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SOME TRIVIAL REMARKS ON ORTHOGONALLY SCATTERED MEASURES AND RELATED GELFAND TRIPLES

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Summary

An orthogonally scattered measure is a Hilbert space valued set function orthogonal on disjoint sets. It has been proved that a set function μ is an orthogonally scattered measure iff there exists a spectral measure E such that $E(\Delta_1)\mu(\Delta_2) = \mu(\Delta_1 \cap \Delta_2)$ for each measurable sets Δ_1, Δ_2. An alternative approach to the theory of the inductive limit space $S_{\mathcal{R}}$ is given in terms of o.s. measures. (cf. the author's paper "On Gelfand Triples Originating from Algebras of Unbounded Operators", EUT Report 84-WSK-02)

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Foreword

In the present etude we study Hilbert space valued measures defined on the pre-ring of bounded Borel subsets of C^n.

Although the justification for such a study arises from our attempt to describe certain inductive limits of Hilbert spaces in terms of so called spectral trajectories, we have found the presented results interesting for their own sake. For this reason the paper contains mostly an expository material and results which are not of an immediate use for the further investigation. On the other hand the methods and topics presented here seem to be rather simple and perhaps likely to infer from well known results of P.Masani and others . Thus we address our paper to those who know the subject better.

1. *C.A.O.S. Measures*

Considering the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ (cf. [EGK] ,[EK]) we notice that they can be embedded into so called spaces of spectral trajectories. If E is the joint spectral measure for a family of mutually commuting self-adjoint operators $A = \{A_1, A_2, \ldots, A_n\}$ and Σ_n is the family of all bounded Borel subsets of C^n then a function $\Sigma_n \ni \Delta \longmapsto \mu(\Delta) \in H$ is called "a spectral trajectory" with "the generating spectral measure E" if for each $\Delta_1, \Delta_2 \in \Sigma_n$

$$E(\Delta_1)\ \mu(\Delta_2) = \mu(\Delta_1 \cap \Delta_2)$$

Thus the function μ must be necessarily orthogonally scattered (o.s.) , i.e. for each $\Delta_1, \Delta_2 \in \Sigma_n$ such that $\Delta_1 \cap \Delta_2 = \emptyset$ $\mu(\Delta_1) \perp \mu(\Delta_2)$. In this way we arrived to the well known concept of orthogonally scattered measures on a pre-ring . We notice also that the functions μ are countably additive (c.a.) on disjoint sets i.e. if $\{\Delta_i\}_{i\in I} \subset \Sigma_n$ is such that $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$ and $\bigcup_{i\in I} \Delta_i \in \Sigma_n$ then $\mu(\bigcup_{i\in I} \Delta_i) = \sum_{i\in I} \mu(\Delta_i)$, where the series is convergent in norm in the Hilbert space H.

We give here the precise definition:

1.1 Definition ([M 1])

Let H be a Hilbert space and Σ be a pre-ring of subsets of a space Λ . Then a set function $\Sigma \ni \Delta \to \mu(\Delta) \in H$ is called:

i) *a countably additive measure* (c.a.m.) on Σ if for any countable family $\{\Delta_i\}_{i \in I}$ of disjoint sets, such that $\Delta_i \in \Sigma$, $\bigcup_{i\in I}\Delta \in \Sigma$, $\sum_{i \in I} \mu(\Delta_i) = \mu(\bigcup_{i \in I} \Delta_i)$, where the series converges in norm in H (unconditionally) .

ii) *a countably additive orthogonally scattered measure* (c.a.o.s.m.) if μ is countably additive and for each pair of disjoint sets $\Delta_1, \Delta_2 \in \Sigma$ $\mu(\Delta_1) \perp \mu(\Delta_2)$.

We notice here that any linear combination of c.a. measures is again a c.a. measure. This is obviously not true for the case of c.a.o.s. measures. However, if for two measures μ and ν, which are c.a.o.s., the measures $\mu + \nu$ and $\mu + i\nu$ are c.a.o.s., then every linear combination of μ and ν is a c.a.o.s.m. too. This leads to the following notion of compatibility.

1.2 Definition

Let Σ be a pre-ring and μ, ν be a couple of c.a.o.s. mesures on Σ. We say that μ and ν are compatible if for each $\Delta_1, \Delta_2 \in \Sigma$, such that $\Delta_1 \cap \Delta_2 = \emptyset$, $\mu(\Delta_1) \perp \nu(\Delta_2)$.

1.2. Proposition

Let μ and ν be a couple of c.a.o.s. measures on a pre-ring Σ, with values in a Hilbert space H. Then the following conditions are equivalent:

i) μ and ν are compatible,

ii) for each $\alpha, \beta \in C^1$ the c.a. measure $\alpha\mu + \beta\nu$ is o.s.,

ii) the c.a. measures $\mu + i\nu$, $\mu + \nu$ are o.s..

Proof:

i) $\Rightarrow$ ii) $\Rightarrow$ iii) is trivial.

iii) $\Rightarrow$ i) Let $\Delta_1, \Delta_2 \in \Sigma$ and $\Delta_1 \cap \Delta_2 = \emptyset$.

Then because $\mu + i\nu$ is o.s. we have:

$$(\mu(\Delta_1) \mid \nu(\Delta_2)) = (\nu(\Delta_1) \mid \mu(\Delta_2))$$

Similarly for $\mu + \nu$ we have:

$$-(\mu(\Delta_1) \mid \nu(\Delta_2)) = (\nu(\Delta_1) \mid \mu(\Delta_2)) .$$

Hence $(\mu(\Delta_1) \mid \nu(\Delta_2)) = 0$.

□

1.4. Corollary

If c.a.o.s. measures μ and ν are compatible then all their linear combinations are mutually compatible c.a.o.s. measures.

Considering again the spectral measure E on Σ_n we can produce a family of mutually compatible c.a.o.s. measures just taking $\mu_x(\Delta) = E(\Delta)x$, where $x \in H$ and $\Delta \in \Sigma_n$. Thus a natural question arises how general is this example. To study this problem we invented the notion of maximal set of mutually compatible measures We say that a linear manifold $\mathcal{N}$ of (mutually compatible) c.a.o.s. measures is maximal if any c.a.o.s.m. on Σ which is compatible with all elements of $\mathcal{N}$ belongs to $\mathcal{N}$. It is clear that such a manifold $\mathcal{N}$ is a maximal element in the family of all linear spaces of mutually compatible c.a.o.s. measures on Σ , ordered by the set inclusion. We are going to associate with any maximal manifold $\mathcal{N}$ certain spectral measure on Σ , which is generating for all elements of $\mathcal{N}$. Let us define the following linear subspaces of H , for each $\Delta \in \Sigma$ putting:

(1.5.) $$\mathcal{N}(\Delta) = \{\mu(\Delta) \mid \mu \in \mathcal{N}\} .$$

1.6 Proposition

Let $\mathcal{N}$ be a maximal family of mutually compatible c.a.o.s. measures on a (pre-)ring Σ. Let $\mathcal{N}(\Delta)$ be as in (1.5).

Then:

i) $\mathcal{N}(\Delta)$ is a closed linear subspace of H.

ii) For any $\Delta, \Delta' \in \Sigma$, $\Delta' \subset \Delta$, we have $\mathcal{N}(\Delta') \subset \mathcal{N}(\Delta)$.

iii) $\bigcup_{\Delta\in\Sigma} \mathcal{N}(\Delta)$ is a dense linear subspace of H .

Proof:

It is clear that for each $\Delta\in\Sigma$ the set $\mathcal{N}(\Delta)$ is a linear subspace of H.

i) Let $x \in \overline{\mathcal{N}(\Delta)}$ and let $\{\mu_n\}_{n\in N} \subset \mathcal{N}$ be such a sequence in $\mathcal{N}$ that $\mu_n(\Delta) \underset{n\to\infty}{\to} x$ in H. Let $\Delta' \subset \Delta$, $\Delta' \in \Sigma$, then we have the following orthogonal decomposition for each $n \in N$:

$$\mu_n(\Delta) = \mu_n(\Delta \setminus \Delta') + \mu_n(\Delta').$$

Observe that the map $\Sigma \ni \Delta \to \|\mu(\Delta)\|^2 \in R^1$ (1.7)

is a c.a. R^1-valued measure on Σ for each c.a.o.s.m. μ. Hence, denoting $\Delta'' = \Delta \setminus \Delta'$, for each $m,n \in N$ we have:

$$\|\mu_n(\Delta) - \mu_m(\Delta)\|^2 = \|\mu_n(\Delta') - \mu_m(\Delta')\|^2 + \\ + \|\mu_n(\Delta'') - \mu_m(\Delta'')\|^2.$$

In this way we see that there exist elements of H

$$x_{\Delta'} = \lim_{n\to\infty} \mu_n(\Delta'), \quad x_{\Delta''} = \lim_{n\to\infty} \mu_n(\Delta'')$$

such that $x = x_{\Delta'} + x_{\Delta''}$,

Observe that $x_{\Delta'} \perp \mathcal{N}(\Delta'')$ and $x_{\Delta''} \perp \mathcal{N}(\Delta')$, hence

$(x_{\Delta'} | x_{\Delta''}) = \lim_{n\to\infty}(x_{\Delta'} | \mu_n(\Delta'')) = 0$, i.e. $x_{\Delta'} \perp x_{\Delta''}$.

