Multiobjective control

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Multiobjective control: A survey

Bas Vroemen

WFW report 96.013

B.G. Vroemen
Faculty of Mechanical Engineering
Eindhoven University of Technology
February 1996

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B. G. Vroemen
Faculty of Mechanical Engineering
Eindhoven University of Technology

February 12, 1996

Trainee project
Supervisor: Bram de Jager

Contents

Nomenclature

Only those symbols are included that are used in more than one section. Symbols that are not included in this list are defined in the same section where they were encountered. Definitions were taken from [Francis 87, Zhou et al. 90, Sznaier 94, Dahleh and Diaz-Bobillo 95, Lancaster and Rodman 95].

ℓ_p	space containing all discrete-time signals that have a finite p-norm ($p = 1, 2, \infty$)
$\mathcal{L}_p$	space containing all continuous-time signals that have a finite p-norm ($p = 1, 2, \infty$)
$\mathcal{P}$	space containing all signals that have a bounded 'power-norm'
$\mathcal{S}$	space containing all signals that have a bounded 'spectrum-norm'
$\mathbb{Z}$	set of all integers
$\mathbb{R}$	the real numbers
$\mathbb{R}^+$	the positive real numbers
$\mathbb{C}$	the complex numbers
$\mathcal{H}_2$	the Hardy space of all complex-valued functions which are analytic in the open right half plane—Re(s) > 0— (or for discrete-time: analytic outside the unit disc) and satisfy $\|.\|_2 < \infty$
$\mathcal{H}_\infty$	the Hardy space of all complex-valued functions which are analytic in the open right half plane—Re(s) > 0— (or for discrete-time: analytic outside the unit disc) and satisfy $\|.\|_\infty < \infty$
$\mathcal{RH}_\infty$	space of real rational functions in $\mathcal{H}_\infty$
$\mathcal{RH}_{\infty,\delta}$	subspace of functions in $\mathcal{RH}_\infty$ which are analytic outside the disc of radius δ ($0 < \delta < 1$), equipped with the norm $\| G(z) \|_{\infty,\delta} = \sup_{0 \leq \theta \leq \pi} \overline{\sigma}(G(\delta e^{j\theta}))$
t	time (discrete or continuous)
τ	time-shift
s	Laplace transform variable
z	shift-operator or z-transform variable
n	dimension of x, the states
m	dimension of u, the control actions
l	dimension of y, the measurements
$q_{(i)}$	dimension of $z_{(i)}$, the regulated outputs ($i = 1, 2$)
$d_{(i)}$	dimension of $w_{(i)}$, the exogenous disturbances ($i = 1, 2$)
G	'plant' with state-space realization (A, B, C, D)
P	observability Gramian

S	controllability Gramian
$\overline{\sigma}$	maximum singular value
θ	phase-shift
$T_{w_{(i)}\to z_{(i)}}$	transfer function from $w_{(i)}$ to $z_{(i)}$ $(i=1,2)$
g_{ij}	impulse response element (i,j) of system with transfer function matrix G
K	controller matrix
p_1, p_2	the two norms of a two-norm optimization problem, where usually the p_1-norm of some transfer matrix is minimized and the p_2-norm of a second (possibly the same) transfer function is constrained
γ	p_2-norm bound, $\gamma \in \mathbb{R}^+$
Δ	norm-bounded uncertainty
Φ	transfer matrix $T_{w\to z}$
$\mathcal{A}$	linear operator from $\ell_{p_1}^{q\times d}$ to $\ell_{p_2}^{n_b\times m_b}$
$\mathcal{A}_{constr.}$	linear operator representing the admissible K's
Q	the free parameter which is used in the Youla-parameterization
K_{nom}	the nominal controller (: K for $Q=0$)
n_c	dimension of x_c, the states of the fixed-order controller
(A_c, B_c, C_c, D_c)	state-space realization of the fixed-order controller
$\tilde{n}$	dimension of $\tilde{x}$, the states of the closed-loop system
$\tilde{q}$	dimension of $\tilde{z}$, the regulated outputs of the closed-loop system
$(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$	state-space realization of the closed-loop system, where $\tilde{B} =: \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 \end{bmatrix}$, $\tilde{C} =: \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix}$ and $\tilde{D} =: \begin{bmatrix} \tilde{d}_{11} & \tilde{d}_{12} \\ \tilde{d}_{21} & \tilde{d}_{22} \end{bmatrix}$
J	$\mathcal{H}_2$ performance functional, defined in (4.10)
$\mathbb{E}$	expectation operator
R_1	$:= C_1^T C_1$
R_2	$:= D_{12}^T D_{12}$
R_3	$:= D_{13}^T D_{13}$
R_{12}	$:= C_1^T D_{12}$
R_{13}	$:= C_1^T D_{13}$
R_{23}	$:= D_{12}^T D_{13}$
$\tilde{S}$	solution to Lyapunov equation (4.12) of the closed-loop system
$\tilde{R}_1$	$:= \tilde{c}_1^T \tilde{c}_1$
$\tilde{V}$	$:= \tilde{B}\tilde{B}^T$
$\mathbf{S}$	solution to Lyapunov equation (4.14) of the auxiliary minimization problem
I_{q_2}, I_{d_2}	unitary matrices $\in \mathbb{R}^{q_2\times q_2}$, $\in \mathbb{R}^{d_2\times d_2}$, respectively
M_{q_2}	$:= I_{q_2} - \gamma^{-2} D_{22} D_{22}^T$
M_{d_2}	$:= I_{d_2} - \gamma^{-2} D_{22}^T D_{22}$
$\mathcal{J}$	auxiliary cost, defined in (4.20)
$\mathcal{F}_l$	*lower* linear fractional transformation
F, L	controller, estimator matrices
V_{ij}	matrices used in Youla-parameterization of $T_{w_1\to z_1} := V_{11} + V_{12} Q V_{21}$
T_{ij}	matrices used in Youla-parameterization of $T_{w_2\to z_2} := T_{11} + T_{12} Q T_{21}$
R_{zz}	autocorrelation matrix
R_{zu}	cross-correlation matrix

S_{zz}	spectral density
S_{zu}	cross-spectral density
$\mathbf{\Delta}$	$:= \{\mathrm{diag}[\Delta_1, \Delta_2, \ldots, \Delta_p], \Vert \Delta_i \Vert \leq 1\}$
D	scaling matrix
M^T	the transpose of M
M^*	the complex-conjugate transpose of M = the complex-conjugate of M^T
$M^\sim$	$:= M^*$ for $\mathrm{Re}(s) = 0$ ($\vert z\vert = 1$ for discrete-time); for $\mathrm{Re}(s) \neq 0$ ($\vert z\vert \neq 1$) it is defined as $M^\sim(s) := M^T(-s)$ ($M^\sim(z) := M^T(\frac{1}{z})$)
$M^\perp$	the orthogonal complement to a nonempty set $M \subseteq \mathbb{C}^n : M^\perp = \{x \in \mathbb{C}^n \vert \langle x, y\rangle = 0$ for all $y \in M\}$ where $\langle x, y\rangle$ is the inner product of x and y
$\mathrm{tr}(M)$	trace of M
stab.	:=stable
stabil.	:=stabilizing
AME	Affine Matrix Equation
AMI	Affine Matrix Inequality
ARE	Algebraic Riccati Equation
ARI	Algebraic Riccati Inequality
CARE	Continuous Algebraic Riccati Equation
DA	Delay Augmentation
DARE	Discrete Algebraic Riccati Equation
ESPR	Extended Strictly Positive Real
FME	Finitely Many Equations
FMV	Finitely Many Variables
LBR	Lossless Bounded Real
LME	Linear Matrix Equation
LMI	Linear Matrix Inequality
LP	Linear Programming
LQG	Linear Quadratic Gaussian
MI	Matrix Inequality
MIMO	Multi Input Multi Output
QMI	Quadratic Matrix Inequality
RMS	Root Mean Square
SISO	Single Input Single Output

Summary

In this report we survey a number of approaches to the multiobjective optimization problem. In practice, this usually boils down to a mixed-norm optimization problem, where traditionally the norms of interest are $\mathcal{H}_2$, $\mathcal{H}_\infty$ and ℓ_1. Specifications such as simultaneous rejection of disturbances having different characteristics (white noise, bounded energy, persistent); good tracking of classes of inputs; satisfaction of bounds on the peak values of some outputs; closed-loop bandwidth etc. cannot be cast into a single-norm form and therefore a mixed-norm formalism combining the $\mathcal{H}_2$, $\mathcal{H}_\infty$ and ℓ_1 norm can be expected to be of considerable interest. Although it would be nice to have all three norms present, most approaches focus on the two-norm problem. Most frequently encountered is the $\mathcal{H}_2/\mathcal{H}_\infty$ mixed norm optimization problem, but combinations of ℓ_1 and the other two norms are starting to get attention as well. Semi-norm formulations, using the 'power-norm' or the 'spectrum-norm' (see Section 2.1) as well as the Extended Strictly Positive Real (ESPR, see Chapter 3) stability criterion are sometimes used as an alternative.

The approaches investigated will be categorized into seven groups (Chapter 5), based on their problem setting (Chapter 3) as well as some techniques (Chapter 4) that are frequently used. An important feature in the problem statement is the number of sets of inputs and outputs respectively. If either one of these is less than the number of norms considered (which is usually two), the class of problems that can be handled is restricted considerably: most problems are not stated with the same sets of inputs (or dually outputs) for, say, the $\mathcal{H}_2$ performance measures as for (say) the $\mathcal{H}_\infty$ norm-bounded uncertainties. Some of the techniques that are utilized are the Youla-parameterization, the auxiliary cost functional ('of Bernstein and Haddad'), Lagrange multiplier techniques, convex optimization and Linear (sometimes called Affine) or alternatively Quadratic Matrix Inequalities (LMI, AMI, QMI). Other methods encountered are Linear Programming (LP), duality theory, Delay Augmentation (DA), homotopy and continuation methods, the entropy cost functional and a lossless bounded real formulation.

Finally, in Chapter 6 a comparison of these approaches will be made in an overview of pros and cons, although it will not be possible to decide which approach would qualify as most promising. This, of course, also depends on what application (number of sets of inputs/outputs etc.) one has in mind. The chapter concludes with mentioning some of the latest publications on this subject.

Chapter 1

Introduction

During the past years, much progress has been made on many single objective control problems. Several important controller synthesis problems have been formulated as optimization problems. In particular, the LQG or $\mathcal{H}_2$, $\mathcal{H}_\infty$ and ℓ_1[1] control theories have provided some basic synthesis tools. The underlying premise behind these theories is that all the design objectives can be translated into minimizing a suitably weighted norm of a closed-loop transfer function matrix.

The LQG approach proved particularly suited to meet performance constraints while guaranteeing closed-loop stability in the presence of disturbances. Despite of this, LQG control was shown to possess no guaranteed robustness margins if applied in conjunction with an observer or Kalman filter. This resulted in the development of $\mathcal{H}_\infty$ control theory which could deal with the problem of robust stability: obtaining closed-loop stability in the presence of system uncertainty. For systems with structured uncertainty the $\mathcal{H}_\infty$ framework can be refined to μ-analysis which has been successfully applied to a number of hard practical control problems (see e.g. [Skogestad et al. 88]).

However, despite its significance, $\mathcal{H}_\infty$ control—being a frequency domain method—cannot directly address time domain specifications. Recently, ℓ_1 optimal control problems have been addressed, where the signals involved are bounded in magnitude. This presents a method to accommodate the time domain specifications, although of course it cannot directly accommodate some common classes of frequency domain specifications (such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ bounds).

Clearly, a single norm is usually not enough to capture different, often conflicting, design specifications. In an attempt to cast the specifications into a single norm form, designers are forced to choose weighting functions, which remains essentially an art. It is therefore natural to expect a mixed-norm formalism to be of considerable interest. Although it would be nice to have all three norms present in a mixed-norm formalism, so far most efforts have been focussed

[1] ℓ_1 denotes the discrete-time case, whereas $\mathcal{L}_1$ is used for continuous-time. In the following, whenever ℓ_p is mentioned, the same can be assumed to be true for the continuous-time case, unless stated otherwise (where p can be 1,2 or ∞). This means that whenever $\mathcal{L}_p$ is mentioned, only the continuous-time case is referred to.

on solving the two-norm problem, mostly being the mixed $\mathcal{H}_2/\mathcal{H}_\infty$, or the $\ell_1/\mathcal{H}_\infty$ problem. One exception to this is formed by the approach followed by [Dahleh and Diaz-Bobillo 95] which makes it possible to minimize the ℓ_1 norm subject to $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ constraints (or even other combinations, such as $\mathcal{H}_2/\ell_1$). The $\mathcal{H}_2/\mathcal{H}_\infty$ problem received by far the greatest deal of attention, due to the simple fact that both the single-norm problems which it combines have been around much longer than the ℓ_1 problem. It is in this $\mathcal{H}_2/\mathcal{H}_\infty$ setting where the problem of designing fixed-order controllers is addressed, yielding a possibly non-convex and therefore complex problem.

The report is organized as follows: In Chapter 2 the notation to be used is introduced, whereas the possible problem statements are given in Chapter 3. Then, in Chapter 4, several methods are listed which are then combined to form the many approaches to solving the problem that are discussed in Chapter 5. Finally, Chapter 6 concludes with a brief comparison of the several approaches and some final remarks.

Throughout this report, the time t, the shift-operator z, and the Laplace transform variable s will often be omitted for clarity.

Chapter 2

Preliminaries

2.1 Signal spaces

- $\ell_p(\mathcal{L}_p)$ denotes the space that contains all the discrete-time (continuous-time) signals that have a finite p-norm, which for $x(t) \in \mathbb{R}^n$, is defined by [Dahleh and Diaz-Bobillo 95]:

$$\| x \|_p = \left(\sum_{t=-\infty}^{\infty} \sum_{i=1}^{n} |x_i(t)|^p \right)^{\frac{1}{p}} = \left(\sum_{t=-\infty}^{\infty} |x(t)|_p^p \right)^{\frac{1}{p}} \tag{2.1}$$

for discrete-time and

$$\| x \|_p = \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} |x_i(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} = \left(\int_{-\infty}^{\infty} |x(t)|_p^p \mathrm{d}t \right)^{\frac{1}{p}} \tag{2.2}$$

for continuous-time. Here we adopted the notation $|.|_p$ to denote the p-norm on $\mathbb{R}^n$ (where $|.|$ denotes the (usual) absolute value). In case $p = 1, 2$ the appropriate expressions can be easily derived from this. For $p = \infty$ this is not so obvious; then the norm is defined as

$$\| x \|_\infty = \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} |x_i(t)| \tag{2.3}$$

and

$$\| x \|_\infty = \sup_{t \in \mathbb{Z}} \max_{1 \leq i \leq n} |x_i(t)| \tag{2.4}$$

for continuous-time and discrete-time respectively. This norm is sometimes referred to as the *peak* of a signal [Boyd and Barratt 91].

This means e.g. that an ℓ_2-signal contains a finite amount of energy and an ℓ_∞-signal attains a finite maximum magnitude; the ℓ_1 space doesn't have such an obvious physical interpretation. The 2-norm is sometimes referred to as the *energy-norm*, whereas the energy of a signal actually is defined to be the square of its 2-norm. The 1-norm is sometimes called the *action* of a signal. The following two signal spaces are defined here for continuous-time only.

- Bounded power signals ($\mathcal{P}$): signals that have a bounded 'power-norm' $\|.\|_{\mathcal{P}}$, defined by [Zhou et al. 90]:

$$\| x \|_{\mathcal{P}} = \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|_2^2 \mathrm{d}t \right)^{1/2} = (\mathrm{tr}\,[R_{xx}(0)])^{1/2} \tag{2.5}$$

where

$$R_{xx}(\tau) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau)x^*(t)\mathrm{d}t \tag{2.6}$$

is the autocorrelation matrix. Here $x^*(t)$ denotes the complex conjugate transpose of $x(t)$. If $x \in \mathcal{P}$ it can be shown that $\| x \|_{\mathcal{P}} \leq \sqrt{n} \| x \|_{\infty}$, where n is the dimension of x. This 'power-norm' is the square root of the average power (also denoted *average-absolute value*; $\| x \|_{aa}$ [Boyd and Barratt 91]) of x. Other terms used for this (semi-)norm are *Root-Mean-Square (RMS)-norm* [Boyd and Barratt 91] and *power semi-norm* [Zhou et al. 96]. In [Doyle et al. 92] it is denoted by *pow(x)*.

- Bounded spectrum signals ($\mathcal{S}$): signals that are in $\mathcal{P}$ and have a bounded 'spectrum-norm' $\|.\|_{\mathcal{S}}$, defined by

$$\| x \|_{\mathcal{S}} := \| S_{xx}(j\omega) \|_{\infty}^{1/2} \tag{2.7}$$

where

$$S_{xx}(j\omega) := \int_{-\infty}^{\infty} R_{xx}(\tau)\mathrm{e}^{-j\omega\tau}\mathrm{d}\tau \tag{2.8}$$

is the Fourier transform of R_{xx} and is called the spectral density of x. Another term for this norm is *spectral density norm* [Zhou et al. 96].

Although, strictly speaking, white noise is not in $\mathcal{S}$ it can be thought of as the limit of a sequence of signals in $\mathcal{S}$ whose spectra in the limit approaches a constant matrix. In the following $\mathcal{S}$ is therefore assumed (as was done in [Zhou et al. 90]) to include white noise, where *the term white noise will be used to describe the case where* $S_{xx} = I$. Both the bounded power and the bounded spectrum norm are semi-norms (since they can be zero for a nonzero (ℓ_2) signal) and were especially used by K. Zhou *et al.* in [Zhou et al. 90]. With the definition of S_{xx} we can also write

$$\| x \|_{\mathcal{P}}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}[S_{xx}(j\omega)]\mathrm{d}\omega \tag{2.9}$$

In case of the continuous-time signal spaces, we have the set inclusions, depicted in Figure A (see [Doyle et al. 92]), whereas for the discrete-time case the $\ell_p(\mathbb{Z})$-spaces are nested with $\ell_\infty(\mathbb{Z})$ as the largest, depicted in Figure B (see [Dahleh and Diaz-Bobillo 95]):

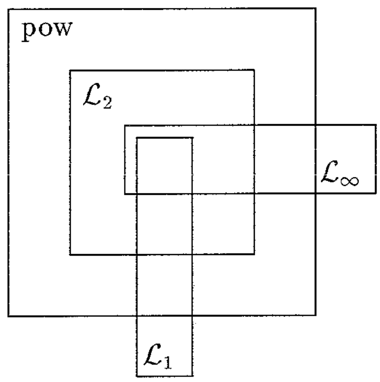


Figure A.

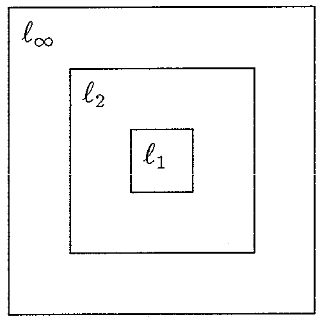


Figure B.

2.2 System norms

Before we define system norms, we first define some (standard) properties:

A system $G(s)$ is called *proper* if

> $G(\infty)$ is finite or, equivalently,
>
> degree numerator $\leq$ degree denominator.

