A construction of generalized eigenprojections based on geometric measure theory

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A CONSTRUCTION OF GENERALIZED EIGENPROJECTIONS BASED ON GEOMETRIC MEASURE THEORY

by

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Abstract

Let M denote a σ-compact locally compact metric space which satisfies certain geometrical conditions. Then for each σ-additive projection valued measure P on M there can be constructed a "canonical" Radon-Nikodym derivative Π: $\alpha \mapsto \Pi_\alpha$, $\alpha \in M$, with respect to a suitable basic measure ρ on M. The family $(\Pi_\alpha)_{\alpha\in M}$ consists of generalized eigen-projections related to the commutative von Neumann algebra generated by the projections $P(\Delta)$, Δ a Borel set of M.

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I.

In this paper M denotes a σ-compact locally compact (and hence separable) metric space. It follows that any positive Borel measure on M is regular (cf. [3], p. 162). In the monograph [2], certain geometrical conditions on M are introduced, which lead to the following result.

0. Theorem

Let μ denote a positive Borel measure on M with the property that bounded Borel sets of M have finite μ-measure, and let f denote a Borel function which is μ-integrable on bounded Borel sets. Then there exists a μ-null set N_f such that for all $\alpha \in M \setminus N_f$ both $\mu(B(\alpha,r)) > 0$, and the limit

$$\tilde{f}(\alpha) = \lim_{r \downarrow 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} f \, d\mu$$

exists. We have $f = \tilde{f}$ μ- almost everywhere.

($B(\alpha,r)$ denotes the closed ball with radius r and centre α.)

Remark: In the previous theorem, the Borel function f can be replaced by a Borel measure ν with the property that bounded Borel sets of M have finite ν-measure. Then a "canonical" Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is obtained, which satisfies

$$\frac{d\nu}{d\mu}(\alpha) = \lim_{r \downarrow 0} \frac{\nu(B(\alpha,r))}{\mu(B(\alpha,r))}$$

μ-almost everywhere.

In the sequel we assume that $\mathcal{M}$ also satisfies Federer's geometrical conditions. As examples of such spaces $\mathcal{M}$ we mention

- finite dimensional vector spaces with metric $d(x,y) = v(x - y)$ where v is any norm,
- Riemannian manifolds (of class ≥ 2) with their usual metric.

Let X denote a separable Hilbert space with inner product $(\cdot,\cdot)$ and let there be given a σ-additive projection valued set function $\mathcal{P}$ on $\mathcal{M}$. So for all Borel sets $\Delta \subset \mathcal{M}$, $\mathcal{P}(\Delta)$ is an orthogonal projection on X. Moreover, if Δ is the disjoint union $\bigcup_{j=1}^{\infty} \Delta_j$, then $\mathcal{P}(\Delta) = \sum_{j=1}^{\infty} \mathcal{P}(\Delta_j)$. In particular $\sum_{j=1}^{\infty} \mathcal{P}(\Delta_j) = \mathcal{I}$ if $\bigcup_{j=1}^{\infty} \Delta_j = \mathcal{M}$.

Now let $\mathcal{R}$ denote a positive bounded linear operator on X with the property that for each bounded Borel set Δ the positive operator $\mathcal{R}\mathcal{P}(\Delta)\mathcal{R}$ is trace class. E.g. for $\mathcal{R}$ any positive Hilbert-Schmidt operator can be taken.

For each bounded Borel set Δ we define $\rho(\Delta) = \mathrm{trace}(\mathcal{R}\mathcal{P}(\Delta)\mathcal{R})$. In a natural way, ρ becomes a σ-finite positive Borel measure on $\mathcal{M}$. Each bounded Borel set of $\mathcal{M}$ has a finite ρ-measure.

We take a fixed orthonormal basis $(v_k)_{k\in\mathbb{N}}$ in X, and for each $k,\ell \in \mathbb{N}$ we define the set function

$$\Phi_{k\ell} : \Delta \mapsto (\mathcal{R}\mathcal{P}(\Delta)\mathcal{R}\, v_\ell, v_k) \ , \quad \Delta \text{ Borel}.$$

The set functions $\Phi_{k\ell}$ are absolutely continuous with respect to ρ.