Let us consider the map:

$\Sigma \ni \Delta' \to \mu(\Delta') = x_{\Delta\cap\Delta'} \in H.$

This map is a finitely additive o.s. measure on Σ .

On the other hand the map:

$$\Sigma \ni \Delta' \to ||\mu(\Delta')||^2$$

is the pointwise limit of c.a. measures $||\mu_n(\,.\,)||^2$, thus by the Vitali- Hahn- Saks theorem it is coutably additive ([DS] Part 1, IV 10.5 p.321). It is easy to observe that any o.s. measure ρ is c.a. iff $||\rho(\,.\,)||^2$ is c.a. Hence μ is a c.a.o.s.m. on Σ .

Now let $\nu \in \mathcal{N}$, $\Delta_1, \Delta_2 \in \Sigma$, $\Delta_1 \cap \Delta_2 = \emptyset$.
Then we have $(\mu(\Delta_1) \mid \nu(\Delta_2)) = \lim_{n\to\infty} (\mu_n(\Delta_1 \cap \Delta) \mid \nu(\Delta_2)) = 0.$
It means that μ is compatible with all elements of $\mathcal{N}$ and by the maximality of $\mathcal{N}$ we have $\mu \in \mathcal{N}$, i.e. $x \in \mathcal{N}(\Delta)$.

ii) Let $\Delta' \subset \Delta$, $\Delta', \Delta'' \in \Sigma$. Then the formula:

$$\mu_{\Delta'}(\Delta'') := \mu(\Delta' \cap \Delta'')$$

gives a c.a.o.s.m. for each $\mu \in \mathcal{N}$. It is obvious that $\mu_{\Delta'} \in \mathcal{N}$ and hence $\mathcal{N}(\Delta') = \{\mu(\Delta \cap \Delta') = \mu_{\Delta'}(\Delta) : \mu \in \mathcal{N}\} \subset \mathcal{N}(\Delta)$.

iii) Because the family Σ is directed by set inclusion we see that in virtue of ii) the set $\bigcup_{\Delta\in\Sigma} \mathcal{N}(\Delta)$ is a linear subspace of H.
Now suppose that $x \in (\bigcup_{\Delta\in\Sigma} \mathcal{N}(\Delta))^{\perp}$ and $x \neq 0$.
Put $\nu(\Delta) = \chi_{\Delta}(\lambda)x$ for some $\lambda \in \Lambda$. Then ν is a c.a.o.s.m. on Σ and it is compatible with all elements of $\mathcal{N}$. Hence $\nu \in \mathcal{N}$. But $x = \nu(\{\lambda\}) \notin \bigcup_{\Delta\in\Sigma} \mathcal{N}(\Delta)$. This is a contradiction.

□

1.8 Corollary

For each c.a.o.s.m. μ on the ring Σ_n of all bounded Borel subsets of C^n the measure $\Sigma_n \ni \Delta \to \|\mu(\Delta)\|^2 =: \rho(\Delta)$ extends to a Borel measure on C^n (in general not finite).

Proof:

It is clear that ρ is a positive finitely additive set function on Σ_n. Let $\{\Delta_i\}_{i\in N} \subset \Sigma_n$ and let $\Delta_i \cap \Delta_j = \emptyset$, for $i \neq j$. Suppose that the series $\sum_{i=1}^{m} \mu(\Delta_i)$ converges in H.
Then $\sum_{i=1}^{\infty} \|\mu(\Delta_i)\|^2 = \|\sum_{i=1}^{\infty} \mu(\Delta_i)\|^2 = \lim_{m\to\infty} \rho(\bigcup_{i=1}^{m} \Delta_i) =$
$= \sum_{i=1}^{\infty} \rho(\Delta_i)$. Hence we can extend ρ onto the whole of the σ-algebra of Borel subsets of C^n, putting $\rho(\Delta) = \sum_{i=1}^{\infty} \rho(\Delta_i)$ (possibly ∞), for any Borel set Δ and where $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$, $\Delta = \bigcup_{i=1}^{\infty} \Delta_i$, $\Delta_i \in \Sigma_n$.

It is clear that this definition does not depend on the choice of $\{\Delta_i\}_{i\in N}$ and gives a countably additive set function, bounded on compact sets, i.e. a Borel measure on C^n.

□

1.9 Remark ([M] Theorem 1.8)

If μ is an o.s. measure on a pre-ring Σ then the set function:

$\Sigma \ni \Delta \to \rho(\Delta) := \|\mu(\Delta)\|^2$

is countably additive iff μ is countably additive.

Now we are ready to reconstruct a generating spectral measure for a given c.a.o.s. measure.

1.10 Proposition

Let μ be a c.a.o.s.m. on the pre-ring of bounded Borel subsets of C^n, Σ_n. Then there exists a spectral measure E on C^n such that:

i) For each $\Delta, \Delta' \in \Sigma_n$

$$E(\Delta')\,\mu(\Delta) = \mu(\Delta \cap \Delta').$$

ii) If μ is bounded, i.e. if $\exists\, c \in R^1$, $c > 0$ $\forall \Delta \in \Sigma_n \|\mu(\Delta)\| \leqq c$, then there exists $x \in H$ such that $\forall \Delta \in \Sigma_n$

$$\mu(\Delta) = E(\Delta)\,x \quad \text{and} \quad \|x\| \leqq c.$$

iii) There exists a Borel function f, bounded on bounded Borel sets in C^n, and there exists $y \in H$, such that for each $\Delta \in \Sigma_n$:

$$\mu(\Delta) = \int_\Delta f(\lambda)\, dE(\lambda)\, y =: A_\mu\, E(\Delta)\, y,$$

where by $A_\mu = \int_{C^n} f(\lambda)\, dE(\lambda)$ we denote the operator, which is spectral with respect to the measure E.

Proof:

Let N be any maximal family of compatible c.a.o.s. measures containing μ. Let $E(\Delta)$, for $\Delta \in \Sigma_n$, be the orthogonal projection on the subspace $N(\Delta)$ in H. Clearly we have $E(\Delta') \leqq E(\Delta)$ for $\Delta' \subset \Delta$, and $E(\Delta \cup \Delta') = E(\Delta) + E(\Delta')$ for $\Delta \cap \Delta' = \emptyset$, $\Delta, \Delta' \in \Sigma_n$. Moreover, because in this case $N(\Delta) \perp N(\Delta')$ so $E(\Delta)\,E(\Delta') = 0$.

Let Γ be a Borel set. Then $\Gamma = \bigcup_{i \in N} \Delta_i$ for some disjoint family $\{\Delta_i\}_{i \in N}$ of bounded Borel subsets of C^n. The sequence of projections $E_m = \sum_{i=1}^{m} E(\Delta_i)$ is increasing and bounded by $\mathbb{1}$. Hence it tends strongly to a projection, which we denote by $E(\Gamma)$. $E(\Gamma)$ does not depend on the choice of the family $\{\Delta_i\}_{i\in N}$. Indeed, let $\Gamma = \bigcup_{i\in N} \Delta_i = \bigcup_{i\in N} \Delta'_i$. Then taking $\Delta_{ij} = \Delta_i \cap \Delta'_j$, we have got the family $\{\Delta_{ij}\}_{i,j\in N}$ of disjoint sets in Σ_n, such that $\Gamma = \bigcup_{i,j\in N} \Delta_{ij_k}$ and

$$\lim_m \sum_{i=1}^{m} E(\Delta_i) = \lim_{m,k} \sum_{i=1}^{m} \sum_{j=1}^{k} E(\Delta_{ij}) = \lim_k \sum_{j=1} E(\Delta_j) .$$

In this way we have constructed the spectral measure E on C^n, such that for each $x \in H$ the measure $\Delta \to \| E(\Delta)\|^2$ is Borel (cf.Corollary 1.8).

i) Let E be the spectral measure constructed above and let $\Delta, \Delta' \in \Sigma_n$. Then $E(\Delta')\mu(\Delta) = E(\Delta')(\mu(\Delta \setminus \Delta') + \mu(\Delta \cap \Delta')) =$

$= E(\Delta')\mu(\Delta' \cap \Delta) = \mu(\Delta \cap \Delta')$ since $N(\Delta \cap \Delta') \subset N(\Delta')$.

ii) Consider the net of vectors $\{\mu(\Delta)\}_{\Delta \in \Sigma_n}$, where the family Σ_n is directed by the set inclusion : $\Delta > \Delta'$ iff $\Delta' \subset \Delta$.

Because the net $\{\mu(\Delta)\}_{\Delta \in \Sigma_n}$ is uniformly bounded by c, thus it has weak cluster points. Let $x \in H$, with $\|x\| \leqq c$, be a cluster point of the net. Let $\Delta \in \Sigma_n$, $z \in H$ and $\varepsilon > 0$ be arbitrary. Then there exists $\Delta' \in \Sigma_n$, $\Delta' > \Delta$, such that

$$|(E(\Delta)z \mid x - \mu(\Delta'))| < \varepsilon .$$

It follows that:

$$|(z \mid E(\Delta)x - \mu(\Delta))| = |(E(\Delta)z \mid x - \mu(\Delta') + \mu(\Delta' \setminus \Delta))| \leqq$$

$$\leqq |(E(\Delta)z \mid x - \mu(\Delta'))| + |(z \mid E(\Delta)\mu(\Delta' \setminus \Delta))| \leqq \varepsilon .$$

Because ε and z were arbitrary we have $\mu(\Delta) = E(\Delta)x$.

iii) We assume now that μ is unbounded. Then we can easily construct such a sequence $\{\Delta_m\} \subset \Sigma_n$ of subsets in C^n

that : $\bigcup_{m=1} \Delta_m = C^n$, $\Delta_m \cap \Delta_k = \emptyset$ for $m \neq k$,
for each $m \in N$ $\mu(\Delta_m) \neq 0$ and for each $\Delta \in \Sigma_n$ there exists $m_o \in N$ such that $\Delta \subset \bigcup_{m=1}^{m_o} \Delta_m$.

