estimator for the variance of imbalance: In both cases the variance of imbalance was estimated to be 8.3. The reason for this is that the potential improvement on the quality of the estimations of totsd (170.5 estimated versus 174.4 simulated in case f0=f1=1 and σ=20) and the service level (only 0.2% misestimated for f0=f1=1 and σ=20) is small (see Table 1).

This does not imply that the overall impact of imbalance is negligible. This impact is thoroughly investigated in Section 5.1. It will appear that neglecting the variance of imbalance when stocknorms are determined may lead to a maximal decrease in service level of approximately 10% in case a depot is present!

σ	f1	f0	σ^2_{imb}		totsd		α	
			est	sim	est	sim	est	sim
5	1	1	8.3	7.5	45.5	45.3	94.4	94.4
5	1	3	8.3	7.9	56.0	55.9	93.8	93.8
5	3	1	75.0	66.0	64.8	63.7	94.3	94.1
5	3	3	75.0	63.2	117.5	116.7	94.4	94.3
10	1	1	8.3	10.7	86.4	86.6	93.7	93.8
10	1	3	8.3	16.6	92.4	93.1	93.4	93.4
10	3	1	75.0	76.1	98.0	98.1	93.6	93.6
10	3	3	75.0	80.3	138.6	138.9	93.1	93.1
20	1	1	8.3	92.8	170.5	174.4	92.9	93.1
20	1	3	8.3	109.3	173.6	178.2	92.8	93.0
20	3	1	75.0	169.6	176.6	180.9	92.9	93.1
20	3	3	75.0	228.6	202.0	208.0	92.4	92.7

Simulation results for a divergent system with depot, N=4 and (ℓ+1)=4. For various values of σ, f0 and f1 the resulting σ^2_{imb}(est), σ^2_{imb}(sim), totsd(est), totsd(sim), $\alpha(\sigma^2_{imb}$(est)) and $\alpha(\sigma^2_{imb}$(sim)) are given.

Table 1.

4.2 The system without depot.

As an indicator for the quality of (8) as an estimator for σ^2_{imb}, the ratio σ^2_{imb}(sim) over σ^2_{imb}(est) will be used again. For the system without depot this ratio appeared to be 3.8 on average, the maximum ratio registered was 41.0(!) and the minimum ratio 0.8; the performance of formula (8) is even worse than the performance of formula (9) in the previous subsection. This bad performance is mainly due to the 6 simulations with σ=20 and f0·f1=1. The remaining 66 simulations showed a ratio σ^2_{imb}(sim) over σ^2_{imb}(est) of 1.30 on average.

The bad performance in case of f0·f1=1 and σ=20 can be explained by looking more closely at formula (8). In case f0·f1 equals 1, the second term in (8) vanishes. This is based on the assumption, that every next period a new order will be allocated to the final products. Due to the large variance of demand it is possible, that this assumption is violated. In that case the variance of imbalance is increased with σ^2(N-1)/N, which is large if σ is large.

Although imbalance is sometimes badly estimated because of the large variance of demand the influence of this on the system's performance might be negligible, because that very same variance of demand may far outweigh the total variance of imbalance in the stocknorm formula. To investigate this, totsd(est) and totsd(sim) are considered.

The 72 simulations showed an average ratio totsd(sim) over totsd(est) of 1.01, with a maximum of 1.13 and a minimum of 0.97. This together with the observation that $\alpha(\sigma^2_{imb}(est))$ equals 93.4% for the 72 simulations, whereas $\alpha(\sigma^2_{imb}(sim))$ is only 0.2% higher on average, leads to the same conclusion as in the previous subsection: Although formula (8) is a bad indicator for the variance of imbalance under certain circumstances, it can be used very well for determination of safety stocknorms.

As in subsection 4.1 a representative subset of the simulation data is given for the system without depot (see Table 2). Each row is based on 5000 simulation runs.

