Analysis of both kinematically and statically velocity fields in plane strain compression

Forming

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Analysis of both kinematically and statically admissible velocity fields in plane strain compression*).

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Summary: In connection with the CIRP co-operative program on upsetting and complementary to the finite elements analysis several classes of velocity fields have been investigated with respect to minimizing or stabilizing power in the system, in the case of non-strainhardening material. Next it is examined whether such fields satisfy local equilibrium as well as body equilibrium. It is shown that the modified elliptic velocity field satisfies all requirements.
Application of this velocity field enables to calculate both the stress - and the strain rate distribution throughout the specimen.
Finally through an incremental procedure the kinematics of deformation can be calculated.

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1. INTRODUCTION

With an eye to the CIRP co-operative work in upsetting [1, 2, 3] several classes of velocity fields have been investigated in order to find whether they stabilize or minimize the process with respect to power on the one hand and satisfy conditions for local and body equilibrium at the other.
For the time being the work is restricted to a situation of plane strain in non-strain hardening (rigid-ideal plastic) material.
It is found that the elliptic velocity field satisfies conditions for minimum power as well as local equilibrium, however, it violates the requirements for body equilibrium. It is shown that modification of the velocity field gives a solution for the latter problem.
From the analysis it follows that upsetting proceeds in three phases. If the compression ratio, which is the width b of the sample over its height h,

$$b^* = \frac{b}{h} < 2 \tag{1}$$

the process is mechanically unstable.
If $2 < b^* < 6$, the velocity field is virtually elliptic. Body equilibrium is provided by stress peaks at the very edges of the specimen.

Finally, if $b^* > 6$, the velocity field is modified elliptic and the system moves to a slightly lower level of power as compared to the elliptic case.
From the velocity fields thus defined both the stress distribution throughout the specimen and the distortion of it can be calculated.

2. THE ELLIPTIC VELOCITY FIELD

In the present analysis reduced (dimensionless) quantities are used

$$\left.\begin{aligned} & x^* = \frac{x}{h} \;;\; z^* = \frac{z}{h} \;;\; b^* = \frac{b}{h} \\ & \sigma^*_{ij} = \frac{\sigma_{ij}}{\bar{\sigma}} \;;\; \dot{\varepsilon}^*_{ij} = \dot{\varepsilon}_{ij}\,\frac{h}{U_0} \\ & U^*_i = \frac{U_i}{U_0} \end{aligned}\right\} \qquad (2)$$

where U_0 is the velocity of the press ram and $\bar{\sigma}$ stands for the effective stress, which as said before is considered to be a constant.
For the sake of simplicity the asterisks will be omitted.
The elliptic velocity field is defined as

$$U_x = x\left[A + B\sqrt{1 - (2Pz)^2}\,\right] \qquad (3)$$

thus being symmetrical with respect to z, if $z = 0$ represents the plane of symmetry. Moreover the function assumes a maximum in $z = 0$, whereas it is antisymmetrical with respect to x.
In this definition the velocity of the upper platen is $-\frac{1}{2}U_0$ with respect to $z = 0$.
The condition for continuity of the material flow requires

$$\tfrac{1}{2}x = \int_0^{\frac{1}{2}} U_x\, dz \qquad (4)$$

through which follows

$$U_x = x\left[1 + B\left\{\sqrt{1 - (2Pz)^2} - 2J\right\}\right] \qquad (5)$$

where

$$2J = \tfrac{1}{2}\left\{\sqrt{1-P^2} + \frac{1}{P}\arcsin P\right\} \qquad (6)$$

From eq. 5 it is derived

$$\dot{\varepsilon}_{xx} = \frac{\partial U_x}{\partial x} = 1 + B\left\{\sqrt{1 - (2Pz)^2} - 2J\right\} \qquad (7)$$

*) Full text in report WPT 0439 EINDHOVEN UNIVERSITY PRESS.

and since in plane strain it holds

$$\dot{\varepsilon}_{zz} = - \dot{\varepsilon}_{xx} \tag{8}$$

it is obtained by integration

$$-U_z = z \left[1 + B\left\{\tfrac{1}{2}\sqrt{1 - (2Pz)^2} - 2J\right\}\right] + \frac{B}{4P} \text{arc sin } 2Pz \tag{9}$$

which satisfies the boundary conditions at z = 0 as well as at $z = \pm \frac{1}{2}$.