Let us denote $r_m = \|\mu(\Delta_m)\|$, $c_m = \max(m, r_m)$ and put

$$y = \sum_{m=1}^{\infty} \frac{1}{c_m^2 r_m^2} \mu(\Delta_m) .$$

Because $\sum_{m=1}^{\infty} \frac{1}{c_m^2 r_m^2} \| \mu(\Delta_m)\|^2 = \| y \|^2 \leqq \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty$

we have $y \in H$.

Now we define the function f on C^n by:

$$f(\lambda) = c_m r_m \qquad \text{for} \qquad \lambda \in \Delta_m .$$

It is clear that f is Borel and bounded on elements of Σ_n.

For each $\Delta \in \Sigma_n$ we have $\Delta \subset \bigcup_{m=1}^{m_o} \Delta_m$ for some $m_o \in N$.

Thus

$$\mu(\Delta) = \sum_{m=1}^{m_o} c_m r_m / c_m r_m \; \mu(\Delta \cap \Delta_m) = \int_{\Delta} f(\lambda)\, dE(\lambda)\, y .$$

The operator $A_\mu = \int_{C^n} f(\lambda)\, dE(\lambda)$ is ess.s.a. with the domain $\mathcal{D}(A_\mu)$ containing the set $\bigcup_{\Delta \in \Sigma_n} E(\Delta)H$.

□

1.11 Proposition

A set of c.a. H-valued measures $\mathcal{N}$ on Σ_n is a maximal family of mutually compatible c.a.o.s. measures on Σ_n if and only if there exists a (unique) spectral measure E on C^n, which is generating for all elements of $\mathcal{N}$.

Proof:

The existence of a measure E and its uniqueness follows from the previous result (cf.1.10).

Let us prove now that for a given spectral measure E on C^n the set G_E of all spectral trajectories generated by E , i.e.:

$$G_E = \{ \mu \mid \mu : \Sigma_n \to H, \quad E(\Delta)\,\mu(\Delta') = \mu(\Delta \cap \Delta') \}$$

is a maximal family of mutually compatible c.a.o.s. measures. Obviously elements of G_E are c.a.o.s. measures. Their compatibility follows directly from the properties of the spectral measure E. The maximality of G_E remains to be proved.

Let ν be a c.a.o.s.m. compatible with all elements of G_E. Then there exists a spectral measure F which is generating for ν and all elements of G_E . If $\Delta, \Delta' \in \Sigma_n$, then for each $\mu \in G_E$

$$F(\Delta')\,\mu(\Delta) = \mu(\Delta \cap \Delta') = E(\Delta')\,\mu(\Delta).$$

Thus $F(\Delta') = E(\Delta')$ on the dense set $\bigcup_{\Delta \in \Sigma_n} E(\Delta)H$ in H. Hence $F \equiv E$ and ν is a spectral trajectory with the generating measure E, i.e. $\nu \in G_E$. Hence $\mathcal{N} = G_E$.

□

The compatibility relation between c.a.o.s. measures leads to the existence of a common generating spectral measure on a given (pre)-ring of subsets of a set Λ . We are interested however which relation between c.a.o.s. measures provides spectral measures which only commute. Let us consider the following example:

Let A and B be a couple of commuting normal operators with the spectral measures E_A and E_B respectively, defined on C^1. Let $x,y \in H$. Put $\mu(\Delta) = E_A(\Delta)x$, $\nu(\Delta') = E_B(\Delta')y$ for Δ , Δ' Borel subsets of C^1. Let E be the joint spectral measure for A and B , defined on C^2. Then the c.a.o.s. measures: $\tilde{\mu}(\tilde{\Delta}) = E(\tilde{\Delta})x$ and $\tilde{\nu}(\tilde{\Delta}) = E(\tilde{\Delta})y$, are "extensions" of μ and ν onto C^2. Observe that these extensions are compatible , although μ and ν are not. We say in this situation that the measures μ and ν are "weakly compatible". To be more precise we have to define at first the notion of an extension of c.a.o.s.m.

1.12 <u>Definition</u>

Let μ be a c.a.o.s.m. on Σ_n. Then a c.a.o.s.m. $\tilde{\mu}$ on the (pre)-ring Σ_{n+m} of (bounded) Borel sets in C^{n+m} is called an extension of μ onto C^{n+m} (onto Σ_{n+m}) if:

i) $\forall \Delta' \in \Sigma_m$ the map

$$\Sigma_n \ni \Delta \to \tilde{\mu}(\Delta \times \Delta') =: \mu_{\Delta'}(\Delta)$$

is a c.a.o.s.m. compatible with μ .

ii) For any increasing family $\{\Delta'_\alpha\} \subset \Sigma_m$, such that $\bigcup_\alpha \Delta'_\alpha = C^m$ the net $\{\mu_{\Delta'_\alpha}(\Delta)\} = \{\tilde{\mu}(\Delta \times \Delta'_\alpha)\}$ tends to $\mu(\Delta)$ in H for each $\Delta \in \Sigma_n$.

Our example of weakly compatible c.a.o.s. measures was based on bounded extensions, constructed via extensions of the spectral measures. We will show now that the general construction is essentially the same.

1.13 Proposition

Let μ be a c.a.o.s.m. on Σ_n in C^n and let $\tilde{\mu}$ be an extension of μ onto C^{n+m}. Then there exist spectral measures E and F on C^n and C^m respectively which commute and, for any Δ, Δ_1, $\Delta_2 \in \Sigma_n$, Δ_1', $\Delta_2' \in \Sigma_m$, the following holds:

$$E(\Delta_1)\,\mu(\Delta_2) = \mu(\Delta_1 \cap \Delta_2)$$

$$F(\Delta'_1)\tilde{\mu}(\Delta \times \Delta_2') = \tilde{\mu}(\Delta \times \Delta_1' \cap \Delta_2').$$

Moreover the measure $\tilde{E}(\Delta \times \Delta') = E(\Delta)\,F(\Delta')$, $\Delta \in \Sigma_n$, $\Delta' \in \Sigma_m$, can be extended to a spectral measure on C^{n+m}, generating for $\tilde{\mu}$.

Proof:

Let $\Sigma_n \ni \Delta \to \mu_{\Delta'}(\Delta) := \tilde{\mu}(\Delta \times \Delta')$ be a c.a.o.s.m. on Σ_n, defined for every $\Delta' \in \Sigma_m$.

The family $\{\mu_{\Delta'}\}_{\Delta' \in \Sigma_m}$ consits of mutually compatible c.a.o.s. measures, which are also compatible with the measure μ. Hence there exists a spectral measure E_o on C^n, generating for all $\mu_{\Delta'}$, $\Delta' \in \Sigma_m$, and μ.

Consider now the family of c.a.o.s. measures $\{\nu_\Delta\}_{\Delta \in \Sigma_n}$ on Σ_n defined by:

$$\nu_\Delta(\Delta') = \tilde{\mu}(\Delta \times \Delta'), \text{ for } \Delta' \in \Sigma_m.$$

Since they are mutually compatible, we can find a spectral measure F_o on C^m, generating for all ν_Δ, $\Delta \in \Sigma_n$.

Let

(1.14) $\quad S_{\tilde{\mu}} = $ closed linear span $\{ \tilde{\mu}(\tilde{\Delta}) : \tilde{\Delta} \in \Sigma_{n+m} \}$

It is easy to see that the spectral mesures E_o, F_o commute on $S_{\tilde{\mu}}$, which is also a reducing subspace for them. Thus we can find spectral measures E and F which coincide with E_o and F_o on $S_{\tilde{\mu}}$ and fulfil demanded conditions.

The measure $\tilde{E}$ defined on $\Sigma_n \times \Sigma_m$ by:

$$\tilde{E}(\Delta \times \Delta') = E(\Delta)\, F(\Delta'), \text{ for } \Delta \in \Sigma_n,\ \Delta' \in \Sigma_m,$$

gives rise to the spectral measure on C^{n+m} (cf. [BVS]).

□

1.15 Corollary

Let $\tilde{\mu}$ be an extension of μ onto C^{n+m}.

i) If μ is bounded with respect to Σ_m, then there exist spectral measures E and F on C^n and C^m respectively, such that:

E is generating for μ, F and E commute, and

$$\tilde{\mu}(\Delta \times \Delta') = F(\Delta')\, \mu(\Delta), \text{ for } \Delta \in \Sigma_n,\ \Delta' \in \Sigma_m.$$

ii) If both μ and $\tilde{\mu}$ are bounded we may choose E and F in such a way that:

$$\tilde{\mu}(\Delta \times \Delta') = E(\Delta)\, F(\Delta')\, x$$

and
$$\mu(\Delta) = E(\Delta)\, x,$$

for all $\Delta \in \Sigma_n, \Delta' \in \Sigma_m$ and some $x \in H$.

