The inverse of the incomplete beta integral

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THE INVERSE OF THE INCOMPLETE BETA INTEGRAL

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LANGUAGE

FORTRAN 77

DESCRIPTION AND PURPOSE

For a given probability $\alpha \in [0,1]$ and $m>0$, $n>0$ the subprogram returns $x_\alpha \in [0,1]$, the percentile of the beta distribution satisfying

$$I_x(m,n) = \frac{1}{B(m,n)} \int_0^{x_\alpha} u^{m-1}(1-u)^{n-1} du = \alpha$$

NUMERICAL METHOD

Algorithm AS 64/AS 109 (Majumder and Chattacharjee, 1973; Cran, *et al.*, 1977) uses an approximation to determine an initial value for x_α and thereafter a modified Newton–Raphson method to produce the required accuracy. The modifications are required to ensure that the returned value lies in the appropriate range. When, for example, $m>1$ and $n<1$ the convergence is very slow because the iteration tries to push the x values outside the interval [0,1]. These difficulties are remarkably easily resolved and a clearer and simpler algorithm obtained by remarking that the integral is a monotone increasing function of x for $x \in [0,1]$. Because the support of the beta function is [0,1] repeated

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bisection converges rapidly to the root without the need for any special precautions. Thus although the algorithm is not uniformly better than AS 64/AS 109 it is reliable and cannot fail. No initial approximations to the solution are needed and it is expressed entirely in terms of the incomplete beta integral.

STRUCTURE

REAL FUNCTION BTAINV(M,N,ALPHA,B,IFAIL)

Formal parameters

M	Real	input: parameter m of the beta integral
N	Real	input: parameter n of the beta integral
ALPHA	Real	input: the probability level
B	Real	input/output: the logarithm of the beta function. If $B>0$ it is used as the value of $B(p,q)$. If $B\leq 0$ the value of $B(p,q)$ is evaluated within the *BETAIN* subprogram and is available through *B* for later use
IFAIL	Integer	output: error flag, *IFAIL*=0 indicates success *IFAIL*=1 P≤0 or Q≤0

AUXILIARY ALGORITHMS

BTAINV uses the function *BETAIN(X,P,Q,B,IFAIL)* to evaluate the incomplete beta function, *BETAIN* in turn requires *BETA*0 and the logarithm of the gamma function.

REFERENCES

Cran, G. W., Martin, K. J. and Thomas, G. E. (1977) Remark AS R19 and algorithm AS 109, *Applied Statistics*, **26**, 111–114.

Majumder, K. L. and Bhattacharjee, G. P. (1973) Algorithm AS 63. The Incomplete Beta Integral, *Applied Statistics*, **22**, 409–411.

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      REAL FUNCTION BTAINV(M,N,ALPHA,B,IFAIL)
      IMPLICIT REAL (A-H,O-Z)

C**********************************************************************
C
C     FUNCTION BTAINV(M,N,ALPHA,B,IFAIL)
C
C**********************************************************************

      REAL P,ALPHA,X0,X1,X2    ,B1,    M,N,
     1                  ZERO,ONE,HALF,TOL
      INTEGER NOUGHT,UNITY,TWO,IFAIL

      PARAMETER ( ZERO = 0.0E0,
     1            ONE = 1.0E0,
     2            TWO = 2,
     3            HALF = 0.5E0,
     4            UNITY = 1,
     5            NOUGHT = 0,

C**********************************************************************
C
C  NSTEP IS SUCH THAT 2**(-NSTEP)<=TOL.
C  I.E X IS IN [X-.5*TOL,X+.5*TOL].  ABOUT 3.3 STEPS PER DECIMAL PLACE
C  ARE NEEDED.  17 STEPS GIVE 5DP, 20 GIVE 6DP AND 23 7DP
C
C**********************************************************************


     6            NSTEP = 23,
     7            TOL = 1.0E-5 )

      P=ALPHA

C**********************************************************************
C
C DEAL WITH TRIVIAL CASES AND ARGUMENT CHECKING
C
C**********************************************************************

      IFAIL = NOUGHT

      IF ( P .LT. ZERO .OR.  P .GT. ONE ) THEN

         IFAIL = UNITY
         GO TO 1000

      ELSEIF ( M .LE. ZERO .OR. N .LE. ZERO ) THEN

         IFAIL = TWO
         GO TO 1000

      ELSEIF ( P .EQ. ZERO ) THEN

         BTAINV = ZERO
         GO TO 1000
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      ELSEIF ( P .EQ. ONE ) THEN

         BTAINV = ONE
         GO TO 1000

      ELSEIF ( M .EQ. ONE ) THEN

         BTAINV = ONE - (ONE - P)**(ONE/N)
         GO TO 1000

      ELSEIF ( N .EQ. ONE ) THEN

         BTAINV = P**(ONE/M)
         GO TO 1000

      ELSE
C*********************************************************************
C
C FIND THE INVERSE ONLY FOR THOSE P THAT ARE NOT ZERO OR ONE
C USE SIMPLE BISECTION.  IF B <= 0 THE VALUE IS RETURNED BY THE
C FIRST EVALUATION OF BETAIN AND THEN CARRIED FORWARD
C
C*********************************************************************

         X0 = ZERO
         X2 = ONE



         DO 10,I=1,NSTEP

            X1 = HALF*(X0 + X2)
            B1 = BETAIN(X1,M,N,B,IFAULT)

            IF ( P .GT. B1) THEN
               X0 = X1

            ELSE
               X2 = X1

            ENDIF

10       CONTINUE

         BTAINV = X1

      ENDIF

1000  CONTINUE

      RETURN
      END
```