Next it follows

$$\frac{\partial U_z}{\partial x} = 0 \tag{10}$$

and hence

$$\dot{\varepsilon}_{xz} = \tfrac{1}{2}\frac{\partial U_x}{\partial z} = - 2BP^2 \frac{xz}{\sqrt{1 - (2Pz)^2}} \tag{11}$$

From the strain rates thus derived the effective strain rate is calculated as

$$\dot{\varepsilon} = \frac{2}{\sqrt{3}} \sqrt{\dot{\varepsilon}_{xx}^2 + \dot{\varepsilon}_{xz}^2} =$$

$$= \frac{2}{\sqrt{3}} \left[\left[1 + B\left\{ \sqrt{1-(2Pz)^2} - 2J \right\} \right]^2 + \frac{\left\{2BP^2xz\right\}^2}{1-(2Pz)^2} \right]^{\frac{1}{2}} \tag{12}$$

According to Levy-von Mises the shear stress now can be found as

$$\tau_{xz} = \frac{2}{3} \frac{\dot{\varepsilon}_{xz}}{\dot{\varepsilon}}$$

$$= -\frac{2}{\sqrt{3}} BP^2 \frac{xz}{\left[\left\{1-(2Pz)^2\right\} \left\{1+B\left(\sqrt{1-(2Pz)^2}-2J\right)\right\}^2 + \left\{2BP^2xz\right\}^2 \right]^{\frac{1}{2}}} \tag{13}$$

By substitution of $z = \frac{1}{2}$ the frictional stress in the contact area between tool and workpiece is calculated to be

$$\tau_0 = -\frac{1}{\sqrt{3}} BP^2 \frac{x}{\left[(1-P^2) \left\{1+\frac{B}{2}\left(\sqrt{1-P^2} - \frac{1}{P} \text{arc sin } P\right)\right\}^2 + \left\{BP^2x\right\}^2 \right]^{\frac{1}{2}}} \tag{14}$$

It is remarked that in order to achieve this result not any assumption with respect to friction has been introduced.
Next it is obvious that for increasing x the limiting value of the frictional stress is $\tau_0 = \tau_{MAX} = 1/\sqrt{3}$.
The relative sliding velocity follows from eq. 5 for $z = \frac{1}{2}$ as

$$U_{xo} = x\left[1 + \frac{B}{2}\left(\sqrt{1-P^2} - \frac{1}{P} \text{arc sin } P\right)\right] \tag{15}$$

Since this quantity cannot be negative for any positive value of x, the kinematical constraint to impose on the parameter B is

$$B \le \frac{2}{\frac{1}{P} \text{arc sin } P - \sqrt{1-P^2}} = B_{MAX} \tag{16}$$

The reduced power in the system is

$$P = P_d + P_f \tag{17}$$

where

$$P_d = \frac{4}{b} \int_{x=0}^{b/2} \int_{z=0}^{1/2} \dot{\varepsilon}\, dxdz \tag{18}$$

the contribution of deformation

and

$$P_f = \frac{4}{b} \int_{0}^{b/2} |\tau_0|\, U_{xo}\, dx \tag{19}$$

the contribution of friction.

Through the relationships obtained up to now the eqs. 18 and 19 can be expressed in terms of the co-ordinates and the parameters B and P.
Variational analysis shows that eq. 17 clearly stabilizes and even minimizes in dependence of the parameters.
A typical example is shown in fig. 1, from which can be seen that the minimum is extremely flat as a function of B. Hence it is difficult to determine the optimum value of the parameter sufficiently accurate. The same holds for the parameter P, which proves to be close to one.