Now we can define the weak compatibility relation between two c.a.o.s. measures.

1.16 Definition

We say that a c.a.o.s.m. μ on C^n (Σ_n) is *weakly compatible* with a c.a.o.s.m. ν on C^m (Σ_m) if there exist their extensions $\tilde{\mu}$ and $\tilde{\nu}$ on C^{n+m} , such that $\tilde{\mu}$ is compatible with $\tilde{\nu} \cdot t_{nm}$, where the map $t_{nm}: C^{n+m} \to C^{n+m}$ is defined by: $t_{nm}(\xi_1,\xi_2,\ldots, \xi_{n+m}) = (\xi_{n+1}, \xi_{n+2},\ldots, \xi_{n+m},\xi_1,\ldots\xi_n)$.

Suppose now that there exist mutually commuting generating spectral measures for a couple of c.a.o.s. measures μ and ν . Then the extensions: $\tilde{\mu} (\Delta \times\Delta') = F(\Delta') \mu (\Delta)$

and $\tilde{\nu} (\Delta'\times\Delta) = E(\Delta) \nu (\Delta')$

are compatible c.a.o.s. measures on C^{n+m}, which are in a sense product measures . Thus it turns out that the measures μ and ν are weakly compatible. Moreover we will show that this construction of extensions is always possible for weakly compatible pairs of c.a.o.s. measures.

1.17 Proposition

Let μ and ν be weakly compatible c.a.o.s. measures on C^n and C^m. Then there exist on C^n and C^m respectively spectral measures E_μ , E_ν generating for μ, ν, which commute.Moreover the extension $\tilde{\mu}$ and $\tilde{\nu}$ of the measures μ and ν admit a common generating spectral measure $\tilde{E}$ on C^{n+m}, such that $\tilde{E}(\Delta \times C^m) = E_\mu(\Delta)$ and $\tilde{E}(C^n\times \Delta') = E_\nu(\Delta')$, for all $\Delta \in \Sigma_n$ and $\Delta' \in \Sigma_m$.

Proof:

Let us consider the family of c.a.o.s. measures defined on Σ_n

by $\Sigma_n \ni \Delta \to \nu_{\Delta'}(\Delta) := \tilde{\nu}(\Delta' \times \Delta)$, where $\Delta' \in \Sigma_m$. The measures $\nu_{\Delta'}$ are mutually compatible and they are also compatible with the measure μ. Moreover, if we define measures $\mu_{\Delta'}$ on Σ_n by

$\Sigma_n \ni \Delta \to \mu_{\Delta'}(\Delta) := \tilde{\mu}(\Delta \times \Delta')$ we obtain a compatible family of c.a.o.s. measures $\{\mu_{\Delta'}, \nu_{\Delta'}, \mu : \Delta' \in \Sigma_m\}$ that admits a common generating spectral measure E_μ on C^n. Similarly we construct the measure E_ν. We have then:

for each $\Delta, \Delta_1, \Delta_2 \in \Sigma_n$, $\Delta', \Delta_1', \Delta_2' \in \Sigma_m$

$$E_\mu(\Delta_1)\,\tilde{\mu}(\Delta_2 \times \Delta') = \tilde{\mu}(\Delta_1 \cap \Delta_2 \times \Delta')$$

$$E_\mu(\Delta_1)\,\tilde{\nu}(\Delta' \times \Delta_2) = \tilde{\nu}(\Delta' \times \Delta_1 \cap \Delta_2)$$

$$E_\nu(\Delta_1')\,\tilde{\mu}(\Delta \times \Delta_2') = \tilde{\mu}(\Delta \times \Delta_1' \cap \Delta_2')$$

$$E_\nu(\Delta_1')\,\tilde{\nu}(\Delta_2' \times \Delta) = \tilde{\nu}(\Delta_1' \cap \Delta_2' \times \Delta)$$

The measures E_μ and E_ν can be chosen to commute. Then extending the measure $\Sigma_n \times \Sigma_m \ni \Delta \times \Delta' \to E_\mu(\Delta) E_\nu(\Delta')$ onto C^{n+m} we obtain the desired measure $\tilde{E}$.

□

1.18 Remark

As an easy consequence of Proposition 1.17 and 1.10 we obtain for any couple of weakly compatible c.a.o.s. measures their description by means of two ess.s.a. operators A and B, which strongly commute and the measures are expressed by the formula : $\mu(\Delta) = A\, E_\mu(\Delta)\, x$,
$\nu(\Delta') = B\, E_\nu(\Delta')\, y$, for some $x, y \in H$ and all $\Delta \in \Sigma_n$, $\Delta' \in \Sigma_m$.

In quantum mechanics we consider maximal systems of mutually commuting observables. To each maximal system we ascribe a spectral measure E on the joint spectrum of the considered observables. Next we can use the measure E in the construction of a family of mutually weakly compatible c.a.o.s. measures on the spectrum. An interesting question is how to reverse this construction, i.e. how to reconstruct a maximal set of observables starting from a given family of c.a.o.s. measures. It appears however that the joint spectral measure of a maximal system of observables necessarily has the property of *non-extendibility* defined below. Obviously not all spectral measures have this property and thus we must impose extra assumptions on an initial family of c.a.o.s. measures. In general spectra of C*-algebras of observables need not be embedded into finite dimensional complex space. This leads to difficulties in a generalization of our theory for systems of infinite number of commuting observables. Thus we restrict ourselves to the finite dimensional case.

At first let us consider an example of a maximal system consisting of two observables A and B. Let their spectral measures be E_A and E_B respectively. Their values on Borel subsets of C^1 belong to the von Neumann algebra generated by A and B, $W^*(A,B,1) = \{ A,B \}''$. By the assumption this algebra is maximal abelian. In particular, if E is any spectral measure defined on C^n, commuting with E_A and E_B, its values must belong to $W^*(A,B,1)$. Let $\Delta \subset C^n$ be a Borel set. Then $E(\Delta) = \int_{C^2} \chi_{\tilde{\Delta}}(\lambda)\, d\tilde{E}(\lambda)$, where $\tilde{E} = E_A \cdot E_B$ is the joint spectral measure for the operators A and B and $\tilde{\Delta}$ is such a Borel subset of C^n that its characteristic function $\chi_{\tilde{\Delta}}$ is the Gelfand transform of the projection $E(\tilde{\Delta})$. The relation $\Delta \to \tilde{\Delta}$ extends to a σ-morphism from Borel subsets of C^n into Borel subsets of C^2, say $\phi : \mathcal{B}(C^n) \to \mathcal{B}(C^2)$, such that $\phi(C^n) = C^2$ and for each $\Delta \in B(C^n)$ $E(\Delta) = \tilde{E}(\phi(\Delta))$.

In such a situation we say that the spectral measure $\tilde{E}$ has no non-trivial extensions onto C^n.

1.19 Definition

A spectral measure E defined on a Lebesgue space Λ (cf.[BVS] [R]) is called *non-extendible* if for any $n \in N$ and each spectral measure F on $\Lambda \times C^n$, such that $F(\Delta \times C^n) = E(\Delta)$ for all $\Delta \in B(\Lambda)$, there exists a σ-set- morphism $\phi : B(C^n) \to B(\Lambda)$, such that $\phi(C^n) = \Lambda$ and for all $\Delta_1 \in B(\Lambda)$, $\Delta_2 \in B(C^n)$

(1.20) $$F(\Delta_1 \times \Delta_2) = E(\Delta_1 \cap \phi(\Delta_2)).$$

In other words E has only trivial extensions .

For a given c.a.o.s.m. in general there may be many generating spectral measures. Thus a notion of non-extendibility cannot be properly defined for an individual c.a.o.s.m.However it is possible for families of c.a.o.s. measures.

1.21 Definition

A family N of c.a.o.s. measures on a (pre)-ring of subsets of the space Λ , Σ , is called non-extendible iff:

i) There exists a unique spectral measure E on Λ generating for all elements of N .

ii) For any $x \in H$ the measure space (Λ , $\sigma(\Sigma)$, $||E(\cdot)x||^2$) is a Lebesgue space.

iii) E is a non-extendible spectral measure.

Usually we assume that $\Lambda = C^n$. Then we have a canonical correspondence between non-extendible families of c.a.o.s. measures and maximal systems of n mutually commuting observables, described above.

We observe also that a non-extendible family of c.a.o.s. measures must be necessarily maximal.

To deal with families of c.a.o.s. measures which are merely weakly compatible we must introduce a notion of common extension of a family of measures. At first we denote by $\Xi (X)$ the set all c.a.o.s. measures on a (pre)-ring Σ of subsets of a space X.

1.22 Definition

Let M be a family of c.a.o.s. measures on a pre-ring Σ of subsets of a space Λ . We say that the family M admits *a common extension* onto $\Lambda \times C^n$, for some $n \in N$, if there exists a map

$$\Psi : M \rightarrow \Xi (\Lambda \times C^n)$$

such that:

i) $\forall \ \mu \in M$ $\Psi (\mu)$ is an extension of μ onto $\Lambda \times C^n$.

ii) The set $\Psi (M)$ is a compatible family of c.a.o.s. measures on $\Lambda \times C^n$.

