H-infinite controller design

H$_\infty$ Controller Design

Algebraic Riccati Equalities
versus Linear Matrix Inequalities

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Summary

Many control problems are formulated as designing a stabilising controller, which minimises the H_∞ norm of a closed-loop transfer function. This is called H_∞ control. The common solution to this problem solves two algebraic Riccati equations and will in this paper be referred to as ARE-method. Another solution to this H_∞ control problem involves solving convex optimisation problems in the form of Linear Matrix Inequalities (abbreviated LMI-method).

Both the solution methods use state-space data, the difference between the ARE-method and LMI-method is that the assumptions (on the state-space data) for the ARE-method are more restrictive than the assumptions for the LMI-method. In this paper both methods will be studied with respect to computation time and numerical reliability.

The main conclusion with respect to numerical problems is: we should not use small perturbations to meet the assumptions on the ARE-method. Instead, it is better to use the LMI-method in that case.

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Chapter 1.

Introduction

In the last fifteen years there has been much research on designing H_∞ controllers for linear systems. In the 70's the design of controllers was mainly focused on the LQG optimal control method. It was in 1981 that Zames formulated the H_∞ optimal control theory. In this theory, the design of a feedback controller is based on minimising the H_∞ norm of the closed-loop transfer function. In the last years there came a lot of simplifications and improvements, which lead to the state-space formulas of Glover and Doyle [1]. This method shows that the existence of an H_∞ controller depends on satisfying certain assumptions and conditions on the solutions to two algebraic Riccati Equations (abbreviated ARE). Although this H_∞ control problem has an analytical solution, another method was developed in the last five years. This method uses linear matrix inequalities. This method is still based on the H_∞ norm of the closed-loop transfer function and also requires a stable closed-loop system. These two things together can be expressed via the bounded real lemma into a linear matrix inequality (abbreviated LMI). The reason for this other approach of the H_∞ control problem is to circumvent a part of the restrictive assumptions for the design method using the ARE. To be more precise, the LMI approach even works when the state-space matrices D_{12} and D_{21} are rank-deficient or when the plant has transmission zeros on the imaginary axis. In these cases the ARE-method cannot be used.

This paper is divided into three parts. The first five chapters contain the theory. Chapter six and seven contain the experiences and results with the examples. Chapter eight will give some conclusions and recommendations. In chapter two, H_∞ control in general will be discussed. It is seen, that it is possible to make a general state-space representation of the plant. The plant can also be augmented by filters which specify performance. In the following chapters we will have a look at the two different ways to analyse the H_∞ suboptimal control problem. The 'standard method' is called the Algebraic Riccati Equality -method and the other method is called the Linear Matrix Inequality-method. The AREs and the LMIs can only be used when a number of assumptions on the state-space representation hold. These assumptions will be presented and explained. At first sight the AREs and LMIs seem to differ a lot, but when the assumptions needed for the ARE method are met and a few other simplifying assumptions are met, then the controllers designed by the ARE method and LMI method are equivalent. In chapter six, two examples will be prescribed and in chapter seven we will see what happens if some assumptions are violated.

The aim of this traineeship is to compare the LMI and ARE methods in a few respects. We will look to the general applicability, because for the LMI method there are fewer assumptions. So the LMI method will be more generally applicable. We will study numerical problems, because they may occur when the plant is a bit manipulated to satisfy the assumptions on the ARE method, as is often done in practice. We will also compare the computation time needed for both methods.

Chapter 2.

H∞ Control

In this chapter, the H∞ control theory for linear time-invariant systems will be explained with the aim to present a global idea of this control theory.

The first step in a control theory is to give a clear representation of the control problem. This leads to the next figure, where the problem is formulated in a block diagram [10, p 413].
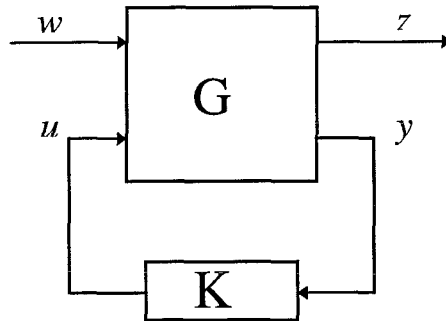
Figure 2-1 Generalised plant G with controller K

In this figure, G stands for the generalised plant, which will be explained below. K stands for a controller that has to be designed. The inputs of G are the exogenous input w, i.e. disturbances or reference signals, and the output from the controller u. The outputs of G are the output to be controlled z and the measured output y, which is available for the controller. The outputs to be controlled are formulated such that they are ideally zero, i.e. tracking errors.

The generalised plant G does not only contain the nominal model, but also weighting functions to penalise the outputs to be controlled and weighting functions characterising the exogenous inputs. These filters give the possibility to characterise the important frequencies in signals. For instance, they can penalise the high frequencies in u to avoid that K generates high-frequency control inputs supplied to an actuator with limited bandwidth.
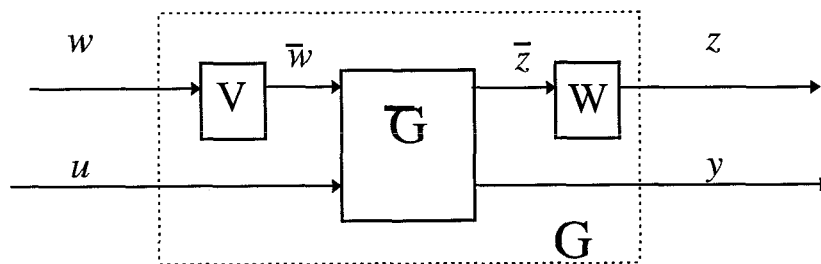
Figure 2-2 Nominal plant with filters

A more detailed look on G is presented in figure 2-2. In this figure w and $\overline{w}$ stands for the unweighted and weighted exogenous input respectively and u are the signals that are generated by the controller. On the right hand side, z and $\overline{z}$ are the weighted and unweighted output to be controlled respectively and the signal y is fed back to the controller.

The aim of controller design is to manipulate the transfer from w to z so a certain norm (the H_∞ norm in this report) of this transfer function is achieved. V and W are used to express the design specifications. The effect of the input weighting V can be explained with the next example. Suppose that the exogenous input is a signal representing a road-disturbance. The controller design accounts for all frequencies, but if this road-disturbance is better modelled as a low frequency signal, the design should emphasise the lower frequencies. To get a better controller design the input signal w can be shaped by using a filter which takes the information of the spectral contents into account. For the road-disturbance this may give a filter in the form of $V = \alpha (1+s/\beta)^{-1}$, with a gain α and a cut-off frequency β. The result of this will be that the controller focuses on suppression of low-frequency ($\omega < \beta$) disturbance inputs. To be complete, it has to be mentioned that the exogenous input usually also contains measurement noises for the output y.