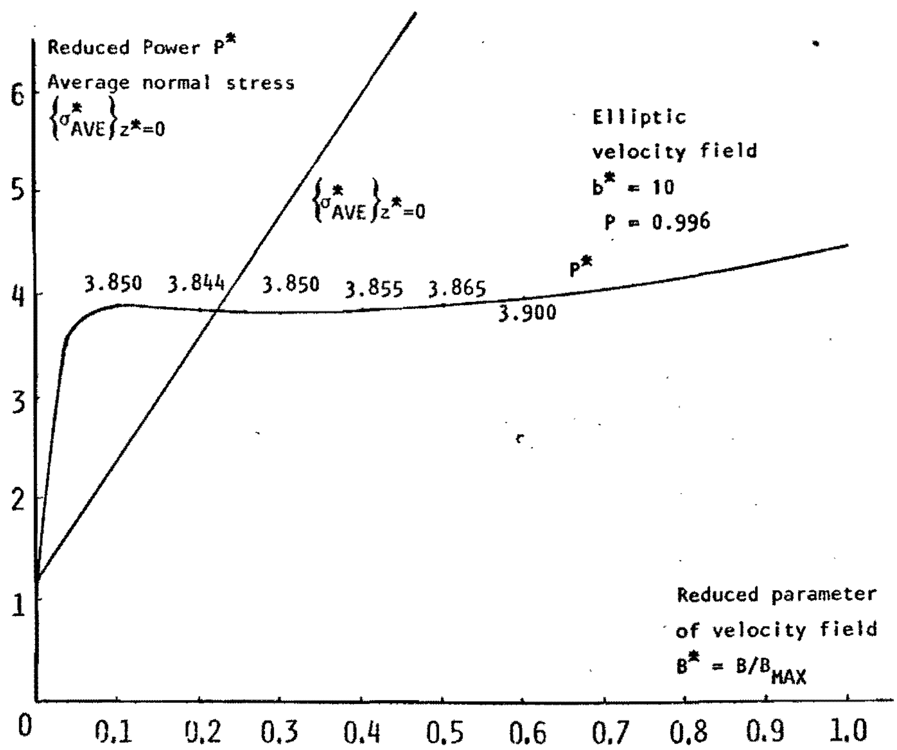

Fig. 1. The reduced power and the reduced average normal stress in the plane z=0 as a function of the reduced parameter B^* of the elliptic velocity field, for a compression ratio $b^* = 10$. The point of intersection determines the particular value of the parameter through which the velocity field satisfies conditions of minimum power as well as of equilibrium with respect to the plane z=0.

However, the reduced power has the same physical meaning as the reduced average normal stress on any plane z, which is

$$\left\{\sigma_{AVE}\right\}_z = \frac{1}{b/2} \int_0^{b/2} \sigma_z \, dx \qquad (20)$$

Now the first differential equation of equilibrium in plane strain is

$$\frac{\partial \sigma_x}{\partial x} = - \frac{\partial \tau_{xz}}{\partial z} \qquad (21)$$

In the plane z=0 it follows from eq. 13 that $\tau_{xz} = 0$ and hence the von Mises plasticity condition reduces to

$$\left\{\sigma_z\right\}_{z=0} = \left\{\sigma_x\right\}_{z=0} - \frac{2}{\sqrt{3}} \qquad (22)$$

The boundary condition is

$$\left\{\sigma_x\right\}_{z=0} = 0 \quad \text{for} \quad x = b/2 \qquad (23)$$

and consequently

$$\left\{\sigma_z\right\}_{z=0} = - \frac{2}{\sqrt{3}} + \int_x^{b/2} \left\{\frac{\partial \tau_{xz}}{\partial z}\right\}_{z=0} dx \qquad (24)$$

and

$$\left\{\sigma_{AVE}\right\}_{z=0} = - \frac{2}{\sqrt{3}} + \frac{1}{b/2} \int_0^{b/2} \int_x^{b/2} \left\{\frac{\partial \tau_{xz}}{\partial z}\right\}_{z=0} dx dx \qquad (25)$$

Application of eq. 13 renders

$$\left\{\sigma_z\right\}_{z=0} = - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \frac{BP^2}{1+B(1-2J)} \left\{\left(\frac{b}{2}\right)^2 - x^2\right\} \qquad (26)$$

and

$$\left\{\sigma_{AVE}\right\}_{z=0} = - \frac{2}{\sqrt{3}} \left\{1 + \frac{1}{3} \frac{BP^2}{1+B(1-2J)} \left(\frac{b}{2}\right)^2\right\} \qquad (27)$$

As also shown in fig. 1 the latter function intersects the curve of reduced power in a well defined way. Now very accurate variational analysis can be performed in order to move the point of intersection to minimum power.
After this having been achieved the parameters B and P are such

that the system satisfies conditions for minimum power as well as conditions for equilibrium with respect to the plane z=0. The final results are visualized in fig. 2.