By Theorem 0, there exists a null set N_1 and there exist Borel functions $\hat{\phi}_{k\ell}$

such that for all $k,\ell \in \mathbb{N}$ and all $\alpha \in M \setminus N_1$

$$\hat{\phi}_{k\ell}(\alpha) = \lim_{r\downarrow 0}\left\{\frac{\Phi_{k\ell}(B(\alpha,r))}{\rho(B(\alpha,r))}\right\} .$$

1. Lemma

Let $\alpha \in M \setminus N_1$. Then for all $k,\ell \in \mathbb{N}$

$$|\hat{\phi}_{k\ell}(\alpha)|^2 \leq \hat{\phi}_{kk}(\alpha)\hat{\phi}_{\ell\ell}(\alpha) .$$

Proof. Consider the estimation,

$$|\hat{\phi}_{k\ell}(\alpha)|^2 = \lim_{r\downarrow 0}\left|\frac{\Phi_{k\ell}(B(\alpha,r))}{\rho(B(\alpha,r))}\right|^2 \leq$$

$$\leq \lim_{r\downarrow 0}\left\{\frac{\Phi_{kk}(B(\alpha,r))}{\rho(B(\alpha,r))}\right\} \lim_{r\downarrow 0}\left\{\frac{\Phi_{\ell\ell}(B(\alpha,r))}{\rho(B(\alpha,r))}\right\} =$$

$$= \hat{\phi}_{kk}(\alpha)\hat{\phi}_{\ell\ell}(\alpha) . \qquad \square$$

The function $\sum_{k=1}^{\infty} \hat{\phi}_{kk}$ is Borel, and the functions $\hat{\phi}_{kk}$ are positive. So for each bounded Borel set Δ, we have

$$\int_{\Delta} (\sum_{k=1}^{\infty} \hat{\phi}_{kk})d\rho = \sum_{k=1}^{\infty} \Phi_{kk}(\Delta) = \rho(\Delta) .$$

Then Theorem 0 yields a null set $N_2 \supset N_1$ such that for all $\alpha \in M \setminus N_2$,

$$\sum_{k=1}^{\infty} \hat{\phi}_{kk}(\alpha) = \lim_{r \downarrow 0} \frac{\int_{B(\alpha,r)} \left(\sum_{k=1}^{\infty} \hat{\phi}_{kk}\right) d\rho}{\rho(B(\alpha,r))} = 1$$

2. <u>Corollary</u>

Let $\alpha \in M \setminus N_2$. Then $\sum_{k,\ell=1}^{\infty} |\hat{\phi}_{k\ell}(\alpha)|^2 < \infty$.

<u>Proof</u>. Consider the estimation

$$\sum_{k,\ell=1}^{\infty} |\hat{\phi}_{k\ell}(\alpha)|^2 \leq \sum_{k=1}^{\infty} \hat{\phi}_{kk}(\alpha) \sum_{\ell=1}^{\infty} \hat{\phi}_{\ell\ell}(\alpha) = 1 \; . \qquad \square$$

3. <u>Definition</u>

The operators $\mathcal{B}_\alpha : X \to X$, $\alpha \in M$, are defined by

$$\begin{cases} \mathcal{B}_\alpha = 0 & \text{for } \alpha \in N_2 \, , \\ \mathcal{B}_\alpha x = \sum_{k,\ell=1}^{\infty} \hat{\phi}_{k\ell}(\alpha)(x,v_\ell)v_k \, , & x \in X, \quad \alpha \in M \setminus N_2 \; . \end{cases}$$

Observe that $\mathcal{B}_\alpha$ is a Hilbert-Schmidt operator for each $\alpha \in M$.

The operators $\mathcal{B}_\alpha$ are related to the set function $\mathcal{P}$ in the following way.