W is called the weighting filter and is used to weight the frequencies of the output signal to be controlled which have to fulfil certain control targets. To be more precise, the filter W could be used as the reciprocal of the upper bound of the controlled output signal. So, when the controlled output signal has to be very small for a certain frequency, than W should be large for that frequency.

The problem presented in figure 2-1 can also be denoted using a transfer function matrix.

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad (1.)$$

For the closed loped system from figure 2-1 this leads to the next transfer function matrix.

$$T_{zw} = T_{zw}(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}, \qquad (2.)$$

relating w to z according to $z = T_{zw}\, w$.

Back to the H_∞ control theory, there are two cases in this control design.

The first case is the H_∞ optimal control problem. This is to design a stabilising controller K which minimises the H_∞ norm of the closed-loop transfer function from w to z :

$$\left\| T_{zw} \right\|_\infty = \sup_\omega \; \sigma_{max} \; \left(T_{zw}(j\omega) \right) \qquad (3.)$$

Here, σ_{max} denotes the maximum singular value of a matrix and sup stands for supremum. The H_∞ norm requires a search for T_{zw} (jω) over ω for the supremum. For SISO T_{zw}, an interpretation of the H_∞ norm is the peak value from the magnitude plot.

The H_∞ optimal control problem gives the theoretical minimum of what can be achieved, but this is not always necessary. In some cases, it satisfies to use a controller which achieves a norm which is smaller than a specified value. This second case will be called H_∞ suboptimal control. We define H_∞ suboptimal control as the problem of finding a stabilising controller K , if there is any, such that $\| T_{zw} (G,K) \|_\infty < \gamma$, with $\gamma > 0$.

Whether the H_∞ optimal control or the H_∞ suboptimal control is used, the controllers found are actually dependent on the state-space data from the generalised plant and on the value γ.

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Chapter 3.

Algebraic Riccati Equality (ARE)

In the previous section, the set-up to design a H_∞ controller has been shown. In the next chapters , the tools to derive the solution to the suboptimal problem will be shown. There are two methods to handle this. The first is known as the Algebraic Riccati Equality (ARE) -method and the second is known as the Linear Matrix Inequality (LMI) -method. In this chapter the ARE method will be presented.

First the generalised plant presented in figure 2-1 has to be put in state-space format:

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{11} w + D_{12} u \quad ; x \in R^{n_x}, w \in R^{n_w}, u \in R^{n_u}, z \in R^{n_z}, y \in R^{n_y} \quad (4.)$$

$$y = C_2 x + D_{21} w + D_{22} u$$

For the ARE controller design, the next assumptions must hold for the state-space formulation to guarantee that an H_∞ controller can be constructed [2].

1. (A, B$_2$) is stabilizable and (A, C$_2$) is detectable.
    The former states that the uncontrollable eigenvalues of A are asymptotically stable and the latter states that the unobservable eigenvalues of A are asymptotically stable. These assumptions are necessary for the existence of a stabilising controller. It also implies that the weighting functions V and W have to be stable, because they are uncontrollable from u and unobservable from y respectively.

2. D_{12} has full column rank (rank (D_{12})= n_u) en D_{21} has full row rank (rank (D_{21}) = n_y).
    The condition on D_{12} implies that all control signals have to be weighted even at infinite frequency, to avoid non-proper controllers which are physically not realisable. The rank condition on D_{21} implies that all measurements are noisy at infinite frequency. According to [2], this is sufficient to ensure proper controllers.

3. $rank\left(\Sigma_1(\omega) = \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \right) = n_x + n_u$, so $\Sigma_1(\omega)$ has full column rank for all $\omega \in$ R.
    This assumption has the following implications, the first and second are equivalent [11, app. B].
    *   rank (G_{12} (jω))= n_u for all $\omega \in$ R,
    * G_{12} (jω) has no transmission zeros on the imaginary axis,
    * $n_z \ge n_u$.
    This is to ensure that T_{zw} is asymptotically stable.

4. $rank\left(\Sigma_2(\omega) = \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right) = n_x + n_y$, so $\Sigma_2(\omega)$ has a full row rank for all $\omega \in$ R.
    This assumption has the following implications, the first and second are equivalent [11, app. B].
    *   rank (G_{21} (jω))= n_y for all $\omega \in$ R,
    * G_{21} (jω) has no transmission zeros on the imaginary axis,
    * $n_w \ge n_y$.
    This is to ensure that T_{zw} is asymptotically stable.

If the assumptions are satisfied there are two AREs that have to be solved to find K. These are the full information ARE and the full control ARE.

An Algebraic Riccati Equality (ARE) has the following structure:

$$X\Gamma + \Gamma^T X - XRX + Q = 0, \qquad\qquad (5.)$$

with Γ, Q and R real matrices with dimension n x n and Q and R symmetric. X is the matrix which has to be computed and will need some properties to give a stabilising solution for (5) [10, p 333-341].

Associated with the ARE (5) is a $2n$ x $2n$ matrix

$$H = \begin{bmatrix} \Gamma & -R \\ -Q & -\Gamma^T \end{bmatrix} \qquad\qquad (6.)$$

This H is called an Hamiltonian matrix. We assume that H has no eigenvalues on the imaginary axis. Then it has n eigenvalues with real part s < 0 and n with real part s > 0, which are symmetric with respect to the imaginary axis. A subspace χ_- (H) can be defined corresponding to the eigenvalues with negative real parts, and a subspace χ_+ (H) can be defined corresponding to the eigenvalues with positive real parts s > 0. χ_- (H) is called the stable eigenspace of the Hamiltonian . By finding a basis for this subspace, stacking the basis vectors up to form a matrix, and partitioning the matrix we get:

$$\chi_-(H) = \mathrm{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Here, X_1 and X_2 are real n x n matrices and Im stands for image or range. If X_1 is non-singular or, equivalently, if the two subspaces

$$\chi_-(H), \quad \mathrm{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

are complementary, the solution to the ARE (4) is obtained via $X = X_2 X_1^{-1}$. The solution X is symmetric and is called stabilising, because $\lambda(\Gamma - RX)$ is stable. Since X is uniquely determined by H, it is denoted as a function 'Ric' of H: $X = \mathrm{Ric}(H)$. Hamiltonians belonging to the domain of Ric ($H \in$ dom. (Ric)) have two properties, which were employed in the above [10 , chap. 13]:

1. H has no eigenvalues on the imaginary axis (stability property) and
2. X_1 is non-singular , i.e. rank (X_1) = n (complementary property).