A system $G(s)$ is called *strictly proper* if

> $G(\infty) = 0$ or, equivalently,
>
> degree numerator $<$ degree denominator.

A system $G(s)$ is called *biproper* if

> G and G^{-1} are both proper or, equivalently,
>
> degree numerator $=$ degree denominator.

A system $G(s)$ is called *non-proper* (or *improper*) if

> it is not restricted to be proper and thus $G(\infty)$ may be infinite or, equivalently,

degree numerator > degree denominator may be true.

A system is called *causal* (or *non-anticipative*) if the output at a certain time instant only depends on the input up to that time instant, including the time instant itself.

If the time instant is excluded, the system is called *strictly causal.*

A system is called *non-causal* (or *acausal*) if it not restricted to be causal and thus the output at a certain time instant may depend on the input after that time instant.

Given a stable strictly proper (in order to keep the norms finite, see e.g. [Doyle et al. 92, p. 16]) transfer function matrix $G(s)$ with state space realization (A, B, C, D), the following performance measures can be defined.

- The $\mathcal{H}_2$ norm of a transfer function $G(s)$ is defined as:

$$\| G(j\omega) \|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}[G^T(-j\omega)G(j\omega)]\mathrm{d}\omega \right)^{1/2} \tag{2.10}$$

for the continuous-time case and

$$\| G(z) \|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}[G(\mathrm{e}^{j\theta})G^T(\mathrm{e}^{-j\theta})]\mathrm{d}\theta \right)^{1/2} \tag{2.11}$$

for the discrete-time case.

The 2-norm can be computed with Lyapunov equations:

$$\| G \|_2 = \mathrm{tr}[SC^TC] = \mathrm{tr}[PBB^T] \qquad (= \mathrm{tr}[CSC^T] = \mathrm{tr}[B^TPB]) \tag{2.12}$$

where S is the controllability Gramian and P is the observability Gramian solving

$$AS + SA^T + BB^T = 0 \qquad A^TP + PA + C^TC = 0 \tag{2.13}$$

- The $\mathcal{H}_\infty$ norm of a transfer function $G(s)$ is defined as:

$$\| G(j\omega) \|_\infty = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(G(j\omega)) \tag{2.14}$$

(where $\overline{\sigma}$ is the maximum singular value) for the continuous-time case and

$$\| G(z) \|_\infty = \sup_{0 \leq \theta \leq \pi} \overline{\sigma}(G(\mathrm{e}^{j\theta})) \tag{2.15}$$

for the discrete-time case.

- The ℓ_1 norm of a transfer function $G(s)$ is not as easy to define as the other two: Recall the 1-norm of a sequence $x(t)$ being $\| x \|_1 = \sum_{t=-\infty}^{\infty} |x(t)|$ (from (2.2) with $n = 1$). Then, given a matrix g with elements g_{ij}, representing a linear operator defined by the usual discrete-time convolution $y = g * u$ (and with a corresponding transfer function matrix G), its 1-norm is defined as:

$$\| G \|_1 = \max_{1 \le i \le m} \sum_{j=0}^{n} \| g_{ij} \|_1 \tag{2.16}$$

for the discrete-time case. The definition for the continuous-time case requires some more notational aspects and can be found in e.g. [Sznaier and Blanchini 94]. Since the interpretation of the 1-norm (see section 2.4) is of much more use to us than the formal definition this will not be repeated here.

2.3 The induced norm

- The induced norm of an operator T is given by:

$$\| T \|_p = \sup_{x \neq 0} \frac{\| Tx \|_b}{\| x \|_a} = \sup_{\| x \|_a \le 1} \| Tx \|_b \tag{2.17}$$

We say: the p-norm is the induced norm from ℓ_a to ℓ_b or the ℓ_a/ℓ_b gain.

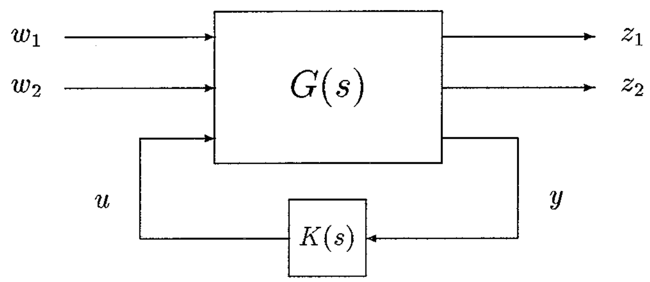
2.4 Norm interpretations

- The $\mathcal{H}_2$ norm:
  1. The induced norm from ℓ_2 to ℓ_∞.
  2. The square root of the average power (=RMS-value or 'power-norm') of the response to a white input signal of unit spectral density or the spectrum/power gain.
  3. The square root of the energy contained in the impulse response.
- The $\mathcal{H}_\infty$ norm:
  1. The induced norm from ℓ_2 to ℓ_2.
  2. The power/power gain.
  3. The spectrum/spectrum gain.
  4. An upper bound on the ℓ_∞/power gain, assuming that the input is restricted to be a persistent sinusoidal signal.
  5. The peak gain of the Bode singular value plot.
- The ℓ_1 norm:
  1. The induced norm from ℓ_∞ to ℓ_∞.

Chapter 3

Statement of the problem

The general problem can be posed as follows. Suppose the plant is given by its transfer function matrix $G(s)$ with three sets of inputs and outputs:



with

$$\dot{x} = Ax + B_1 w_1 + B_2 w_2 + B_3 u \tag{3.1}$$

$$z_1 = C_1 x + D_{11} w_1 + D_{12} w_2 + D_{13} u \tag{3.2}$$

$$z_2 = C_2 x + D_{21} w_1 + D_{22} w_2 + D_{23} u \tag{3.3}$$

$$y = C_3 x + D_{31} w_1 + D_{32} w_2 + D_{33} u \tag{3.4}$$

or equivalently, using packed notation

$$G = \left[\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right] \tag{3.5}$$

Here '=' means that both representations (G and the packed notation) describe the same system, but of course they are not identical. In the following, it is assumed that, whenever two different representations are said to be 'equal', the reader is aware of this.

Furthermore

$$\begin{array}{llll} n = \dim(x) & q_1 = \dim(z_1) & d_1 = \dim(w_1) & l = \dim(y) \\ m = \dim(u) & q_2 = \dim(z_2) & d_2 = \dim(w_2) & \end{array} \tag{3.6}$$

In the system equations (3.1)–(3.4) u represent the control actions, w ($= [w_1, w_2]$) the exogenous disturbances, y the measurements and z ($= [z_1, z_2]$) the regulated outputs. The signal sets $[w_1, z_1]$ are related to performance criteria (measured by what we will call the p_1-norm), whereas $[w_2, z_2]$ are related to ('p_2-') norm constraints. These two norms will usually be either $\mathcal{H}_2$ and $\mathcal{H}_\infty$, ℓ_1 and $\mathcal{H}_\infty$ or $\mathcal{H}_2$ and ℓ_1 (seldom used).

In control problems involving $\mathcal{H}_2$ minimization, D_{11} is always taken to be zero to prevent the $\mathcal{H}_2$ norm from growing infinitely.

For the system as defined above the mixed two-norm problem, as encountered in the literature, can be written as (when $T_{w_i \to z_i}$ denotes the transfer function from w_i to z_i (i=1,2) and $\gamma \in \mathbb{R}$ is the positive p_2-norm bound):

1. *Find an internally stabilizing controller which minimizes* $\| T_{w_1 \to z_1} \|_{p_1}$ *while maintaining* $\| T_{w_2 \to z_2} \|_{p_2} \leq \gamma$

   where p_1 (=2 or 1) can denote either the $\mathcal{H}_2$ or ℓ_1 norm and p_2 (=∞ or 2) denotes the $\mathcal{H}_\infty$ or the $\mathcal{H}_2$ norm.

   Another formulation, used by [Steinbuch and Bosgra 94, Stoorvogel 93], is the following:

2. *Minimize the* p_1*-norm of the transfer function from* w_1 *to* z_1 *using the internally stabilizing controller* $K(s)$*, while maximizing the* p_1*-norm of that same transfer function over the allowable uncertainties:*

$$\sup_{\|\Delta\|_{p_2} \leq 1/\gamma} \min_{K(s)} \| T_{w_1 \to z_1}(K, \Delta) \|_{p_1}$$

   where, in case of the problem addressed by [Steinbuch and Bosgra 94, Stoorvogel 93] $p_1 = 2$ and $p_2 = \infty$.

Finally, [Elia et al. 93, Dahleh and Diaz-Bobillo 95, Voulgaris 94] use the following formulation, where $\Phi = T_{w \to z}$[1]:

3. *Find an internally stabilizing controller $K(s)$ which minimizes $\| \Phi \|_{p_1}$ and satisfies a set of linear constraints given by $\mathcal{A}$ and b:*

$$\inf_{K(s) stabil.} \| \Phi \|_{p_1} \qquad \textit{such that} \qquad \mathcal{A}\Phi \leq b$$

with $\mathcal{A}$ a linear operator from $\ell_{p_1}^{q \times d}$ to $\ell_{p_2}^{n_b \times m_b}$, and $b \in \ell_{p_2}^{n_b \times m_b}$ a fixed element (possibly containing the 'γ-bound').

In this last formulation the constraints consist of performance constraints and feasibility constraints, the latter representing the conditions for Φ so it can be written as $G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$ (i.e. Φ is *feasible*) where G is partitioned according to

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

This approach can also handle the three-norm problem. The description of this problem can be found in Section 5.1 and will not be treated here any further. In the p_2-norm constraint some approaches instead of using $\| T_{w_2 \to z_2} \|_{p_2} \leq \gamma$ use the strict inequality, but this doesn't influence the rest of the approach essentially.

As mentioned before, most approaches focus on solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem, while the other two problems ($\ell_1/\mathcal{H}_\infty$ and $\mathcal{H}_2/\ell_1$) so far have received little attention. The ($\mathcal{H}_2/\ell_1$) problem actually is a special case of the approach followed by [Dahleh and Diaz-Bobillo 95, Voulgaris 94, Elia et al. 93] which provides a method (originating from ℓ_1 optimal control theory) that either minimizes or constrains the ℓ_1 norm combined with $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ norm minimization or constraints. Apart from this, only Sznaier [Sznaier 94, Sznaier and Blanchini 94, Sznaier 93] addresses the mixed $\ell_1/\mathcal{H}_\infty$ problem, both for the discrete-time and the continuous-time case. The widest variety can be found in the approaches to the $\mathcal{H}_2/\mathcal{H}_\infty$ problem, eventually to be divided into 5 categories. One other distinction can be made based on the number of sets of in- and outputs used in the statement of the problem. The distinction discrete-time versus continuous-time, however non-trivial it might be, will not be made explicitly since it doesn't essentially alter the approach used.

Finally, as a counterpart of the $\mathcal{H}_\infty$ norm constraint can be mentioned the Extended Strictly Positive Real (ESPR) stability criterion (see e.g. [Shim 94]). Positive realness is an old, but very important concept in system and control theory and is used in various areas, like network analysis, adaptive control, nonlinear control and robust control. It is well-known that positive realness is closely related to absolute stability. This criterion will however not be treated here.

[1] Since in their formulation $w_1 = w_2 =: w$ and $z_1 = z_2 =: z$, although different linear constraints can be defined for different closed-loop maps $\Phi = T_{w_j \to z_i}$, i.e. on the map between the j^{th} input set and the i^{th} output set.

Another, totally different, approach to the mixed norm problem, is based on the so-called 'behavioral setting'. This methodology can be characterized by the fact that all variables are considered a priori on an equal footing, without a distinction between inputs and outputs, and the behavior is defined as a subset of the possible time trajectories. Because of the fact that this setting, which is so unlike the others, is hardly ever encountered (but is becoming popular), it will not be treated here, but can be found in e.g. [Paganini et al. 94] and references therein.

Chapter 4

Solution of the problem

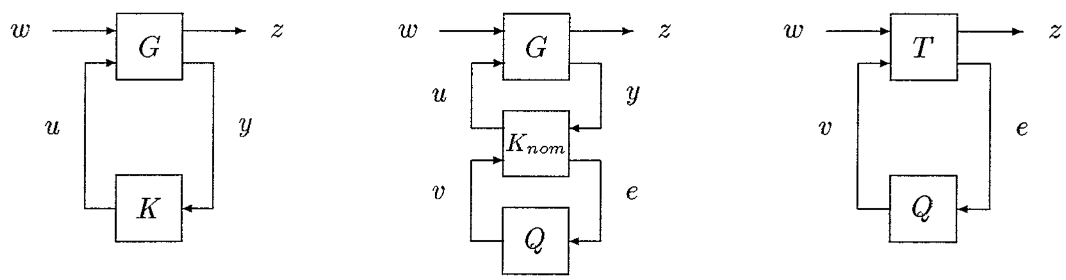
While in the statement of the problem we could write down a generalized formulation, in the problem solution the various approaches followed differ too much to cast them into one setting. However, different approaches sometimes appear to be more or less related and often make use of the same methodologies. This enables us to (partly) describe these approaches as combinations of a number of the following methods (which will be done in the next chapter):

4.1 The Youla- or Q-parameterization

The set of all stabilizing controllers can be parameterized in terms of a free stable parameter Q as

$$K = \mathcal{F}_l(K_{nom}, Q)$$

where K_{nom} is depicted in the following picture:



The design objective is to minimize the transfer function from w to z:

$$\min_{K\ stabil.} \| G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \| \Leftrightarrow \min_{Q\ stab.} \| T_{11} + T_{12}QT_{21} \|$$

So now the optimization problem is parameterized in terms of Q as well: this is the Q- or Youla-parameterization.

This parameterization can be used to cast the problem into a convex optimization problem (see Section 4.5), although this might be an infinite-dimensional problem which, in order to obtain a tractable problem requires several approximations (see [Dahleh and Diaz-Bobillo 95, pp. 43,44]).

4.2 Fixed-order controllers versus full-order controllers

The problem of synthesizing full-order controllers is a well-studied problem. However, these approaches cannot handle a constraint such as fixed or reduced controller order. To describe this problem we have to consider ${n_c}^{\text{th}}$-order dynamic compensators

$$\dot{x}_c = A_c x_c + B_c y \tag{4.1}$$

$$u = C_c x_c + D_c y \tag{4.2}$$

With this the closed-loop system (3.1)–(3.4)+(4.1)–(4.2) can be written as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{w} \tag{4.3}$$

$$\tilde{z} = \tilde{C}\tilde{x} + \tilde{D}\tilde{w} \tag{4.4}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \tilde{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \tilde{n} = n + n_c\,, \tag{4.5}$$

$$\tilde{A} = \begin{bmatrix} A + B_3(I - D_c D_{33})^{-1} D_c C_3 & B_3(I - D_c D_{33})^{-1} C_c \\ B_c(I - D_{33} D_c)^{-1} C_3 & A_c + B_c(I - D_{33} D_c)^{-1} D_{33} C_c \end{bmatrix}, \tag{4.6}$$

$$\tilde{B} = \begin{bmatrix} B_1 + B_3(I - D_c D_{33})^{-1} D_c D_{31} & B_2 + B_3(I - D_c D_{33})^{-1} D_c D_{32} \\ B_c(I - D_{33} D_c)^{-1} D_{31} & B_c(I - D_{33} D_c)^{-1} D_{32} \end{bmatrix} := \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 \end{bmatrix}, \tag{4.7}$$

$$\tilde{C} = \begin{bmatrix} C_1 + D_{13}(I - D_cD_{33})^{-1}D_cC_3 & D_{13}(I - D_cD_{33})^{-1}C_c \\ C_2 + D_{23}(I - D_cD_{33})^{-1}D_cC_3 & D_{23}(I - D_cD_{33})^{-1}C_c \end{bmatrix} := \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} \quad \text{and} \quad (4.8)$$

$$\begin{aligned} \tilde{D} &= \begin{bmatrix} D_{11} + D_{13}(I - D_cD_{33})^{-1}D_cD_{31} & D_{12} + D_{13}(I - D_cD_{33})^{-1}D_cD_{32} \\ D_{21} + D_{23}(I - D_cD_{33})^{-1}D_cD_{31} & D_{22} + D_{23}(I - D_cD_{33})^{-1}D_cD_{32} \end{bmatrix} \\ &:= \begin{bmatrix} \tilde{d}_{11} & \tilde{d}_{12} \\ \tilde{d}_{21} & \tilde{d}_{22} \end{bmatrix} \end{aligned} \quad (4.9)$$

4.3 The auxiliary cost

Since the auxiliary cost or performance measure of Bernstein and Haddad is used only in the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem, this section will be specialized to this particular problem.

With the closed-loop system given by (4.3)–(4.4) the LQG controller synthesis problem with an $\mathcal{H}_\infty$ constraint can be stated as follows:

Find an $n_c{}^{\text{th}}$ order dynamic compensator described by (4.1)–(4.2) which satisfies the following criteria

1. the closed-loop system (4.3)–(4.4) is asymptotically stable, i.e. $\tilde{A}$ is asymptotically stable;
2. the closed-loop transfer function $T_{w_2 \to z_2} := \tilde{c}_2(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{b}_2 + \tilde{d}_{22}$
   satisfies the constraint $\| T_{w_2 \to z_2} \|_\infty \le \gamma$
   where $\gamma > 0$ is a given constant, and
3. the performance functional

$$\begin{aligned} J(A_c, B_c, C_c, D_c) \;:=\; & \lim_{t\to\infty} \mathbb{E}\left\{ \int_0^t [x^T R_1 x + 2x^T R_{13} u + u^T R_3 u \right. \\ & + w_2^T R_2 w_2 + 2w_2^T R_{12} x + 2w_2^T R_{23} u]\, \mathrm{d}t \} \end{aligned} \quad (4.10)$$

   is minimized, where $\mathbb{E}$ is the expected value, $R_1 = C_1^T C_1 \in \mathbb{R}^{n\times n}$, $R_3 = D_{13}^T D_{13} \in \mathbb{R}^{m\times m}$, $R_{13} = C_1^T D_{13} \in \mathbb{R}^{n\times m}$, $R_2 = D_{12}^T D_{12} \in \mathbb{R}^{d_2\times d_2}$, $R_{12} = C_1^T D_{12} \in \mathbb{R}^{d_2\times n}$ and $R_{23} = D_{12}^T D_{13} \in \mathbb{R}^{d_2\times m}$.

Then, for a given compensator the performance (4.10) is given by

$$J(A_c, B_c, C_c, D_c) = \mathrm{tr}[\tilde{S}\tilde{R}_1] \quad (4.11)$$

where $\tilde{R}_1 = \tilde{c}_1^T \tilde{c}_1$ and $\tilde{S}$ satisfies the Lyapunov equation

$$\tilde{A}\tilde{S} + \tilde{S}\tilde{A}^T + \tilde{V} = 0 \tag{4.12}$$

with $\tilde{V} = \tilde{B}\tilde{B}^T$. Note that (4.11) and (4.12) are similar to (2.12) and (2.13).