4. Lemma

Let $\alpha \in M \setminus N_2$. Then we have

$$\lim_{r \downarrow 0} \left\| \mathcal{B}_\alpha - \frac{\mathcal{R}P(B(\alpha,r))\mathcal{R}}{\rho(B(\alpha,r))} \right\|_{HS} = 0$$

with $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm.

Proof

For all $r > 0$,

$$\left\| \mathcal{B}_\alpha - \frac{\mathcal{R}P(B(\alpha,r))\mathcal{R}}{\rho(B(\alpha,r))} \right\|^2 = \sum_{k,\ell=1}^{\infty} \left| \hat{\phi}_{k\ell}(\alpha) - \frac{\Phi_{k\ell}(B(\alpha,r))}{\rho(B(\alpha,r))} \right|^2 .$$

Let $\varepsilon > 0$. Take a fixed $A \in \mathbb{N}$ so large that

$$(*) \qquad \sum_{k=A+1}^{\infty} \hat{\phi}_{kk}(\alpha) < \varepsilon^2/4 .$$

Next, take $r_0 > 0$ so small that for all r, $0 < r < r_0$, and all $k,\ell \in \mathbb{N}$ with $k,\ell \leqq A$.

$$(**) \qquad \left| \hat{\phi}_{k\ell}(\alpha) - \frac{\Phi_{k\ell}(B(\alpha,r))}{\rho(B(\alpha,r))} \right| < \varepsilon/A$$

and also

$$(***) \qquad \sum_{k=A+1}^{\infty} \frac{\Phi_{kk}(B(\alpha,r))}{\rho(B(\alpha,r))} < \varepsilon^2 .$$

Then we obtain the following estimation

$$\Big(\sum_{k=1}^{A}\sum_{\ell=1}^{A} + 2\sum_{k=A+1}^{\infty}\sum_{\ell=1}^{\infty}\Big)\left|\hat{\phi}_{k\ell}(\alpha) - \frac{\Phi_{k\ell}(B(\alpha,r))}{\rho(B(\alpha,r))}\right|^2$$

$$\leq \varepsilon^2 + 4\sum_{k=A+1}^{\infty}\sum_{\ell=1}^{\infty}\Big(|\hat{\phi}_{k\ell}(\alpha)| + \frac{|\Phi_{k\ell}(B(\alpha,r))|^2}{\rho(B(\alpha,r))^2}\Big).$$

By (*)

$$\sum_{k=A+1}^{\infty}\sum_{\ell=1}^{\infty}|\hat{\phi}_{k\ell}(\alpha)|^2 \leq \sum_{k=A+1}^{\infty}\hat{\phi}_{kk}(\alpha) \leq \varepsilon^2/4$$

and by (***)

$$\sum_{k=A+1}^{\infty}\sum_{\ell=1}^{\infty}\frac{|\Phi_{k\ell}(B(\alpha,r))|^2}{\rho(B(\alpha,r))^2} \leq \sum_{k=A+1}^{\infty}\frac{\Phi_{kk}(B(\alpha,r))}{\rho(B(\alpha,r))} < \varepsilon^2 .$$

Thus it follows that

$$\Big\|\mathcal{B}_\alpha - \frac{\mathcal{R}\mathcal{P}(B(\alpha,r))\mathcal{R}}{\rho(B(\alpha,r))}\Big\|_{HS} < \varepsilon\sqrt{6}$$

for all r with $0 < r < r_0$. □

In a natural way, the projection valued set function $\mathcal{P}$ can be linked to the function-algebra $L_\infty(\mathcal{M},\rho)$. To show this, let $x,y \in X$. Then the finite measure $\mu_{x,y}$ is defined by $\mu_{x,y}(\Delta) = (\mathcal{P}(\Delta)x,y)$ where Δ is any Borel set. We have $\int_{\mathcal{M}} d\mu_{x,y} = (x,y)$. Clearly, $\mu_{x,y}$ is absolutely continuous with respect to ρ.