The full information ARE is given by the following formula. Full information refers to the case that the controller is provided with the states x and the exogenous input w.

$$X\left(A\det(r) - B_1 D_{12}^T D_{12} D_{11}^T C_1 - B_1 D_{11}^T D_{12} D_{12}^T C_1 - B_2 D_{12}^T D_{11} D_{11}^T C_1 + B_2 \left(D_{11}^T D_{11} - \gamma^2 I_{n_w}\right)D_{12}^T C_1\right) +$$

$$\left(A^T \det(r) - C_1^T D_{11} D_{12}^T D_{12} B_1^T - C_1^T D_{11} D_{11}^T D_{12} B_2^T - C_1^T D_{12} D_{12}^T D_{11} B_1^T + C_1^T D_{12}\left(D_{11}^T D_{11} - \gamma^2 I_{n_w}\right)B_2^T\right)X \qquad (7.)$$

$$-X\left(B_1 D_{12}^T D_{12} B_1^T - B_1 D_{11}^T D_{12} B_2^T - B_2 D_{12}^T D_{11} B_1^T + B_2\left(D_{11}^T D_{11} - \gamma^2 I_{n_w}\right)B_2^T\right)X +$$

$$C_1^T C_1 \det(r) - C_1^T D_{11} D_{12} D_{12}^T D_{12} D_{11}^T C_1 + C_1^T D_{11} D_{11}^T D_{12} D_{12}^T C_1 + C_1^T D_{12} D_{12}^T D_{11} D_{11}^T C_1 - C_1^T D_{12}\left(D_{11}^T D_{11} - \gamma^2 I_{n_w}\right)D_{12}^T C_1 = 0$$

The Hamiltonian H_X associated to this ARE follows by careful comparison of (7) with (5) and (6).

The full control ARE is given by the following formula. Full control refers to the situation that the controller has full access to the states x and to the controlled output z.

$$Y\left(A^T \det(\hat{r}) - C_1^T D_{21} D_{21}^T D_{11} B_1^T + C_1^T D_{11} D_{21}^T D_{21} B_1^T + C_2^T D_{21} D_{11}^T D_{11} B_1^T - C_2^T\left(D_{11} D_{11}^T - \gamma^2 I_{n_z}\right) D_{21} B_1^T\right) +$$

$$\left(A \det(\hat{r}) - B_1 D_{11}^T D_{21} D_{21}^T C_1 + B_1 D_{11}^T D_{11} D_{211}^T C_2 - B_1 D_{21}^T D_{21} D_{11}^T C_1 + B_1 D_{21}^T\left(D_{11} D_{11}^T - \gamma^2 I_{n_z}\right) C_2\right) Y \qquad (8.)$$

$$-Y\left(C_1^T D_{21} D_{21}^T C_1 - C_1^T D_{11} D_{21}^T C_2 - C_2^T D_{21} D_{11}^T C_1 - C_2^T\left(D_{11} D_{11}^T - \gamma^2 I_{n_z}\right) C_2\right) Y +$$

$$B_1 B_1^T \det(\hat{r}) - B_1 D_{11}^T D_{21} D_{21}^T D_{11} B_1^T + B_1 D_{11}^T D_{11} D_{21}^T D_{21} B_1^T + B_1 D_{21}^T D_{21} D_{11}^T D_{11} B_1^T - B_1 D_{21}^T\left(D_{11} D_{11}^T - \gamma^2 I_{n_z}\right) D_{21} B_1^T = 0$$

There is also an Hamiltonian H_Y associated to this ARE, it follows by comparison (8) with (5) and (6).
In (7) and (8):

$$\det(r) = \det\left(\begin{bmatrix} D_{11}^T D_{11} - \gamma^2 I_{n_w} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix}\right);$$

$$\det(\hat{r}) = \det\left(\begin{bmatrix} D_{11} D_{11}^T - \gamma^2 I_{n_z} & D_{11} D_{21}^T \\ D_{21} D_{11}^T & D_{21} D_{21}^T \end{bmatrix}\right).$$

We denote the solution of (7) as X_∞ and the solution of (8) as Y_∞. Then there exists an internally stabilising controller K (jω) such that $\| F_l (G, K) \|_\infty < \gamma$ if and only if :

1. the Hamiltonians H_X and $H_Y \in$ dom. (Ric) and
2. there exist $X_\infty \geq 0$ and $Y_\infty \geq 0$ satisfying formula (7) and (8)
3. such that $\rho (X_\infty Y_\infty) < \gamma^2$,

In 3 ρ denotes the maximal modulus of the eigenvalues from matrix ($X_\infty Y_\infty$) and is called the spectral radius.

When these three conditions are satisfied then a controller can be derived using X_∞ and Y_∞, see [2]. From (7) and (8), it is obvious that the controller K depends on the state-space data in (5) and on the required performance level γ.

Chapter 4.

Linear Matrix Inequality (LMI).

In this chapter we will examine the other solution method (LMI-method) mentioned in chapter 3.

Linear Matrix Inequalities have the following structure [12]:

$$F(x) \triangleq F_0 + \sum_{j=1}^{m} x_j F_j > 0 \qquad (9.)$$

where $x \in R^m$ is the variable and the symmetric matrices $F_i = F_i^T \in R^{n*n}$, i=0,...,m, are given data. The strict inequality symbol means that F(x) is positive definite, i.e. u^T F(x) u > 0 for all $u \neq 0$. Nonstrict LMIs have the form F(x) $\geq$ 0.

According to [4], the constraint that the H_∞ norm of the closed-loop (2) must be smaller than γ and that the closed-loop system must be internally stable, can be converted into a matrix inequality via the bounded real lemma. This means for a closed-loop like in figure 2-1:

$$\begin{pmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma\, I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma\, I \end{pmatrix} < 0 \qquad (10.)$$

This is an LMI with the variable X_{cl} with dimensions 2n x 2n . If and only if the matrix inequality (10) is feasible and the variable $X_{cl} > 0$, then there are internally stabilising controllers satisfying $\| T_{zw} (G,K) \|_\infty < \gamma$. The subscript 'cl' denotes closed-loop. The closed-loop matrices depend on the state-space matrices of the open loop G and on the state-space matrices of the controller K:
$x_k = A_k x_k + B_k y$; $u = C_k x_k + D_k y$ .The closed-loop state space matrices are then given by:

$$A_{cl} = \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{pmatrix}; B_{cl} = \begin{pmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{pmatrix};$$

$$C_{cl} = \begin{pmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{pmatrix}; D_{cl} = D_{11} + D_{12} D_K D_{21}. \qquad (11.)$$

In (10) the controller matrices A_k, B_k ,C_k , D_k enter A_{cl}, B_{cl} , C_{cl} and D_{cl} . So, for controller construction is (10) not directly useful. Instead, for a given controller, this LMI in X_{cl} could be used to check if the controller achieves internally stability and a H_∞ norm requirement. For the purpose of controller construction, (10) is first rewritten as another LMI in the variable X_{cl} [6, p 1008], with the unknown controller matrices A_k, B_k ,C_k and D_k stored in Ω_k .

$$Z_{X_{cl}} + P_{X_{cl}}^T \Omega_K Q + Q^T \Omega_K^T P_{X_{cl}} < 0; \qquad \Omega_K = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \qquad (12.)$$