LEMMA 4.3.1: Let (A_c, B_c, C_c, D_c) be given and assume there exists an $\mathbf{S} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ satisfying

$$\mathbf{S} \text{ is positive-semidefinite } (\mathbf{S} \geq 0), \tag{4.13}$$

and

$$\tilde{A}\mathbf{S} + \mathbf{S}\tilde{A}^T + \gamma^{-2}(\tilde{B}D_{22}^T + \mathbf{S}\tilde{c}_2^T)M_{q_2}^{-1}(\tilde{B}D_{22}^T + \mathbf{S}\tilde{c}_2^T)^T + \tilde{V} = 0 \tag{4.14}$$

where $M_{q_2} := I_{q_2} - \gamma^{-2}D_{22}D_{22}^T$ is positive-definite. Then

$$(\tilde{A}, \tilde{B}) \text{ is stabilizable} \tag{4.15}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \tag{4.16}$$

In this case

$$\|T_{w_2\rightarrow z_2}\|_\infty \leq \gamma \quad , \tag{4.17}$$

$$\tilde{S} \leq \mathbf{S} \quad (\mathbf{S}-\tilde{S} \text{ is nonnegative-definite}). \tag{4.18}$$

Consequently

$$J(A_c, B_c, C_c, D_c) \leq \mathcal{J}(A_c, B_c, C_c, D_c, \mathbf{S}) \tag{4.19}$$

where

$$\mathcal{J}(A_c, B_c, C_c, D_c, \mathbf{S}) := \text{tr}[\mathbf{S}\tilde{R}_1] \tag{4.20}$$

Hence, the satisfaction of (4.13) and (4.14) along with the generic condition (4.15) leads to:

1. closed-loop stability
2. pre-specified $\mathcal{H}_\infty$ attenuation
3. an upper bound for the $\mathcal{H}_2$ performance criterion which is known as the *auxiliary cost* or *performance measure* (or *index*) *of Bernstein and Haddad.*

This leads to the following optimization problem:

- **Auxiliary minimization problem:**

  Determine $(A_c, B_c, C_c, D_c, \mathbf{S})$ which minimizes the auxiliary cost $\mathcal{J}(A_c, B_c, C_c, D_c, \mathbf{S})$ subject to (4.14) with $\mathbf{S} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ nonnegative-definite.

REMARK 4.3.1: The notation used in this section and the previous one can be easily converted to the notation in [Haddad and Bernstein 90], keeping in mind that in [Haddad and Bernstein 90] $w_1 = w_2 =: w^1$ (and $z_1 =: z$ and $z_2 =: z_\infty$) and therefore, if we leave out w_1, $B_1 = 0$, $D_{21} = 0$ and $D_{31} = 0$. Furthermore, D_{12} and D_c (and D_{11}) are taken to be zero. If in addition to this we set $B_2 D_{32}^T = 0$, $R_{13} = 0$, $D_{33} = 0$, $D_{22} = 0$ and $C_2^T M_{q_2}^{-1} D_{23} = 0$ the results from [Bernstein and Haddad 89] can be obtained. Be aware that some notations used here are similar to the ones there, but may have a totally different definition. Although for the problem stated above no solution is given, it was formulated this way to allow for all existing approaches using the performance measure of Bernstein and Haddad to derive the appropriate expressions.

4.4 Lagrange multipliers

One way of solving the auxiliary minimization problem posed in Section 4.3 is by using Lagrange multipliers as was done in [Haddad and Bernstein 90, Bernstein and Haddad 89]. Likewise, we will take D_c to be zero from now on. Derivation of the necessary conditions requires technical assumptions. Specifically, we restrict $(A_c, B_c, C_c, \mathbf{S})$ to the open set

$$\begin{aligned}\mathcal{X} \;:=\; & \{(A_c, B_c, C_c, \mathbf{S}) : \mathbf{S} \text{ is positive-definite}, \\ & \tilde{A} + \gamma^{-2}\tilde{B}D_{22}^T M_{q_2}^{-1}\tilde{c}_2 + \gamma^{-2}\mathbf{S}\tilde{c}_2^T M_{q_2}^{-1}\tilde{c}_2 \text{ is asymptotically stable}, \\ & \text{and } (A_c, B_c, C_c) \text{ is controllable and observable}\} \end{aligned} \tag{4.21}$$

Then, to optimize $\mathcal{J}(A_c, B_c, C_c, \mathbf{S})$ over the open set $\mathcal{X}$ subject to the constraint that positive-definite $\mathbf{S}$ satisfies (4.14), the following Lagrangian is formed:

$$\begin{aligned}\mathcal{L}(A_c, B_c, C_c, \mathbf{S}, \mathcal{M}) \;:=\; & \mathrm{tr}\{\mathbf{S}\tilde{R}_1 + [\tilde{A}\mathbf{S} + \mathbf{S}\tilde{A}^T \\ & +\gamma^{-2}(\tilde{B}D_{22}^T + \mathbf{S}\tilde{c}_2^T)M_{q_2}^{-1}(\tilde{B}D_{22}^T + \mathbf{S}\tilde{c}_2^T)^T + \tilde{V}]\mathcal{M}\}\end{aligned} \tag{4.22}$$

where $\mathcal{M} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ is a Lagrange multiplier.

Setting $\frac{\partial \mathcal{L}}{\partial \mathbf{S}} = 0$ yields

$$0 = \left(\tilde{A} + \gamma^{-2}[\mathbf{S}\tilde{c}_2^T M_{q_2}^{-1}\tilde{c}_2 + \tilde{B}D_{22}^T M_{q_2}^{-1}\tilde{c}_2]\right)^T \mathcal{M} + \mathcal{M}\left(\tilde{A} + \gamma^{-2}[\mathbf{S}\tilde{c}_2^T M_{q_2}^{-1}\tilde{c}_2 + \tilde{B}D_{22}^T M_{q_2}^{-1}\tilde{c}_2]\right) + \tilde{R}_1 \tag{4.23}$$

[1] This w has a dual interpretation being standard white noise as well as an $\mathcal{L}_2$ signal.

The partial derivatives $\frac{\partial \mathcal{L}}{\partial A_c}$, $\frac{\partial \mathcal{L}}{\partial B_c}$ and $\frac{\partial \mathcal{L}}{\partial C_c}$ are then set to zero yielding three matrix equations that one way or another can lead to the final results (see [Bernstein and Haddad 89, Haddad and Bernstein 90, Ge et al. 94]).

There are more approaches utilizing Lagrange multiplier techniques, not necessarily in this setting. The concept of forming the Lagrangian and then setting its partial derivatives to zero, however, is used frequently and this section should therefore serve as an example.

4.5 Convex optimization

DEFINITION 4.5.1: A set $\mathcal{C}$ is convex if for every x_1 and x_2 in $\mathcal{C}$, $\alpha x_1 + (1-\alpha)x_2$ is also in $\mathcal{C}$ for all $0 < \alpha < 1$.

- **Convex optimization:**

  An optimization of the form

$$\inf_K \| \Phi(K) \| \qquad \text{subject to} \qquad K \in \mathcal{A}_{constr}$$

  is called convex if the set $\mathcal{A}_{constr}$ representing the admissible K's is a convex one. If this set is characterized by linear constraints $\mathcal{A}\Phi \leq b$, it is always convex.

The solution to many convex optimization problems can be computed in a time which is comparable to the time required to evaluate a 'closed-form' solution for a similar problem. Nowadays, a control engineering problem that reduces to solving two Algebraic Riccati Equations (ARE's) is generally regarded as 'solved'. When a control engineering problem reduces to solving even a large number of convex Algebraic Riccati Inequalities (ARI's) the growing belief is this should also be regarded as 'solved', even though there is no 'analytic' solution (see [Boyd et al. 93]). Hence a large number of approaches focuses on making the optimization a convex one, mostly by using some suitable parameterization.

There are effective and powerful algorithms for the solution of these problems, that is, algorithms that compute the global optimum, with non-heuristic[2] stopping criteria. A number of general algorithms exist, for example the ellipsoid algorithm (see e.g. [Boyd and Barratt 91, Bland et al. 81]) and the more recently developed extremely efficient interior point methods for solving LMI (Linear Matrix Inequality)-based problems, based on the work of Nesterov and Nemirovsky [Nesterov and Nemirovsky 93].

4.6 Matrix Inequalities versus Algebraic Riccati Equations

The Algebraic Riccati Equations we consider have the general form:

$$XDX + XA + BX + C = 0 \tag{4.24}$$

[2]Not using informal methods or reasoning from experience in case no precise algorithm was known.

where the coefficients A, B, C, D are real or complex $n \times n$ matrices and $n \times n$ matrix solutions X are to be found.

In control theory, they take a symmetric form:

$$XDX + XA + A^*X + C = 0 \tag{4.25}$$

where C and D are hermitian matrices ($C^* = C$, $D^* = D$).

For discrete systems, equation (4.25) takes the form:

$$X = A^*XA + E_1 - A^*XB(B^*XB + E_2)^{-1}B^*XA \tag{4.26}$$

Here A and E_1 have the size of X, say $n \times n$, but E_2 may have size $m \times m$, say, in which case B is $n \times m$. Equation (4.25) is described as a 'continuous algebraic Riccati equation', or CARE, and equation (4.26) is known as a 'discrete algebraic Riccati equation', or DARE.

When we make special choices for the matrices A, B, C and D we can obtain the Sylvester and Stein equations:

$$XA - BX = C \tag{4.27}$$

and

$$X - BXA = C \tag{4.28}$$

respectively. Their symmetric forms (when $B = A^*$, $C^* = C$) are most important. Another special form of the Sylvester equation is the Lyapunov equation:

$$XA + A^*X = C \tag{4.29}$$

where C is hermitian.

The Sylvester, Stein and Lyapunov equations are Linear Matrix Equations (LME's). There appears to be some confusion over the terms Affine Matrix Inequality (AMI) and Linear Matrix Inequality (LMI) (or, alternatively, Equations: AME and LME). Inequalities of the form

$$F(x) = F_0 + \sum_{i=0}^{n} x_i F_i > 0 \quad , x \in \mathbb{R}^n,\ F_i = F_i^T$$

that are actually affine in x (and, consequently, are sometimes called AMI), are generally referred to as LMI's. Multiple LMI's $F_1(x) > 0, \ldots, F_r(x) > 0$ can be expressed as the single LMI

$$\begin{bmatrix} F_1(x) & & 0 \\ & \ddots & \\ 0 & & F_r(x) \end{bmatrix} > 0$$

Nonlinear (convex) inequalities are converted to LMI form using Schur complements (see e.g. [Wortelboer 94, p. 23]). The basic idea is as follows: the LMI

$$\begin{bmatrix} \mathcal{U}(x) & \mathcal{V}(x) \\ \mathcal{V}^T(x) & \mathcal{W}(x) \end{bmatrix} > 0 \tag{4.30}$$

where $\mathcal{U}(x) = \mathcal{U}^T(x)$, $\mathcal{W}(x) = \mathcal{W}^T(x)$, and $\mathcal{V}(x)$ depends affinely on x is equivalent to

$$\mathcal{W}(x) > 0 \ , \ \ \mathcal{U}(x) - \mathcal{V}(x)\mathcal{W}^{-1}(x)\mathcal{V}^T(x) > 0 \tag{4.31}$$

In other words, the set of nonlinear inequalities (4.31) can be represented as the LMI (4.30). More practical use of LMI's can be found in [Boyd et al. 93] and references therein. Finally when the equality in the ARE becomes an inequality we have a Quadratic Matrix Inequality (QMI). Most of what was treated in this section was taken from [Lancaster and Rodman 95].

Chapter 5

Survey of approaches

As was mentioned in the previous chapter, we will now describe a number of approaches to the solution of the mixed-norm optimization problem, frequently using methods that were mentioned in that chapter. This survey can of course not be exhaustive, but an attempt was made to (briefly) describe the approaches most frequently encountered in the literature.

The following classification was used:

$\ell_1/\frac{\mathcal{H}_2}{\mathcal{H}_\infty}$	$\ell_1/\mathcal{H}_\infty$	$\mathcal{H}_2/\mathcal{H}_\infty$				
		MI's	ARE's			
			$w_1 = w_2$ $z_1 = z_2$	$w_1 = w_2$ $z_1 \neq z_2$	$w_1 \neq w_2$ $z_1 = z_2$	$w_1 \neq w_2$ $z_1 \neq z_2$
5.1	5.2	5.3	5.4	5.5	5.6	5.7

It must be stressed that this classification is fairly arbitrary and other classifications can be equally sufficient. In fact, there may be some approaches that don't actually fit in any one of these classes. However, for the approaches most regularly encountered, this classification should suffice.

5.1 $\ell_1/\frac{\mathcal{H}_2}{\mathcal{H}_\infty}$: a linear programming approach

This approach uses the problem statement (3) from Chapter 3, where most commonly $w_1 = w_2 =: w$ and $z_1 = z_2 =: z$, although different linear constraints can be defined for different closed-loop maps $T_{w_i \to z_j}$, i.e. on the map between the i^{th} input set and the j^{th} output set. Using that formulation, either p_1 or p_2 is taken to be 1 and the remaining p=1, 2 or ∞. Most common is the ℓ_1 minimization combined with $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ constraints

[Dahleh and Diaz-Bobillo 95, Elia et al. 93]. The $\mathcal{H}_2/\ell_1$ problem is not so often encountered [Voulgaris 94]. All these approaches use the technique of *Linear Programming (LP)* combined with *duality theory*. An LP problem is an optimization problem in $\mathbb{R}^n$, where the objective function is linear in the unknowns, and the unknowns have to satisfy a set of linear equality and/or inequality constraints. This can be stated in the following standard form:

$$\min_x c^T x$$

$$\text{subject to}$$

$$Ax = b$$

$$x_i \geq 0 \qquad i = 1, \ldots, n \tag{5.1}$$

where $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

It should be noted that any LP problem can be transformed into the above form. To bring the objective function $\| \Phi \|_1$ (from problem statement (3), Chapter 3) into linear form and to avoid the nonlinearity built into the norm (i.e. the absolute value function), a standard change of variables is used in LP. Let $\Phi = \Phi^+ - \Phi^-$, where Φ^+ and Φ^- are sequences of $q \times d$ matrices with nonnegative entries. Then, when $\phi_{ij}(t)$ denote the elements of the impulse response matrix, replace the ℓ_1[1] norm of Φ by

$$\max_i \sum_{j=1}^{d} \sum_{t=0}^{\infty} \left[\phi_{ij}^+(t) + \phi_{ij}^-(t)\right]$$

which is linear in (Φ^+, Φ^-). This expression equals the norm only if, for every (i, j, t) at least one of $\phi_{ij}^+(t)$, $\phi_{ij}^-(t)$ is zero[2]. This is illustrated in Figure 5.1.

With this change of variables, the ℓ_1 minimization can be restated as follows:

$$\nu^o = \inf_{\Phi^+, \Phi^-} \nu$$

$$\text{subject to}$$

$$\sum_{j=1}^{d} \sum_{t=0}^{\infty} \left[\phi_{ij}^+(t) + \phi_{ij}^-(t)\right] \leq \nu \qquad \text{for } i = 1, \ldots, q$$

$$\Phi = \Phi^+ - \Phi^- \quad \text{is feasible (see Chapter 3 for definition).}$$

[1] From now on, in this section ℓ_1 refers to the discrete-time case only.

[2] Since an optimal value of $\phi_{ij} = \phi_{ij}^+ - \phi_{ij}^-$ can always be achieved with $\phi_{ij}^{+\prime} = \phi_{ij}^+ - \phi_{min}$ and $\phi_{ij}^{-\prime} = \phi_{ij}^- - \phi_{min}$ (where $\phi_{min} = \min(\phi_{ij}^+, \phi_{ij}^-)$) resulting in $\phi_{ij}{}' = \phi_{ij}^{+\prime} - \phi_{ij}^{-\prime} = \phi_{ij}^+ - \phi_{min} - (\phi_{ij}^- - \phi_{min}) = \phi_{ij}^+ - \phi_{ij}^- = \phi_{ij}$ and thereby reducing the sum of ϕ_{ij}^+ and ϕ_{ij}^- with $2\phi_{min}$.

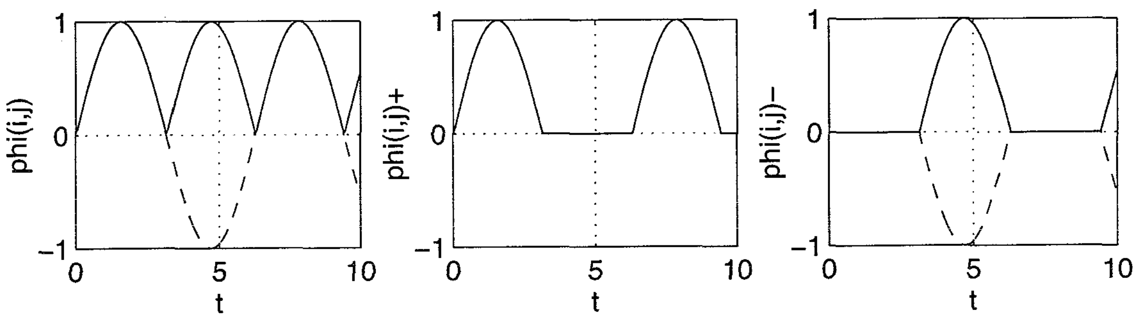


Figure 5.1: Change of variables for a scalar-valued sequence ϕ, where $\phi = \phi^+ - \phi^-(= \sin(t))$ and $\|\phi\|_1 = \phi^+ + \phi^-(= |\sin(t)|)$.

Finally, a compact representation of the ℓ_1 norm can be obtained by defining an operator $\mathcal{A}_{\ell_1} : \ell_1^{q\times d} \to \mathbb{R}^q$ such that

$$(\mathcal{A}_{\ell_1}\Phi)_i = \sum_{j=1}^{d}\sum_{t=0}^{\infty}\phi_{ij}(t) \qquad \text{for } i = 1,\dots,q$$

and a vector with all elements equal to one, $\mathbf{1} \in \mathbb{R}^q$. It follows that

$$\sum_{j=1}^{d}\sum_{t=0}^{\infty}(\phi_{ij}^+(t) + \phi_{ij}^-(t)) \le \nu \quad \text{for } i = 1,\dots,q \quad \Leftrightarrow \quad \mathcal{A}_{\ell_1}(\Phi^+ + \Phi^-) \le \mathbf{1}\nu.$$

Realizing that a large class of specifications can be expressed in terms of linear constraints leads to the following approach. The idea followed is to simply augment the constraint of the linear program, derived from the ℓ_1 optimal control, with the linear specifications constraints and solve the new linear program. With this we can augment the linear operator $\mathcal{A}_{\ell_1}$ with somewhat similar linear operators to get one operator constraint, resulting in a typical augmented operator such as

$$\begin{bmatrix} \mathcal{A}_{\ell_1} & \mathcal{A}_{\ell_1} \\ \mathcal{A}_{\mathcal{H}_\infty} & -\mathcal{A}_{\mathcal{H}_\infty} \\ \mathcal{A}_{temp} & -\mathcal{A}_{temp} \end{bmatrix} \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} \le \begin{bmatrix} \nu\mathbf{1} \\ \gamma\mathbf{1} \\ b_{temp} \end{bmatrix} \tag{5.2}$$

where $\mathcal{A}_{temp}$ and b_{temp} reflect the time domain (**templ**ate) constraints. If these operators should apply on different sets of inputs and outputs (which was said to be a possibility, however unexploited) naturally all dimensions of the appropriate operators and '$\mathbf{1}$-vectors' would change accordingly. The $\mathbf{1}$-vector in the $\mathcal{H}_\infty$ constraint will generally be of a dimension far greater than q since the infinite-dimensional constraint has to be approximated by a finite number of constraints (by sampling the unit circle, see [Dahleh and Diaz-Bobillo 95, pp. 43,44]).