Let f denote a Borel function on $\mathcal{M}$ which is bounded on bounded Borel sets. Then we define the operator T_f by

$$D(T_f) = \{x \in X \mid \int_{\mathcal{M}} |f|^2 d\mu_{x,x} < \infty\}$$

and for $x \in D(T_f)$

$$(T_f\, x,y) = \int_{\mathcal{M}} f\, d\mu_{x,y}, \quad y \in X .$$

Observe that T_f is a normal operator in X. Since f is bounded on bounded Borel sets we derive for each $r > 0$, $\alpha \in \mathcal{M}$ and $x \in X$,

$$|(T_f\, P(B(\alpha,r))x,x)| \leq \int_{\mathcal{M}} |f| \chi_{B(\alpha,r)} d\mu_{x,x} \leq$$

$$\leq \left(\sup_{\lambda \in B(\alpha,r)} |f(\lambda)| \right) (P(B(\alpha,r))x,x) .$$

So $RT_f P(B(\alpha,r))R$ is a trace class operator.

5. Lemma

There exists a null set N_3 such that for all $\alpha \in \mathcal{M} \setminus N_3$

$$\lim_{r \downarrow 0} \left\| f(\alpha) \mathcal{B}_\alpha - \frac{RT_f P(B(\alpha,r))R}{\rho(B(\alpha,r))} \right\|_{HS} = 0 .$$

Proof

Following Lemma 4, we are ready if we can prove that there exists a null set $N_3 \supset N_2$ such that for all $\alpha \in M \setminus N_3$

$$\lim_{r\downarrow 0} \left\| f(\alpha)\, \frac{RP(B(\alpha,r))R}{\rho(B(\alpha,r))} - \frac{RT_f\, P(B(\alpha,r))R}{\rho(B(\alpha,r))} \right\|_{HS} = 0 \; .$$

Therefore we estimate as follows

$$\sum_{k,\ell=1}^{\infty} \rho(B(\alpha,r))^{-2} \left| \int_{B(\alpha,r)} (f(\alpha) - f(\lambda))\, d\mu_{Rv_k,Rv_\ell}(\lambda) \right|^2 \leq$$

$$\leq \rho(B(\alpha,r))^{-2} \left(\int_{B(\alpha,r)} |f(\alpha) - f(\lambda)|^2\, d\rho(\lambda) \right) \left(\int_{B(\alpha,r)} \left(\sum_{k,\ell=1}^{\infty} |\hat{\phi}_{k\ell}(\lambda)|^2 \right) d\rho(\lambda) \right)$$

$$\leq \rho(B(\alpha,r))^{-1} \int_{B(\alpha,r)} |f(\alpha) - f(\lambda)|^2\, d\rho(\lambda)$$

because

$$\sum_{k,\ell=1}^{\infty} |\hat{\phi}_{k\ell}(\lambda)|^2 \leq \left(\sum_{k=1}^{\infty} \hat{\phi}_{kk}(\lambda) \right)^2 = 1 \; .$$

Now there exists a null set $N_3 \supset N_2$ such that the latter expression tends to zero as $r \downarrow 0$ for all $\alpha \in M \setminus N_3$.

II.

In the second part of this paper we employ the above auxiliary results in the announced construction of generalized eigenprojections.

We consider the triple of Hilbert spaces

$$R(X) \subseteq X \subseteq R^{-1}(X) \; .$$

Here $R(X)$ is the Hilbert space with inner product $(\cdot,\cdot)_1$,

$$(u,w)_1 = (R^{-1}u,\ R^{-1}w) \ , \quad u,w \in R(X) \ ,$$

and $R^{-1}(X)$ is the completion of X with respect to the inner product $(\cdot,\cdot)_{-1}$,

$$(x,y)_{-1} = (Rx,\ Ry) \ .$$

The spaces $R(X)$ and $R^{-1}(X)$ are in duality through the pairing $\langle\cdot,\cdot\rangle$,

$$\langle w,G\rangle = (R^{-1}w,\ RG), \quad w \in R(X),\ G \in R^{-1}(X) \ .$$

6. <u>Definition</u>

For each $\alpha \in M$, we define the operator $\pi_\alpha : R(X) \to R^{-1}(X)$ by

$$R\pi_\alpha w = B_\alpha R^{-1} w \ , \qquad w \in R(X) \ .$$

Cf. Definition 3.