The reason for this other notation is that the matrices Z_{Xcl}, P_{Xcl} and Q do not depend anymore on the controller matrices A_k, B_k ,C_k and D_k . The matrix Q only depends on the open loop plant data. The matrices P_{Xcl} and Z_{Xcl} depend on the variable X_{cl} and the open loop plant data. The exact contents of the matrices Z_{xcl} , P_{Xcl} and Q can be found in [4, p 426-427]. Because the matrix Ω_k is not known we can not solve (12). However [4] shows that Ω_k can be eliminated from (12) to obtain necessary and sufficient conditions for solvability of (12). These conditions only depend on X_{cl} and on the plant matrices. It appears that (12) is solvable, if and only if there is a $X_{cl} > 0$ such that :

$$W_P^T \ Y_{X_{cl}^{-1}} \ W_P < 0;$$

$$W_Q^T \ Z_{X_{cl}} \ W_Q < 0; \tag{13.}$$

Here W_P and W_Q denote any bases of the nullspaces of P and Q respectively, which only contain plant data, see [4]. The reason for the introduction of (13) is that the formulas are independent of the controller matrices. The first formula from (13) is rearranged from a formula with the nullspace of matrix $P_{X_{cl}}$ and matrix Z (which depends on X_{cl}) to a formula which depends on the nullspace of matrix P and on matrix Y (which depends on X_{cl}^{-1}) [4,p 426]. To obtain a simpler solvability requirement in terms of LMIs, the following

$$X_{cl} =: \begin{pmatrix} S & N \\ N^T & * \end{pmatrix} \quad and \quad X_{cl}^{-1} =: \begin{pmatrix} R & M \\ M^T & * \end{pmatrix} \tag{14.}$$

are used, with R, S, M and N with dimensions n x n. Now, the matrix inequalities (13) can be rearranged to the two LMIs (15) and (16) and $X_{cl} > 0$ can be rearranged to the LMI (17). The three LMIs only depend on the plant data and variables R , S.

$$\begin{pmatrix} N_{12} & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{pmatrix} \begin{pmatrix} N_{12} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{15.}$$

$$\begin{pmatrix} N_{21} & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A^T S + SA & SB_1 & C_1^T \\ B_1^T S & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} N_{21} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{16.}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{17.}$$

In these LMIs R and S are the unknown (we will see later, that under certain assumptions $R = \gamma X_\infty^{-1}$ and $S = \gamma Y_\infty^{-1}$), N_{12} and N_{21} denote orthonormal bases of the nullspaces of (B_2^T, D_{12}^T) and (C_2, D_{21}) respectively. These orthonormal bases can be spanned by the vectors belonging to the singular values which are equal to zero. The computation of the vectors and singular values can be done by using the singular value decomposition.

For the LMIs there is only one assumption on the plant state-space representation, which is the first assumption from the previous chapter. When this assumption holds, two things have to be done to construct a suboptimal controller [6].

1. Solve the above system of three LMIs, where the unknown variables are two symmetric matrices R and S, with the dimensions equal to the size of the plant order.

2. Given R, S and the plant state-space data compute the controller by solving another LMI called the 'controller LMI'. This will not be detailed here, see [4] for the solution.

The result is a suboptimal stabilising controller fulfilling $\| T_{zw} \|_\infty < \gamma$ [6, p 1010].

Chapter 5.

Resemblance between Algebraic Riccati Equalities and Linear Matrix Inequalities

In this chapter we will see that in certain cases H_∞ suboptimal controller design using the ARE-method or the LMI-method is similar.

We start with the formulas (7) , (8) from the ARE-method and use the next assumptions [1, p 834]:

- (A, B_1) is stabilizable, (A, C_1) is detectable. This assumption is made for a technical reason. Together with the assumption that (A, B_2 ,C_2) is detectable and stabilizable it guarantees that the Hamiltonians associated with the AREs (7) and (8) belong to dom (Ric) [1, p834]

- $D_{11} = 0, \quad D_{12}^T\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \quad D_{21}\begin{bmatrix} B_1^T & D_{21}^T \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}.$ (18.)

In short $D_{11} = 0$ means that there is no direct influence from the exogenous input w, to the output to be controlled z. The last two assumptions in (18) mean that in the state representation (4) C_1 x and D_{12} u are orthogonal so that the penalty on z ($= C_1$ x $+ D_{12}$ u) includes a non-singular normalised penalty on the controller output u. Under these more restrictive assumptions, ARE (7) reduces to:

$$X_\infty A + A^T X_\infty + X_\infty\left(\gamma^{-2} B_1 B_1^T - B_2 B_2^T\right)X_\infty + C_1^T C_1 = 0 \qquad (19.)$$

and ARE (8) to:

$$Y_\infty A^T + A Y_\infty + Y_\infty\left(\gamma^{-2} C_1^T C_1 - C_2^T C_2\right)Y_\infty + B_1 B_1^T = 0 \qquad (20.)$$

If we solve these AREs, with X_∞ and Y_∞ we can form a controller [1] by algebraic manipulation. With the notation from $x_k = A_k x_k + B_k$ y ; u $= C_k x_k + D_k$ y , this gives the following controller:

$$A_k = A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 ; B_k = -Z_\infty L_\infty ; C_k = F_\infty ; D_k = 0.$$

$$with \quad F_\infty = -B_2^T X_\infty \qquad L_\infty = -Y_\infty C_2^T \qquad Z_\infty = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1}. \qquad (21.)$$

The H_∞ controller K_{sub} displayed in (21) is called the central or the minimum entropy controller [10, p 419].

Next we consider the formulas (15) ,(16) from the LMI-method and use the restrictive assumptions above. First, we will transform (15) and (16) [4, pp 427-428] into

$$W_{12}^T\left\{\hat{A}R + R\hat{A}^T - \gamma\ \hat{B}_2\hat{B}_2^T + \begin{pmatrix} \hat{C}_1 R \\ \hat{B}_1^T \end{pmatrix}^T\begin{pmatrix} \gamma\ I & -\hat{D}_{11} \\ -\hat{D}_{11}^T & \gamma\ I \end{pmatrix}^{-1}\begin{pmatrix} \hat{C}_1 R \\ \hat{B}_1^T \end{pmatrix}\right\}W_{12} < 0 \qquad (22.)$$

$$W_{21}^T\left\{\tilde{A}^T S + S\tilde{A} - \gamma\ \tilde{C}_2^T\tilde{C}_2 + \begin{pmatrix} \tilde{B}_1^T S \\ \tilde{C}_1 \end{pmatrix}^T\begin{pmatrix} \gamma\ I & -\tilde{D}_{11}^T \\ -\tilde{D}_{11} & \gamma\ I \end{pmatrix}^{-1}\begin{pmatrix} \tilde{B}_1^T S \\ \tilde{C}_1 \end{pmatrix}\right\}W_{21} < 0 \qquad (23.)$$

with the following shorthands:

10

$$\hat{B}_2 = B_2 D_{12}^+ \qquad \hat{A} = A - \hat{B}_2 C_1 \qquad \hat{B}_1 = B_1 - \hat{B}_2 D_{11}$$
$$\hat{C}_1 = \left(I - D_{12} D_{12}^+\right)C_1 \qquad \hat{D}_{11} = \left(I - D_{12} D_{12}^+\right)D_{11} \qquad \text{and} \qquad (24.)$$

$$\tilde{C}_2 = D_{12}^+ C_2 \qquad \tilde{A} = A - B_1 \tilde{C}_2 \qquad \tilde{C}_1 = C_1 - D_{11} \tilde{C}_2$$
$$\tilde{B}_1 = B_1 \left(I - D_{21}^+ D_{21}\right)C_1 \qquad \tilde{D}_{11} = D_{11}\left(I - D_{21}^+ D_{21}\right) \qquad (25.)$$

Here, $^+$ denotes the pseudo-inverse of a matrix and W_{12} and W_{21} denote the bases of the nullspaces of $(I - D_{12}{}^+ D_{12}) B_2{}^T$ and of $(I - D_{12} D_{12}{}^+) B_2{}^T$.

