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Abstract An introduction is given on Simple Cell Mapping and Interpolated Cell Mapping. Both methods are applied to a nonlinear dynamic system, in order to determine the attractors and basins of attraction. Newly discovered attractors are presented.

1 Introduction

Essential for nonlinear systems is the possibility of coexistence of different steady state solutions (attractors) for the same set of system parameters. In that case, the long term behaviour is determined by the initial state of the system. For each attractor a basin of attraction exists, which is the set of all initial states leading to that attractor. Another characteristic of nonlinear systems is given by the possible occurance of chaotic behaviour, as response to the system's periodic load.

When studying the global behaviour of a nonlinear dynamic system, one can make use of periodic solvers, such as "shooting" or "finite difference" methods. For a certain start-estimate, these methods provide a nearby periodic solution in most cases. By means of a "path-following" method, the evolution of this periodic solution can be followed when a particular system parameter is varied. However, no information can be obtained about possible chaotic attractors or other, coexisting, periodic attractors.

Using the *Cell Mapping* method however, one can find all attractors—periodic and chaotic—as well as their basins of attraction. In some cases, also unstable and saddle solutions can be found. This method was introduced by Hsu [3,2] and is based on a discretization of the system's state space in so-called *cells*. We distinguish between *Simple Cell Mapping* (SCM), *Generalized Cell Mapping* (GCM) (not treated here), and *Interpolated Cell Mapping* (ICM) (introduced by Tongue [4]).

The SCM and ICM methods are briefly explained. Application of the methods to a beam with nonlinear support is performed. Some newly discovered attractors are presented.

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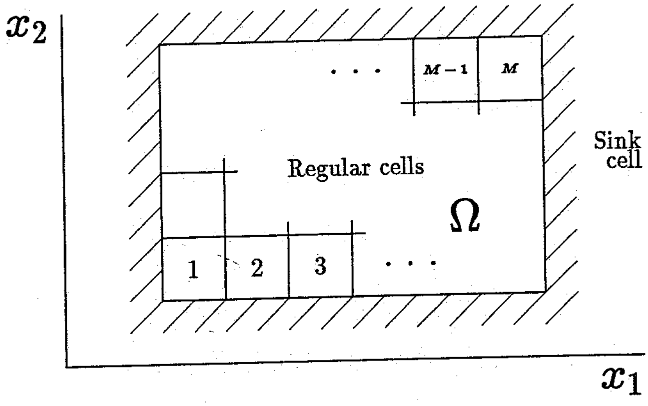

Figure 1: Discretization of a two-dimensional state space.

2 Simple Cell Mapping

Under the SCM method, a particular region Ω in the state space, called the *region of interest*, is discretized into a finite number of cells, say M, with index $1, .., M$. The remaining part of the state space is called the *sink cell* and has index 0 (see Fig. 1). Each cell represents an indivisible state entity. The state of the system is described by a cell index $\xi \in \{0, .., M\}$ instead of a state vector $\mathbf{x} = (x_1, .., x_N)$, where N is the state space dimension.

The evolution of a system can be described as a sequence of cells, by inspecting its state at discrete equidistant times. Let $\xi(n)$ denote the cell containing the system's state at $t = nT$, $n = 0, 1, ..$, with T the time between two state inspections. The system evolution is then governed by

$$\xi(n+1) = C(\xi(n)), \tag{1}$$

where $C : \mathbb{N} \to \mathbb{N}$ is called a *Simple Cell Mapping*. For periodic systems, T should be chosen equal to (a multiple of) the system's period to obtain a SCM C which is independent of n.

We distinguish between cells which are *periodic*, i.e. cells ξ^* with $C^m(\xi^*) = \xi^*$, for some $m \in \mathbb{N}$ which is called the *period* of ξ^*, and cells which are not. These cells are called *transient cells*, and have the chance of being mapped onto a periodic cell, in a finite number of steps, or onto the sink cell. By definition, the system will then stay there forever.

Groups of periodic cells represent the system's solutions. Although aperiodic motion cannot occur because of the finite number of cells, chaotic motion can be expected when dealing with periodic groups of relatively long period. Transient cells which are mapped onto a periodic group represent the basin of attraction corresponding to that group. For more details about the SCM method, see Hsu [2].

2.1 Application: Beam with Nonlinear Support

We consider a beam system with a central nonlinear support. This system was studied by Fey [1], who investigated the long term behaviour by means of CMS (component mode synthesis) methods and finite difference techniques. Here, no attention is paid to these techniques.