Eventually, this will be combined with the feasibility- (or interpolation-) constraints (see [Dahleh and Diaz-Bobillo 95, pp. 123–126]), again using ('similar') linear operators:

$$\begin{bmatrix} \mathcal{A}_{feas} & -\mathcal{A}_{feas} \\ -\mathcal{A}_{feas} & \mathcal{A}_{feas} \end{bmatrix} \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} \le \begin{bmatrix} b_{feas} \\ -b_{feas} \end{bmatrix} \tag{5.3}$$

which is equivalent to $\mathcal{A}_{feas}\Phi = b_{feas}$ apart from the fact that the LP problem size has doubled. Linear programming problems can be solved using the efficient Simplex method (see [Dahleh and Diaz-Bobillo 95, pp. 195–200]). Another important issue in linear minimization problems (and thus LP) is *duality theory.* Given the standard form minimization (5.1), which we will call the 'primal problem', it is always possible to define an associated linear maximization problem, known as the 'dual problem'. The corresponding primal-dual pair is given by

$$\begin{array}{llll} (primal) & \min_{x} c^T x & (dual) & \max_{\eta} \eta^T b \\ & \text{subject to} & & \text{subject to} \\ & Ax = b & & \eta^T A \leq c^T \\ x_i \geq 0 & i = 1, \ldots, n & & \end{array} \tag{5.4}$$

where η is the vector of dual variables $\in \mathbb{R}^m$ (i.e. in 'dual space'). The *equality* constraints in the primal problem can easily be derived from the *inequality* constraints (5.2) and (5.3) by using so-called slack-variables[3]. It can be shown that the primal problem has an optimal solution if and only if the dual problem has an optimal solution, and further both achieve the same optimal value. Duality theory is used for instance in the solution of the multiblock problem (i.e. a problem in which $d > l$ and/or $q > m$, whereas for a one-block problem $d = l$ and $q = m$). For multiblock problems, both the primal and the dual problem have infinitely many variables *and* constraints (whereas one-block problems have finitely many (primal) constraints but still infinitely many variables; however, the underlying problem can—by looking at the structure of the dual problem—be shown to be finite-dimensional). In principle, one can attempt to get approximate solutions by an appropriate truncation of the original problem. There are basically three approximation methods:

1. Finitely Many Variables (FMV): provides a suboptimal polynomial feasible solution by constraining the number of (primal) variables to be finite.
2. Finitely Many Equations (FME): provides a superoptimal infeasible solution by including only a finite number of (primal) equality constraints. It is to be combined with FMV to get an idea of the achieved accuracy.
3. Delay Augmentation (DA): provides both a suboptimal and a superoptimal solution by embedding the problem into a one-block problem through augmenting the operators U and V with delays (where $\Phi = H - UQV$ is an equivalent form of the Youla-parameterization as used in [Dahleh and Diaz-Bobillo 95]).

For a more thorough treatment on these methods the reader is referred to [Dahleh and Diaz-Bobillo 95, Chapter 12].

The FME/FMV method does have a few drawbacks:

[3] In general, given a set of m inequalities of the form $Ax \leq b$, then $x \in \mathbb{R}^n$ satisfies the set if and only if there exists a nonnegative vector of slack-variables, $y \in \mathbb{R}^m$, such that $Ax + y = b$.

- FME/FMV requires existence of polynomial feasible solutions, and
- FME/FMV results in controllers of high order, related to the order of the approximation.

The DA method is used much more often since it doesn't necessarily suffer from order-inflation when in- and outputs are (re)ordered properly (depending on which rows of Φ are 'partially dominant', see [Dahleh and Diaz-Bobillo 95, p. 302] for definition). Another (earlier mentioned) drawback of the LP approach, when combined with $\mathcal{H}_\infty$ constraints, is that the infinite-dimensional $\mathcal{H}_\infty$ constraints have to be replaced by a finite number of constraints by sampling the unit circle. This may prevent finding a solution if the performance specifications are tight. Moreover, it has been recently shown [Venkatesh and Dahleh 93] that, for a class of problems, the approximations obtained by sampling the unit circle will fail to converge to the solution, even when the number of sampling points tends to infinity. Note that in this approach, according to [Elia et al. 93], the solution is obtained by solving LP's instead of convex or non-convex optimization and neither does it use Lagrange multiplier techniques.

5.2 $\ell_1/\mathcal{H}_\infty$: using the Youla-parameterization

This approach considers one of the same problems as the approach mentioned in the previous section: the $\ell_1/\mathcal{H}_\infty$ problem. However, it utilizes the more general description where $w_1 \neq w_2$ and $z_1 \neq z_2$ in both the discrete-time- and the continuous-time case (see [Sznaier 93](SISO) and [Sznaier 94](MIMO) for discrete-time and [Sznaier and Blanchini 94](MIMO) for continuous-time).

The main result shows that a suboptimal solution to the $\ell_1/\mathcal{H}_\infty$ problem, with performance arbitrarily close to the optimum, can be obtained by solving a finite-dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem. First, a brief description of the discrete-time problem will be given, after which the continuous-time problem can be solved using the discrete-time results.

Derivation of these results requires more preliminaries:

By $\mathcal{H}_\infty$ we denote the space of stable transfer function matrices $G(z)(G(s)) \in \ell_\infty(\mathcal{L}_\infty)$ which are analytic[4] outside the unit disk (or for continuous-time: analytic in Re(s) ≥ 0). $\mathcal{RH}_\infty$ denotes the subspace of real rational transfer matrices of $\mathcal{H}_\infty$. Similarly, $\mathcal{RH}_{\infty,\delta}$ denotes the subspace of transfer matrices in $\mathcal{RH}_\infty$ which are analytic outside the disc of radius δ, $0 < \delta < 1$, equipped with the norm

$$\| G(z) \|_{\infty,\delta} := \sup_{0 \leq \theta \leq \pi} \overline{\sigma}\left(G(\delta e^{j\theta})\right)$$

(compare this definition to (2.15)).

For the system G with state-space realization (3.5), the following assumptions are made:

[4] Having a complex derivative at every point of its domain, and in consequence possessing derivatives of all orders and agreeing with its Taylor series locally.

1. D_{13} has full column rank
2. D_{31} has full row rank
3. (A, B_3) and (C_3, A) are stabilizable and detectable, respectively.

Then, the set of all internally stabilizing controllers can be parameterized in terms of a free parameter $Q \in \mathcal{RH}_\infty$, resulting in the Youla-parameterization:

$$K = \mathcal{F}_l(K_{nom}, Q) \tag{5.5}$$

where $\mathcal{F}_l$ denotes the lower linear fractional transformation: $\mathcal{F}_l = K_{nom11} + K_{nom12}Q(I - K_{nom22}Q)^{-1}K_{nom21}$ where K_{nom} is partitioned according to the following state-space realization

$$\left[\begin{array}{c|cc} A + B_3F + LC_3 + LD_{33}F & -L & B_3 + LD_{33} \\ \hline F & 0 & I \\ -(C_3 + D_{33}F) & I & -D_{33} \end{array}\right] =: \begin{bmatrix} K_{nom11} & K_{nom12} \\ K_{nom21} & K_{nom22} \end{bmatrix} \tag{5.6}$$

and where F and L are selected such that $A + B_3F$ and $A + LC_3$ are stable. By using this parameterization, the closed-loop transfer matrices can be written as:

$$T_{w_1 \to z_1} = V_{11} + V_{12}QV_{21} \tag{5.7}$$

$$T_{w_2 \to z_2} = T_{11} + T_{12}QT_{21} \tag{5.8}$$

where V_{ij}, T_{ij} are stable transfer matrices. The discrete-time $\ell_1/\mathcal{H}_\infty$ problem can now be precisely stated as:

- **Problem 1:** *(Mixed $\ell_1/\mathcal{H}_\infty$ control problem)*

  1. Find the optimal value of the performance measure:

  $$\mu^o = \inf_{Q \in \mathcal{RH}_\infty} \| T_{w_1 \to z_1} \|_1 = \inf_{Q \in \mathcal{RH}_\infty} \| V_{11} + V_{12}QV_{21} \|_1 \tag{5.9}$$

  subject to

  $$\| T_{11} + T_{12}QT_{21} \|_\infty \leq \gamma$$

2. Given $\varepsilon > 0$, synthesize a controller such that

$$\| T_{w_2 \to z_2} \|_\infty \leq \gamma \quad \text{and} \quad \| T_{w_1 \to z_1} \|_1 \leq \mu^o + \varepsilon$$

It is known that it is possible to select F and L such that T_{12} is inner[5] and T_{21} is co-inner[6]. If T_{12} (T_{21}) is not square, we can choose $T_{12\perp}$ ($T_{21\perp}$) such that $T_{12a} = [T_{12}\ T_{12\perp}]$ ($T_{21a}^{\sim} = [T_{21}^{\sim}\ T_{21\perp}^{\sim}]$) is a unitary matrix.

This fact can be used to reduce $\| T_{w_2 \to z_2} \|_\infty$ to the form:

$$\| T_{w_2 \to z_2} \|_\infty = \left\| T_{11} + T_{12a} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} T_{21a} \right\|_\infty = \left\| R + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \tag{5.10}$$

where $R^{\sim} = T_{12a}^{\sim} T_{11} T_{21a}^{\sim}$ has a state-space realization

$$R^{\sim} =: \left[\begin{array}{c|cc} \hat{A} & B_a & B_b \\ \hline C_a & D_{aa} & D_{ab} \\ C_b & D_{ba} & D_{bb} \end{array} \right] \tag{5.11}$$

In the sequel for simplicity we will call

$$B_e = \begin{bmatrix} B_a & B_b \end{bmatrix}$$

$$C_e = \begin{bmatrix} C_a \\ C_b \end{bmatrix}$$

$$D_{er} = \begin{bmatrix} D_{aa} & D_{ab} \end{bmatrix}$$

$$D_{ec} = \begin{bmatrix} D_{aa} \\ D_{ba} \end{bmatrix}$$

We will also assume that $\gamma = 1$ (this does not entail any loss of generality, since it can always be accomplished by scaling the input matrix B_2). It can be shown that problem 1 can be solved by considering a sequence of modified problems:

- **Problem 2:** *(Mixed $\ell_1/\mathcal{H}_{\infty,\delta}$ control problem)*

  1. Given T_{ij}, $V_{ij} \in \mathcal{RH}_{\infty,\delta}$, find

$$\mu_\delta = \inf_{Q \in \mathcal{RH}_{\infty,\delta}} \| V_{11} + V_{12} Q V_{21} \|_1 \tag{5.12}$$

  subject to

$$\left\| R + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right\|_{\infty,\delta} \leq 1$$

  where $\delta < 1$ and $R \in \mathcal{RH}_{\infty,\delta}$

[5] A square system G is called inner if $G^{\sim} G = I$ *and* G is stable, where $G^{\sim} := G^*$ for $\text{Re}(s) = 0$ ($|z| = 1$); for $\text{Re}(s) \neq 0$ ($|z| \neq 1$) the definition is $G^{\sim}(s) := G^T(-s)$ ($G^{\sim}(z) := G^T(\frac{1}{z})$). In engineering terminology, an inner function is stable and all-pass with unit magnitude.

[6] A matrix G is said to be co-inner if G^T is inner.

2. Given $\varepsilon > 0$, synthesize a controller yielding a cost μ_δ^ε such that $\mu_\delta \leq \mu_\delta^\varepsilon \leq \mu_\delta + \varepsilon$.

LEMMA 5.2.1: Consider an increasing sequence $\delta_i \to 1$. Then $\mu_{\delta_i} \to \mu^o$.

Next, if $(\ell_1/\mathcal{H}_{\infty,\delta})$ is feasible, it can be shown that a rational suboptimal solution, arbitrarily close to the optimum, can be found by solving a truncated problem. Moreover, solving this truncated problem only entails solving a finite-dimensional optimization problem and an unconstrained 4-block (i.e. $d > l$ and $q > m$) $\mathcal{H}_\infty$ problem.

THEOREM 5.2.1: Let $R^\sim$ have a state-space realization as in (5.11). Then, a suboptimal solution to the mixed $\ell_1/\mathcal{H}_{\infty,\delta}$ control problem, with cost μ_δ^ε, $\mu_\delta \leq \mu_\delta^\varepsilon \leq \mu_\delta + \varepsilon$ is given by $Q^o = Q_F^o + z^{-N} Q_R^o$ where $Q_F^o = \sum_{i=0}^{N-1} Q(i) z^{-i}$;

$$\underline{Q} = \begin{bmatrix} Q(0) & 0 & \cdots & 0 \\ Q(1) & Q(0) & \cdots & 0 \\ \vdots & & \ddots & \\ Q(N-1) & \cdots & & Q(0) \end{bmatrix}$$

solves the following finite-dimensional convex optimization problem

$$\underline{Q}^o = \operatorname*{argmin}_{\substack{\underline{Q} \in \mathbb{R}^{Nm \times Nl} \\ \| \mathcal{Q} \|_2 \leq 1}} \| \underline{v}_1 + \nu_{12} \underline{Q} \nu_{21} \|_1$$

and Q_R^0 solves the approximation problem

$$Q_R^o(z) = \operatorname*{argmin}_{Q_R \in \mathcal{RH}_{\infty,\delta}} \| T_{11}(z) + T_{12}(z) Q_F^o T_{21}(z) + z^{-N} T_{12}(z) Q_R(z) T_{21}(z) \|_{\infty,\delta}$$

where

$$\mathcal{Q}(\underline{Q}) = \begin{bmatrix} \hat{y}\hat{A}^N\hat{x} & \hat{y}\hat{A}^{N-1}B_a & \cdots & \hat{y}\hat{A}B_a & \hat{y}B_a & \hat{y}\hat{A}^{N-1}B_b & \hat{y}\hat{A}^{N-2}B_b & \cdots & \hat{y}\hat{A}B_b & \hat{y}B_b \\ C_a\hat{A}^{N-1}\hat{x} & C_a\hat{A}^{N-2}B_a & \cdots & C_aB_a & D_{aa} & C_a\hat{A}^{N-2}B_b & C_a\hat{A}^{N-3}B_b & \cdots & C_aB_b & D_{ab} \\ C_a\hat{A}^{N-2}\hat{x} & C_a\hat{A}^{N-3}B_a & \cdots & D_{aa} & 0 & C_a\hat{A}^{N-3}B_b & C_a\hat{A}^{N-4}B_b & \cdots & D_{ab} & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ C_a\hat{x} & D_{aa} & 0 & \cdots & 0 & D_{ab} & 0 & 0 & \cdots & 0 \\ C_b\hat{A}^{N-1}\hat{x} & C_b\hat{A}^{N-2}B_a & \cdots & C_bB_a & D_{ba} & C_b\hat{A}^{N-2}B_b & C_b\hat{A}^{N-3}B_b & \cdots & C_bB_b & -Q^T(0) \\ C_b\hat{A}^{N-2}\hat{x} & C_b\hat{A}^{N-3}B_a & \cdots & D_{ba} & 0 & C_b\hat{A}^{N-3}B_b & C_b\hat{A}^{N-4}B_b & \cdots & -Q^T(0) & -Q^T(1) \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ C_b\hat{x} & D_{ba} & 0 & \cdots & 0 & -Q^T(0) & -Q^T(1) & -Q^T(2) & \cdots & -Q^T(N-1) \end{bmatrix}$$

$$\underline{v}_1 = \begin{bmatrix} V_{11}(0) \\ \vdots \\ V_{11}(N-1) \end{bmatrix}$$

$$\nu_{12} = \begin{bmatrix} V_{12}(0) & 0 & \cdots & 0 \\ V_{12}(1) & V_{12}(0) & \cdots & 0 \\ \vdots & & \ddots & \\ V_{12}(N-1) & \cdots & & V_{12}(0) \end{bmatrix} \quad \nu_{21} = \begin{bmatrix} V_{21}(0) \\ V_{21}(1) \\ \vdots \\ V_{21}(N-1) \end{bmatrix}$$

$$N = \left\lceil \frac{\log \varepsilon(1-\delta) - \log h}{\log \delta} \right\rceil \quad h = \sqrt{n_1}\left(\| V_{11} \|_{\infty,\delta} + \| V_{12} \|_{\infty,\delta} \| V_{21} \|_{\infty,\delta} (1 + \| R \|_{\infty,\delta}) \right)$$

$$(\text{where} \quad V_{11} \in \mathbb{R}^{m_1 \times n_1})$$

$$\hat{x} = \hat{X}^{1/2} \ , \ \hat{y} = \hat{Y}^{1/2}$$

where $\hat{X} > 0$ and $\hat{Y} > 0$ are the solutions to the following (uncoupled) ARE's:

$$\begin{aligned} \hat{X} &= \hat{A}\hat{X}\hat{A}^T + \gamma^{-2} B_e B_e^T \\ &+ (\hat{A}\hat{X}C_a^T + \gamma^{-2} B_e D_{er}^T)(I - \gamma^{-2} D_{er} D_{er}^T - C_1 \hat{X} C_1^T)^{-1} (C_a \hat{X} \hat{A}^T + \gamma^{-2} D_{er} B_e^T) \\ \hat{Y} &= \hat{A}^T \hat{Y} \hat{A} + C_e^T C_e \\ &+ (\hat{A}^T \hat{Y} B_a + C_e^T D_{ec})(I - D_{ec}^T D_{ec} - B_1^T \hat{Y} B_1)^{-1} (B_a^T \hat{Y} \hat{A} + D_{ec}^T C_a) \end{aligned}$$

and where $Q(k)$, $V_{ij}(k)$ denote the k^{th} element of the impulse response of $Q(z)$, $V_{ij}(z)$ respectively.

This is the main result of [Sznaier 94] from which an iterative algorithm can be derived and can be found in the same article but is omitted here. In [Sznaier 94] $\underline{v}_1$, ν_{12} and ν_{21} are defined somewhat differently but it is not clear whether this would yield different results. It is not likely that it should since in [Sznaier 93], which also handles the discrete-time case (but SISO), definitions similar to those used here are encountered.

For the continuous-time case the proposed method is based upon solving an auxiliary discrete-time $\ell_1/\mathcal{H}_\infty$ problem, obtained using the simple transformation $z = 1+\psi s$ and then transforming back the resulting controller to the s-domain. To do this the Euler Approximating System (EAS) is introduced, which can be shown to have ℓ_1 and $\mathcal{H}_\infty$ bounds that are upper bounds of the corresponding continuous-time quantities. Moreover, these bounds are non-increasing with ψ and converge to the exact value as $\psi \to 0$.

DEFINITION 5.2.1: Consider the (continuous-time) system G represented by (3.5). Its Euler Approximating System (EAS) is defined as the following discrete system:

$$\left[\begin{array}{c|ccc} I + \psi A & \psi B_1 & \psi B_2 & \psi B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array} \right]$$

where $\psi > 0$.

THEOREM 5.2.2: Consider a strictly decreasing sequence $\psi_i \to 0$ and the corresponding EAS(ψ_i).