Observe that $\pi_\alpha : R(X) \to R^{-1}(X)$ is continuous.

7. <u>Theorem</u>

I. For all $\alpha \in M \setminus N_2$ and for all $w \in R(X)$

$$\lim_{r\downarrow 0} \left\| \pi_\alpha w - \frac{P(B(\alpha,r))}{\rho(B(\alpha,r))} w \right\|_{-1} = 0 \ .$$

II. Let $f : M \to \mathbb{C}$ be a Borel function which is bounded on bounded Borel sets. Then there exists a null set $N_f \supset N_2$ such that for all $\alpha \in M \setminus N_f$ and all $w \in R(X)$

$$\lim_{r\downarrow 0} \left\| f(\alpha)\ \pi_\alpha w - T_f \frac{P(B(\alpha,r))}{\rho(P(\alpha,r))} w \right\|_{-1} = 0 \ .$$

Proof

The proof of I follows from Lemma 4 and the inequality

$$\left\|\pi_\alpha w - \frac{P(B(\alpha,r))}{\rho(B(\alpha,r))} w\right\|_{-1} \leq \left\|R\pi_\alpha R - \frac{RP(B(\alpha, r))R}{\rho(B(\alpha,r))}\right\|_{HS} \|R^{-1}w\| .$$

The proof of II follows from Lemma 5 and the inequality

$$\left\|f(\alpha)\pi_\alpha w - \frac{T_f P(B(\alpha,r))w}{\rho(B(\alpha,r))}\right\|_{-1} \leq$$

$$\leq \left\|f(\alpha)R\pi_\alpha R - \frac{RT_f P(B(\alpha,r))R}{\rho(B(\alpha,r))}\right\|_{HS} \|R^{-1}w\| .$$

□

8. Corollary

Let the operator RT_fR^{-1} be closable in X. Then T_f is closable as an operator from $R^{-1}(X)$ into $R^{-1}(X)$. For its closure $\bar{T}_f$ we have

$$\bar{T}_f\pi_\alpha w = f(\alpha)\pi_\alpha w$$

with $w \in R(X)$ and $\alpha \in M \setminus N_f$.

The results stated in Theorem 7 and Corollary 8 indicate that the mappings $\pi_\alpha : R(X) \to R^{-1}(X)$ give rise to ("candidate") generalized eigenspaces $\pi_\alpha R(X)$ for the commutative von Neumann algebra $\{T_f \mid f \in L_\infty(M,\rho)\}$.

Finally, we explain in which way the operators π_α, $\alpha \in M$, can be seen as generalized projections.

9. Lemma

Let $w \in \mathcal{R}(X)$. Then in weak sense

$$w = \int_M \Pi_\alpha w \, d\rho(\alpha) \ .$$

So for all $v \in \mathcal{R}(X)$,

$$(v,w) = \int_M \langle v, \Pi_\alpha w\rangle d\rho(\alpha) \ .$$

Proof. Let Δ be a bounded Borel set. For all $v \in \mathcal{R}(X)$,

$$\sum_{k,\ell=1}^{\infty} |\phi_{k\ell}(\alpha)(\mathcal{R}^{-1}v, v_\ell)(v_k, \mathcal{R}^{-1}w)| \leq$$

$$\leq \Big(\sum_{k,\ell=1}^{\infty} |\phi_{k\ell}(\alpha)|^2\Big)^{\frac{1}{2}} \|\mathcal{R}^{-1}w\| \|\mathcal{R}^{-1}v\| \ ,$$

and hence by Fubini's theorem

$$\int_\Delta \langle v, \Pi_\alpha w\rangle d\rho(\alpha) = \sum_{k,\ell=1}^{\infty} \Phi_{k\ell}(\Delta)(\mathcal{R}^{-1}v, v_\ell)(v_k, \mathcal{R}^{-1}w) =$$

$$= (\mathcal{P}(\Delta)v, w) \ .$$

Since M can be written as the disjoint union of bounded Borel sets it follows that

$$\int_M \langle v, \Pi_\alpha w\rangle d\rho(\alpha) = (v,w) \ . \qquad \square$$