1.23 Definition

We say that an extension $\tilde{\mu}$ of a c.a.o.s. measure μ on a space Λ onto the space $\Lambda \times C^n$ is trivial if there exists a σ-set-morphism $\phi : \Sigma_n \rightarrow \Sigma$, such that $\phi (C^n) = \Lambda$ and for each $\Delta' \in \Sigma_n, \Delta \in \Sigma$

$$\tilde{\mu} (\Delta \times \Delta') = \mu (\Delta \cap \phi (\Delta')) .$$

A simple result follows.

1.24 Proposition

A family M of c.a.o.s. measures on the pre-ring Σ_m of bounded Borel subsets of C^m is non-extendible iff it is maximal compatible and admits only trivial common extensions.

1.25 Corollary

There is a canonical correspondence between maximal systems of n mutually (strongly) commuting normal operators in H and families of c.a.o.s. measures in C^n having none but trivial common extensions.

We say that a c.a.o.s.m. ν is basic or cyclic if the set $\{ \nu (\Delta) : \Delta \in \Sigma \}$ is total in H .

1.26 Remark

i) There exists only one generating spectral measure for a basic c.a.o.s.m.

ii) For any basic c.a.o.s.m. ν there is the unique maximal family of compatible c.a.o.s. measures containing ν .

1.27 Proposition

Let N be a maximal family of compatible c.a.o.s. measures on C^n. Then N contains a basic measure if and only if it is non-extendible.

We notice now that elements of a family of c.a.o.s. measures which admits common extensions are mutually weakly compatible. Thus we arrived to the main result of this section.

1.28 Theorem

There is a canonical correspondence between maximal systems of 2n mutually commuting observables and non-extendible families of mutually (weakly) compatible c.a.o.s. measures on the ring Σ_n of bounded Borel subsets of C^n.

Proof:

By Proposition 1.27 we may assume that we are given a basic measure on C^n, say μ . Let E be the unique generating spectral measure for a non-extendible family of c.a.o.s. measures M containig μ , defined on C^n. Let us consider the family of s.a. operators A_k, k = 1,2,...,2n, defined by:

$$A_k = \int_{C^n} \mathrm{Re}\ \lambda_k \ dE(\lambda) \qquad \text{for } k = 1,2,\dots,n$$

$$A_k = -i\int_{C^n} \mathrm{Im}\ \lambda_{k-n} \ dE(\lambda) \quad \text{for } k = n,\dots,2n.$$

The operators A_k have the common dense domain $\bigcup_{\Delta\in\Sigma_n} E(\Delta)H$.

The C*-algebra generated by operators 1, $(A_k - i\mathbb{1})^{-1}$ will be denoted by A . We will show that A is maximal abelian. By the Segal theorem ([T], Theorem 5, Sect.5, [Ma] Ch.VIII, Sect. 4 Theorem 1) this is equivalent to the existence of of a cyclic vector. By Proposition 1.10 the c.a.o.s.m. μ is of the form $\mu(\Delta) = A\,E(\Delta)\,x$ for some $x\in H$ and where A is an ess.s.a. operator affiliated with the von Neumann algebra $W^*(E)$ generated by the spectral projections $E(\Delta)$.

Clearly $E(\Delta') A E(\Delta) = A E(\Delta' \cap \Delta)$ for all $\Delta, \Delta' \in \Sigma_n$. Moreover $A E(\Delta) \in W^*(E)$.

We will show that the vector x is a cyclic vector for $\mathcal{A}$. At first we show that this vector is cyclic for $W^*(E)$. Indeed, the linear span of the set $\{ A E(\Delta) x : \Delta \in \Sigma_n \}$ is dense in H, so must be the set $W^*(E) x$. Thus $W^*(E)$ is a maximal abelian C*algebra.

Clearly $\mathcal{A} \subset W^*(E)$. Let $U \in \mathcal{A}'$ and let $U x = 0$. Since U commutes with all $(A_k - i1)^{-1}$, it commutes with the spectral measure E. Thus $U \in W^*(E)'$ and it follows that $U = 0$. It follows that x is separating for $\mathcal{A}'$ ([T]). In particular it means that x is cyclic for $\mathcal{A}$. In this way we have shown that $\{ A_k \}$ is a so called complete system of observables since the algebra $\mathcal{A}$ is a maximal abelian C*- algebra, generated by n normal generators (or 2n s.a.).

To prove the converse statement it is enough to take as a family of c.a.o.s. measures $\mathcal{M}$ the unique maximal family of compatible c.a.o.s. measures containing the c.a.o.s.m. defined by:

$$\Sigma_n \ni \Delta \rightarrow E(\Delta)\,\omega ,$$

where E is the joint spectral measure of a given family of observables and ω is the cyclic vector associated with them.

□

1.29. Corollary

There is a canonical correspondence between basic c.a.o.s. measures on C^n and maximal systems of 2n strongly commuting observables (possibly unbounded).

2. Duality

Let $\mathcal{N}$ be a maximal family of mutually compatible c.a.o.s. measures on a ring Σ of subsets of Λ. Let $||\cdot||_\Delta$ denote the seminorm on $\mathcal{N}$, defined by $||\mu||_\Delta = ||\mu(\Delta)||_H, \mu \in \mathcal{N}$. Let τ be the l.c. topology generated by these seminorms on $\mathcal{N}$.

2.1 Proposition

The l.c. topological vector space $(\mathcal{N}, \tau)$ is a projective limit of the family of Hilbert spaces $\mathcal{N}(\Delta)$, with the system of projections given by: $\pi_{\Delta'\Delta}: \mathcal{N}(\Delta) \to \mathcal{N}(\Delta')$, where for $\mu \in \mathcal{N}$ and $\Delta' \subset \Delta$ $\pi_{\Delta'\Delta}\, \mu(\Delta) = E(\Delta')\mu(\Delta) = \mu(\Delta')$, and where E is the spectral measure associated with $\mathcal{N}$.

Proof:

σ_{pr} - the projective limit topology on $\mathcal{N}$ is defined as the weakest l.c. topology for which all projections $\pi_\Delta : \mathcal{N} \to \mathcal{N}(\Delta)$ defined by $\mathcal{N} \ni \mu \overset{\pi_\Delta}{\to} \mu(\Delta) \in \mathcal{N}(\Delta)$ are still continuous. From this it follows that τ is stronger than σ_{pr}.

On the other hand let $\{\mu_\alpha\}_{\alpha \in I} \subset \mathcal{N}$ be a null net with respect to σ_{pr} topolgy. For each projection π_Δ, $\Delta \in \Sigma$, the net $\pi_\Delta \mu_\alpha = \mu_\alpha(\Delta)$ tends to 0 in $\mathcal{N}(\Delta)$. Hence $\mu_\alpha \to 0$ in the topology τ. Thus σ_{pr} is equivalent to τ.

□

2.2. Corollary

Each family $\mathcal{N}$ which is maximal with respect to the set-inclusion of families of mutually compatible c.a.o.s. measures, when endowed with the topology τ is a complete l.c. topological vector space.

2.3. Proposition

Let N be a maximal family of mutually compatible c.a.o.s. measures on a pre-ring Σ . Then N endowed with the topology σ_{pr} of projective limit of the family $N(\Delta)$,$\Delta \in \Sigma$, is a complete, barreled, reflexive, Mackey l.c. topological vector space.

Proof:

The completeness follows from general properties of projective limits of complete spaces. Similarly N is semi-reflexive as a projective limit of Hilbert spaces.

To show the reflexivity we should prove that N is infra-barreled, i.e. every convex, circled, closed subset of N , absorbing all bounded sets in N is a neighborhood of 0.

Let U be such a barrel absorbing all bounded sets in N . Suppose at the contrary that U does not contain any neighborhood of 0 in N , in particular , that there exists a sequence of elements of Σ , say $\{\Delta_i\}_{i \in N}$, such that $U \cap N(\Delta_i) \neq N(\Delta_i)$ for infinitely many indices. In the projective topology of $\prod_i N(\Delta_i)$ the set $U \cap \prod_i N(\Delta_i)$ is not a neighborhood of 0 and it does not absorb all bounded sets in $\prod_i N(\Delta_i)$. This yields a contradiction since the projective topology in $\prod_i N(\Delta_i)$ is induced by the topology of N .

By the Theorem 5.6. and Corollary 5.3 Ch. IV Sect. 5 in [Sch] N is reflexive. By the way we infer that N is barreled. Again following [Sch] we see that N is Mackey.

□

2.4. Corollary

The strong dual of the space (N, σ_{pr}) , N'_β, is a reflexive, barreled, Mackey space.

We will show now that the strong dual of N has also a nice representation which connects our present approach with our previous theory of inductive-projective limits of Hilbert spaces (cf. [EGK] , [EK]).

Let E be the spectral measure associated with a maximal family N of compatible c.a.o.s. measures on a (pre-) ring Σ .

Let $S = \bigcup_{\Delta\in\Sigma} E(\Delta) H$ be the inductive limit of the family $E(\Delta) H$ of Hilbert spaces. The family of embeddings is given by the natural embeddings $E(\Delta) H \subset S$. The topology τ_{ind} on S is the strongest l.c. topology in S for which all these embeddigns are still continuous. Clearly S is a Hausdorff strict inductive limit.

2.5. Proposition

The inductive limit space S is a complete, reflexive, barreled, bornological l.c. topological vector space.

Proof:

The result is just a quotation of 6.6 Ch. II Sect.6, 5.8 Ch.IV Sect.6 and Corollary 1, 8.2 Ch.II Sect.9 of [Sch] .