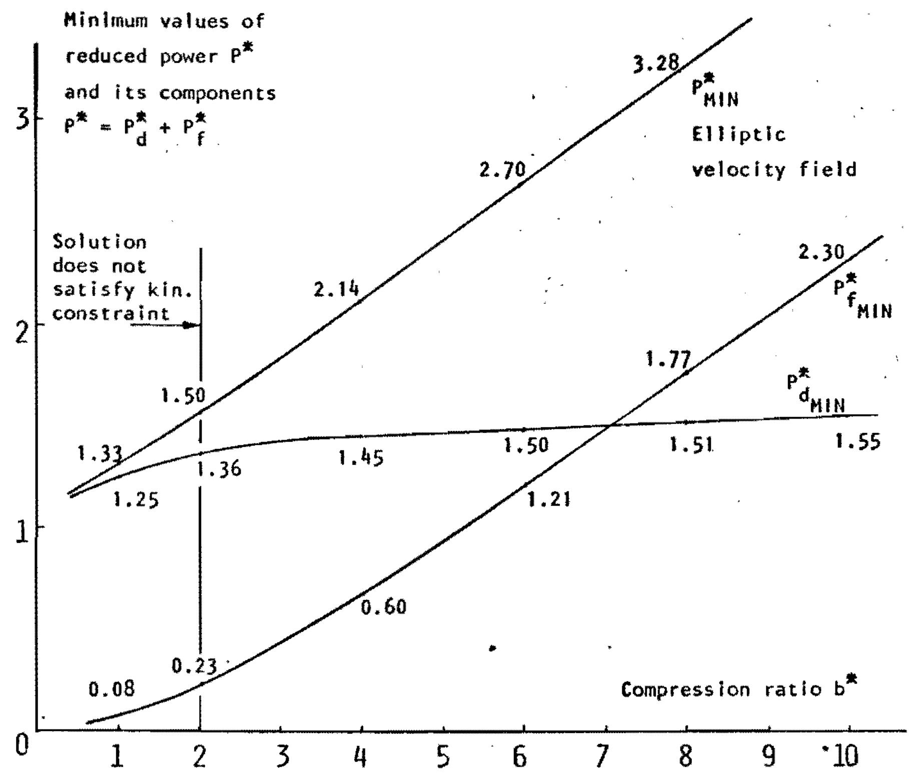

Fig. 2. Minimum values of reduced power and its components due to deformation and due to friction, respectively, as a function of compression ratio. Elliptic velocity field.

There exists a hyperbolic relationship between the parameter B and the compression ratio b

$$B \cdot b = 3 \tag{28}$$

and analogeous

$$b(1-P) = \frac{1}{3^3} = 0.037 \tag{29}$$

These relationships make it possible to calculate stress and strain solely as a function of the instantaneous compression ratio.

It is remarked that

1. if the compression ratio b < 1.8 the kinematical constraint is violated. In the present model the situation is mechanically unstable.

2. the linearity as well as the inclination of the curve in fig. 2 agree very well with the results reported by Unksow [4] and Prandtl [5].
3. when the compression ratio increases the effect of friction becomes increasingly dominant.

The second equation of equilibrium in plane strain is

$$\frac{\partial \sigma_z}{\partial z} = - \frac{\partial \tau_{xz}}{\partial x} \tag{30}$$

Hence the normal stress in the contact plane between tool and workpiece is

$$\left\{\sigma_z\right\}_{z=\frac{1}{2}} = - \int_0^{\frac{1}{2}} \left\{\frac{\partial \tau_{xz}}{\partial x}\right\}_x dz + \left\{\sigma_z\right\}_{z=0} \tag{31}$$

According to eq. 13 τ_{xz} is a steadily decreasing function of x. This implies that in eq. 31 the absolute value of the normal stress in the contact plane in any point x is less than in the corresponding point in the plane z=0. Consequently it holds

$$\left\{\sigma_{AVE}\right\}_{z=0} \neq \left\{\sigma_{AVE}\right\}_{z=\frac{1}{2}} \tag{32}$$

which states that the elliptic velocity field does not satisfy body equilibrium.

3. THE MODIFIED ELLIPTIC VELOCITY FIELD

The problem thus encountered can be met by modifying the velocity field in such a way that τ_{xz} is a non-steady function of x. It is proposed

$$U_x = x \left[1+B\left\{1 - \left(\frac{\mu x}{b/2}\right)^2\right\}^n \left\{\sqrt{1-(2Pz)^2} - 2J\right\}\right] \tag{33}$$

where

$$\begin{cases} n \geq 0 \\ \mu \approx 1 \end{cases}$$

The latter quantity is introduced merely to avoid instability of the computation at the very edge x = b/2.
Performing the analysis in the same way as in the previous section results in rather complicated expressions for stress and strain (see report WPT 0439). The major conclusion is that the modified elliptic velocity field satisfies body equilibrium in two different cases.

$$\left.\begin{aligned} n &= 3b^2\ 10^{-4} \\ B &= 3/b \\ P &= 1 - 1/3^3\ b \end{aligned}\right\} \tag{34}$$

for $2 < b < 6$

which because of the small value of the parameter n is named the quasi-elliptic velocity field.
A typical example of stress distribution is shown in fig. 3. Obviously conditions for body equilibrium are satisfied by development of a stress peak at the very edge of the specimen, as also observed in experiments [6].