Remark: If R is Hilbert-Schmidt, the integral $\int_M \pi_\alpha w d\rho(\alpha)$ exists in strong sense. So in addition we have

$$\int_M \|R\pi_\alpha w\| d\rho(\alpha) < \infty .$$

In $\pi_\alpha R(X)$ we define the sesquilinear form $(\cdot,\cdot)_\alpha$ by

$$(F,G)_\alpha = \langle v, \pi_\alpha w\rangle ,$$

where $F = \pi_\alpha v$, $G = \pi_\alpha w$. $(F,G)_\alpha$ does not depend on the choice of v and w. It can be shown easily that $(\cdot,\cdot)_\alpha$ is a well-defined non-degenerate sesquilinear form in $\pi_\alpha R(X)$. By X_α we denote the completion of $\pi_\alpha R(X)$ with respect to this sequilinear form.

10. Theorem

I. The Hilbert space X_α with inner product $(\cdot,\cdot)_\alpha$ is a Hilbert subspace of $R^{-1}(X)$. π_α maps $R(X)$ continuously into X_α.

II. Let f be a Borel function which is bounded on bounded Borel sets. Suppose the operator T_f is closable in $R^{-1}(X)$ with closure $\bar{T}_f$. Then there exists a null set N_f such that for each $\alpha \in M \setminus N_f$ and all $G \in X_\alpha$ we have

$$\bar{T}_f G = f(\alpha)G .$$

Proof.

I. Let $G \in \pi_\alpha R(X)$, $G = \pi_\alpha w$. We estimate as follows

$$\|RG\|^2 = \langle R^2\pi_\alpha w,\ \pi_\alpha w\rangle \leq$$

$$\leq \langle R^2\pi_\alpha w,\ \pi_\alpha R^2\pi_\alpha w\rangle^{\frac{1}{2}} \langle w,\ \pi_\alpha w\rangle^{\frac{1}{2}} \leq$$

$$\leq \|R\pi_\alpha R\|^{\frac{1}{2}} \|R\pi_\alpha w\| \ \|\pi_\alpha w\|_\alpha \ .$$

It follows that

$$\|G\|_{-1} \leq \|R\pi_\alpha R\|^{\frac{1}{2}} \|G\|_\alpha \ .$$

Hence X_α can be seen as a subspace of $R^{-1}(X)$.

II. By Corollary 8, there exists a null set N_f such that for all $\alpha \in M \setminus N_f$ and for all $w \in R(X)$

$$\bar{T}_f\ \pi_\alpha w = f(\alpha)\pi_\alpha w \ .$$

Let $\alpha \in M \setminus N_f$. Since $X_\alpha \hookrightarrow R^{-1}(X)$ and $\pi_\alpha R(X)$ is dense in X_α it follows that for all $G \in X_\alpha$, $G \in \mathrm{Dom}(\bar{T}_f)$ and $\bar{T}_f\ G = f(\alpha)G_\alpha$. □

11. Corollary

Let $\pi_\alpha^+ : X_\alpha \to R^{-1}(X)$ denote the adjoint of π_α.
Then $\pi_\alpha^+\pi_\alpha = \pi_\alpha$.

Proof

Let $w, v \in R(X)$. We have

$$\langle w, \pi_\alpha v\rangle = (\pi_\alpha w, \pi_\alpha v)_\alpha = \langle w,\ \pi_\alpha^+\pi_\alpha v\rangle \ .$$ □

Let $(u_k)_{k\in\mathbb{N}}$ denote an orthonormal basis in X which is contained in $\mathcal{R}(X)$. For each $\alpha \in \mathcal{M}$, the sequence $(\pi_\alpha u_k)_{k\in\mathbb{N}}$ is total in X_α. So the spaces X_α, $\alpha \in \mathcal{M}$, establish a measurable field of Hilbert spaces. Its field structure S is defined by

$$\phi \in S \Leftrightarrow \text{the functions } \alpha \mapsto (\phi(\alpha), \pi_\alpha u_k)_\alpha \text{ are Borel functions.}$$