σ	f1	f0	σ^2_{imb} est	sim	totsd est	sim	α est	sim
5	1	1	9.4	11.4	45.6	46.0	94.1	94.2
5	1	3	28.1	29.2	58.7	58.9	94.1	94.1
5	3	1	103.1	105.3	68.2	68.5	94.4	94.4
5	3	3	159.4	163.9	123.1	123.4	94.6	94.6
10	1	1	9.4	29.4	86.5	88.3	93.2	93.5
10	1	3	84.4	89.1	98.7	99.1	93.5	93.6
10	3	1	159.4	165.5	104.6	105.1	93.7	93.7
10	3	3	384.4	377.7	155.4	155.1	93.8	93.8
20	1	1	9.4	278.5	170.5	182.7	92.1	93.0
20	1	3	309.4	426.6	187.0	191.9	92.7	93.0
20	3	1	384.4	507.2	190.0	195.2	92.7	93.1
20	3	3	1284.4	1072.3	245.3	238.2	93.5	93.1

Simulation results for a system <u>without</u> depot, N=4 and $(\ell+1)=4$. For various values of σ, f0 and f1 the resulting $\sigma^2_{imb}(est)$, $\sigma^2_{imb}(sim)$, totsd(est), totsd(sim), $\alpha(\sigma^2_{imb}(est))$ and $\alpha(\sigma^2_{imb}(sim))$ are given.

Table 2.

Finally the consequences of using the estimator for the variance of imbalance for a system with depot (that is $Q^2/12$) as an estimator for the system without depot are investigated. As might be expected, using $Q^2/12$ as estimator for the system without depot leads to poor results in case Q_{comm} and σ are high. In those 6 cases a 2.7% drop in service level is due to mis-estimating σ^2_{imb}. For all other systems $Q^2/12$ is a reasonable estimator.

5. Mathematical analysis of the impact of imbalance
on the service level.

The question posed in this Section is: "What is the largest possible decrease in service level due to neglecting imbalance when the stocknorms are determined?". To answer this question a mathematical worst case analysis is performed using the results of the previous sections.

It will be shown that in general imbalance has hardly any impact on the service level. The exceptions are

1. systems with large time between two orders for the final products and demand, which has a relatively small variance.

or 2. systems without depot, large time between two orders for the common part and very uncertain demand.

To come up with these results the service level is calculated for a system which does have a positive variance of imbalance, but whose stocknorm is based on the assumption that imbalance is negligible. This service level is compared with the target service level, which would have been achieved if there had not been any imbalance.

First of all extra variables are defined and old variables are redefined:

$$\text{totm} := \Sigma(\ell_{comm}+\ell+1)\mu - Q_{comm}/2$$

$$\text{totsd(est)} := \sqrt{[\frac{Q_{comm}^2}{12} + \ell_{comm}\Sigma\sigma^2 + \{\Sigma\sqrt{[(\ell+1)\sigma^2 + \sigma_{imb}^2(\text{est})]}\}^2]}$$

$$\text{totsd(0)} := \sqrt{[\frac{Q_{comm}^2}{12} + \ell_{comm}\Sigma\sigma^2 + \{\Sigma\sqrt{(\ell+1)\sigma^2}\}^2]}$$

Thus totsd(0) represents the total system standard deviation if the variance of imbalance equals zero.

The stocknorm for a divergent system corresponding with a service level α may now be re-written as

$$r_{comm} = \text{totm} + k.\text{totsd(est)}$$

if totsd(est) is assumed to be the true total standard deviation.
Here k is solved from

$$\Phi(k) = \beta \qquad \text{if } Q_{comm}^2/12 < 4.(\text{totsd}^2(\text{est}) - Q_{comm}^2/12)$$

$$k = \sqrt{12}.(\beta-0.5) \quad \text{elsewhere.}$$

with $\beta = \alpha$ in case of no depot and

$\beta = 1/3 + (2/3)\alpha$ in case a depot is present.

Finally from here on Q_{comm} and Q will be expressed as $f0f1N\mu$
resp. $f1\mu$.

If it is assumed that imbalance is negligible, the appropriate
stocknorm $r_{comm}(0)$ equals

$$r_{comm}(0) = totm + k \cdot totsd(0)$$

with $\quad\quad\quad \Phi(k) = \beta$.