A one degree of freedom model of the beam system is given by the following equation:

$$M\ddot{x} + b\dot{x} + (1 + 6H(-x))kx = F\cos(2\pi ft), \tag{2}$$

with $M = 1.0358$ kg, $b = 116.61\xi$ Ns/m, $k = 3282.2$ N/m, $F = 19.693$ N, and $H(x)$ the Heavyside function:

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Equation (2) describes the motion of a periodically forced beam which is supported by a one-sided linear spring. The stiffness of the spring equals six times the beam stiffness. For $f \epsilon [0, 40]$ Hz and $\xi = 0.01, 0.05, 0.1$, periodic motions were obtained by means of the abovementioned methods, combined with a path-following technique (see [1]). For instance, a second order and a fourth order subharmonic solution were found for $f = 21.5$ Hz and $f = 8.34$ Hz resepectively (with $\xi = 0.01$).

We applied the SCM method to the beam system (2) for $f = 21.5$ Hz and $f = 8.34$ Hz, with $\xi = 0.01$. For Ω, we had $|x| \leq 0.01$, $|\dot{x}| \leq 0.86$ and $-0.0025 \leq x \leq 0.0125$, $|\dot{x}| < 0.6$ respectively. For discretization, 201×201 and 151×101 cells were used respectively.

Image cells were determined by means of the *center point method*: For each regular cell, a numerical integration of equation (2) was performed with the cell's center point as initial condition. The integration was executed over a fixed number of forcing periods T $(= 1/f)$. The cell containing the calculated trajectory's end point was taken to be the image cell. Here, integration was carried out over four and five forcing periods respectively.

2.2 Results

Fig. 2 and 3 show the results obtained by SCM for the abovementioned parameter values. These figures represent Poincaré sections of the state space, made at $t = 0, \tau, 2\tau, ..$, with $\tau = 4T$ and $\tau = 5T$ respectively. Notice that only a part of the considered Ω's is shown.

For $f = 21.5$ Hz, a $P-4$ group was found, representing the second order subharmonic solution. Further, a $P-10$ group was found. The basins of attraction of both attractors are given by the corresponding transient cells (see Fig. 2).

For $f = 8.34$ Hz, a $P-6$ group was found, representing the fourth order subharmonic solution. Additionally, a $P-12$ group was found. Basins of attraction of both attractors are shown in Fig. 3.

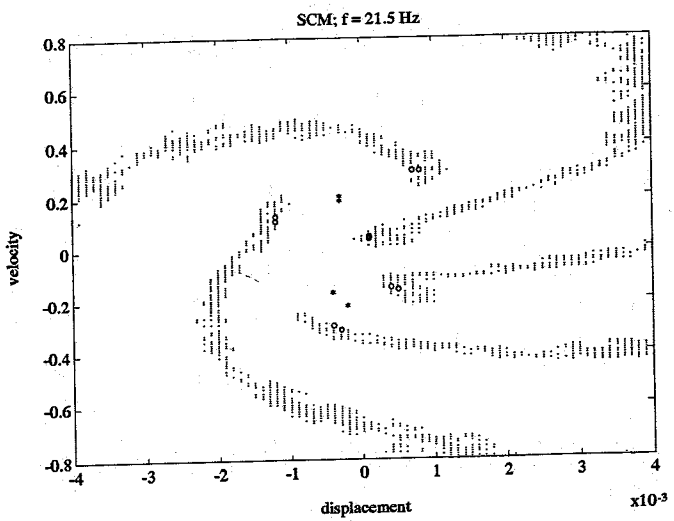

Figure 2: $P-4$ group $(*)$ and transient cells (white); $P-10$ group $(\circ)$ and transient cells $(\cdot)$.

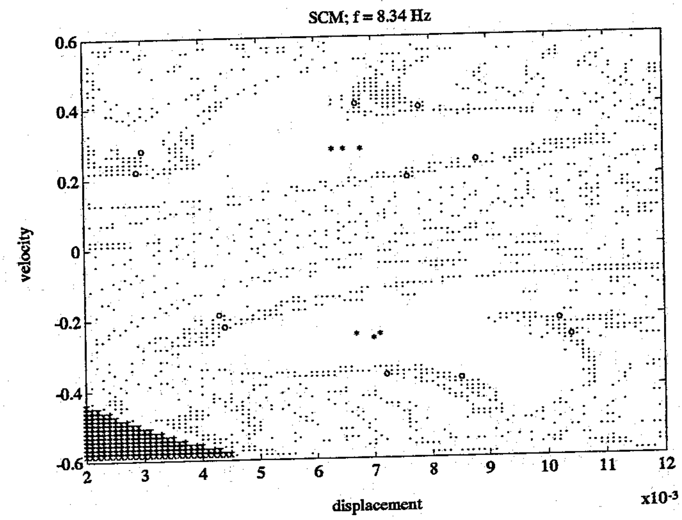

Figure 3: $P-6$ group $(*)$ and transient cells (white); $P-12$ group $(\circ)$ and transient cells $(\cdot)$; transient cells leading to the sink cell (x).