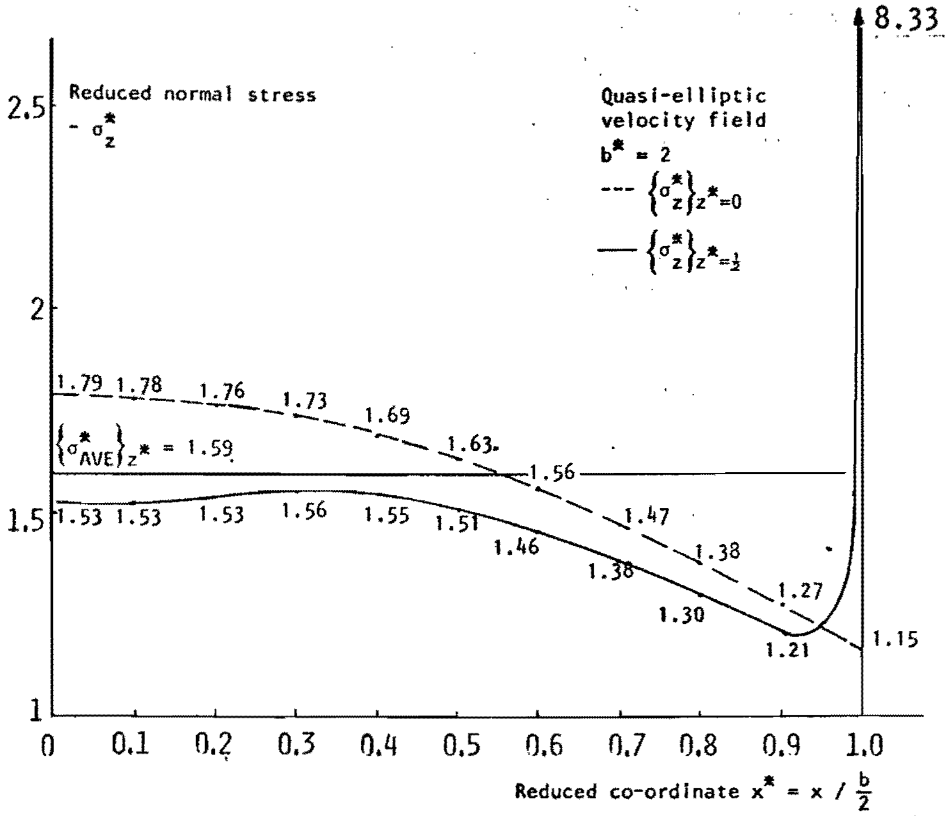

Fig. 3. The distribution of normal stress in the plane z=0 as well as in the contact plane between tool and workpiece for the quasi-elliptic velocity field.

The second solution is

$$\left.\begin{aligned} & n \approx 1 \\ & B = \frac{2\pi + \mathrm{Si}\ (b/7\pi)}{b} \quad {}^{*)} \\ & P = 1 - \left(\frac{2}{b}\right)^6 \\ & \text{for } b > 6 \end{aligned}\right| \qquad (35)$$

which is the modified elliptic velocity field.
The basic equations of the modified elliptic field are

$$U_x = x\left[1+B\left\{1-\left(\frac{x}{b/2}\right)^2\right\}\left\{\sqrt{1-(2Pz)^2}-2J\right\}\right] \tag{36}$$

$$-U_z = z+B\left\{1-3\left(\frac{x}{b/2}\right)^2\right\}\left\{\frac{1}{2}z\sqrt{1-(2Pz)^2}+\frac{1}{4P}\arcsin 2Pz-2Jz\right\} \tag{37}$$

$$\dot{\varepsilon}_{xx} = 1+B\left\{1-3\left(\frac{x}{b/2}\right)^2\right\}\left\{\sqrt{1-(2Pz)^2}-2J\right\} \tag{38}$$

$$\frac{1}{2}\frac{\partial U_x}{\partial z} = -2BP^2\left\{1-\left(\frac{x}{b/2}\right)^2\right\}\frac{xz}{\sqrt{1-(2Pz)^2}} \tag{39}$$

$$\frac{1}{2}\frac{\partial U_z}{\partial x} = 3B\frac{x}{(b/2)^2}\left\{\frac{1}{2}z\sqrt{1-(2Pz)^2}+\frac{1}{4P}\arcsin 2Pz-2Jz\right\} \tag{40}$$