The restrictive assumptions on the formulas (22) - (25) will give us, with the simplification where W_{12} and W_{21} are identity matrices:

$$I\left\{AR + RA^T + \left(\gamma^{-1} B_1 B_1^T - \gamma\ B_2 B_2^T\right) + \gamma^{-1} R C_1^T C_1 R\right\}I < 0 \qquad (26.)$$

$$I\left\{A^T S + SA + \left(\gamma^{-1} C_1^T C_1 - \gamma\ C_2^T C_2\right) + \gamma^{-1} S B_1 B_1^T S\right\}I < 0 \qquad (27.)$$

With the substitutes $R = \gamma X_\infty^{-1}$ and $S = \gamma Y_\infty^{-1}$ it is shown in [6, p 1012] that the left hand sides of these inequalities are the same as the left hand sides of the AREs in (19) and (20).

Now we can derive a suboptimal controller satisfying the restrictive assumptions and by simplifying 'controller LMI' formula [6, p1010] we take $D_k = 0$ and then it follows that

$$A_K = A + \left(\gamma^{-2} B_1 B_1^T - B_2 B_2^T\right)X_\infty + \left(\gamma^{-2} Y_\infty X_\infty - I\right)^{-1} Y_\infty C_2^T C_2 ,$$
$$B_K = -\left(\gamma^{-2} Y_\infty X_\infty - I\right)^{-1} Y_\infty C_2^T , \qquad C_K = -B_2^T X_\infty \qquad (28.).$$

With a few changes we can rewrite this to:

$$A_K = A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_K + Z_K L_K C_2 \qquad B_K = -Z_K L_K$$
$$C_K = F_K \qquad D_K = 0 \qquad with \qquad (29.)$$
$$F_K = -B_2^T X_\infty \qquad L_K = -Y_\infty C_2^T \qquad Z_K = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1}$$

We see that the computed H_∞ suboptimal controllers in (21) and (29) are the same under the simplifying assumptions from (18) and the one above (18).

Chapter 6.

Problem formulations

A tractor-semitrailer

In the next two chapters, we will study some examples to see what can happen when some assumptions are violated. In this chapter, we will present two examples. The first example is an active suspension control problem for a tractor-semitrailer model [8]. This model has degrees of freedom in the vertical plane and gravity is eliminated. The four degrees of freedom are the displacement of the front axle s_1, the displacement of the rear axle s_2, the displacement of the Centre Of Mass (COM) of the truck-chassis s_3 and the rotation around the COM of the truck-chassis s_4. The vector x from the state-space representation (4) consists of $x = [\ s\ \ \dot{s}\ ]'$.
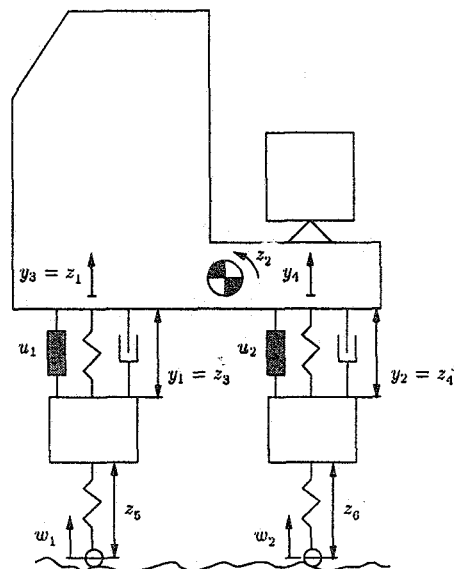
Figure 6-1 The tractor-semitrailer with 4 degrees of freedom

The aim of the controller design is to achieve an improvement of performance in comparison to a passive suspension concerning drive comfort and handling. With respect to the driver-comfort the accelerations of the truck must be kept small. Which frequencies have to be weighted depends on the human sensitivity to certain frequencies. To achieve this goal the suspension of the truck is supplied with two actuators. This makes it possible to get a better grip on the behaviour of the suspension, as well as on suspension deflections (which must be limited due to the space limitation) and on the dynamic tire deflections (which must be kept low to get a good handling).

The model has four sensors and two actuators. Sensors y_1 and y_2 measure the suspension deflections, y_3 and y_4 measure the chassis accelerations. The actuators u_1 and u_2 are placed between the axles and the truck-chassis. The exogenous input w contains two signals representing the road-disturbance and four signals representing the measurement-noises. The output to be controlled z contains eight signals including two signals which are used to weight the controller output (i.e. to avoid actuator saturation). The other six controlled outputs are the vertical chassis acceleration z_1, rotational chassis acceleration z_2, front and rear suspension deflection z_3 and z_4, and front and rear tire deflection z_5 and z_6. The model does not have tire-damping.

The filter V which specifies the road-disturbance is a low pass bandfilter and the filters shaping the white measurement-noises with zero-mean are constant gains. The filter W weighting the vertical chassis accelerations and rotational chassis accelerations are emphasise the lower frequencies 10^0 to 10^2 [rad/s]. The filters concerning the deflections are chosen as constant over the frequency-range. The parameters of the truck-model and the other filters can be found in appendix I, while the filters V and W and their parameters can be found in appendix II.

Mass-spring-damper system

Because the nominal plant of the truck together with the filters is a complex model, we will also use the simple model in Fig. 6-2 to get a clear idea of the problems concerning the transmission zeros. This model can be interpreted as the front or rear of the truck.