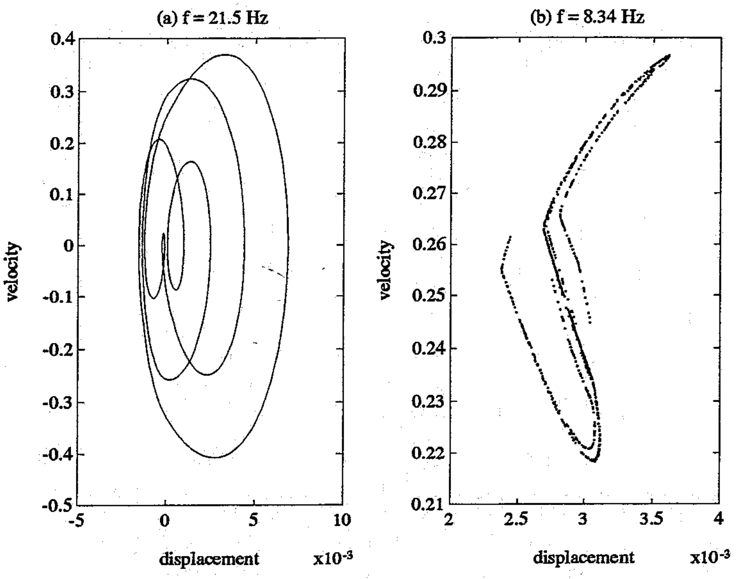

Figure 4: Newly discovered attractors: (a) Orbit of 5-th order subharmonic solution. (b) Poincaré section of chaotic attractor.

Accurate numerical integration was used to identify the new groups. Taking the center point of one of the $P-5$ cells as initial state ($f = 21.5$ Hz), a fifth order subharmonic solution was obtained (see Fig. 4a). An initial state corresponding to the $P-12$ group ($f = 8.34$ Hz) yielded a trajectory that settled on a chaotic attractor. This chaotic attractor consists of six pieces, which are being "visited" one after another by the system. Figure 4b shows one of these pieces. This figure was obtained by plotting the trajectory at $t = 6nT$, $n = 0, 1, .., 1000$.

2.3 Remarks

Due to the discretization error under SCM, the period and the position of the discovered groups will not always match the properties of the actual attractors (as we have seen in § 2.2). However, groups with a large number of transient cells imply the presence of an attractor. For these groups, a regular numerical integration over only a short time interval will provide the exact period and position of the corresponding attractor.

3 Interpolated Cell Mapping

The ICM method was introduced by Tongue [4] as an improvement on SCM, taking only slightly more CPU-time. Just as under SCM, a numerical integration is carried out over a fixed number of forcing periods for a huge number of initial states. These states should no longer be seen as cell center points but as interpolation points (grid points). The system evolution is no longer described by a sequence of cells but by ordinary state coordinates. When the state of the system at $t = nT$ is given by $\mathbf{x}_n = (x_1(nT), .., x_N(nT))$, the state $\mathbf{x}_{n+1}$ at $t = (n+1)T$ is obtained by a bilinear interpolation of $\mathbf{x}_n$ between the image points of the gridpoints surrounding $\mathbf{x}_n$. For an arbitrary initial state, a complete trajectory can be constructed in this way. For a more detailed explanation of the ICM method, we refer to Tongue [4].

3.1 Application: Beam with Nonlinear Support

We applied the ICM method to the beam system (2) for the same values of f. As an interpolation grid, the center points of the SCM cells were used. Since for these states integration had already been done, only interpolation had to be carried out to construct the trajectories emanating from the grid points. Interpolation was continued until convergence to a periodic attractor was obtained (convergence criterium: $5 \cdot 10^{-5}$). Trajectories that did not show convergence within 100 interpolation steps were supposed to be chaotic; the corresponding end points, called *chaotic points*, were considered to lie on a chaotic attractor.

3.2 Results

Fig. 5 and 6 show the results obtained by ICM. Discovered groups represent Poincaré sections of interpolated trajectories that showed periodicity according to the imposed criterium.

For $f = 21.5$ Hz, a $P-2$ and a $P-5$ group were found, representing the second and fifth order subharmonic solution respectively. In Fig. 5, they are shown together with their basins of attraction.