Assuming that the true relationship between the stocknorm and the
service level is given by

$$r_{comm} = totm + k \cdot totsd(est),$$

it becomes clear that the service level which corresponds with $r_{comm}(0)$
equals $\Phi(k \cdot totsd(0)/totsd(est))$ rather than $\Phi(k)$, since

$$r_{comm}(0) = totm + k \cdot totsd(0)$$
$$= totm + k \cdot totsd(0)/totsd(est) \cdot totsd(est).$$

Obviously, if the ratio $totsd(0)/totsd(est)$ decreases, the actual
service level corresponding with $r_{comm}(0)$ will deviate more and more
from the target service level α. Therefore this ratio, and particularly
its maximum value, is analyzed mathematically below. This is done
separately for the systems with and without depot.

5.1 The system with depot.

In case a depot is present, $totsd(est)$ and $totsd(0)$ can be written as :
$$totsd^2(est) = \ell_{comm}N\sigma^2 + f0^2f1^2N^2\mu^2/12 + N^2(\ell+1)\sigma^2 + N^2f1^2\mu^2/12.$$
$$totsd^2(0) = \ell_{comm}N\sigma^2 + f0^2f1^2N^2\mu^2/12 + N^2(\ell+1)\sigma^2.$$

So

$$\frac{totsd^2(est)}{totsd^2(0)} = \frac{\ell_{comm}/N + (\ell+1) + f1^2(\mu/\sigma)^2(f0^2+1)/12}{\ell_{comm}/N + (\ell+1) + f1^2(\mu/\sigma)^2f0^2/12}$$

This ratio is maximal if ℓ_{comm}, $(\ell+1)$ and $f0$ are minimal and $N, f1$ and
(μ/σ) are maximal. Since $\ell_{comm}/N+\ell+1\geq 1$ it follows that
$$totsd(est)/totsd(0) \leq \sqrt{[(1+(f0^2+1)b)/(1+f0^2b)]}$$
with $b = f1^2(\mu/\sigma)^2/12$.

So $totsd(est)/totsd(0)$ is maximal if $f1\to\infty$, $\mu/\sigma\to\infty$ and $f0=1$.

In general the maximal ratio equals $\sqrt{[(f0^2+1)/f0^2]}$.

Note that the maximum ratio equals $\sqrt{2}$ and is achieved if $f0$ equals 1.

This result is going to be used to find a lowerbound on the expected
service level if the stocknorm is based on the assumption that

imbalance is negligible. To do so it is necessary to know that in case f0=1 the appropriate equation for the safety factor is $\Phi(k)=\beta$. This fact can be proven easily and the proof is therefore left out here.

Assuming imbalance is negligible, the stocknorm for a system with f0=1 equals

$$r_{comm}(0) = totm + k \cdot totsd(0)$$

with

$$\Phi(k) = 1/3 + 2/3\alpha.$$

Since in reality there is imbalance, the actual service level γ corresponding with this stocknorm $r_{comm}(0)$ can be solved from

$$r_{comm}(0) = totm + k' \cdot totsd(est)$$

with

$$\Phi(k') = 1/3 + (2/3)\gamma.$$

Obviously

$$k' = k \cdot totsd(0)/totsd(est).$$

So γ can be solved from

$$\Phi(k \cdot totsd(0)/totsd(est)) = 1/3 + (2/3)\gamma.$$

This yields:

$$\gamma = 1.5 \cdot \Phi(k \cdot totsd(0)/totsd(est)) - 0.5.$$

Since

$$totsd(0)/totsd(est) \geq 1/\sqrt{2}$$

we have

$$\underline{\gamma \geq 1.5 \cdot \Phi(k/\sqrt{2}) - 0.5}.$$

For α=95% the corresponding k equals $\Phi^{-1}(0.333+0.667 \cdot 0.95)=1.83$ and thus $\gamma \geq 85.3\%$ if $f0 \geq 1$.

If f0 gets larger than 1 this lowerbound rapidly increases:

$$\gamma \geq 92.4\% \text{ if } f0 \geq 2$$

and $\gamma \geq 93.9\%$ if $f0 \geq 3$.