Let:

$$\mu_{\psi_i} = \inf_{\substack{Q \in \mathcal{RH}_\infty \\ \| T_{w_2 \to z_2} \|_\infty \leq 1}} \| T_{w_1 \to z_1}(z, \psi_i)^{EAS} \|_1$$

and let μ^o be defined as in Problem 1. Then the sequence μ_{ψ_i} is non-increasing and such that $\mu_{\psi_i} \to \mu^o$.

Theorem 5.2.2 shows that the $\mathcal{L}_1/\mathcal{H}_\infty$ problem can be solved by solving a sequence of discrete-time $\ell_1/\mathcal{H}_\infty$ problems, each one having the form:

$$\mu_E = \inf_{\substack{Q \in \mathcal{RH}_\infty(T) \\ \| T_{11}+T_{12}QT_{21} \|_\infty \leq 1}} \| V_{11} + V_{12}QV_{21} \|_1$$

where T_{ij}, $T_{ij} \in \mathcal{RH}_\infty(T)$. It is not clear from [Sznaier and Blanchini 94] what the 'T' in $\mathcal{RH}_\infty(T)$ stands for. There $\mathcal{RH}_\infty(T)$ is defined as the set of real-rational functions in $\mathcal{H}_\infty(T)$, where $\mathcal{H}_\infty(T)$ denotes the set of stable complex functions $G(z) \in \mathcal{L}_\infty(T)$ and where $\mathcal{L}_\infty(T)$ denotes the Lebesgue space of complex valued transfer function matrices which are essentially bounded on the unit circle with the ∞-norm as defined in (2.15).

LEMMA 5.2.2: A suboptimal rational solution can be obtained by solving a discrete-time mixed $\ell_1/\mathcal{H}_\infty$ control problem for the corresponding EAS, with $\delta = 1 - \psi^2$. Moreover, if $K(z)$ denotes the controller for the EAS, the suboptimal continuous-time controller is given by $K(\psi s + 1)$.

The approach that was presented here is a departure from previous approaches to solving this type of problems, where several approximations, such as replacing the infinite-dimensional $\mathcal{H}_\infty$ constraint by a finite number of constraints by sampling the unit circle, were required to obtain a tractable mathematical problem. Perhaps the most severe limitation of the proposed method is that it may result in very high order controllers (roughly N), necessitating some type of model reduction. Note however that this disadvantage is shared by some widely used design methods, such as μ-synthesis or ℓ_1-optimal control theory, that may also produce controllers of very high order, the latter method especially. Application of some well established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order.

5.3 $\mathcal{H}_2/\mathcal{H}_\infty$: convex optimization using matrix inequalities

The approach described here has received a great deal of attention (see e.g. [Boyd and Barratt 91, Boyd et al. 93, Geromel et al. 92, Khargonekar and Rotea 91, Halikias 94, Scherer 95]). However, although all these approaches use matrix inequalities (MI's) to arrive at a convex optimization problem there still exists a wide variety among these approaches. For instance, some, but not all approaches use the Youla-parameterization; some set $w_1 = w_2$ where others take all four sets of inputs and outputs to be different; the MI's involved can be LMI's (most common), QMI's or AMI's (see e.g. [Feron et al. 92], which by the way only considers the $\mathcal{H}_2$ control problem); and some approaches use the performance measure of Bernstein and Haddad where others don't. A good example of a method using the performance measure of Bernstein and Haddad is [Khargonekar and Rotea 91]. They consider problem (1) from Chapter 3 with $w_1 = w_2 := w$ and $z_1 \neq z_2$ and use the performance measure (4.20) where S solves (4.13) (with (4.14)). However, they take a suboptimal approach. When $\nu(G)$ denotes the optimal performance measure (for K an 'admissible' (: proper and internally stabilizing) controller satisfying the $\mathcal{H}_\infty$ norm constraint[7]) the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is formulated as: *Compute $\nu(G)$ and, given any $\alpha > \nu(G)$, find a controller $K \in \mathcal{A}_\infty$ such that the auxiliary cost $\mathcal{J} < \alpha$.*

Given the plant G_{sf} ('sf' denotes the state-feedback controller):

$$G_{sf} := \begin{cases} \dot{x} &= Ax + Bw + B_3 u \\ z_1 &= C_1 x + D_{13} u \\ z_2 &= C_2 x + D_{23} u \\ y &= x \end{cases},$$

one could choose to use B_2 instead of B, thereby saying $w := w_2$, but this is no more than a matter of notation, since 'w_2' is the only w present and has the dual interpretation mentioned in Section 4.3.

The key idea is to replace the search over the admissible static state-feedback gain matrices K. This is done by introducing the change of variables $K = WY^{-1}$ (which essentially is an over-parameterization), where Y is the solution to the quadratic matrix inequality that characterizes the infinity norm constraint (5.13).

Without loss of generality it is assumed that $\gamma = 1$. Let $W \in \mathbb{R}^{m \times l}$ and symmetric positive-definite $Y \in \mathbb{R}^{l \times l}$ (and $l = n$ since $y = x$) be given and define $\mathcal{Z}(W,Y) \in \mathbb{R}^{n \times n}$:

$$\mathcal{Z}(W,Y) := AY + YA^T + B_3 W + W^T B_3 + BB^T + (C_2 Y + D_{23} W)^T (C_2 Y + D_{23} W) \tag{5.13}$$

We define also

$$f(W,Y) := \operatorname{tr}\left[(C_1 Y + D_{13} W) Y^{-1} (C_1 Y + D_{13} W)^T\right]$$

Finally, define the set Ω of real matrices (W,Y):

$$\Omega(G_{sf}) := \{(W,Y) \,|\, Y = Y^T > 0, \mathcal{Z}(W,Y) < 0\}$$

and consider the optimization problem

$$\zeta(G_{sf}) := \inf_{(W,Y) \in \Omega(G_{sf})} f(W,Y) \tag{5.14}$$

[7] $K \in \mathcal{A}_\infty$ as defined in [1] where '∞' denotes the $\mathcal{H}_\infty$ constraint.

Furthermore $\mathcal{A}_{\infty,m}$ denotes the subset of $\mathcal{A}_{\infty}$ for which K is memoryless (or static).

THEOREM 5.3.1: Consider the system G_{sf} defined in the above, along with the definition of $f(W,Y)$, $\Omega(G_{sf})$ and $\zeta(G_{sf})$ as stated in the above. Then,

$$\mathcal{A}_{\infty,m}(G_{sf}) \neq \emptyset \Leftrightarrow \Omega(G_{sf}) \neq \emptyset$$

and in case one (and thus both) of these sets is indeed nonempty (and when $\nu_m(G_{sf})$ denotes the optimal performance measure for the state-feedback problem with $K \in \mathcal{A}_{\infty,m}(G_{sf})$),

$$\nu_m(G_{sf}) = \zeta(G_{sf})$$

Furthermore, given any $\alpha > \nu_m(G_{sf})$, there exists $(W,Y) \in \Omega(G_{sf})$ such that the state-feedback gain $K := WY^{-1}$ satisfies:

$$K \in \mathcal{A}_{\infty,m}(G_{sf}) \quad \text{and} \quad \mathcal{J} \leq f(W,Y) \leq \alpha$$

At first sight this theorem doesn't seem very attractive. The calculation of $\zeta(G_{sf})$ involves a search over the set $\Omega(G_{sf})$, whereas $\nu_m(G_{sf})$ can be computed by solving a nonlinear programming problem with only the real matrix K as the decision variable. However, the over-parameterization introduced with the change of variables $K = WY^{-1}$ (which causes the dimension of $\Omega(G_{sf})$ to exceed the number of free parameters in K) can be shown to be most useful since the optimization problem defined in (5.14) is a convex problem. This can be shown to be true based on the fact that both the set Ω and the function $f : \Omega \rightarrow \mathbb{R}^+$ are convex. On the other hand, the set of feasible static state-feedback gains, $\mathcal{A}_{\infty,m}(G_{sf})$ is not necessarily convex.

Finally, for the full-information problem, where $y = \begin{bmatrix} x^T & w^T \end{bmatrix}^T$, they show that the use of dynamic full-information controllers can not improve upon the performance over all memoryless state-feedback controllers. A fact worth noting is that based on this very article, there also exists an $\mathcal{H}_\infty$/ESPR-'version' ([Shim 94], also see Chapter 3) yielding similar results. Another example in this category, [Halikias 94], uses the Youla-parameterization along with certain results from *superoptimal interpolation theory* (see [Halikias 94] and references therein), through which the problem can be formulated as a *multi-disk minimization* in terms of a free parameter of reduced dimension which can be tackled via a number of convex programming techniques (described in e.g. [Boyd and Barratt 91, Dorato 91]).

LMI-based convex optimization problems are treated extensively in control literature and it does seem to have great potential, since there exist effective and powerful algorithms for the solution of these problems, as was described earlier in Section 4.5.

5.4 $\mathcal{H}_2/\mathcal{H}_\infty$: optimizing an entropy cost functional

This section refers to the work done mainly by Mustafa and Glover in [Mustafa 89, Glover and Mustafa 89, Mustafa and Glover 88, Mustafa et al. 91]. It is shown that the auxiliary performance index of Bernstein and Haddad can be interpreted nicely as an entropy expression, yielding the central $\mathcal{H}_\infty$ controller for the full-order case. They address the problem where $w_1 = w_2 =: w$ and $z_1 = z_2 =: z$, resulting in the matrices corresponding to the w and z that are being left out to be zero. So, with the arbitrary choice $w := w_2$, $z := z_2$, B_1, D_{21}, D_{31} (and D_{11}) are zero and furthermore D_{22}, D_{33} and D_c are *taken to be* zero.

Or equivalently

$$\begin{aligned} \dot{x} &= Ax + B_2 w_2 + B_3 u \\ z_2 &= C_2 x + D_{23} u \\ y &= C_3 x + D_{32} w_2 \end{aligned}$$

Then, if we define the entropy of $T_{w\to z}$ ($=T_{w_2 \to z_2}$):

DEFINITION 5.4.1: The entropy of $T_{w\to z}$, where $\| T_{w\to z} \|_\infty < \gamma$, is defined by

$$\mathcal{I}(T_{w\to z}, \gamma) := \lim_{s_0 \to \infty} \left\{ \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln \left| \det \left(I - \gamma^2 T_{w\to z}(j\omega) T^*_{w\to z}(j\omega) \right) \right| \left[\frac{s_0}{|s_0 - j\omega|} \right]^2 d\omega \right\} \quad ,$$

where $s_0 \in \mathbb{R}^+$,

the first of the problems of interest can be stated:

- **Problem A:** *The maximum entropy/$\mathcal{H}_\infty$ control problem* [Glover and Mustafa 89, Mustafa and Glover 88]. Find, for the plant G, a feedback controller K such that:
  1. K stabilizes G
  2. The closed-loop transfer function $T_{w\to z} = \mathcal{F}_l(G, K)$ satisfies the $\mathcal{H}_\infty$ norm bound $\| T_{w\to z} \|_\infty < \gamma$, where $\gamma \in \mathbb{R}$ is given
  3. The closed-loop entropy $\mathcal{I}(T_{w\to z}, \gamma)$ is maximized.

REMARK 5.4.1: Problem A is equivalent to the *Risk Sensitive Linear Quadratic Gaussian* control problem of [Whittle 81, Bensoussan and Van Schuppen 85]. This link was established in [Glover and Doyle 88].

If we recall the performance functional J, defined in (4.10) and (4.11), we have (the proof of this may be found in [Mustafa and Glover 88]):

PROPOSITION 5.4.1: $\quad -\mathcal{I}(T_{w\to z}, \gamma) \geq J(T_{w\to z})$

Next, recall $\mathcal{J}$ denoting the auxiliary cost as defined in (4.20). Then the second problem of interest is:

- **Problem B:** *The combined $\mathcal{H}_\infty/LQG$ control problem* [Bernstein and Haddad 89]. Find, for the plant G, a feedback controller K such that:
  1. K stabilizes G
  2. The closed-loop transfer function $T_{w\to z} = \mathcal{F}_l(G, K)$ satisfies the $\mathcal{H}_\infty$ norm bound $\| T_{w\to z} \|_\infty < \gamma$, where $\gamma \in \mathbb{R}$ is given
  3. The auxiliary cost $\mathcal{J}(T_{w\to z}, \gamma)$ is minimized.

and from (4.19) we know that

PROPOSITION 5.4.2: $\mathcal{J}(T_{w\to z}, \gamma) \geq J(T_{w\to z})$

For completeness we will state the well-known LQG problem associated with G:

- **Problem C:** *The LQG control problem.* Find, for the plant G, a feedback controller K such that:
  1. K stabilizes G
  2. The LQG cost $J(T_{w\to z})$ is minimized.

After a few mild assumptions have been made (see [Mustafa 89]) the key result is established:

THEOREM 5.4.1: For any $T_{w\to z} \in \mathcal{RH}_\infty$ with $\| T_{w\to z} \|_\infty < \gamma$, minus the entropy equals the auxiliary cost, i.e.
$-\mathcal{I}(T_{w\to z}, \gamma) = \mathcal{J}(T_{w\to z}, \gamma)$.

where the auxiliary cost is defined by (4.20), with $\mathbf{S}$ the *positive*-definite solution to (4.14). This is in contrast with [Bernstein and Haddad 89], where $\mathbf{S}$ must be positive-*semi*definite. Furthermore $\mathbf{S}$ is insisted on being the *stabilizing* solution $\mathbf{S}_s$ to (4.14), a condition which is not mentioned in [Bernstein and Haddad 89].

Next, they state the state-space realization of the controller which solves problems A and B, expressed in terms of the stabilizing solutions, denoted X_∞ and Y_∞ to two algebraic Riccati equations, followed by the maximum value of the entropy and the minimum value of the auxiliary cost, respectively. While the maximum entropy can be expressed in terms of X_∞ and Y_∞, the minimum auxiliary cost in addition to this requires the solution $\bar{\mathbf{S}}$, to a third algebraic Riccati equation coupled to the other two. Since the two optimal values were said to be equal in theorem 1, we will be able to discard of the (yet to be stated) minimum auxiliary cost expression and the corresponding algebraic Riccati equation as redundant.

PROPOSITION 5.4.3: The controller which solves the equivalent problems A and B has a state-space realization

$$K = \left[\begin{array}{c|c} A + Y_\infty(\gamma^{-2}C_2^TC_2 - C_3^TC_3) - B_3B_3^TX_\infty Z & Y_\infty C_3^T \\ \hline -B_3^TX_\infty Z & 0 \end{array}\right]$$

where $X_\infty \geq 0$, $Y_\infty \geq 0$ are the stabilizing solutions to the ARE's

$$0 = X_\infty A + A^TX_\infty + C_2^TC_2 + X_\infty(\gamma^{-2}B_2B_2^T - B_3B_3^T)X_\infty$$
$$0 = Y_\infty A^T + AY_\infty + B_2B_2^T + Y_\infty(\gamma^{-2}C_2^TC_2 - C_3^TC_3)Y_\infty$$

and where
$Z := (I - \gamma^{-2}Y_\infty X_\infty)^{-1}$

In saying the stabilizing solutions, we mean the solutions X_∞ and Y_∞ such that

$$A + (\gamma^{-2}B_2B_2^T - B_3B_3^T)X_\infty \quad \text{is asymptotically stable and}$$
$$A + Y_\infty(\gamma^{-2}C_2^TC_2 - C_3^TC_3) \quad \text{is asymptotically stable.}$$

and

PROPOSITION 5.4.4: Minus the maximum value of the entropy is given by
$-\mathcal{I}_{max}(T_{w\to z}, \gamma) = \mathrm{tr}\,[X_\infty B_2B_2^T + X_\infty ZY_\infty X_\infty B_3B_3^T]$

PROPOSITION 5.4.5: The minimum value of the auxiliary cost is given by
$\mathcal{J}_{min}(T_{w\to z}, \gamma) = \mathrm{tr}\,[Y_\infty C_2^TC_2 + \bar{\mathrm{S}}_s\bar{R}]$,
where $\bar{\mathrm{S}}_s > 0$ is the stabilizing solution to the algebraic Riccati equation
$0 = \bar{A}\bar{\mathrm{S}} + \bar{\mathrm{S}}\bar{A}^T + \gamma^2\bar{\mathrm{S}}\bar{R}\bar{\mathrm{S}} + Y_\infty C_3^TC_3Y_\infty$
and
$\bar{A} := A - B_3B_3^TX_\infty Z + \gamma^{-2}Y_\infty C_2^TC_2$
$\bar{R} := C_2^TC_2 + X_\infty ZB_3B_3^TZ^TX_\infty$
In saying the stabilizing solution, we mean the solution $\bar{\mathrm{S}}_s$, such that $\bar{A} + \gamma^{-2}\bar{\mathrm{S}}_s\bar{R}$ is asymptotically stable.

As mentioned before, we can discard of this last proposition and therefore there is no need to solve the third coupled algebraic Riccati equation. Although this is an attractive feature, it must be remembered that this approach addressed the problem where both sets of inputs and outputs are equal.

5.5 $\mathcal{H}_2/\mathcal{H}_\infty$: fixed-order controller design using the auxiliary cost

This class of approaches refers mainly to the work done by Bernstein and Haddad in [Bernstein and Haddad 89, Haddad and Bernstein 90, Haddad et al. 91] and some methods based on it [Ge et al. 94]. The general setup was described in Sections 4.2–4.4 which ended with setting the partial derivatives $\frac{\partial \mathcal{L}}{\partial A_c}$, $\frac{\partial \mathcal{L}}{\partial B_c}$ and $\frac{\partial \mathcal{L}}{\partial C_c}$ to zero. The results of this can be obtained by various matrix manipulations (which can be found in [Bernstein and Haddad 89] addressing the simplified problem as mentioned in Section 4.2) and will be stated here for the most general case being the mixed-norm reduced- (or fixed-) order dynamic compensation problem (solutions to the problems of finding fixed- as well as full-order controllers for both the $\mathcal{H}_2/\mathcal{H}_\infty$ and the pure $\mathcal{H}_\infty$ problem can be found in [Haddad and Bernstein 90]). Here the same problem setting is used as in [Haddad and Bernstein 90], where B_1, D_{21}, D_{31}, D_{12} and D_c (and D_{11}) are taken to be zero.

First, for arbitrary positive-semidefinite S,Σ, $\hat{S} \in \mathbb{R}^{n\times n}$ and $\alpha, \beta \geq 0$[8] we define the matrices

$$\begin{aligned} S_a &= SC_3^T + B_2 M_{d_2}^{-1} D_{32}^T \\ \Sigma_a &= \left[B_3^T + \gamma^{-2} R_{23\infty}^T B_2^T + \gamma^{-2} R_{13\infty}^T (S + \hat{S})\right] \Sigma + R_{13}^T \\ \Lambda &= \left(\alpha^2 I_n + \beta^2 \gamma^{-2} \hat{S}\Sigma\right)^{-1} \end{aligned}$$

where

$$R_{23\infty} := D_{22}^T M_{q_2}^{-1} D_{23} \quad R_{13\infty} := C_2^T M_{q_2}^{-1} D_{23} \quad M_{d_2} = I_{d_2} - \gamma^{-2} D_{22}^T D_{22}$$

Furthermore

$$R_{21\infty} := D_{22}^T M_{q_2}^{-1} C_2 \quad R_{1\infty} := C_2^T M_{q_2}^{-1} C_2 \quad R_{3\infty} := D_{23}^T M_{q_2}^{-1} D_{23}$$
$$V_{1\infty} := B_2 M_{d_2}^{-1} B_2^T \quad V_{2\infty} := D_{32} M_{d_2}^{-1} D_{32}^T$$

Next, the following lemma is required for the statement of the main theorem:

LEMMA 5.5.1: Let positive-semidefinite matrices $\hat{S}$, $\hat{\Sigma} \in \mathbb{R}^{n\times n}$ and suppose $\text{rank}[\hat{S}\hat{\Sigma}] = n_c$. Then there exist $n_c \times n$ Ψ, Γ, and $n_c \times n_c$ invertible Υ, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that

$$\hat{S}\hat{\Sigma} = \Psi^T \Upsilon \Gamma \quad , \quad \Gamma \Psi^T = I_{n_c}.$$

Furthermore, the $n \times n$ matrices

$$\kappa = \Psi^T \Gamma \quad , \quad \kappa_\perp = I_n - \kappa$$

are idempotent[9] and have rank n_c and $n - n_c$.