For $f = 8.34$ Hz, a $P-4$ group was found, representing the fourth order subharmonic solution. Further, a large number of chaotic points were found, together forming a chaotic attractor. The initial states corresponding to these points form the basin of attraction (the white area in Fig. 6).

3.3 Remarks

As opposed to SCM, the periodic groups obtained by ICM have the correct period and position in state space. The chaotic points approximate the actual chaotic attractor very well. Fig. 4b corresponds to the upper left piece of the chaotic attractor in Fig. 6. Further, the basin boundaries have been determined more accurately under ICM.

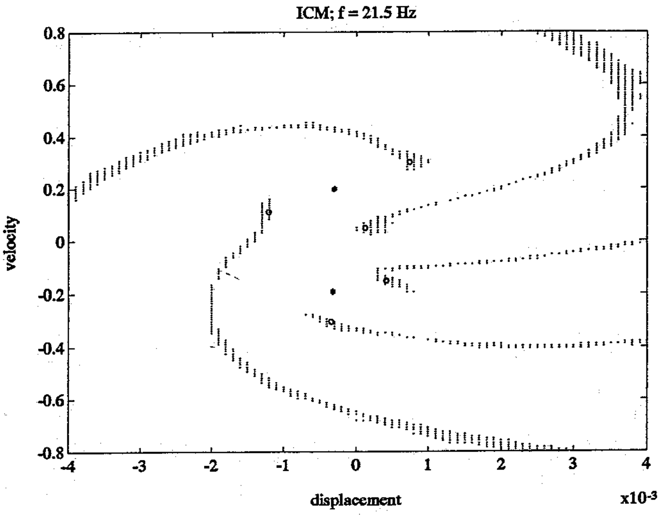

Figure 5: $P-2$ group ($*$) and basin of attraction (white); $P-5$ group ($\circ$) and basin of attraction ($\cdot$).

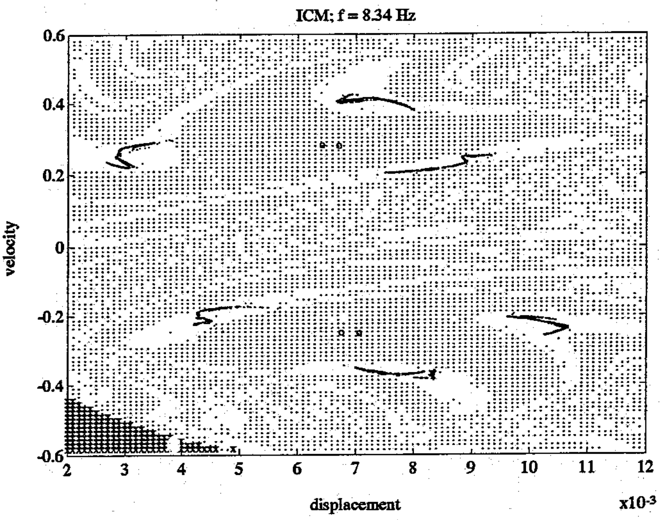

Figure 6: $P-4$ group ($\circ$) and basin of attraction ($\cdot$); chaotic attractor ($\cdot\cdot$) and basin of attraction (white); initial states leading to the sink cell (x).

4 Conclusion

Two types of Cell Mapping methods have been applied to a beam with nonlinear support. For two system frequencies, the system's attractors and basins of attraction have been determined. In addition to periodic solutions obtained by other techniques, a fifth order subharmonic solution was found for $f = 21.5$ Hz and a chaotic attractor was found for $f = 8.34$ Hz.

Under ICM, the attractors and basins of attraction have been determined more accurately compared to SCM. For this, some extra amount of CPU-time was needed. Numerical integration proved the correctness of the chaotic attractor and the basins of attraction, as they were found under ICM.

To obtain a complete picture of the dynamic behaviour of a nonlinear system, the Cell Mapping methods are very suited. For a particular value of a system parameter all existing attractors can be found in general. Using these attractors as start-estimates, path-following techniques may be used to locate the attractors for other values of the considered parameter.

References

[1] R.H.B. Fey. *Steady-State Behaviour of Reduced Dynamic Systems with Local Nonlinearities.* PhD thesis, Eindhoven University of Technology, The Netherlands, 1991.

[2] C.S. Hsu. *Cell to Cell Mapping; A Method of Global Analysis for Nonlinear Systems.* Springer-Verlag, 1987.

[3] C.S. Hsu. A theory of cell-to-cell mapping dynamical systems. *Journal of Applied Mechanics*, 47:931–939, 1980.

[4] B.H. Tongue. Interpolated cell mapping of dynamical systems. *Journal of Applied Mechanics*, 55:461–466, 1988.