These figures are the worst results obtainable. Usually performance is far better. To show this simulation is used. All 72 simulations from subsection 4.3 were performed once again, now using zero as an estimator for the variance of imbalance. These simulations led to an average service-level of 92.5%, which is only 0.7% less compared with the service-level achieved by using formula (9) as an estimator for the variance of imbalance.

The worst results were indeed obtained for simulations with small coefficient of variation, small f0 (=Q_{comm}/NQ) and high f1 (=Q/μ). In those cases the service-level was up to 3.5% lower compared with the

simulations in which formula (9) was used. The reason for this is that in these cases there is only a little amount of safety stock available due to the small coëfficient of variation, while imbalance is considerable due to the large lot-sizes Q.

Note however, that in most cases (neglecting) imbalance has little impact on the expected service-level. The simple explanation for this is that the variance of imbalance is only one of many variance factors and it is large only if other variance factors are large too.

5.2 The system without depot.

For the determination of the ratio totsd(est)/totsd(0) for a system without depot two cases will be distinguished. The reason for this is that based on approximation (8) for the variance of imbalance no fixed upperbound can be found for totsd(est)/totsd(0) unless σ/μ is given. For most cases however another approximation can be used, which does yield a fixed upperbound.

-1. no high Q_{comm} or no high σ/μ.

In subsection 4.2 it has been shown empirically, that in these cases $Q^2/12$ is a reasonable estimator for σ^2_{imb}. That means that totsd(est)/totsd(0) is the same as in the case with depot and all results derived there with respect to totsd(0)/totsd(est) apply here as well.

-2. high Q_{comm} and σ/μ.

Under these conditions the second term in formula (8) dominates and σ^2_{imb} may be approximated by $0.5 \cdot f0f1(N-1)\sigma^2/N$. Since f0f1 is high, f0f1-1 is approximated by f0f1. This results in:

$$\frac{totsd^2(est)}{totsd^2(0)} = \frac{\ell_{comm}/N + (\ell+1) + f0^2f1^2(\mu/\sigma)^2/12 + 0.5 \cdot f0f1(N-1)/N}{\ell_{comm}/N + (\ell+1) + f0^2f1^2(\mu/\sigma)^2/12}$$

Optimizing with respect to f0f1 yields a maximal ratio if $f0^2f1^2 = 12(\ell_{comm}/N + \ell+1)/(\mu/\sigma)^2$.
This maximal ratio equals $1 + \dfrac{(N-1) \cdot \dfrac{0.5\sqrt{3}}{N \cdot (\mu/\sigma) \cdot \sqrt{(\ell_{comm}/N+\ell+1)}}}{}$.
For a given (μ/σ) this ratio is maximal if $N \to \infty$, $\ell_{comm} \to 0$ and $\ell \to 0$: The maximal ratio equals $1 + 0.5\sqrt{3}(\sigma/\mu)$ and has no fixed upperbound for all possible systems since if $(\mu/\sigma) \to 0$ this ratio will get very large.

From the simulations with zero as an estimator for the variance of
imbalance it appeared that neglecting imbalance in case no depot is
available decreased the service level with 2% on average (91.6% in
stead of 93.6%). The influence of imbalance was particularly large

a) when σ/μ was small and f1 was large.

b) when σ/μ, Q_{comm} and N were large.

The largest drop in service level due to neglecting imbalance was 5.7%
(for small σ/μ (=0.5) and large f1 (=3)).

6. Comparing the system with depot and the system without depot.

Comparing formulas (8) and (9) and formulas (2') and (3") the
essential differences between a system with depot and without depot
become visible: A system with depot has less variance of imbalance
(compare the columns σ^2_{imb}(est) in Tables 1 and 2 e.g.), which
contributes to a higher service level. The price which has to be paid
for this is the fact that inventory has to be retained in the depot.
As long as this inventory remains in the depot, it does not contribute
to a higher service level. So in order to achieve the target service
level total inventory has to be raised. This effect is quantified by
formula (3"'). The dilemma that having a depot has a positive as well
as a negative effect on service level has been mentioned by Eppen and
Schrage [4].