□

2.6. Theorem

The spaces N and S are in duality that makes them representations of strong duals of each other.

Proof:

The duality is defined as follows:

Let $\mu \in \mathcal{N}$, $s \in S$. Then

(2.7.) $< \mu \mid s > \; = \; (x \mid \mu(\Delta))_H$,

where $s = E(\Delta) x$ for some $\Delta \in \Sigma$, $x \in H$.

The definition 2.7. does not depend on the decomposition of s into the form $E(\Delta) x$. Indeed, let $s = E(\Delta) x = E(\Delta') x' =$ $= E(\Delta \cap \Delta') x = E(\Delta \cap \Delta') x'$. Then

$$(x \mid \mu(\Delta))_H = (E(\Delta) x \mid \mu(\Delta))_H = (E(\Delta \cap \Delta') x \mid \mu(\Delta \cap \Delta'))_H =$$
$$= (E(\Delta') x' \mid \mu (\Delta'))_H = (x' \mid \mu(\Delta'))_H .$$

Hence the formula 2.7. gives a continuous embedding of S into $\mathcal{N}'_\beta$ and of $\mathcal{N}$ into S'_β . Because of the reflexivity these embeddings are equivalent. We are going to show that actually they are equalities.

Let $\phi \in S'_\beta$. Then its restriction to every $E(\Delta) H$ is a continuous linear functional on the Hilbert space $\mathcal{N} (\Delta) = E(\Delta) H$. Thus there exists a vector, say $\phi(\Delta) \in \mathcal{N}(\Delta)$, such that

$$(\phi(\Delta) \mid \mu (\Delta))_{\mathcal{N}(\Delta)} = < \phi \mid E(\Delta) \mu>.$$

Because for every $\Delta, \Delta' \in \Sigma$ we have $E(\Delta') \phi (\Delta) = \phi (\Delta' \cap \Delta)$ the function $\Sigma \ni \Delta \to \phi (\Delta) \in H$ is a spectral trajectory, i.e. it is a c.a.o.s. compatible with all elements of Σ . Hence $\phi \in \mathcal{N}$ and $S'_\beta \subset \mathcal{N}$. Because this embedding is continuous we have eventually $\mathcal{N} = S'_\beta$ and $S = \mathcal{N}'_\beta$ as l.c. topological vector spaces.

□

There are properties of spaces S and $\mathcal{N}$ which can be easily described in terms of the spectral measure E.

2.8. Corollary

i) If the algebra $W^*(E(\Delta); \Delta \in \Sigma)$ is countably decomposable then the space S is of the type (LF).

ii) For each $\Delta \in \Sigma$ $E(\Delta)$ is finite dimensional iff the space S is Montel. Then N is Montel too.

Proof:

i) By the assumption there exists an at most countable sequence of elements of Σ, $\{\Delta_i\}_{i \in N}$, such that $\forall \Delta \in \Sigma$ $\exists$ $\Delta_i \in \Sigma$ with the property that $E(\Delta) \leq E(\Delta_i)$.

Thus it is easy to see that S is a strict inductive limit of the sequence of Hilbert spaces $E(\Delta_i) H$.

ii) The result follows from the fact that the unit ball is compact only in a finitely dimensional Hilbert space.

□

If N is a family of c.a.o.s. measures on C^n then for each $\mu \in N$ there exists an operator A_μ, spectral with respect to E, and a vector $x_\mu \in H$ such that for every $\Delta \in \Sigma$ $\mu(\Delta) = A_\mu E(\Delta) x_\mu$ (cf.Proposition 1.10).

Let Θ denote the collection of all operators obtained in this way. Let $\Omega = \{ E(\Delta) : \Delta \in \Sigma \}$. Then, following the results of [EK-2] we can prove that the inductive limit topology on S is given by the family of seminorms $S \ni s \to \| Ls \|$, where $L \in \Omega^{cc}$ (the strong bicommutant of Ω). In particular it follows that the space S is Hausdorff.

On the other hand it is easy to see that for each $L \in \Omega^{cc}$ the measure $\Sigma \ni \Delta \to L E(\Delta) x \in H$, where $x \in H$, belongs to N.

Thus we have $\Omega^{cc} \subset \Theta$.

Let us consider the topology τ_Θ on $S_\Omega = S$ defined by means of the family of seminorms $S \ni s \to ||A s||$, $A \in \Theta$.

The topology τ_Θ is stronger than τ_{ind}.

On the other hand each set of the form:

$$T_A = \{ \mu_{A,x} : \mu_{A,x}(\Delta) = A E(\Delta) x, || x || \leqq 1 \}$$

is bounded in N. Let $s \in E(\Delta) H$. Since $|| A s|| = \sup_{||x|| \leqq 1} |(A s| x)| =$

$$= \sup_{||x|| \leqq 1} |(\mu_{A,x}(\Delta) | s)| = \sup_{\mu \in T_A} |<\mu | s>|$$

the topology τ_Θ is weaker then the Mackey topology on S_Ω, i.e. weaker than τ_{ind}. Thus $\tau_\Theta \sim \tau_{ind}$. Thus we arrived to the following result:

2.9. Proposition

Let Σ_n be the ring of bounded Borel subsets of C^n, N be a maximal family of mutually compatible measures, and let Θ be the collection of spectral operators associated with N.

Then the strong topology on the dual N'_β of N is generated by the family of seminorms

$$N' \ni s \to || A s || \text{ where } A \in \Theta .$$

This topology is equivalent to the inductive limit topology on N, induced by the family of Hilbert spaces $\{ N(\Delta) : \Delta \in \Sigma_n\}$.

The following problem arises:

As we have seen in Proposition 1.10, all measures in N can be described by simply constructed operators A_μ . The collection Θ consists of apparently more operators than just $\{ A_\mu : \mu \in N\}$.It would be desirable to avoid the abundance of the elements of Θ in the description of the topology τ_{ind} in N'.

Obviously the Mackey topolgy in $N'$is stronger than the topology of uniform convergence on (bounded) sets of the form:

$$T_{A_\mu} = \{ \mu_{A_\mu, x} : \mu_{A_\mu, x}(\Delta) = A_\mu E(\Delta) x , ||x|| \leq 1\}$$

for a fixed $A_\mu \in \Theta$.

On the other hand changing A_μ we can enrich this topology up to equivalence with τ_{ind}.

Let us consider now the particular case of a family N containing a basic c.a.o.s. measure μ . It has been mentioned before that in such a case the von Neumann algebra $W^*(E(\Delta) ; \Delta \in \Sigma)$ has a cyclic vector and hence it is a maximal abelian C*-algebra. It is also clear that the maximal family of compatible measures containing μ is unique as well as the associated spectral measure E . We have shown in Proposition 1.18 in [EK-2] that the strong bicommutant Ω^{cc} of the family $\Omega = \{ E(\Delta) : \Delta \in \Sigma_n\}$ is monotonuously generated by the von Neumann algebra $W^*(E(\Delta) ; \Delta \in \Sigma_n)$. In such a case the topology on N'_β is generated by a family of Σ_n- finite Borel functions in the following sense:

the seminorms of the form

$$N'_\beta \ni s \rightarrow \left\| \int_{C^n} f(\lambda)\, dE(\lambda)\, s \right\|$$

where f is a Borel function on C^n such that

$$\forall \quad \Delta \in \Sigma_n \qquad \sup_{\lambda \in \Delta} |f(\lambda)| < \infty \quad ,$$

generate a l.c. topology on N'_β equivalent to the strong dual topology β.

This simple situation may be modified if we take instead of a maximal family of compatible measures some other family (smaller), containing the basic measure μ . Under certain conditions we will show that the dual of this family can be represented in the form $S_{\mathcal{R}}$, for some adequate generating family of operators $\mathcal{R}$, commuting with the spectral measure E .

Let us recall the definition of a generating family of operators.

2.10. Definition ([EK-2] Def.1.1.)

Let $\mathcal{R}$ be a family of bounded operators in a Hilbert space H. Then $\mathcal{R}$ is called a generating family of operators if it has the following properties:

i) $\forall\ a \in \mathcal{R}\quad 0 \leqslant a \leqslant 1$ (positivity and boundedness)

ii) $\forall\ a,b \in \mathcal{R}\quad ab = ba$ (commutativity)

iii) $\forall\ a,b, \in \mathcal{R}\ \exists\ c \in \mathcal{R}\quad a \leqslant c$ and $b \leqslant c$ (directedness)

iv) $\forall\ a \in \mathcal{R}\ \exists b \in \mathcal{R}\quad a \leqslant b^2$ (sub-semi-group property)

For each $a \in \mathcal{R}$ put $aH = \{ax \mid x \in H\}$. aH becomes a Hilbert space when endowed with the scalar product

$(a x \mid a y)_a = (r(a) x \mid r(a) y)_H$

where r(a) is the right (hence left) support of a (cf. [Sa]).

2.11. Definition

By $S_{\mathcal{R}}$ we denote the inductive limit of the family $\{ aH : a \in \mathcal{R} \}$ of Hilbert spaces defined above for the generating family of operators $\mathcal{R}$.