This simplified model is a system with two masses m, two springs with stiffness k, one damper with damping b and an actuator u. In this system w, x_1 and x_2 are the displacements. Here w is the exogenous input and u is the output from a controller. We are interested in the acceleration of mass 2. Minimisation of this acceleration is the control target z. The measurement y is the acceleration of mass 1, which is not disturbed with measurement noises. We use the representation according to the state space representation (4).
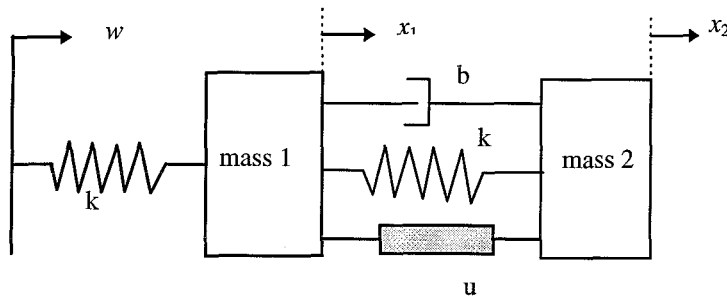
Figure 6-2 System with two masses, two springs and one damper and actuator

The shaping filter V and weighting filter W are identity matrices, so this will result in the following state-space matrices for the generalised plant:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2k}{m} & \frac{k}{m} & \frac{-b}{m} & \frac{b}{m} \\ \frac{k}{m} & \frac{-k}{m} & \frac{b}{m} & \frac{-b}{m} \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{k}{m} \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{m} \\ \frac{1}{m} \end{bmatrix}$$

$$C_1 = \begin{bmatrix} \frac{k}{m} & \frac{-k}{m} & \frac{b}{m} & \frac{-b}{m} \end{bmatrix} \quad D_{11} = [0] \quad D_{12} = \begin{bmatrix} \frac{1}{m} \end{bmatrix} \qquad (30.)$$

$$C_2 = \begin{bmatrix} \frac{-2k}{m} & \frac{k}{m} & \frac{-b}{m} & \frac{b}{m} \end{bmatrix} \quad D_{21} = \begin{bmatrix} \frac{k}{m} \end{bmatrix} \quad D_{22} = \begin{bmatrix} \frac{-1}{m} \end{bmatrix}$$

This system can be abbreviated to (A, B, C, D), with B=[B$_1$,B$_2$], C=[C$_1$;C$_2$], D=[D$_{11}$,D$_{12}$;D$_{21}$,D$_{22}$]. In the next chapter we will have a better look on the G$_{12}$ transfer from u to z and G$_{21}$ from w to y. With the parameters (m=1, b=1, k=1) the eigenvalues of A are -0.9567±1.2272j and -0.0433±0.6412j , so there are no unstable poles. What we also can see is that there are transmission zeros for G$_{12}$ and G$_{21}$. The transmission zeros for G$_{12}$ are [0 ± j; 0; 0] , the transmission zeros for G$_{21}$ are [0; 0; -0.500 ± 0.8660 j], so there are transmission zeros on the imaginary axis. Except for the transmission zeros there are no further violations on the assumptions for the ARE-method.

Chapter 7.

Difference between Algebraic Riccati Equalities and Linear Matrix Inequalities

It was shown in chapters 3 and 4 that there are more assumptions on the state-space representation for a design using the AREs than there are for a design using the LMIs. In this chapter we will see the differences between the ARE-method and the LMI-method. To start with, the 'active suspension control problem' presented in the previous chapter will be studied.

We will examine what happens when one of the assumptions for the ARE-method is violated. The second assumption from chapter 3 will be used as an example. It says that D_{12} must have full column rank and that D_{21} must have full row rank. If we take a look at the state-space representation we see that matrix D_{12} represents about the direct coupling between the control input u and the output to be controlled z and D_{21} represents the direct coupling between the exogenous input w and the measurements y.

We are going to manipulate the matrices D_{21} and D_{12} so they don't have a full rank. For instance, take D_{21} as $[0^{4 \times 4} \ \varepsilon \ I^{4 \times 4}]$ and decrease the value of ε to zero. As long as ε is not zero then D_{21} has a full row rank. The decrease of ε can be interpreted as a reduction of the sensor noise, so there will be better measurements and this will result in a lower H_∞ norm of the closed-loop T_{zw} , which is an improvement of performance. Whenever ε becomes zero the second assumption on the ARE is violated. The LMI-method is not sensitive to the violation of the second assumption from the ARE.

From MATLAB we use two routines which can compute the H_∞ suboptimal controller by using the AREs. They are called hinfsyn from the Mu-Analysis & Synthesis-toolbox and hinfric from the LMI Control-toolbox. There is one routine which computes the H_∞ suboptimal controller by using the LMIs, which is called hinflmi, also from the LMI Control-toolbox. These three routines are used to analyse the possible differences between the LMIs and the AREs.

In table 7-1, the results with decreasing ε are documented for the three MATLAB routines. The table shows the relation between decreasing ε and performance γ while the routines worked with tolerance 1e-5 and shows also the computation time in seconds needed to solve the problem.

ε	hinfsyn		hinfric		hinflmi	
	γ	time	γ	time	γ	time
1	0.7151	23.4	0.7151	32.6	0.7151	141.0
10^{-1}	0.7135	26.2	0.7135	30.2	0.7135	146.0
10^{-2}	0.7132	26.9	0.7132	29.0	0.7132	174.0
10^{-3}	0.7132	30.0	0.7132	29.5	0.7132	163.9
10^{-4}	0.7131	33.8	0.7131	30.0	0.7132	176.0
10^{-5}	0.7131	25.6	inf.	-	0.7132	171.1
10^{-6}	0.7131	27.0	inf.	-	0.7132	165.1
10^{-7}	inf.	-	inf.	-	0.7132	164.0
10^{-8}	inf.	-	inf.	-	0.7132	164.0
10^{-9}	inf.	-	inf.	-	0.7132	164.0
10^{-10}	inf.	-	inf.	-	0.7132	164.0
0	D21 has not full row rank	-	inf.	-	0.7132	164.0

Table 7-1 Performance levels and computation times for decreasing ε values.

We see in Fig. 7-1 that the performance of T_{zw} improves for reduced ε, because the value γ decreases for all routines in the beginning. However below a certain ε the value γ becomes infinite for the Riccati methods. In the table, this is denoted as inf.. Theoretically the rank from matrix D_{21} at that moment is