$$\dot{\varepsilon}_{xz} = \frac{1}{2}\frac{\partial U_x}{\partial z}+\frac{1}{2}\frac{\partial U_z}{\partial x}$$

$$\tau_0 = -\frac{1}{\sqrt{3}}\frac{BP^2\left\{1-\left(\frac{x}{b/2}\right)^2\right\}x}{\left[(1-P^2)\left\{1+\frac{B}{2}\left\{1-3\left(\frac{x}{b/2}\right)^2\right\}\left\{\sqrt{1-P^2}-\frac{1}{P}\arcsin P\right\}\right\}^2+\left\{BP^2\left\{1-\left(\frac{x}{b/2}\right)^2\right\}x\right\}^2\right]^{\frac{1}{2}}} \tag{41}$$

$$U_{xo} = x\left[1+\frac{B}{2}\left\{1-\left(\frac{x}{b/2}\right)^2\right\}\left\{\sqrt{1-P^2}-\frac{1}{P}\arcsin P\right\}\right] \tag{42}$$

$$\left\{\frac{\partial\tau_{xz}}{\partial z}\right\}_{z=0} = \frac{1}{\sqrt{3}}\frac{Bx\left[-2P^2\left\{1-\left(\frac{x}{b/2}\right)^2\right\}+3\frac{1-2J}{(b/2)^2}\right]}{1+B\left\{1-3\left(\frac{x}{b/2}\right)^2\right\}\left\{1-2J\right\}} \tag{43}$$

$$\left\{\frac{\partial\tau_{xz}}{\partial x}\right\}_x = \frac{2}{3}\cdot\frac{1}{\dot{\varepsilon}}\left[\frac{\partial\dot{\varepsilon}_{xz}}{\partial x}-\frac{4}{3}\frac{\dot{\varepsilon}_{xz}}{\dot{\varepsilon}^2}\left\{\dot{\varepsilon}_{xx}\frac{\partial\dot{\varepsilon}_{xx}}{\partial x}+\dot{\varepsilon}_{xz}\frac{\partial\dot{\varepsilon}_{xz}}{\partial x}\right\}\right] \tag{44}$$

where

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*) $Si\left(\frac{b}{7\pi}\right) = \int_0^{b/7\pi}\frac{\sin t}{t}\,dt$; for $b \le 10$ $Si(b/7\pi) \approx b/7\pi$.

$$\frac{\partial \dot{\varepsilon}_{xx}}{\partial x} = -\ 6B\ \frac{x}{(b/2)^2}\ \left\{\sqrt{1-(2Pz)^2}\ -\ 2J\right\} \tag{45}$$

$$\frac{\partial \dot{\varepsilon}_{xz}}{\partial x} = B\left[-\ P^2\ \frac{z}{\sqrt{1-(2Pz)^2}}\ \left\{1\ -\ 3\left(\frac{x}{b/2}\right)^2\right\} + \right.$$

$$\left. + \frac{3}{(b/2)^2}\left\{\frac{1}{2}\ z\ \sqrt{1-(2Pz)^2}\ +\ \frac{1}{4P}\ \mathrm{arc\ sin}\ 2Pz - 2Jz\right\}\right] \tag{46}$$

Through these equations the optimizing procedure can be performed analogeous to the way as shown earlier.
It appears that the power in the modified elliptic field is slightly less than in the quasi-elliptic field, as shown in fig. 4.