We can put $S_{\mathcal{R}} = \bigcup_{a\in\mathcal{R}} aH$ with the l.c. topology defined as the strongest topology on $S_{\mathcal{R}}$ for which all embeddings $g_a: aH \to S_{\mathcal{R}}$ are still continuous. $S_{\mathcal{R}}$ is Hausdorff for the embedding $S_{\mathcal{R}} \subset H$ is continuous. We notice that for each $a\in \mathcal{R}$ the map $H \ni x \to ax \in S_{\mathcal{R}}$ is continuous.

For a given generating family $\mathcal{R}$ of operators we will consider the space $S_{\mathcal{R}}$ as the dual of certain space of o.s. measures on the spectrum Λ of the W*-algebra generated by $\mathcal{R}$. Known examples of such a situation suggest that we must properly choose the (pre-) ring Σ of subsets of Λ . Thus put

(2.12) $\Sigma = \{\Delta \subset \Lambda, \Delta \text{ is a Borel set}, \exists\ a\in \mathcal{R}, \exists\ c\in R^1,\ c > 0,\ \chi_\Delta(\lambda) \leq c\,\hat{a}(\lambda)\}$

where $\hat{a}$ is the Gelfand transform of the operator a considered as an element of the C*-algebra $W^*(\mathcal{R})$ generated by $\mathcal{R}$ and $\mathbf{1}$, χ_Δ is the characteristic function of the set Δ .

Σ is a ring of sets since $\mathcal{R}$ is directed and it is easy to see that all Borel subsets of elements of Σ belong to Σ .

Let E be the joint spectral measure of the family $\mathcal{R}$. Let us denote by S_Ω the inductive limit of Hilbert spaces $\{ E(\Delta) H | \Delta\in\Sigma\}$ introduced before. By the previous results each continuous linear functional on S_Ω can be represented as a c.a.o.s.m. on the spectrum Λ of $W^*(\mathcal{R})$. It follows from 2.12. that for each $\Delta\in\Sigma$ there exists $b\in\mathcal{R}$ such that $b^{-1} E(\Delta)$ is bounded . Thus we have $S_\Omega \subset S_{\mathcal{R}}$. Moreover by the spectral theorem it is easy to see that S_Ω is dense in $S_{\mathcal{R}}$ in the inductive limit topology. The embedding $S_\Omega \subset S_{\mathcal{R}}$ is continuous.

Indeed , each Hilbert space $E(\Delta) H$, $\Delta\in\Sigma$, is a subspace of some Hilbert space bH , $b\in \mathcal{R}$. Hence, if a set U is open in $S_{\mathcal{R}}$

then $U \cap bH$ is open for each $b \in \mathcal{R}$ and thus $U \cap E(\Delta) H$ is open for each $\Delta \in \Sigma$. In this way we see that $U \cap S_\Omega$ is open in the inductive limit topology in S_Ω .

Then it follows that $S'_\mathcal{R} \subset S'_\Omega$, where the embedding is continuous in the strong dual topologies. In particular it means that each continuous linear functional on $S_\mathcal{R}$ can be represented as a c.a.o.s.m. on Σ . It is given by the (unique) extension of a c.a.o.s.m. to an "integral" defined on elements of $\mathcal{R}$. This concept is explained by the following lemma.

2.13. Lemma

Let $\ell \in S'_\mathcal{R}$. Then there exists a c.a.o.s.m. μ on Σ such that for each $a \in \mathcal{R}$ there exists a vector $\mu(a) \in H$ with the properties:

i) $\forall \; s \in S_\mathcal{R} \quad \ell(s) = (\mu(a) | x)_H$, where $s = a x$,

ii) $\forall \; \Delta \in \Sigma \quad E(\Delta) \mu(a) = a \mu(\Delta)$.

Proof:

For each $a \in \mathcal{R}$ the map $H \ni x \rightarrow \ell(a x)$ is continuous linear. Thus there exists the vector $\mu(a) \in H$ fulfilling i). On the other hand $\ell |_{S_\Omega}$ is continuous and hence can be represnted as a c.a.o.s.m.μ on Σ .
We have then $\ell(E(\Delta) | a x) = (\mu(\Delta) | a x)_H = (\mu(a) | E(\Delta) x)_H$.
The last relation holds for all $x \in H$ so $E(\Delta) \mu(a) = a \mu(\Delta)$, since a is s.a.

□

We call the element $\mu(a)$ of H an integral with respect to a c.a.o.s.m. μ since it is an extension of a linear functional $\ell|_{S_\Omega}$ defined on " simple functions " S_Ω onto wider class of "functions" $S_\mathcal{R}$.

2.14. Definition

Let $\mathcal{R}$ be a generating family of operators and μ be a spectral trajectory with respect to the spectral measure E associated with $\mathcal{R}$.

Then μ is called a $\mathcal{R}$-bounded c.a.o.s.m. on Σ iff

for each $a \in \mathcal{R}$ the c.a.o.s. measure $\Sigma \ni \Delta \to a\,\mu(\Delta) \in H$

is bounded. The set of $\mathcal{R}$-bounded c.a.o.s.measures is denoted by $T_{\mathcal{R}}$.

2.15. Remark

If a c.a.o.s.m. μ is $\mathcal{R}$-bounded then for each $a \in \mathcal{R}$ there exists the vector $\mu(a) \in H$ such that 2.13.ii) holds. Moreover

$$\| \mu(a) \| = \sup_{\Delta \in \Sigma} \| a\,\mu(\Delta) \| .$$

The set $T_{\mathcal{R}}$ is a linear set consisting of mutually compatible c.a.o.s. measures on the ring Σ. Let us introduce in $T_{\mathcal{R}}$ a l.c. topology generated by the family of seminorms:

(2.16.) $$T_{\mathcal{R}} \ni \mu \to \| \mu(a) \| =: \| \mu \|_a , \text{ where } a \in \mathcal{R} .$$

Let us denote now the topological dual of $T_{\mathcal{R}}$ endowed with the topology 2.16. by $T'_{\mathcal{R}}$. We have the following algebraic result.

2.17. Theorem

The following dualities take place:

i) $S_{\mathcal{R}} \equiv T'_{\mathcal{R}}$

ii) $S'_{\mathcal{R}} \equiv T_{\mathcal{R}}$

Proof:

At first we establish the notation.

$\langle \, | \, \rangle_S$ denotes the duality between $S_{\mathcal{R}}$ and $S'_{\mathcal{R}}$, $\langle \, | \, \rangle_T$ duality between $T_{\mathcal{R}}$ and $T'_{\mathcal{R}}$. We will prove the existence of the following embeddings:

$S_{\mathcal{R}} \overset{\alpha_1}{\subset} T'_{\mathcal{R}} \overset{\alpha_2}{\subset} S_{\mathcal{R}}$ and $T_{\mathcal{R}} \overset{\beta_1}{\subset} S'_{\mathcal{R}} \overset{\beta_2}{\subset} T_{\mathcal{R}}$

and the relations:

(2.18.) $\alpha_2 \cdot \alpha_1 = id_S$

(2.19.) $\alpha_1 \cdot \alpha_2 = id_{T'}$

(2.20.) $\beta_2 \cdot \beta_1 = id_T$

(2.21.) $\beta_1 \cdot \beta_2 = id_{S'}$.

i) At first we will show the existence of the embedding α_1.

For each $s \in S_{\mathcal{R}}$ we define a linear functional on the space $T_{\mathcal{R}}$ by:

(*) $<\mu \mid \alpha_1(s)>_T = (\mu(a) \mid x)$, where $s = a\,x$, $a \in \mathcal{R}$, $x \in H$, $\mu \in T_{\mathcal{R}}$.

To see that this definition does not depend on the decomposition of s put $a\,x = a'\,x' = s$, with $a' \in \mathcal{R}$, $x' \in H$.

Then for each $\Delta \in \Sigma$, $(E(\Delta)\,\mu(a) \mid x) = (a\,\mu(\Delta) \mid x) =$

$= (a'\,\mu(\Delta) \mid x') = (E(\Delta)\,\mu(a') \mid x')$. Thus we have $(\mu(a) \mid x) =$

$= (\mu(a') \mid x')$ for all $\mu \in T_{\mathcal{R}}$.

The continuity of the functional (*) follows from the estimation:

$|<\mu \mid \alpha_1(a\,x)>_T| = |(\mu(a) \mid x)| \leqslant \|\mu(a)\|\,\|x\| = \|\mu\|_a\ \|x\|$.

To show the existence of the embedding α_2 we have to find out a proper representation of every $\varphi \in T'_{\mathcal{R}}$ in the space $S_{\mathcal{R}}$.

Let $\varphi \in T'_{\mathcal{R}}$. By the continuity of φ and directedness of $\mathcal{R}$ we can choose $a \in \mathcal{R}$ such that for all $\mu \in T_{\mathcal{R}}$ $|\varphi(\mu)| \leqslant c\|\mu\|_a$, for some constant $c > 0$. We notice that if $\mu, \nu \in T$ and $r(a)\,\mu(\Delta) =$ $= r(a)\,\nu(\Delta)$ for each $\Delta \in \Sigma$ then $\varphi(\mu) = \varphi(\nu)$. Indeed: we have $|\varphi(\mu - \nu)| \leqslant c \cdot \|\mu - \nu\|_a =$

$$= c \cdot \sup_{\Delta \in \Sigma} \|a\,r(a)(\mu(\Delta) - \nu(\Delta))\| = 0.$$

Observe that φ defines a continuous (bounded) linear functional $\tilde{\varphi}$ on the linear manifold $\{\mu(a) \mid \mu \in T_{\mathcal{R}}\}$ by $\tilde{\varphi}(\mu(a)) = \varphi(\mu)$.