So the direct integral $\mathcal{H} = \int_{\mathcal{M}}^{\oplus} X_\alpha d\rho(\alpha)$ is well-defined.
(For the general theory of direct integrals, see [1], p. 161-172.)
The vector fields $\alpha \mapsto \pi_\alpha u_k$, $\alpha \in \mathcal{M}$, $k \in \mathbb{N}$, give rise to an orthonormal system $(\phi_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$. (We recall that the elements of $\mathcal{H}$ are equivalence classes of square integrable vector fields.) We define the isometry $\mathcal{U} : X \to \mathcal{H}$ by

$$\mathcal{U}x = \sum_{k=1}^{\infty} (x,u_k)\phi_k \ , \qquad x \in X \ .$$

Then for all $x,y \in X$ we have

$$(x,y) = \int_{\mathcal{M}} d\mu_{x,y} = \int_{\mathcal{M}} ((\mathcal{U}x)(\alpha), \ (\mathcal{U}y)(\alpha))_\alpha \ d\rho(\alpha) \ .$$

It follows that for all $x,y \in X$ and all $f \in L_\infty(\mathcal{M},\rho)$

$$(T_f \ x,y) = \int_{\mathcal{M}} f d\mu_{x,y} = \int_{\mathcal{M}} f(\alpha)((\mathcal{U}x)(\alpha), \ (\mathcal{U}y)(\alpha))_\alpha \ d\rho(\alpha) \ ,$$

and hence we can write

$$\mathcal{U}T_f x = \int_M^{\oplus} f(\alpha)(Ux)(\alpha)\,d\rho(\alpha) \ .$$

12. Lemma

The operator $\mathcal{U} : X \to H$ is unitary.

Proof . We show that the set $\mathcal{U}(\{T_f\, u_k | k \in \mathbb{N},\ f \in L_\infty(M,\rho)\})$ is total in H. Let ϕ be a square integrable vector field such that for all $f \in L_\infty(M,\rho)$ and all $k \in \mathbb{N}$

$$0 = (\phi,\ T_f u_k)_H = \int_M f(\alpha)\ (\phi(\alpha),\ \pi_\alpha u_k)_\alpha d\rho(\alpha) \ .$$

Since $f \in L_\infty(M,\rho)$ is arbitrary taken, $(\phi(\alpha),\ \pi_\alpha\ u_k)_\alpha$ vanishes except on a set $\tilde{N}_k$ of measure zero. Taking $\tilde{N} = \bigcup_{k=1}^{\infty} \tilde{N}_k$ this yields $\phi(\alpha) = 0$ on $M \setminus \tilde{N}$, and hence

$$\int_M \|\phi(\alpha)\|_\alpha^2\ d\rho(\alpha) = 0 \ . \qquad \square$$

Now the mappings π_α, $\alpha \in M$, can be seen as generalized projections as follows: Let $w \in R(X)$. The vector field $\alpha \mapsto \pi_\alpha\ w$ is a representant of the class $\mathcal{U}w$. These representants $\alpha \mapsto \pi_\alpha\ w$, $w \in R(X)$, are canonical. Indeed, there exists a null set N $(= N_2)$ such that for all $w \in R(X)$, and for all $\alpha \in M \setminus N$,

$$\lim_{r \downarrow 0} \left\| \pi_\alpha w - \rho(B(\alpha,r))^{-1} \int_{B(\alpha,r)} \pi_\lambda w \, d\rho(\lambda) \right\|_{-1} = 0 \ .$$

(Cf. Theorem 7.)

So the family $(\pi_\alpha)_{\alpha \in M}$ selects a canonical representant out of each class $\mathcal{U}w$, $w \in \mathcal{R}(X)$. In this sense, each π_α "projects" $\mathcal{R}(X)$ densely into X_α.

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