This dilemma can be quantified if the estimators for the variance of
imbalance are used: To achieve the same service level the stocknorms
for the system with depot resp. without depot are

totm + k_{depot}·totsd$_{depot}$(est) and totm + $k_{no\ depot}$·totsd$_{no\ depot}$(est),

where $k_{(no)\ depot}$ is the safety factor for the system with(out) depot.
So in order to determine the system with the lowest stocknorm it should
be investigated whether the following ratio is smaller or larger than
one:

$$\frac{k_{depot} \cdot totsd_{depot}(est)}{k_{no\ depot} \cdot totsd_{no\ depot}(est)} .$$

Both factors $k_{depot}/k_{no\ depot}$ and totsd$_{depot}$(est)/totsd$_{no\ depot}$(est)
will be investigated.

The factor $k_{depot}/k_{no\ depot}$ depends on whether k is solved from

$$\Phi(k) = \beta \qquad (a)$$

or from

$$k = \sqrt{12}.(\beta-0.5) \qquad (b)$$

Figure 6 shows the ratio $k_{depot}/k_{no\ depot}$ in both cases for various values of α. Note that as α increases, the relative difference in the safety factors gets smaller and thus it will sooner become economical to have a depot. It should be mentioned here that the formula $\beta=1/3+(2/3)\alpha$ is a rough approximation. From [3] it is known, that the service level α in a 2-stage system also depends on the ratio of the variances in the first and second echelon.
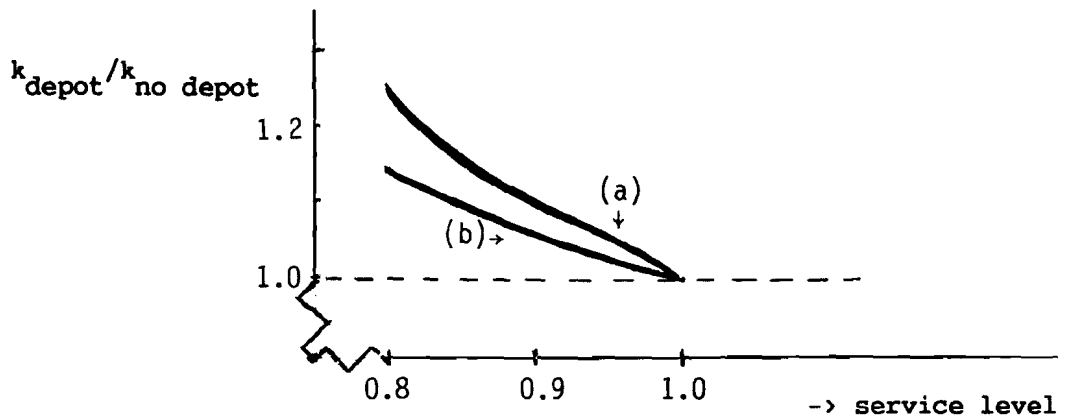

Figure 6.

To evaluate $totsd_{depot}(est)/totsd_{no\ depot}(est)$ two cases are distinguished:

-1. no high Q_{comm} or no high σ/μ.

Using again the result that in this case $Q^2/12$ is a reasonable estimator for σ^2_{imb} in case of no depot, $totsd_{no\ depot}(est)$ becomes equal to $totsd_{depot}(est)$. Since k_{depot} is always larger than $k_{no\ depot}$, in these cases it does not pay (in terms of service level) to have a depot.

-2. high Q_{comm} and σ/μ.

In these cases σ^2_{imb} may be estimated with $0.5 f0 f1 (N-1)\sigma^2/N$.

Then the ratio $totsd_{depot}(est)/totsd_{no\ depot}(est)$ is equal to

$$\sqrt{\frac{\ell_{comm}/N + (\ell+1) + f0^2 f1^2 (\mu/\sigma)^2/12 + f1^2 (\mu/\sigma)^2/12}{\ell_{comm}/N + (\ell+1) + f0^2 f1^2 (\mu/\sigma)^2/12 + 0.5 \cdot f0 f1 (N-1)/N}}$$

The nominator and denominator are equal except for the last two terms. It seems that this ratio will tend to decrease if f0 and N and σ increase.