full and it will only loose its rank when the value of ε becomes exactly zero. So, this phenomenon is due to numerical problems.
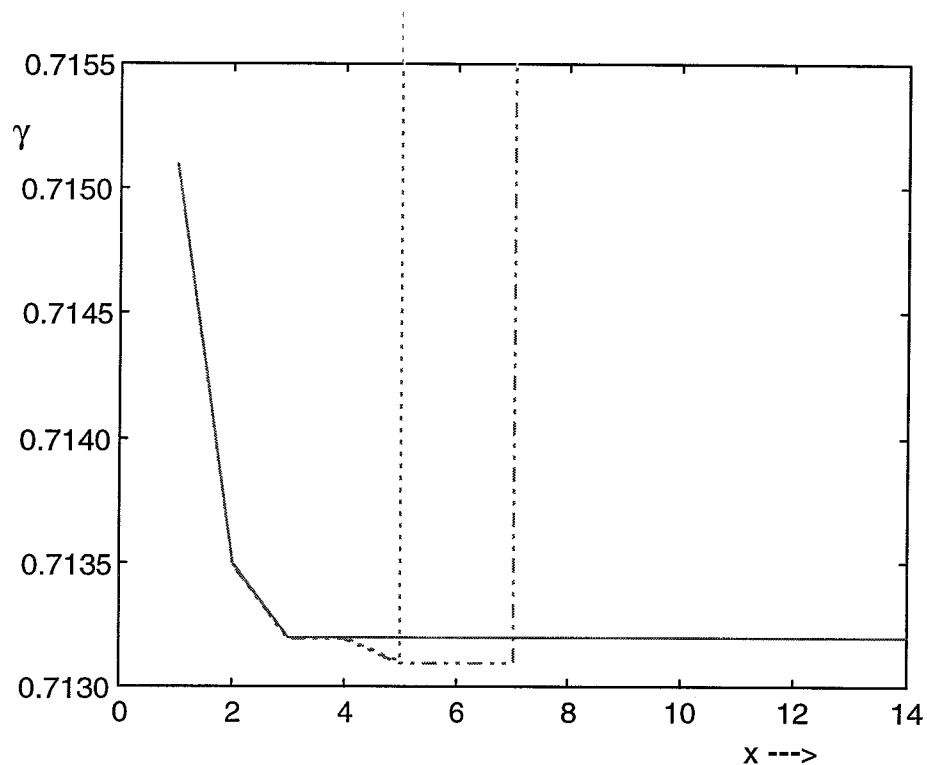
Figure 7-1 the closed-loop performance γ of the generalised plant when decreasing pivot of D_{21} to $\varepsilon * 10^{1-x}$.

We can also look at D_{12}, which represents the direct coupling between the control input u and the output to be controlled z. There is an analogy between checking D_{21} and D_{12}. In this case we have to assure that D_{12} has full column rank. We use for D_{12} the matrix $[\ 0^{6x2}\ ;\ \varepsilon\ I^{2x2}\ ]$ and decrease the value of ε from one to zero.

ε	hinfsyn		hinfric		hinflmi	
	γ	time	γ	time	γ	time
1	0.7151	31.6	0.7151	32.6	0.7151	141.0
10^{-1}	0.4569	33.7	0.4569	40.1	0.4569	133.9
10^{-2}	0.4514	30.2	0.4514	37.0	0.4514	138.3
10^{-3}	0.4511	30.1	0.4511	37.7	0.4512	288.4
10^{-4}	0.4510	29.9	0.4510	37.3	0.4511	348.4
10^{-5}	0.4510	30.5	0.4510	36.4	0.4511	351.2
10^{-6}	inf.	-	0.4510	36.5	0.4510	266.1
10^{-7}	inf.	-	0.5863	147.2	0.4510	528.2
10^{-8}	inf.	-	0.5863	148.0	0.4510	150.8
10^{-9}	inf.	-	0.5863	147.4	0.4510	145.3
10^{-10}	inf.	-	0.5863	147.0	0.4510	61.3
0	D21 has not full row rank	-	0.5863	147.2	0.4510	62.1

Table 7-2 shows the relation between decreasing ε and performance γ while the routines worked with tolerance 1e-5 and shows also the computation time in seconds needed to solve the problem

The result from this case shows similar trends to the result from the case before. The LMI-method shows better results than the ARE-method. We can see that by decreasing ε, the performance γ becomes worse. The reason why this happens is that by decreasing ε, we say that the controller is allowed to manipulate the closed-loop transfer with larger amplitudes. This will make it easier to control the closed-loop transfer and this will result in a smaller closed-loop H_∞ norm. With respect to the computation time we can say that in the first case (D_{21}) the LMI-method is six times slower than the ARE-methods. In the second case (D_{12}) the hinfsyn routine is about seven times faster than the hinflmi routine and the hinfric routine is about three times faster than the hinflmi routine.

The other assumptions on the state space representation which must hold for the ARE are assumption 3 and 4 from chapter 3. For the generalised plant this means that the transfer functions G_{12} and G_{21} don't have transmission zeros on the imaginary axis. For the active suspension problem there are no transmission zeros for G_{12} and G_{21} . To observe what can happen in situations where there are transmission zeros, the mass-spring-damper example is studied.

In the case of the two mass-spring-damper there are transmission zeros. To see what can happen due to this transmission zeros this system is also changed with a small number. We have manipulated the system to (A + ε I, B, C, D) and we varied ε in the range between ε = -1 and ε = 1.

When ε = 1 the system is unstable, because all the poles are shifted into the right half space. When ε decreases to the absolute value of the real part of the eigenvalue closest to the imaginary axis the system remains unstable. When ε is smaller than this value the system becomes stable. (i.e. ε < 0.0433)
The following will happen when we use the hinfsyn routine in the situation where ε is smaller than -10^{-4} the γ is equal to zero until ε reaches -10^{-4} . In the range from -1.00×10^{-4} to 1.00×10^{-4} the performance γ is not steady and changes between zero and infinity. In the situation that ε is larger than 1.00×10^{-4} , γ will increase from zero to a value of 480.
For the hinfric routine almost the same things happen. When ε is in the range from -10^{-4} to -10^{-6} the γ will change from zero to infinity. In the range -10^{-6} to zero γ will be of a constant value which is 8.9 . If ε is between zero and 10^{-4} the γ changes from zero to infinity. When ε is bigger then 10^{-4} , γ will increase from zero to 480.
For the hinflmi routine, γ will be zero when ε is smaller -10^{-4} , then in the range -10^{-4} to zero will be constant 1.00×10^{-4}. When ε passes zero the value of γ will also increase to 480.

A part of these observations can be explained. The H_∞ norm of the open loop from the system (A, B, C, D) has a value 3.4. This means that in the manipulated closed-loop system (i.e. ε < 0.0433) it will be very easy to minimise the output z with respect to the disturbances w. In this closed-loop system (with a controller bounded by the weighting function) it is possible to achieve that the disturbances w have no influence on the output z. The reason why the output from the hinfsyn and the hinfric routines change from zero to infinity in a very small range for ε is not clear, it can be due to numerical problems.

Chapter 8.

Conclusions and recommendations

The H_∞ -controller computation can be accomplished in various manners. One manner is with the use of Algebraic Riccati Equations and another method uses Linear Matrix Inequalities.
In MATLAB there are at the moment several routines which are based on the ARE and one routine which is based on the LMIs. The routines based on the ARE are the hinfsyn-routine from the Mu-Analysis and Synthesis toolbox , the hinfric-routine from the LMI Control-toolbox and the hinf-routine from the Robust Control Toolbox. We only used the first and second routine. The routine based on the LMIs comes also from the LMI Control-toolbox and is called hinflmi.

With regard to the computation-time all the routines are dependent on the tolerance. The reason for this is that the routines use an iteration-algorithm for γ. So with a small tolerance more iterations are needed to get the final result for γ_{opt}. Also the value from a greater γ_{max} has influence on the computation time, since the γ-iteration starts at a higher value (γ_{max}) there are more iterations needed to get the final result for the same γ_{max} and tolerance . There is a difference between the computation-time for the methods based on the ARE and LMIs. The LMI method takes more time. To be more precise, it is two to six times slower than the ARE methods.