[8] Where for simplicity it is assumed that $R_3 := D_{13}^T D_{13} =: \alpha^2 \hat{R}_3$ and $R_{3\infty} := D_{23}^T M_{q_2}^{-1} D_{23} =: \beta^2 \hat{R}_3$, where the nonnegative scalars α, β are design variables such that $\alpha^2 + \beta^2 \neq 0$.

[9] Having the property that it is equal to its own square (e.g. the identity matrix; $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$).

THEOREM 5.5.1: Let $n_c \leq n$, suppose there exist nonnegative-definite matrices S, Σ, $\hat{S}$ and $\hat{\Sigma} \in \mathbb{R}^{n\times n}$ satisfying

(ARE1):

$$0 = (A+\gamma^{-2}B_2R_{21\infty})S + S(A+\gamma^{-2}B_2R_{21\infty})^T + \gamma^{-2}SR_{1\infty}S + V_{1\infty} - S_aV_{2\infty}^{-1}S_a^T + \kappa_\perp S_aV_{2\infty}^{-1}S_a^T\kappa_\perp^T \quad ,$$

(ARE2):

$$\begin{aligned} 0 = &\left(A+\gamma^{-2}[S+\hat{S}]R_{1\infty} + \gamma^{-2}B_2R_{21\infty} - \gamma^{-2}\hat{S}\Lambda^T\Sigma_a^T\hat{R}_3^{-1}R_{13\infty}^T\right)^T\Sigma \\ &+\Sigma\left(A+\gamma^{-2}[S+\hat{S}]R_{1\infty} + \gamma^{-2}B_2R_{21\infty} - \gamma^{-2}\hat{S}\Lambda^T\Sigma_a^T\hat{R}_3^{-1}R_{13\infty}^T\right) \\ &+R_1 - \Lambda^T\Sigma_a^T\hat{R}_3^{-1}\Sigma_a\Lambda + \kappa_\perp^T\Lambda^T\Sigma_a^T\hat{R}_3^{-1}\Sigma_a\Lambda\kappa_\perp \quad , \end{aligned}$$

(ARE3):

$$\begin{aligned} 0 = &\left(A - B_3\hat{R}_3^{-1}\Sigma_a\Lambda + \gamma^{-2}S[R_{1\infty} - R_{13\infty}\hat{R}_3^{-1}\Sigma_a\Lambda] + \gamma^{-2}B_2[R_{21\infty} - R_{23\infty}\hat{R}_3^{-1}\Sigma_a\Lambda]\right)\hat{S} \\ &+\hat{S}\left(A - B_3\hat{R}_3^{-1}\Sigma_a\Lambda + \gamma^{-2}S[R_{1\infty} - R_{13\infty}\hat{R}_3^{-1}\Sigma_a\Lambda] + \gamma^{-2}B_2\left[R_{21\infty} - R_{23\infty}\hat{R}_3^{-1}\Sigma_a\Lambda\right]\right)^T \\ &+\gamma^{-2}\hat{S}\left(R_{1\infty} - R_{13\infty}\hat{R}_3^{-1}\Sigma_a\Lambda - \Lambda^T\Sigma_a^T\hat{R}_3^{-1}R_{13\infty}^T + \beta^2\Lambda^T\Sigma_a^T\hat{R}_3^{-1}\Sigma_a\Lambda\right)\hat{S} \\ &+S_aV_{2\infty}^{-1}S_a^T - \kappa_\perp S_aV_{2\infty}^{-1}S_a^T\kappa_\perp^T \quad , \end{aligned}$$

(ARE4):

$$\begin{aligned} 0 = &(A - S_aV_{2\infty}^{-1}C_3 + \gamma^{-2}B_2R_{21\infty} + \gamma^{-2}SR_{1\infty} - \gamma^{-2}S_aV_{2\infty}^{-1}D_{32}R_{21\infty})^T\hat{\Sigma} \\ &+\hat{\Sigma}(A - S_aV_{2\infty}^{-1}C_3 + \gamma^{-2}B_2R_{21\infty} + \gamma^{-2}SR_{1\infty} - \gamma^{-2}S_aV_{2\infty}^{-1}D_{32}R_{21\infty}) \\ &+\Lambda^T\Sigma_a^T\hat{R}_3^{-1}\Sigma_a\Lambda - \kappa_\perp^T\Lambda^T\Sigma_a^T\hat{R}_3^{-1}\Sigma_a\Lambda\kappa_\perp \quad , \end{aligned}$$

$\mathrm{rank}[\hat{S}] = rank[\hat{\Sigma}] = rank[\hat{S}\hat{\Sigma}] = n_c$

and let $(A_c, B_c, C_c, \mathbf{S})$ be given by

$$\begin{aligned} A_c &= \Gamma\Big[A - B_3\hat{R}_3^{-1}\Sigma_a\Lambda - S_aV_{2\infty}^{-1}C_3 + S_aV_{2\infty}^{-1}D_{33}\hat{R}_3^{-1}\Sigma_a\Lambda + \gamma^{-2}\big(SR_{1\infty} + B_2R_{21\infty} \\ &\quad -B_2R_{23\infty}\hat{R}_3^{-1}\Sigma_a\Lambda - SR_{13\infty}\hat{R}_3^{-1}\Sigma_a\Lambda - S_aV_{2\infty}^{-1}D_{32}R_{21\infty} + S_aV_{2\infty}^{-1}D_{32}R_{23\infty}\hat{R}_3^{-1}\Sigma_a\Lambda\big)\Big]\Psi^T \\ B_c &= \Gamma S_aV_{2\infty}^{-1} \\ C_c &= -\hat{R}_3^{-1}\Sigma_a\Lambda\Psi^T \\ \mathbf{S} &= \begin{bmatrix} S+\hat{S} & \hat{S}\Gamma^T \\ \Gamma\hat{S} & \Gamma\hat{S}\Gamma^T \end{bmatrix} \end{aligned}$$

Then, $(\tilde{A}, \tilde{B})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $T_{w_2\to z_2}$ satisfies the $\mathcal{H}_\infty$ disturbance attenuation constraint (4.17) and the $\mathcal{H}_2$ performance criterion (4.10) satisfies the bound

$$J(A_c, B_c, C_c) \leq \mathrm{tr}\left[(S+\hat{S})R_1 - 2R_{13}\hat{R}_3^{-1}\Sigma_a\Lambda\hat{S} + \Lambda^T\Sigma_a^T\hat{R}_3^{-1}R_3\hat{R}_3^{-1}\Sigma_a\Lambda\hat{S}\right]$$

From these results the full-order case results can be derived by setting $n_c = n$ (see [Haddad and Bernstein 90]) so that $\kappa = \Psi = \Gamma = I$ and $\kappa_\perp = 0$. In this case the last ('additional') term in each of (ARE1)–(ARE4) can be deleted and (ARE4) becomes superfluous.

Another way of solving the problem, when leaving off from the point where the partial derivatives $\frac{\partial\mathcal{L}}{\partial A_c}$, $\frac{\partial\mathcal{L}}{\partial B_c}$ and $\frac{\partial\mathcal{L}}{\partial C_c}$ were set to zero, is by using homotopy techniques (see [Ge et al. 94] and references in [Bernstein and Haddad 89, Haddad and Bernstein 90]). These techniques have been developed to account for the additional terms in the equations (ARE1)–(ARE4) which existing Riccati equation solvers cannot handle. Homotopy methods utilize the solution of a related easily solved problem as the starting point. In the case of full-order $\mathcal{H}_2/\mathcal{H}_\infty$ control with unequalized weights, the starting point is provided by the standard LQG solution. The approach followed in [Ge et al. 94] is based on [Bernstein and Haddad 89] and combines it with so-called *probability-one homotopy algorithms* (this name will become clear in the following). Also, in [Ge et al. 94] the number of parameters (which determine A_c, B_c and C_c) is reduced from $n_c(n_c+m+l)$ $(= n_c\times n_c + n_c\times l + m\times n_c)$ to $n_c(m+l)$ $(= n_c + n_c\times l + (m-1)n_c)$ by using the 'Ly, Bryson and Cannon'-parameterization [Ly et al. 94], which, like all (mostly canonical) realizations involving a minimal number of independent parameters, cannot provide a smooth, global representation of all MIMO (or in case of the 'Ly, Bryson and Cannon'-parameterization even SISO) systems. It does however provide a generic representation which is particularly suited for parametric optimization (see [Ge et al. 94] and [Ly et al. 94] for further information). The (reduced number of) parameters are now cast into one vector ξ. Furthermore, the relevant matrices $A(\lambda)$, $B(\lambda)$, etc. and the $\mathcal{H}_\infty$-norm bound $\gamma(\lambda)$ are defined as

$$A(\lambda) = A_0 + \lambda(A_f - A_0) \quad B(\lambda) = B_0 + \lambda(B_f - B_0) \quad \gamma(\lambda) = \gamma_0 + \lambda(\gamma_f - \gamma_0) \quad \text{etc.}$$

where λ is a variable step-size that is to be computed each step and A_0 and A_f denote the starting and ending point of $A(\lambda)$ for each step (and likewise for the other matrices and $\gamma(\lambda)$).

The homotopy map $\rho(\xi,\lambda)$ is then defined basically as the combination of the three partial derivatives $\frac{\partial\mathcal{L}}{\partial A_c}$, $\frac{\partial\mathcal{L}}{\partial B_c}$ and $\frac{\partial\mathcal{L}}{\partial C_c}$, where only those elements corresponding to the parameter elements of A_c, B_c and C_c are present. The numerical algorithm which computes ξ_f for which $\lambda_f = 1$ and $\rho(\xi_f,\lambda_f) = 0$ starts with $\lambda_0 = 0$ and γ_0 such that ${\gamma_0}^{-2}$ is approximately zero. The initial ξ_0 is chosen such that $\rho(\xi_0, 0) = 0$ and can be derived from the LQG solution in the full-order case, but has to be computed from an initialization scheme for the reduced-order problem. In practice, it may be difficult to find the initial point ξ_0 such that $\rho(\xi_0,0) = 0$. A somewhat more artificial homotopy then, letting ξ_0 be the chosen initial point, is the Newton homotopy map defined as $\bar{\rho}(\xi,\lambda) = \rho(\xi,\lambda) - (1-\lambda)\rho(\xi_0,0)$. To guarantee a full rank Jacobian matrix (denoted by $D\rho(\xi,\lambda)$) along the whole homotopy zero curve, except possibly at the solution corresponding to $\lambda = 1$, define the homotopy map to be $\hat{\rho}(\xi,\lambda) = \rho(\xi,\lambda) - (1-\lambda)(\xi - \xi_0)$. Once the initial point is chosen, the rest of the computation is as follows:

1. Set $\lambda := 0$, $\xi := \xi_0$.
2. Compute $\mathbf{S}$ and $\mathcal{M}$ according to (4.14) and (4.23).
3. Evaluate the homotopy map $\rho(\xi,\lambda)$ or $\hat{\rho}(\xi,\lambda)$ and the Jacobian of the homotopy map $D\rho(\xi,\lambda)$ or $D\hat{\rho}(\xi,\lambda)$.
4. Predict the next point $(\xi(0),\lambda(0))$ on the homotopy zero curve using e.g. a Hermite cubic interpolant.
5. For $k := 0, 1, 2, \ldots$ until convergence do

$$(\xi(k+1),\lambda(k+1)) = [D\rho(\xi(k),\lambda(k))]^{\dagger}\rho(\xi(k),\lambda(k)),$$

where $[D\rho(\xi,\lambda)]^\dagger$ is the Moore-Penrose inverse[10] of $D\rho(\xi,\lambda)$.
Let $(\xi_1,\lambda_1) = \lim_{k\to\infty}(\xi(k),\lambda(k))$.

6. If $\lambda_1 < 1$, then set $\xi := \xi_1$, $\lambda := \lambda_1$, and go to step 2.

7. If $\lambda_1 \geq 1$, compute the solution $\overline{\xi}$ at $\lambda = 1$.

The standard classical *continuation* techniques solve $\rho(\xi,\overline{\lambda}+\Delta\lambda) = 0$ for fixed $\Delta\lambda > 0$, given a solution $(\overline{\xi},\overline{\lambda}) : \rho(\overline{\xi},\overline{\lambda}) = 0$. It is implicitly assumed that $\xi = \xi(\lambda)$, i.e. the zero curve γ of $\rho(\xi,\lambda)$ being tracked in (ξ,λ) space is monotone in γ. Other tacit assumptions are that γ does not bifurcate or otherwise contain singularities. The more general homotopy methods which are used in [Ge et al. 94] make no such assumptions, and include mechanisms to deal with bifurcations and turning points. In particular, homotopy methods do not assume that the zero curve γ is monotone in λ. A continuation or homotopy algorithm is not *a priori* globally convergent (where globally convergent means that the zero curve γ reaches a solution $\overline{\xi}$, $\rho(\overline{\xi},1) = 0$ from an *arbitrary* starting point ξ_0, $\rho(\xi_0,0) = 0$). However, probability-one homotopy methods are provably globally convergent under mild assumptions [Watson et al. 87], and their zero curve γ is guaranteed to contain no singularities with probability one. Interestingly, these particular algorithms are implemented in software-packages such as HOMPACK [Watson et al. 87].

5.6 $\mathcal{H}_2/\mathcal{H}_\infty$: using a bounded power characterization

In this section the semi-norms as defined in Section 2.1 are used to obtain both necessary and sufficient conditions for optimality. Unlike most of the other approaches the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is stated in terms of signal sets. The problem addressed [Doyle et al. 89, Zhou et al. 90] sets $w_1 \neq w_2$, $z_1 = z_2 =: z$ where w_1 is assumed to be fixed and white, and w_2 is assumed to be bounded in power. The design objective is to minimize the power of the output error signal z, i.e. compute

$$\sup_{w_1\in\mathcal{S},w_2\in\mathcal{P}} \| z \|_{\mathcal{P}}^2 \tag{5.15}$$

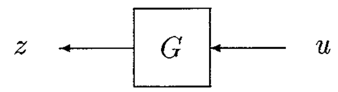
with $\mathcal{S}$ and $\mathcal{P}$ as defined in Section 2.1. It will be seen that if only w_1 is present, the problem reduces to the standard $\mathcal{H}_2$ problem. Similarly, if only w_2 is present we obtain the standard $\mathcal{H}_\infty$ problem. To describe the approach followed, we must first define some more properties:

If we consider a linear system G with convolution kernel (impulse response) $g(t)$, input u and output z

[10]The Moore-Penrose inverse of an $m_M \times n_M$ matrix M is the unique $n_M \times m_M$ matrix $M^\dagger$ satisfying the conditions:

(a) $M^\dagger M M^\dagger = M^\dagger \quad , \quad M M^\dagger M = M$,

(b) $(M^\dagger M)^* = M^\dagger M \quad , \quad (M M^\dagger)^* = M M^\dagger$.



the following (standard) properties are defined

$$\begin{aligned} R_{zu}(\tau) &= g(\tau) * R_{uu}(\tau) \\ R_{zz}(\tau) &= g(\tau) * R_{uu}(\tau) * g^*(-\tau) \\ S_{zu}(j\omega) &= G(j\omega)S_{uu}(j\omega) \\ S_{zz}(j\omega) &= G(j\omega)S_{uu}(j\omega)G^*(j\omega) \end{aligned}$$

Denote the cross spectral density of w_1 and w_2 by $S_{w_1w_2}(j\omega)$. Now assume G is stable and partition G compatibly with w_1 and w_2 as $\begin{bmatrix} G_1 & G_2 \end{bmatrix}$, where G_1 is assumed strictly proper (otherwise the output signal can have unbounded power).

Now we can compute the power spectral of the output z. To do that let

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Then the spectral density matrix of w can be computed as

$$S_{ww} = \begin{bmatrix} S_{w_1w_1} & S_{w_1w_2} \\ S^*_{w_1w_2} & S_{w_2w_2} \end{bmatrix}$$

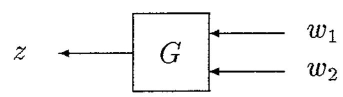
Using this formula and the earlier defined expression for S_{zz}, we get

$$S_{zz} = \begin{bmatrix} G_1(j\omega) & G_2(j\omega) \end{bmatrix} \begin{bmatrix} S_{w_1w_1} & S_{w_1w_2} \\ S^*_{w_1w_2} & S_{w_2w_2} \end{bmatrix} \begin{bmatrix} G_1(j\omega)^* \\ G_2(j\omega)^* \end{bmatrix}$$

and, according to (2.9)

$$\| z \|_{\mathcal{P}}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}[S_{zz}(j\omega)] \mathrm{d}\omega$$

These relations form the basis for the mixed-norm performance analysis, where we examine the norms induced on G with inputs w_1 and w_2 (so this is a system without a controller or uncertainties).



Consecutively, there will be treated:

1. The orthogonal case, i.e. $S_{w_1 w_2} = 0$
2. The white and causal case, i.e. w_1 is assumed to be white with $S_{w_1 w_1} = I$ *and* $S_{w_2 w_1} = \mathsf{S}(s)$ with $\mathsf{S}(s)$ strictly causal (i.e. we assume that $w_2(t)$ can be generated from w_1 through a strictly causal filter)
3. The non-white and non-causal case
4. The white and non-causal case
5. The non-white and causal case

The 4$^{\text{th}}$ problem appears to be equal to the 3$^{\text{rd}}$ problem, i.e. the worst-case signal w_1 in the 3$^{\text{rd}}$ problem is shown to be white. The 5$^{\text{th}}$ problem is not solved in the paper, but it can be shown that the worst-case w_1 is not necessarily white.

1. **The orthogonal case**

   Here we have

   $$\sup_{w_1 \in \mathcal{S}, w_2 \in \mathcal{P}} \| z \|_{\mathcal{P}}^2 = \| G_1 \|_2^2 + \| G_2 \|_\infty^2$$

   and the worst-case signal w_1 is white noise with unit spectral density, $S_{w_1 w_1} = I$.