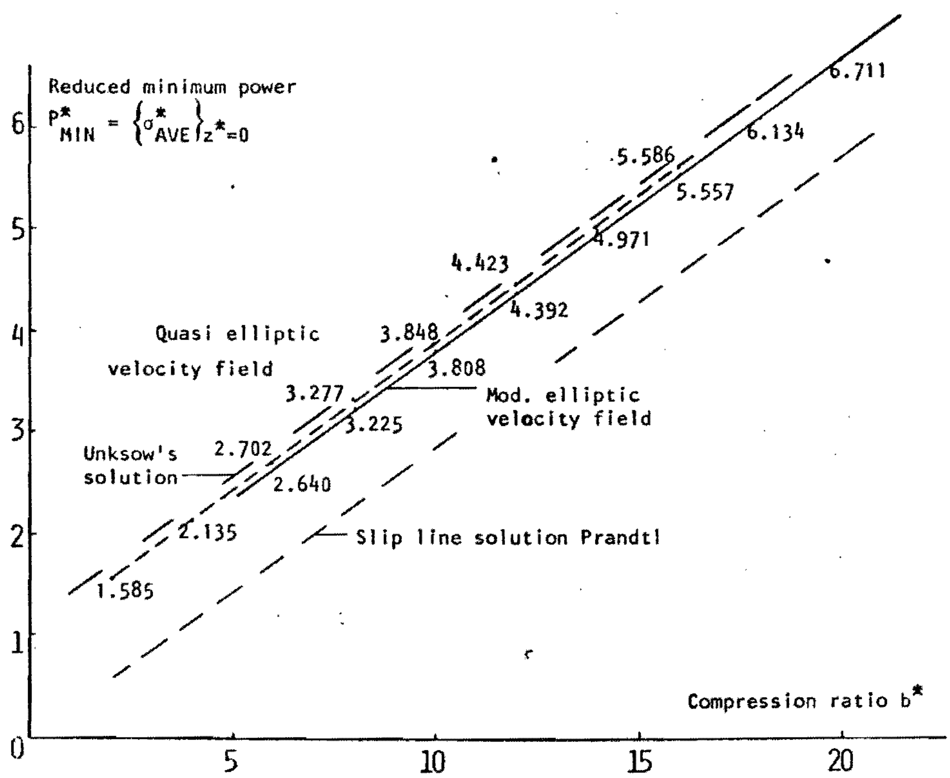

Fig. 4. Reduced minimum power and average reduced normal stress in the plane z=0 as a function of compression ratio in both the case of an quasi-elliptic and a modified elliptic velocity field.
A comparison is given with Prandtl's slip line

solution [5] and Unksow's solution [4] for rotational symmetry.

However, if the compression ratio $b < 6$ the modified field violates the kinematical constraint eq. 16, whereas the quasi-elliptic field satisfies it. For these reasons the different modes of compression, as mentioned before, must be distinguished. Typical examples of stress distributions are shown in figs. 5 and 6.

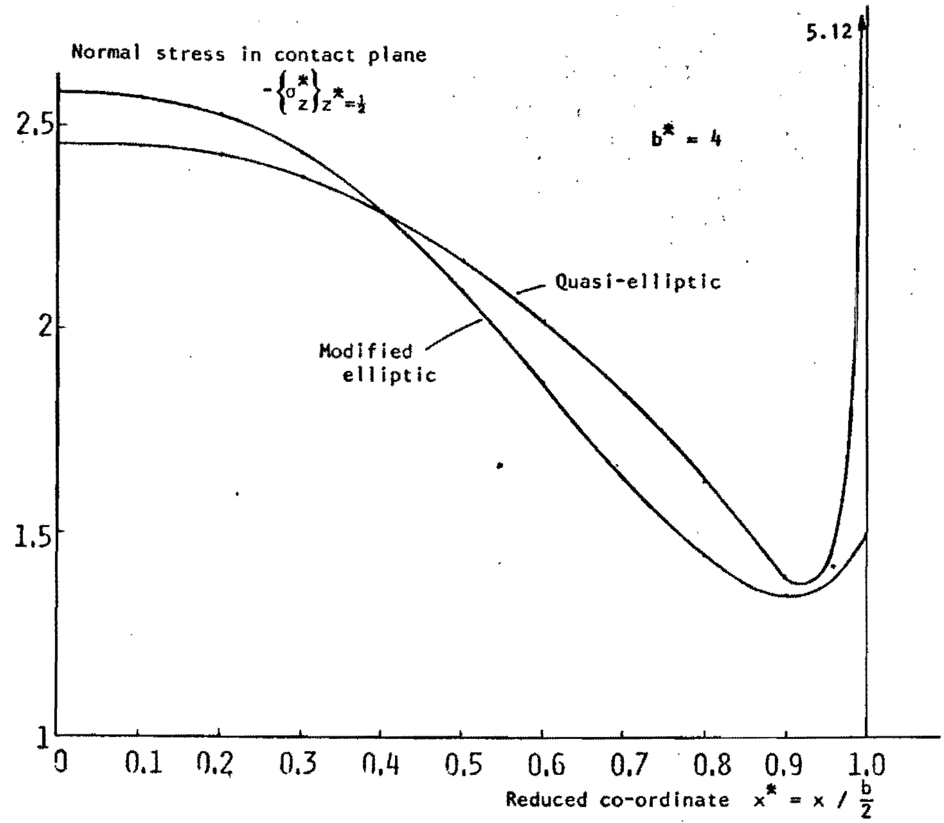

Fig. 5. The distribution of normal stress in the contact plane between tool and workpiece in the transition region between the quasi-elliptic velocity field and the modified elliptic velocity field.