With regard to the performance there are two situations. The first situation is that the generalised plant fulfils all the assumptions from chapter 3. In this case the performances achieved with the controllers from both methods are the same. The second situation is when the generalised plant has rank-deficiencies or has transmission zeros on the cross transfer matrices G_{12} or G_{21} . The performance of the closed-loop transfer with the controller designed with the LMIs is more reliable then the performance achieved by the controllers designed with ARE routines. This is because the LMI-method can handle situations where the assumptions are violated and small manipulations to meet the assumptions are not needed. The performance achieved by the controllers designed by the ARE is numerically not steady enough when this generalised plant is disturbed with small changes.

To avoid numerical problems it is better to use the method using the LMIs. The disadvantage of this that the computational time is at least three times higher than the time required by the ARE methods. On the other hand if the state-space representation data is fulfilling the rank-conditions and there are no numerical problems the solution from the ARE-method H_∞ control design is reliable and can reduce a lot of computation-time.

What we recommend for future research is to investigate a wider class of problems. Here, we have only looked at the problem of an active suspension for a truck-semitrailer and a simple mass-spring-damper system. To honestly compare the ARE and LMI-method, other problems should be studied and especially with respect to the assumptions for the AREs. We also have to find out if there are other control problems with situations that can give (numerical) problems.

With respect to the computation time the LMI-method is still slower than the ARE-method, it has to be checked if there are better algorithms to solve the LMIs.

When a problem does not meet the assumptions for the ARE controller design, it is better to use the LMI-method instead of using small perturbations to meet the assumptions on the state-space matrices.

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APPENDIX I   model of the truck (with y1 y2 y3 y4 and u1 u2)

% SYSTEM PARAMETERS:

% Masses and inertias (loaded semitrailer):
maf=1.0e3;   mar=1.5e3;
Mch=7.0e3;   Jch=1.1e4;   Mt=1.2e4;

% Geometric parameters:
a=0.46;   b=3.04; c=2.44;

% Suspension stiffness:
ktf=2.5e6;   ktr=5.0e6;
ksf=5.0e5;   ksr=5.0e5;

% Suspension damping:
bsf=5.0e4;   bsr=5.0e4;

% MATRICES FOR "MECHANICAL SYSTEM EQUATION"
% M*sddot+B*sdot+K*s=E1*v+E3*u,

M=[maf 0 0 0
   0 mar 0 0
   0 0 Mch+Mt c*Mt
   0 0 c*Mt Jch+c^2*Mt];

B=[btf+bsf 0 -bsf a*bsf
   0 btr+bsr -bsr -b*bsr
   -bsf -bsr bsf+bsr -a*bsf+b*bsr
   a*bsf -b*bsr -a*bsf+b*bsr a^2*bsf+b^2*bsr];

K=[ktf+ksf 0 -ksf a*ksf
   0 ktr+ksr -ksr -b*ksr
   -ksf -ksr ksf+ksr -a*ksr+b*ksr
   a*ksf -b*ksr -a*ksf+b*ksr a^2*ksf+b^2*ksr];

E1=[ktf 0
    0 ktr
    zeros(2)];

E2=[btf 0
    0 btr
    zeros(2)];

E3=[1 0
    0 1
    -1 -1
    a -b];

```
% STATE DIFFERENTIAL EQUATIONS:

A=[zeros(4) eye(4)
  -M\K -M\B ];

B1=[zeros(4,2) zeros(4,4)
    M\E1 zeros(4,4)];

B2=[zeros(4,2)
    M\E3   ];

C1=[A(7,:)-a*A(8,:)
   A(8,:)
   -1  0 1 -a zeros(1,4)
    0 -1 1  b zeros(1,4)
    1  0 0  0 zeros(1,4)
    0  1 0  0 zeros(1,4)
   zeros(2,8)  ];

D11=[B1(7,:)-a*B1(8,:)
    B1(8,:)
    zeros(2,2) zeros(2,4)
    -1*eye(2,2) zeros(2,4)
    zeros(2,2) zeros(2,4)];

D12=[B2(7,:)-a*B2(8,:)
    B2(8,:)
    zeros(4,2)
    eye(2,2) ];

C2=[-1  0 1 -a zeros(1,4)
    0 -1 1  b zeros(1,4)
    A(7,:)-a*A(8,:)
    A(7,:)+b*A(8,:)];

D21=[zeros(2) eye(2,4)
    B1(7,1:2)-a*B1(8,1:2) 0 0 1 0
    B1(7,1:2)+b*B1(8,1:2) 0 0 0 1];

D22=[zeros(2)
    B2(7,:)-a*B2(8,:)
    B2(7,:)+b*B2(8,:)];

A;B=[B1 B2];C=[C1; C2];D=[D11 D12;D21 D22];
```

APPENDIX II   filters for the model y1 y2 y3 y4 and u1 u2 and z1 z2 z3 z4 z5 z6 (z7 z8) and w1 w2

values of the variables

$\rho_1 = 10$; $\rho_2 = 10$; $\rho_3 = 100$; $\rho_4 = 100$; $\rho_5 = 5e\text{-}5$

$\omega_1 = 20 * \pi$; $\omega_2 = 10*\pi$; $\omega_3 = 4* \pi$; $\omega_4 = 10* \pi$; $\omega_5 = 100* \omega_4$; $\omega_{10} = 0.4$; $\omega_{20} = 1$;

$\omega_0 = .5* \pi$; $v = 25$; $v_0 = 8.0E\text{-}3$; $\zeta = 1$; $\Theta = 1e\text{-}1$;

% weighting functions w1....w6 with w1 w2 y1 y2 y3 y4

$$Vw_{1,2} = \frac{v_0}{s / \omega_0 + 1}$$

$$Vw_{3,4} = 2.6 * 10^{-3} \Theta$$

$$Vw_{5,6} = 150 Vw_{3,4}$$

% weighting functions z1....z8 with z1 z2 z3 z4 z5 z6 u1 u2

$$Wz_1 = \rho_1 \omega_1^2 \frac{s / \omega_2 + \omega_{10}}{s^2 + 2\zeta\omega_1 s + \omega_1^2}$$

$$Wz_2 = \rho_2 \frac{\omega_{20}}{s / \omega_3 + 1}$$

$$Wz_{3,4} = \rho_3$$

$$Wz_{5,6} = \rho_4 \qquad and \qquad for \qquad u1 \& u2$$

$$Wz_{7,8} = \rho_5 \frac{s / \omega_4 + 1}{s / \omega_5 + 1}$$

here are the filters

Filter Vw1,2

Frequentie rad/s

Filter Wz1,rho1=10

Frequentie rad/s

Filter Wz2,rho2=10

Frequentie rad/s

Filter Wz7,8,rho5=5e-5

Frequentie rad/s