2. **The white and causal case**

   This case is the main focus of this paper. As was said, w_1 is assumed to be white with $S_{w_1 w_1} = I$ and $w_2 \in \mathcal{P}$. Furthermore $S_{w_2 w_1} = \mathsf{S}(s)$ with $\mathsf{S}(s)$ strictly proper. When the system equations are

   $$\begin{aligned} \dot{x} &= Ax + B_1 w_1 + B_2 w_2 \\ z_2 &= C_2 x + D_{22} w_2 \end{aligned}$$

   and suppose $\| G_2 \|_\infty < \gamma$, with $\gamma > 0$. Denote

   $$X = \text{Ric} \begin{bmatrix} A + B_2 \gamma^{-2} M_{d_2}^{-1} D_{22}^T C_2 & B_2 \gamma^{-2} M_{d_2}^{-1} B_2^T \\ -C_2^T [I + D_{22} \gamma^{-2} M_{d_2}^{-1} D_{22}^T] C_2 & -[A + B_2 \gamma^{-2} M_{d_2}^{-1} D_{22}^T C_2]^T \end{bmatrix} \geq 0$$

   where $X = \text{Ric}(H)$ (with H the Hamiltonian belonging to the corresponding ARE) uses the 'Ric'-operator, uniquely determining X by H. Furthermore, H is said to belong to the domain of Ric: $H \in \text{dom}(\text{Ric})$.

   Then

   THEOREM 5.6.1:

   $$\sup_{w_2 \in \mathcal{P}} \left\{ \| z \|_{\mathcal{P}}^2 - \gamma^2 \| w_2 \|_{\mathcal{P}}^2 \right\} = \text{tr}[B_1^T X B_1]$$

   with a worst-case signal $w_2 = \gamma^{-2} M_{d_2}^{-1} (D_{22}^T C_2 + B_2^T X) x$.

Finally to compute (5.15), we have to find γ such that the worst-case $w_2 \in \mathcal{P}$, this is given in the following theorem:

THEOREM 5.6.2: Let γ_0 be such that $\| \gamma_0^{-2} M_{d_2}^{-1}(\gamma_0)\,(D_{22}^T C_2 + B_2^T X(\gamma_0))\, x \|_{\mathcal{P}} = 1$
Then

$$\sup_{w_2 \in \mathcal{P}} \| z \|_{\mathcal{P}}^2 = \mathrm{tr}\,[B_1^T X(\gamma_0) B_1 + \gamma_0^2]$$

3. **The non-white and non-causal case**

Here we examine the case when w_1 is not restricted to be white and w_2 is not restricted to be a causal function of w_1. Again we assume $\| G_2 \|_\infty < \gamma$, $\gamma > 0$. Without loss of generality we assume that the spectral densities of w_1 and w_2 have the following decompositions:

$$\begin{aligned} S_{w_1 w_1} &= S_{11} S_{11}^* \\ S_{w_1 w_2} &= S_{11} S_{12}^* \\ S_{w_2 w_2} &= S_{12} S_{12}^* + S_{22} S_{22}^* \end{aligned}$$

where S_{11} can be restricted to be a stable and minimum phase transfer matrix, in fact, w_1 can be thought of as the output of the stable system S_{11} with a unit density white input. Then the following result can be shown to be true:

THEOREM 5.6.3: Let γ be such that $\| w_2 \|_{\mathcal{P}} = \left\| (\gamma^2 I - G_2^* G_2)^{-1} G_2^* G_1 \right\|_2 = 1$
Then

$$\sup_{w_1 \in \mathcal{S}, w_2 \in \mathcal{P}} \| z \|_{\mathcal{P}}^2 = \left\| \gamma^2 (\gamma^2 I - G_2 G_2^*)^{-1} G_1 \right\|_2$$

with the worst-case signal w_1 white with unit spectral density ($S_{w_1 w_1} = I$) and w_2 having spectral density $S_{w_2 w_2} = S_{12} S_{12}^*$ where $S_{12} = (\gamma^2 I - G_2^* G_2)^{-1} G_2^* G_1 S_{11}$

Note that from the expression for S_{12} it is seen that the worst-case signal w_2 can be generated from passing w_1 through the non-causal linear system $(\gamma^2 I - G_2^* G_2)^{-1} G_2^* G_1$.

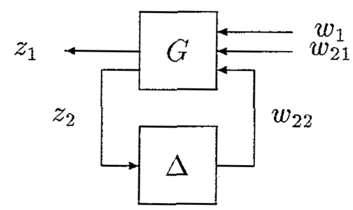
4. **The white and non-causal case**

As was seen from the previous case the worst-case w_1 was white, so the two problems are identical.

5. **The non-white and causal case**

This problem so far remains unsolved. However, examples exist which show that in this case, the worst-case w_1 is not white.

Now we will analyze the system performance when the system model has structured norm-bounded perturbations, as in the following diagram



where G is partitioned according to the inputs and outputs as

$$G = \left[\begin{array}{c|cc} G_{11} & G_{121} & G_{122} \\ G_{21} & G_{221} & G_{222} \end{array}\right] =: \left[\begin{array}{cc} G_1 & G_2 \end{array}\right]$$

and $G_1 = \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix}$ is strictly proper.

The uncertainty is structured such that $\Delta \in \mathbf{\Delta}$ where

$$\mathbf{\Delta} = \{\text{diag}[\Delta_1, \Delta_2, \ldots, \Delta_p], \| \Delta_i \|_\infty \leq 1\}$$

Again we assume that $w_1 \in \mathcal{S}$ and $w_2 \in \mathcal{P}$. The robust performance problem in this setting concerns the following question:

when does

$$\frac{\| z_2 \|_{\mathcal{P}}^2}{\| w_{21} \|_{\mathcal{P}}^2 + \| w_1 \|_{\mathcal{S}}^2} \leq 1 \qquad \forall \Delta \in \mathbf{\Delta} \tag{5.16}$$

hold?

A sufficient condition for this problem can be obtained using the mixed-norm analysis results that were stated in the above. Define a set of scaling matrices[11]

$$\mathcal{D} = \left\{\text{diag}[\mathsf{d}_1 I_1, \mathsf{d}_2 I_2, \ldots, \mathsf{d}_p I_p] \mid \mathsf{d}_i, \mathsf{d}_i^{-1} \in \mathcal{H}_\infty\right\}$$

Then $\mathsf{D}\Delta\mathsf{D}^{-1} = \Delta$ for all $\Delta \in \mathbf{\Delta}$ and $\mathsf{D} \in \mathcal{D}$, and let

$$z := \begin{bmatrix} z_1 \\ \mathsf{D} z_2 \end{bmatrix} \qquad w_2 := \begin{bmatrix} w_{21} \\ \mathsf{D} w_{22} \end{bmatrix}$$

Then we have

$$z = \left[\begin{array}{c|cc} G_{11} & G_{121} & G_{122}\mathsf{D}^{-1} \\ \mathsf{D}G_{21} & \mathsf{D}G_{221} & \mathsf{D}G_{222}\mathsf{D}^{-1} \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} =: \begin{bmatrix} \hat{G}_1 & \hat{G}_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

[11]These scaling matrices will be denoted by D and not by D to avoid confusion with the matrix D, which connects u (and possibly w) with y.

Now consider a mixed-norm analysis problem

$$J_m = \inf_{D \in \mathcal{D}} \sup_{w_1 \in \mathcal{S}, w_2 \in \mathcal{P}} \frac{\| z \|_{\mathcal{P}}^2}{\| w_2 \|_{\mathcal{P}}^2 + \| w_1 \|_{\mathcal{S}}^2} \tag{5.17}$$

THEOREM 5.6.4: (5.16) holds, if $J_m \leq 1$.

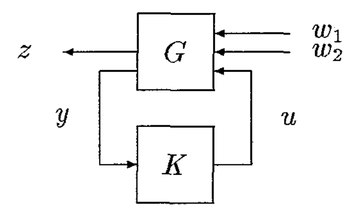
Now let w_1 be such that $\| w_1 \|_{\mathcal{S}} \leq 1$. Then the test $J_m \leq 1$ for a given D$\in \mathcal{D}$ is equivalent to

$$\sup_{w_1 \in \mathcal{S}, w_2 \in \mathcal{P}} \left\{ \| z \|_{\mathcal{P}}^2 - \| w_2 \|_{\mathcal{P}}^2 \right\} \leq 1.$$

To get the least conservative test possible, a search on D is required. Furthermore, (5.16) has two special cases:

- $w_1 = 0$: the so-called *robust $\mathcal{H}_\infty$ performance problem*, reducing to μ-analysis[12].
- $w_{21} = 0$: we shall call this the *robust $\mathcal{H}_2$ performance problem.*

Finally[13] we consider the synthesis problem, when the system is subjected to mixed disturbance signals and is described by the following diagram



where both G and K are assumed to be real-rational and proper. When we only consider the white and causal case, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem can be stated as: find an internally stabilizing controller K such that

$$\min_{K \text{ stabil.}} \sup_{w_2 \in \mathcal{P}} \left\{ \| z \|_{\mathcal{P}}^2 - \gamma^2 \| w_2 \|_{\mathcal{P}}^2 \right\} \tag{5.18}$$

is solved.

A both necessary and sufficient condition for this problem to be solvable is that there exists a K such that $\| T_{w_2 \to z} \|_\infty < \gamma$, i.e. the corresponding $\mathcal{H}_\infty$ problem ($w_1 = 0$) is solvable.

After a few mild assumptions have been made (see [Zhou et al. 90]), we can state the final result:

[12] Selecting the best D scalings for the mixed problem is not as simple as for μ-synthesis where these matrices can be taken to be constant for all frequencies.

[13] The fourth case, in which both Δ and K are present, is not treated in the paper.

THEOREM 5.6.5: Given $\gamma > 0$ and the plant G, there exists a controller K which solves problem (5.18) if and only if the following conditions hold:

1. $E_\infty \in \text{dom(Ric)}$ and $X_\infty := \text{Ric}(E_\infty) \geq 0$ where

$$E_\infty := \begin{bmatrix} A - B_3 D_{23}^T C_2 & \gamma^{-2} B_2 B_2^T - B_3 B_3^T \\ -C_2^T D_{23}^\perp (D_{23}^\perp)^T C_2 & -(A - B_3 D_{23}^T C_2)^T \end{bmatrix}$$

2. There exist $\bar{L}$, $\bar{Y}$ and $\bar{P}$ which satisfy

$$\bar{L} D_{31} D_{31}^T + B_1 D_{31}^T + \bar{P} C_3^T + \gamma^{-2} \bar{P} \bar{Y} \bar{L} D_{32} D_{32}^T + \gamma^{-2} \bar{P} \bar{Y} B_2 D_{32}^T = 0$$

$$\bar{Y}(\check{A} + \bar{L} C_3) + (\check{A} + \bar{L} C_3)^T \bar{Y} + \bar{Y} \tilde{R} \bar{Y} + F_\infty^T F_\infty = 0$$

$$\bar{Y} \geq 0 \qquad \text{and} \qquad \check{A} + \bar{L} C_3 + \tilde{R} \bar{Y} \text{ is stable.}$$

$$(\check{A} + \bar{L} C_3 + \tilde{R} \bar{Y}) \bar{P} + \bar{P} (\check{A} + \bar{L} C_3 + \tilde{R} \bar{Y})^T + (B_1 + \bar{L} D_{31})(B_1 + \bar{L} D_{31})^T = 0$$

Moreover, when these conditions hold, one such controller is

$$K(s) := \left[\begin{array}{c|c} \check{A} + B_3 F_\infty + \bar{L} C_3 & -\bar{L} \\ \hline F_\infty & 0 \end{array}\right]$$

where $\tilde{R} = \gamma^{-2}(B_2 + \bar{L} D_{32})(B_2 + \bar{L} D_{32})^T \qquad \check{A} = A + \gamma^{-2} B_2 B_2^T X_\infty$
and $F_\infty = -(D_{23}^T C_2 + B_3^T X_\infty)$.

The results presented here (according to the authors) turn out to have a superficial similarity with the results of [Bernstein and Haddad 89] that hints at deeper connections. It would therefore be useful to compare these results. However, this is not done in this report.

5.7 $\mathcal{H}_2/\mathcal{H}_\infty$: minimizing the worst-case $\mathcal{H}_2$-norm

Finally, we describe methods which use problem statement (2) from Chapter 3 (where p_1=2, p_2=∞) *and* four different sets of inputs and outputs w_1, w_2, z_1 and z_2 [Steinbuch and Bosgra 94, Stoorvogel 93]. This still allows for a considerable variety in the approach followed. In [Steinbuch and Bosgra 94] a 'lossless bounded real formulation' (this will be explained in the following) is used to parameterize the uncertainty $\Delta(s)$, thereby reducing the original constrained optimization to an unconstrained one. [Stoorvogel 93] uses a Lagrange multiplier φ for the same purpose. Both methods result in an optimization scheme of a form similar to the D-K-iteration in μ-synthesis (see also Section 5.6). To illustrate this both methods will be described in the following, starting with [Steinbuch and Bosgra 94]. They developed a parameterization for stable strictly proper $\mathcal{H}_\infty$ norm-bounded uncertainties using LMI's and exploit the situation when the worst-case perturbation is lossless bounded real.

- **Inequality formulation:**

THEOREM 5.7.1: Let $(F_\Delta, G_\Delta, H_\Delta, J_\Delta)$ be an asymptotically stable minimal realization of the transfer function $\Delta(s) = H_\Delta(sI - F_\Delta)^{-1}G_\Delta + J_\Delta$. Then the following statements are equivalent:

1. $\| \Delta \|_\infty < 1$
2. $\exists X > 0$ such that $\begin{bmatrix} F_\Delta^T X + XF_\Delta & XG_\Delta & H_\Delta^T \\ G_\Delta^T X & -I & J_\Delta^T \\ H_\Delta & J_\Delta & -I \end{bmatrix} < 0$
3. $\exists X > 0$ such that
   (a) $F_\Delta^T X + XF_\Delta + \begin{bmatrix} XG_\Delta & H_\Delta^T \end{bmatrix} \begin{bmatrix} I & -J_\Delta^T \\ -J_\Delta & I \end{bmatrix}^{-1} \begin{bmatrix} G_\Delta^T X \\ H_\Delta \end{bmatrix} < 0$
   (b) $\begin{bmatrix} -I & J_\Delta^T \\ J_\Delta & -I \end{bmatrix} < 0$

In the sequel we will denote the set of all transfer functions with $\| \Delta \|_\infty < 1$ as $\boldsymbol{\Delta}$. Theorem 5.7.1 directly leads to the following parameterization which characterizes all real rational causal stable transfer functions $\Delta(s)$ of order n_Δ having $\| \Delta \|_\infty < 1$.

1. Choose J_Δ such that

$$\begin{bmatrix} -I & J_\Delta^T \\ J_\Delta & -I \end{bmatrix} < 0 \tag{5.19}$$

2. Let G_Δ and H_Δ be matrices of appropriate dimensions containing the free parameters and let $F_\Delta = F_s + F_k$ with $F_s = F_s^T$ and $F_k = -F_k^T$, such that

$$F_s < -\frac{1}{2} \begin{bmatrix} G_\Delta & H_\Delta^T \end{bmatrix} \begin{bmatrix} I & -J_\Delta^T \\ -J_\Delta & I \end{bmatrix}^{-1} \begin{bmatrix} G_\Delta^T \\ H_\Delta \end{bmatrix} \tag{5.20}$$

and

$$F_k = \operatorname{diag} \begin{bmatrix} 0 & -a_i \\ a_i & 0 \end{bmatrix} \quad , \quad i = 1, 2, \dots, \frac{1}{2}n \tag{5.21}$$

for n even and

$$F_k = \begin{bmatrix} \operatorname{diag} \begin{bmatrix} 0 & -a_i \\ a_i & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad i = 1, 2, \dots, \frac{1}{2}(n-1) \tag{5.22}$$

for n uneven.

- **Lossless bounded real formulation:**

When the inequality constraints (5.19) and (5.20) are both active then the worst-case perturbation is lossless bounded real. This means that the perturbation is on its bounds at all frequencies and for all its singular values. Now we need two definitions:

DEFINITION 5.7.1: The real rational function $\Theta(s)$, $s \in \mathbb{C}$ is *lossless positive real* if $\Theta(s) + \Theta^{\sim}(s) = 0$.

DEFINITION 5.7.2: The real rational function $\Delta(s)$, $s \in \mathbb{C}$ is *lossless bounded real (LBR)* (or *inner*, see Section 5.2) if $\Delta^{\sim}(s)\Delta(s) = I$. The set of all such $\Delta(s)$ is denoted $\mathbf{\Delta}_{LBR}$.

LEMMA 5.7.1: Let $\Delta(s) = (I - \Theta(s))(I + \Theta(s))^{-1}$ then $\Delta(s)$ is lossless bounded real if and only if $\Theta(s)$ is lossless positive real.

LEMMA 5.7.2: Let $\Theta(s) = \bar{H}_\Delta(sI - \bar{F}_\Delta)^{-1}\bar{G}_\Delta + \bar{J}_\Delta$, with $\bar{F}_\Delta + \bar{F}_\Delta^T = 0$, $\bar{G}_\Delta = \bar{H}_\Delta^T$ (so it is assumed that $d_2 = q_2$) and $\bar{J}_\Delta + \bar{J}_\Delta^T = 0$, with $\bar{F}_\Delta \in \mathbb{R}^{n_\Delta \times n_\Delta}$ and $\bar{J}_\Delta \in \mathbb{R}^{d_2 \times d_2}$ ($=\mathbb{R}^{q_2 \times q_2}$), and with $\bar{H}_\Delta$ and $\bar{G}_\Delta$ of compatible dimensions. Then the real matrices $\bar{F}_\Delta$, $\bar{H}_\Delta$ and $\bar{J}_\Delta$ parameterize all lossless bounded real transfer functions Θ with state dimension n_Δ.

LEMMA 5.7.3: Let $\Delta(s) = (I - \Theta(s))\,(I + \Theta(s))^{-1}$ with $\Theta(s) = \bar{H}_\Delta\,(sI - \bar{F}_\Delta)^{-1}$ $\bar{G}_\Delta + \bar{J}_\Delta$ with $\bar{F}_\Delta$, $\bar{H}_\Delta$ and $\bar{J}_\Delta$ as defined in the previous lemma. Then a state-space realization for $\Delta(s)$ is given by:

$$\left[\begin{array}{c|c} F_\Delta & G_\Delta \\ \hline H_\Delta & J_\Delta \end{array}\right] = \left[\begin{array}{c|c} \bar{F}_\Delta - \bar{H}_\Delta^T(I + \bar{J}_\Delta)^{-1}\bar{H}_\Delta & -\sqrt{2}\bar{H}_\Delta^T(I + \bar{J}_\Delta)^{-1} \\ \hline \sqrt{2}(I + \bar{J}_\Delta)^{-1}\bar{H}_\Delta & (I - \bar{J}_\Delta)(I + \bar{J}_\Delta)^{-1} \end{array}\right] \tag{5.23}$$

And this is a parameterization for all stable lossless bounded real $\Delta(s)$.

Since matrices $\bar{F}_\Delta$ and $\bar{J}_\Delta$ are skew-symmetric, we further reduce the number of free variables and state the main result (for the parameterization):

THEOREM 5.7.2: Define the matrices θ_Δ and ϕ_Δ as upper triangular real matrices, with zero on their diagonal, and with appropriate dimensions, such that $\bar{F}_\Delta = \theta_\Delta - \theta_\Delta^T$, and $\bar{J}_\Delta = \phi_\Delta - \phi_\Delta^T$, then the triple $(\theta_\Delta, \phi_\Delta, \bar{H}_\Delta)$ parameterizes all stable lossless bounded real transfer functions $\Delta(s) = H_\Delta(sI - F_\Delta)^{-1}G_\Delta + J_\Delta$ with H_Δ, F_Δ, G_Δ and J_Δ defined by (5.23).