It is remarked that in case $b \approx 5$, which is close to the transition from the one mode to the other, the formerly high negative value of σ_x in the plane $z=0$ close to the edge of the specimen, rapidly changes in a positive (tensile) stress.

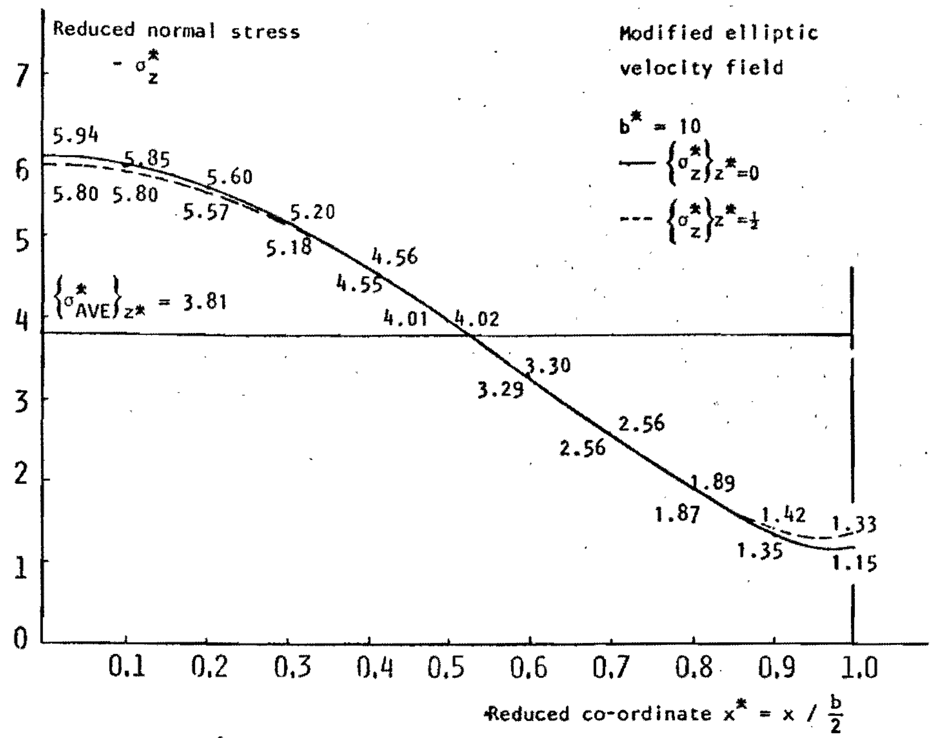

Fig. 6. The distribution of normal stress in the plane z=0 as well as in the contact plane between tool and workpiece for the modified elliptic velocity field.

4. THE KINEMATICS OF DEFORMATION

Once the velocity field being known also both the field of incremental displacements and the local incremental effective strain can be computed.
In relation to stepwise decrease of the height h of the sample large displacements and hence the deformation of the sample can be visualized. At the same time the local effective strain, obtained by integration of its local incremental value along the path of a material point, can be computed. An example is shown in fig. 7.
It is remarked that in the present model barreling mainly develops in the quasi-elliptic phase, because the low value of the parameter n brings about a considerable difference between $\dot{U}_x$ at $\{z = 0, x = b/2\}$ and that velocity at $\{z = \frac{1}{2}h, x = b/2\}$.
As soon as the modified elliptic mode is reached, the barreling previously developed moves further outward without any remarkable change.

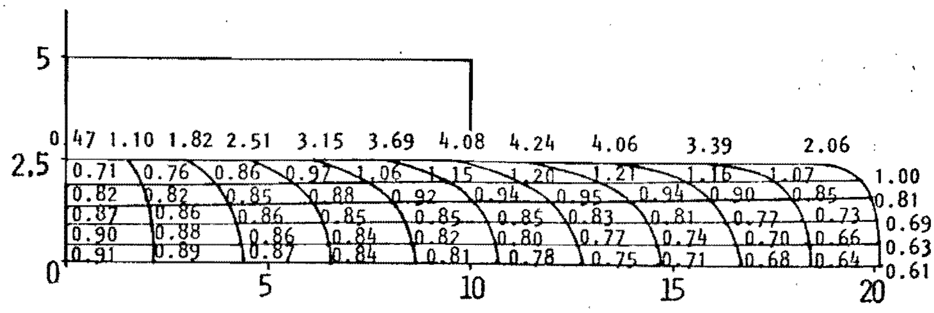

Fig. 7. The distortion and the local effective strains of a sample of the original dimensions $h_o = 10$, $b_o = 20$ after 50% reduction in height.

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