$\tilde{\varphi}$ is well defined and bounded in the Hilbert space $r(a)H$ in which the set $\{ \mu(a): \mu \in T_{\mathcal{R}} \}$ is dense. Thus we can represent $\tilde{\varphi}$ (hence φ) by a vector $v \in r(a)H$ such that for each $\mu \in T_{\mathcal{R}}$ $\tilde{\varphi}(\mu(a)) =$ $= (v \mid \mu(a))$. Now put

$$(**) \qquad \alpha_2(\varphi) = a v \in S_{\mathcal{R}} .$$

Then we have:

for each $\mu \in T_{\mathcal{R}}$ $\varphi(\mu) = (v \mid \mu(a)) = \langle \mu \mid \alpha_2(\varphi) \rangle_T$.

To see that α_2 is well defined suppose that φ has two representants of this form, i.e. that there exist $a, a' \in \mathcal{R}$ and $v, v' \in H$ such that $\varphi(\mu) = (v \mid \mu(a)) = (v' \mid \mu(a'))$ for all $\mu \in T_{\mathcal{R}}$. Because the measures μ_Δ defined by $\Sigma \ni \Delta' \to \mu_\Delta(\Delta') := \mu(\Delta \cap \Delta')$ belong to $T_{\mathcal{R}}$ for any $\mu \in T_{\mathcal{R}}$, we have:

$$(v \mid \mu_\Delta(a)) = (a v \mid \mu(\Delta)) = (v' \mid \mu_\Delta(a')) = (a' v' \mid \mu(\Delta))$$

for all $\Delta \in \Sigma$, and all $\mathcal{R}$-bounded measures μ , in particular it holds for all measures of the form $\Sigma \ni \Delta \to E(\Delta) y$, $y \in H$. It follows that $a v = a' v'$.

Now we will show the relations 2.18. and 2.19.

Let $a x = s \in S_{\mathcal{R}}$ and $\alpha_2 \cdot \alpha_1 (s) = b y$ .Then for each $\mu \in T_{\mathcal{R}}$ we have: $(\mu(a) \mid x) = (\mu(b) \mid y)$. In virtue of Remark 2.15. we have :

$$(E(\Delta)\mu(a) \mid x) = (a \mu(\Delta) \mid x) = (\mu(\Delta) \mid a x) = (\mu(\Delta) \mid b y)$$

Thus $b y = a x$, i.e. 2.18. holds.

Let $\varphi \in T'_{\mathcal{R}}$. Let us compute $\alpha_1 \cdot \alpha_2(\varphi) = \alpha_1(b y)$, where for each $\mu \in T_{\mathcal{R}}$ $\langle \varphi \mid \mu \rangle_T = (\mu(b) \mid y)$. Then $\langle \alpha_1(b y) \mid \mu \rangle_T =$ $= (\mu(b) \mid y) = \langle \varphi \mid \mu \rangle_T$. Hence $\alpha_1 \cdot \alpha_2(\varphi) = \varphi$, i.e. 2.19. holds.

In this way we proved the relation i), i.e. $S_{\mathcal{R}} \equiv T'_{\mathcal{R}}$.

ii) In virtue of the considerations preceding Lemma 2.13. every element of the dual $S'_{\mathcal{R}}$ of $S_{\mathcal{R}}$ can be regarded as a c.a.o.s.m. on the joint spectrum Λ of the family $\mathcal{R}$. Let $\ell \in S'_{\mathcal{R}}$. We put:

(***) $\beta_2(\ell) = \mu$, where the existence of the c.a.o.s.m. μ is established by Lemma 2.13. We will show that $\mu \in T_{\mathcal{R}}$. For this it is enough to notice that $\sup_{\Delta\in\Sigma} \| a\,\mu(\Delta)\| =$

$= \sup_{\Delta\in\Sigma} \| E(\Delta)\,\mu(a)\| = \|\mu(a)\| < \infty$.

Now we will show the existence of $\beta_1 : T_{\mathcal{R}} \to S'_{\mathcal{R}}$. Let $\mu \in T_{\mathcal{R}}$. By Remark 2.15. there exists $\mu(a) \in H$ such that $\forall\, \Delta\in\Sigma$ $E(\Delta)\,\mu(a) = a\,\mu(\Delta)$. Let $\beta_1(\mu)$ be the linear functional defined on $S_{\mathcal{R}}$ by:

(****) $\langle \beta_1(\mu) \mid s \rangle = (\mu(a) \mid x)$, where $s = a\,x \in S_{\mathcal{R}}$, with $a \in \mathcal{R}$ and $x \in H$.

As before we can show that this definition does not depend on the decomposition of s into the form $a\,x$.

To show that $\beta_1(\mu) \in S'_{\mathcal{R}}$ we notice that the set: $U = \{ s \in S_{\mathcal{R}} \mid |\langle\beta_1(\mu) \mid s\rangle| < 1 \}$ is open in $S_{\mathcal{R}}$. Indeed, let $b, b' \in \mathcal{R}$ be such that $b^{\frac{1}{2}} \leq b'$. Then $\forall\; y \in H$, $\forall\, \Delta \in \Sigma$

$|(E(\Delta)\,\mu(b) \mid y)| = |(b^{\frac{1}{2}}\mu(\Delta) \mid b^{\frac{1}{2}} y)| \leq$

$\leq \| b^{\frac{1}{2}}\mu(\Delta)\|\,\|b^{\frac{1}{2}} y\| \leq \|b^{\frac{1}{2}}\|\,\|r(b)\,y\|\,\|b'\,\mu(\Delta)\| \leq$

$\leq \| b^{\frac{1}{2}}\|\,\|r(b)\,y\|\,\|\mu(b')\|$.

So $|\langle \beta_1(\mu) \mid s \rangle| = \lim_{\Delta \nearrow \Lambda} |(E(\Delta)\,\mu(b) \mid y)| \leq$

$\leq \| b^{\frac{1}{2}}\|\,\|\mu(b')\|\,\|r(b)\,y\|$.

It means that $\forall\; b \in \mathcal{R}\ \exists\, \varepsilon_b > 0$ $(\varepsilon_b < (\|b^{\frac{1}{2}}\|\|\mu(b')\|)^{-1})$ such that $\{ s \in bH \mid \|s\|_b < \varepsilon_b \} \subset U \cap bH$.

So $U \cap bH$ is open in the Hilbert space bH thus U is open in $S_{\mathcal{R}}$. This proves continuity of the functional $\beta_1(\mu)$ on $S_{\mathcal{R}}$.

In order to prove the relations 2.20. and 2.21. we can use arguments similar to those in proving 2.18. and 2.19. Namely we have:

For each $\mu \in T_{\mathcal{R}}$ and all $\Delta \in \Sigma$ $((\beta_2 \cdot \beta_1)(\mu)(\Delta) | x) =$
$= \langle \beta_1(\mu) \mid E(\Delta) x \rangle = (\mu(\Delta) | x)$, so $\beta_2 \cdot \beta_1(\mu) = \mu$.

Now for any $\ell \in S'_{\mathcal{R}}$, for all $a\, x \in S_{\mathcal{R}}$ and $\Delta \in \Sigma$, we have

$$\langle \beta_1 \cdot \beta_2(\ell) \mid a E(\Delta) x \rangle = (a\, \beta_2(\ell)(\Delta) \mid x) =$$
$$= \langle \ell \mid a E(\Delta) x \rangle, \text{ so } \beta_1 \cdot \beta_2(\ell) = \ell .$$

□

2.22. Corollary

The relations i) and ii) of Theorem 2.17. are adjoint to each other in the sense that: $\forall\, s \in S_{\mathcal{R}}, \forall \mu \in T_{\mathcal{R}}$,

$$\langle \alpha_1(s) \mid \mu \rangle_T = \langle \beta_1(\mu) \mid s \rangle_S$$

and $\forall\, \varphi \in T'_{\mathcal{R}}, \forall \ell \in S'_{\mathcal{R}}$

$$\langle \ell \mid \alpha_2(\varphi) \rangle_S = \langle \varphi \mid \beta_2(\ell) \rangle_T .$$

2.23. Conjecture

The space $T_{\mathcal{R}}$ with the topology 2.16. is identical with the projective limit of the family of normed spaces $\{T_a \mid a \in \mathcal{R}\}$ of a-bounded c.a.o.s.measures on the joint spectrum of the family $\mathcal{R}$. Under conditions similar to those imposed on $\mathcal{R}$ in our paper [EK-2] the dualities 2.17. yield reflexivity of the spaces $T_{\mathcal{R}}$ and $S_{\mathcal{R}}$, turning them into topological duals of each other.

Final Remarks

In this hastily prepared paper we included certain ideas concerning c.a.o.s. measures that are already known in wider context. The authors are grateful to Prof.P.Masani for pointing out very rich bibliography on the subject which we unfortunately ignored while preparing this paper.

Concerning connections with our previous works on generalized functions spaces the idea of a possibility of an introduction of "spectral trajectories" into the theory belongs to Prof. Jan de Graaf.

In our paper we used the notion of a pre-ring of subsets , which seems to be too general for our goals. The reader should assume that all pre-rings in our paper are in fact rings of subsets.

Also the idea of c.a.o.s. measures defined on such abstract spaces as spectra of C*-algebras needs more careful investigation.

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