Now consider the system with feedback. This can be described by (3.1)–(3.4) where D_{22}, D_{33} (and D_{11}) are taken to be zero. The perturbed system, where $\Delta(s) = H_\Delta(sI - F_\Delta)^{-1}G_\Delta + J_\Delta$, is then given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} A + B_2 J_\Delta C_2 & B_2 H_\Delta \\ G_\Delta C_2 & F_\Delta \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B_1 + B_2 J_\Delta D_{21} \\ G_\Delta D_{21} \end{bmatrix} w_1 + \begin{bmatrix} B_3 + B_2 J_\Delta D_{23} \\ G_\Delta D_{23} \end{bmatrix} u \\ &=: A_p x_p + B_{p1} w_1 + B_{p3} u \end{aligned} \tag{5.24}$$

$$z_1 = \left[\begin{array}{cc} C_1 + D_{12}J_\Delta C_2 & D_{12}H_\Delta \end{array} \right] \left[\begin{array}{c} x \\ p \end{array} \right] + \left[\begin{array}{c} D_{13} + D_{12}J_\Delta D_{23} \end{array} \right] u =: C_{p1}x_p + D_{p13}u \tag{5.25}$$

$$y = \left[\begin{array}{cc} C_3 + D_{32}J_\Delta C_2 & D_{32}H_\Delta \end{array} \right] \left[\begin{array}{c} x \\ p \end{array} \right] + \left[\begin{array}{c} D_{31} + D_{32}J_\Delta D_{21} \end{array} \right] w_1 =: C_{p3}x_p + D_{p31}w_1 \tag{5.26}$$

Notice that $D_{12}J_\Delta D_{21}$ needs to be 0 for $\| T_{w_1 \to z_1}(\Delta) \|_2 < \infty$ and that $D_{32}J_\Delta D_{23} = 0$ is assumed for simplicity.

The design problem (where $\mathbf{\Delta}_{(LBR)}$ denotes either $\mathbf{\Delta}$ or $\mathbf{\Delta}_{LBR}$) is

$$\sup_{\Delta \in \mathbf{\Delta}_{(LBR)}} \min_{K(s)} \| T_{w_1 \to z_1}(K, \Delta) \|_2$$

If we assumed that $\Delta_k(s)$ would qualify as the worst-case uncertainty, we could determine the feedback law $K^\star(s)$ that would be $\mathcal{H}_2$-optimal. By computing an $\mathcal{H}_2$-optimal $K^\star(s)$ for each $\Delta_k(s)$, we iterate over $\Delta_k(s)$ until it satisfies the conditions for a worst-case disturbance. This is the '*D*-*K*'-like procedure which we mentioned in the foregoing.

When (5.24)–(5.26) is assumed to be stable and $\Delta_k(s)$ (and thereby F_Δ, G_Δ, H_Δ, J_Δ) is assumed to be fixed, the optimization problem

$$\min_K \| T_{w_1 \to z_1}(u = K(s)y) \|_2$$

can be solved as a standard $\mathcal{H}_2$ or LQG type of problem:

$$(A_p - B_p D_{p1}^T C_{p1})^T X + X(A_p - B_p D_{p1}^T C_{p1}) - X B_p B_p^T X + C_{p1}^T (I - D_{p1} D_{p1}^T) C_{p1} = 0 \tag{5.27}$$

$$(A_p - B_{p1} D_{1p}^T C_p) Y + Y(A_p - B_{p1} D_{1p}^T C_p)^T - Y C_p^T C_p Y + B_{p1}(I - D_{1p}^T D_{1p}) B_{p1}^T = 0 \tag{5.28}$$

and the $\mathcal{H}_2$ optimal control law $u = K^\star(s)y$ is defined by

$$\begin{array}{rcl} \dot{v} & = & [A_p - B_p(B_p^T X + D_{p1}^T C_{p1})]v + [Y C_p^T + B_{p1} D_{1p}^T]y \\ u & = & -[B_p^T X + D_{p1}^T C_{p1}]v \end{array} \tag{5.29}$$

The optimization problem including the uncertainty $\Delta \in \mathbf{\Delta}$ can now be formulated as a **constrained** optimization problem over a standard $\mathcal{H}_2$ optimal control problem:

$$\max_{(F_s,F_k,G_\Delta,H_\Delta,J_\Delta)} \| T_{w_1\to z_1}(u = K^\star(s)y) \|_2 \tag{5.30}$$

with $K^\star$ the solution to (5.27)–(5.29), and F_s, F_k, G_Δ, H_Δ, J_Δ according to (5.19)–(5.22). If $\Delta \in \mathbf{\Delta}_{LBR}$ the optimization problem can be formulated as an **unconstrained** optimization problem:

$$\max_{(\theta_\Delta,\phi_\Delta,\bar{H}_\Delta)} \| T_{w_1\to z_1}(u = K^\star(s)y) \|_2 \tag{5.31}$$

with $K^\star$ the solution to (5.27)–(5.29) , and with $(F_\Delta, G_\Delta, H_\Delta, J_\Delta)$ defined by (5.23), where $\bar{F}_\Delta = \theta_\Delta - \theta_\Delta^T$, $\bar{J}_\Delta = \phi_\Delta - \phi_\Delta^T$. In the paper it is not mentioned whether this iteration *always* converges, or, if it doesn't always, under what conditions it does.

In [Stoorvogel 93], which will be described briefly, two interpretations of the $\mathcal{H}_2$ norm are used, i.e.:

1. The square root of the energy contained in the impulse response and
2. The RMS-value of the response to a white noise input.

The approach is said to be conservative in the sense that the disturbance system is not assumed to be causal. Furthermore, the uncertainty is assumed to be unstructured, although frequency dependent weights can be incorporated. For state-space realization (3.1)–(3.4) it is assumed that A is stable and $\| T_{w_2\to z_2} \|_\infty < 1$. Furthermore D_{21} (and D_{11}) is taken to be zero, so the system without feedback has the state-space realization

$$G : \begin{cases} \dot{x} &= Ax + B_1 w_1 + B_2 w_2 \\ z_1 &= C_1 x + D_{12} w_2 \\ z_2 &= C_2 x + D_{22} w_2 \end{cases} \tag{5.32}$$

In this paper several different maximal $\mathcal{H}_2$ costs are defined namely $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{B}_1^\varphi$, $\mathcal{B}_{st}$, $\mathcal{B}_2$ and $\mathcal{B}_{ac}$.

$\mathcal{B}$ is the maximal $\mathcal{H}_2$ norm for the case where $\Delta(s)$ is assumed to be causal

$$\mathcal{B}(x_0) := \sup_{w_2\in\mathcal{L}_2} \{ \| z_1 \|_2^2 \mid x(0) = x_0, \int_0^t \|z_2(\tau)\|_2^2 - \| w_2(\tau) \|_2^2 \, \mathrm{d}\tau \geq 0 \;\; \forall t > 0 \}$$

where x_0 denotes the nonzero initial condition of x, representing an impulse on $x = 0$. This corresponds to definition (1.) of the $\mathcal{H}_2$ norm where the input is an impulse. This problem is particularly hard to solve and at this moment the solution is unknown. $\mathcal{B}_1$ denotes the case where $\Delta(s)$ is non-causal

$$\mathcal{B}_1(x_0) := \sup_{w_2\in\mathcal{L}_2} \{ \| z_1 \|_2^2 \mid x(0) = x_0, \| w_2 \|_2 \leq \|z_2\|_2 \}$$

It is shown that this $\mathcal{B}_1$ depends on the particular basis of the input space chosen for the impulse input. To get rid of this dependence $\mathcal{B}_2$ will eventually be introduced. First $\mathcal{B}_1^{\varphi}$ is defined as

$$\mathcal{B}_1^{\varphi}(x_0) := \sup_{w_2 \in \mathcal{L}_2} \{\| z_1 \|_2^2 + \varphi(\| z_2 \|_2^2 - \| w_2 \|_2^2)\}$$

where $\varphi \geq 0$ is the Lagrange multiplier. $\mathcal{B}_1^{\varphi}$ is shown to be equal to $x_0^T P(\varphi) x_0$ where $P(\varphi)$ solves the algebraic Riccati equation:

$$\begin{aligned} 0 \;=\; & (B_2^T P + D_{12}^T C_1 + \varphi D_{22}^T C_2)^T (\varphi I - D_{12}^T D_{12} - \varphi D_{22}^T D_{22})^{-1} (B_2^T P + D_{12}^T C_1 + \varphi D_{22}^T C_2) \\ & + A^T P + PA + C_1^T C_1 + \varphi C_2^T C_2 \end{aligned} \tag{5.33}$$

such that

$$A + B_2(\varphi I - D_{12}^T D_{12} - \varphi D_{22}^T D_{22})^{-1}(B_2^T P + D_{12}^T C_1 + \varphi D_{22}^T C_2) \quad (=: A_{cl} \text{ in [Stoorvogel 93]})$$

$$\text{is asymptotically stable.} \tag{5.34}$$

Such a P exists if and only if

$$\varphi > \varphi_{min} = \inf \left\{ \varrho^2 \;\middle|\; \left\| \begin{bmatrix} C_1 \\ \varrho C_2 \end{bmatrix} (sI - A)^{-1} B_2 + \begin{bmatrix} D_{12} \\ \varrho D_{22} \end{bmatrix} \right\|_\infty \leq \varrho \right\} \tag{5.35}$$

$\mathcal{B}_{st}$ is the worst-case $\mathcal{H}_2$ norm over all static linear time-varying disturbance systems. Finally, $\mathcal{B}_2$ is derived as an upper bound for $\mathcal{B}_{st}$. However, it turns out to be quite a crude bound since it is also an upper bound for $\mathcal{B}_1$. Using the second definition of the $\mathcal{H}_2$ norm, it is shown that

$$\mathcal{B}_{st} \leq \mathcal{B}_2 := \inf \{\mathrm{tr}[B_1^T P(\varphi) B_1] \mid \varphi > \varphi_{min}\}$$

Ultimately, the relation of $\mathcal{B}_2$ to the auxiliary cost of a related problem is investigated. Therefore, the following related system (still without feedback) is defined:

$$G_\varphi : \left\{ \begin{array}{rcl} \dot{x} & = & Ax + B_1 w_1 + B_2 w_2 \\ z & = & \begin{bmatrix} C_1 \\ \sqrt{\varphi} C_2 \end{bmatrix} x + \begin{bmatrix} D_{12} \\ \sqrt{\varphi} D_{22} \end{bmatrix} w_2 \end{array} \right. \tag{5.36}$$

When $\mathcal{B}_{ac}$ is defined as the auxiliary cost of the system (see Section 4.3), we have

$$\mathrm{tr}[B_1 P(\varphi) B_1] = \mathcal{B}_{ac}(\varphi, G_\varphi)$$

(where $P(\varphi)$ again solves (5.33) subject to (5.34) and (5.35)) and therefore

$$\mathcal{B}_2 = \inf_{\varphi > \varphi_{min}} \mathcal{B}_{ac}(\varphi, G_\varphi)$$

Then it can be shown that $\mathcal{B}_{st} \leq \mathcal{B} \leq \mathcal{B}_1 \leq \mathcal{B}_2 \leq \mathcal{B}_{ac}$ and $\mathcal{B}_1 \leq \mathcal{B}_1^{\varphi}$.

Now augment G to include a controller K:

$$G : \begin{cases} \dot{x} &= Ax + B_1 w_1 + B_2 w_2 + B_3 u \\ z_1 &= C_1 x + D_{12} w_2 + D_{13} u \\ z_2 &= C_2 x + D_{22} w_2 + D_{23} u \\ y &= C_3 x + D_{31} w_1 + D_{32} w_2 \end{cases} \tag{5.37}$$

We will minimize

$$\mathcal{B}_2(G \times K) = \inf_{\varphi} \mathcal{B}_2(\varphi, G \times K)$$

(this equality is shown to be true in [Stoorvogel 93]) where K is a stabilizing controller of the form:

$$K : \begin{cases} \dot{v} &= H_{11} v + H_{12} y \\ u &= H_{21} v + H_{22} y \end{cases} \tag{5.38}$$

Next, for *fixed* φ, we define the following related system (with feedback):

$$G_\varphi : \begin{cases} \dot{x} &= Ax + B_1 w_1 + B_2 w_2 + B_3 u \\ z &= \begin{bmatrix} C_1 \\ \sqrt{\varphi} C_2 \end{bmatrix} x + \begin{bmatrix} D_{12} \\ \sqrt{\varphi} D_{22} \end{bmatrix} w_2 + \begin{bmatrix} D_{13} \\ \sqrt{\varphi} D_{23} \end{bmatrix} u \\ y &= C_3 x + D_{31} w_1 + D_{32} w_2 \end{cases} \tag{5.39}$$

Minimization of $\mathcal{B}_2(\varphi, G \times K)$ over all stabilizing controllers is shown to be equivalent to minimization of $(1+\varphi)\mathcal{B}_{ac}(\varphi, G_\varphi \times K)$ over all stabilizing controllers. The problem of minimizing $\mathcal{B}_{ac}(\varphi, G_\varphi \times K)$ has been discussed in literature (see [Bernstein and Haddad 89, Doyle et al. 89, Khargonekar and Rotea 91, Zhou et al. 90]). Next some assumptions are stated through which the results of [Khargonekar and Rotea 91] are applicable, where the problem is reduced to a convex optimization over a finite-dimensional space as was seen in Section 4.5. Here, an additional parameter search over φ must be carried out to obtain the smallest worst-case $\mathcal{H}_2$ measure $\mathcal{B}_2$, i.e. we apply the following scheme:

$$\begin{aligned} \inf_K \mathcal{B}_2(G \times K) &= \inf_K \inf_\varphi \mathcal{B}_2(\varphi, G \times K) \\ &= \inf_\varphi \inf_K (1+\varphi)\mathcal{B}_{ac}(\varphi, G_\varphi \times K) \\ &= \inf_\varphi (1+\varphi) \inf_K \mathcal{B}_{ac}(\varphi, G_\varphi \times K) \end{aligned}$$

As was mentioned before, this scheme has the form of a 'φ-K'-iteration similar to the D-K-iteration, well-known from μ-synthesis.

Obviously, the two methods described differ in many ways, but on the other hand they do have things in common, one of which is the important feature of not having the drawback of equalized input sets and/or output sets: a problem setting that only allows for a restricted class of problems.

Finally, an approach that may be classified into this category is [Rotea and Khargonekar 91]. Here, again $w_1 \neq w_2$, $z_1 \neq z_2$ and the solution is obtained by solving ARE's (not LMI's).

On the other hand, they do not use a 'D-K' type of optimization to achieve the worst-case uncertainty. However, necessary and sufficient conditions are derived for an unconstrained optimization problem, whereas the original optimization problem is a constrained one, for which these conditions are sufficient. So again, the optimization is reduced from constrained to unconstrained.

Chapter 6

Conclusions

We have surveyed a large number of approaches to solve the mixed-norm optimization problem. It was seen that all but one focus on solving the two-norm problem, although this one approach (see Section 5.1), which considers a three-norm problem setting does not really exploit this possibility and ends up giving no more than a vague description of what the methodology would look like. It was also seen that the $\mathcal{H}_2/\mathcal{H}_\infty$ problem received the greatest deal of attention. This is due to the fact that the need for a mixed-norm formalism originates from the separate $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theories not being able to accommodate all practical design specifications. To accommodate bounded-magnitude signals, the ℓ_1 optimal control theory was developed, but not until a few years ago, which explains the relatively small number of approaches to this problem. Most of the approaches tend to have an ad hoc character, but the same is said for μ-synthesis [Zhou et al. 90], which has been successfully applied in recent years. All methods have their pros and cons, and differ in complexity depending on how general the problem is posed. A list of this can be found on the following page. This list will not be complete, but serves as an overview of what was discussed throughout Chapter 5.

It is not clear which one of these approaches qualifies as most promising. The future will point out which methods are best suited for practical application, but all efforts will undoubtedly contribute to what must become a clean closed-loop solution to the mixed-norm optimization problem.

Recently (December 1995), a number of articles were published in *Proceedings of the* 34^{th} *Conference of Decision and Control* that with regard to this survey deserve our attention, but couldn't be included in this report. This concerns a.o. an article by M. Sznaier, M. Holmes and J. Bu [Sznaier et al. 95] that addresses the mixed $\mathcal{H}_2/\mathcal{L}_1$ control problem where $w_1 = w_2$, $z_1 \neq z_2$. They utilize LMI's to arrive at a convex optimization problem (combined with a 'line-search'=one-dimensional minimization). Another article worth mentioning is [Kapila and Haddad 95] by V. Kapila and W. M. Haddad, that considers the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ stabilization problem, where $w_1 = w_2$ and $z_1 = z_2$. It is recommended that a survey of mixed-norm optimization techniques is carried out every few years, since the work in this area is growing and for some time to come not finished.

APPROACH	PRO	CONTRA
1. $\ell_1/\frac{\mathcal{H}_2}{\mathcal{H}_\infty}$: a linear programming approach	-accommodates bounded-magnitude signals -possibly three-norm optimization -possibly $w_1 \neq w_2$, $z_1 \neq z_2$ -can be solved using the efficient DA method that doesn't (necessarily) suffer from order inflation	-(mostly) $w_1 = w_2$, $z_1 = z_2$ -specifications such as $\mathcal{H}_\infty$ constraints must be approximated; when this is done by sampling the unit circle this may prevent a solution from being found
2. $\ell_1/\mathcal{H}_\infty$: using the Youla-parametrization	-accommodates persistent bounded signals -$w_1 \neq w_2$, $z_1 \neq z_2$ -convex optimization -provides an iterative algorithm	-may result in very high order controllers
3. $\mathcal{H}_2/\mathcal{H}_\infty$: convex optimization using matrix inequalities	-convex optimization -LMI based optimization can be solved with very efficient algorithms -some approaches take $w_1 \neq w_2$, $z_1 \neq z_2$	-not all approaches take $w_1 \neq w_2$, $z_1 \neq z_2$
4. $\mathcal{H}_2/\mathcal{H}_\infty$: optimizing an entropy cost functional	-relatively simple solution	-$w_1 = w_2$, $z_1 = z_2$
5. $\mathcal{H}_2/\mathcal{H}_\infty$: fixed-order controller design using the auxiliary cost	-fixed (reduced) order controller design -$z_1 \neq z_2$ -provides an algorithm -homotopy techniques can effectively solve the ARE's	-$w_1 = w_2$ -existing Riccati equation solvers cannot handle the ARE's that result from the fixed-order controller design
6. $\mathcal{H}_2/\mathcal{H}_\infty$: using a bounded power characterization	-induced norm interpretation, instead of an ad hoc upper bound -$w_1 \neq w_2$ -(structured uncertainty)	-$z_1 = z_2$
7. $\mathcal{H}_2/\mathcal{H}_\infty$: minimizing the worst-case $\mathcal{H}_2$-norm	-$w_1 \neq w_2$, $z_1 \neq z_2$ -[Stoorvogel:] convex optimization -[Steinbuch and Bosgra:] lossless bounded real formulation leads to unconstrained optimization	-[Stoorvogel:] uncertainty not assumed to be causal -[Stoorvogel:] unstructured uncertainty

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