From simulations it appeared that for none of the 24 divergent systems with N=2 (the same systems as simulated in Section 4) it was worthwhile to have a depot. The stocknorm in these simulations aimed for a service level of 95% for the system with depot. For the system without depot the system stocknorm was chosen equal to the stocknorm for the system with depot in order to attain equal average system inventories. For larger N a depot was only worthwhile if Q_{comm} and σ were high.

This comparison is not as black and white as it seems. The simulations for the system with depot were performed using stocknorms based on equal safety factors for both echelons. The system without depot can be seen as a system with depot and an infinitely large safety factor for the first echelon. All that can be concluded from the above is that if a depot is available, the stocknorms for the first echelon should generally be chosen high, unless the variance of imbalance is very large. In the precise determination of the stocknorms added value should also be taken into account: The higher the added value in the last production process, the lower its stocknorm.

7. Conclusions.

a. In this paper estimators for the variance of imbalance have been derived, which can be used very well to gain insight in the role of imbalance in divergent systems and to determine stocknorms.

b. Imbalance has a significant influence on the system's performance in the following cases only:
   - in case the coefficient of variation (of the forecast error) of demand is relative small (small σ/μ), the leadtime for the final products is small (small $\ell+1$) but the variance of imbalance is large (large Q/μ).
   - in case of no depot and large imbalance, that is if Q_{comm} and σ^2 are large in a system without depot.

c. In the stationary situations with unlimited capacity and identical products as considered in this paper it is hardly ever advantageous to be very cautious with pushing inventory out of the depot: Pushing extra inventory out of the depot generally has a larger positive effect on the service level than the negative effect caused by the

decreased possibility to keep the final products balanced. The only exception here are divergent systems with high added value or high storage costs in the last production process and/or systems with very large variance of imbalance; the latter are systems with many products, large coëfficient of variation for demand and a large lot-size for the common part.

Further research is necessary to investigate the influence of non-stationarity of the demand process and limited capacity. The case of non-identical products is investigated by the author. It appeared that for non-identical products a more elaborate definition of imbalance is needed and that the system is far more complex. Results for this case will be part of a dissertation.

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Appendix.

Derivation of formula (7).

For the system without depot the variance of imbalance just after an order is allocated equals

$$\frac{Q^2}{12} \cdot \frac{(N-1)}{N} \cdot \frac{(N+2)}{N}.$$

This is due to the fact that the inventory levels I_j are uniformly distributed on the interval $[\min\{I_i\}, \min\{I_i\}+Q]$; a result which is analogous to the fact that in a 1-echelon system with orderpoint r and lot-size Q the inventory position after ordering is uniformly distributed between r and $r+Q$.

In terms of variance of I_j the situation with $I_j \sim u[\min\{I_i\},\min\{I_i\}+Q]$ $j=1,..,N$ is equivalent with the situation that $N-1$ products chosen randomly from the N products are uniformly distributed on $[0,Q]$ and one product equals zero.

Imbalance is defined as $\text{var}(I_j-\bar{I})$ and can be calculated as follows:

$$\text{var}(I_j-\bar{I}) = E(I_j-\bar{I})^2 =$$

$$= E[I_j^2] - (2/N) \cdot \{E[I_j^2] + E[I_j \cdot \sum_{i \neq j} I_i]\} + (1/N^2) \cdot \{E[\sum I_j \cdot \sum_{i \neq j} I_i] + E[\sum I_i^2]\}$$

$$= (1-1/N) \cdot E[I_j^2] - 1/N \cdot E[I_j \cdot \sum_{i \neq j} I_i]$$

$$= (1-1/N)^2 Q^2/3 - 1/N \cdot (N-1)(N-2)Q^2/(4N)$$

$$= \frac{Q^2}{12} \cdot \frac{(N-1)}{N} \cdot \frac{(N+2)}{N}$$

Q.E.D.