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Pore scale model for non-isothermal flow and mineral precipitation and dissolution in a thin strip

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Abstract

Motivated by rock-fluid interactions occurring in a geothermal reservoir, we present a two-dimensional pore scale model of a thin strip consisting of void space and grains, with fluid flow through the void space. Ions in the fluid are allowed to precipitate onto the grains, while minerals in the grains are allowed to dissolve into the fluid, and we take into account the possible change in aperture that these two processes cause. We include temperature dependence and possible effects of the temperature in both fluid properties and in the mineral precipitation and dissolution reactions. For the pore scale model equations, we investigate the limit as the thin strip become infinitely thin, deriving one-dimensional effective equations.

1 Introduction

Geochemistry has a substantial impact in exploiting geothermal systems. In a geothermal reservoir, the injected water and the in-situ brine have different temperatures and chemical compositions and flow through highly heterogeneous regions. Due to the varying chemical properties of the rocks, the temperature and the flow regimes can change significantly. As a consequence of flow and geochemical reactions, composition of reservoir fluids as well as reservoir rock properties will develop dynamically with time. Minerals dissolving and precipitating onto the reservoir matrix, can change the porosity and hence the permeability of the system substantially. Mineral solubility can change by the cooling of the rock, or by the different ion content in the in-situ brine and in the injected water. The interaction between altering temperature, solute transport with mineral dissolution and precipitation, and fluid flow is highly coupled and challenging to model appropriately as the relevant physical processes jointly affect each other [2]. The effect of changing porosity through the production period of the geothermal reservoir, may have severe impact on operating conditions, as pores may close and block flow paths, or new pores may open to create enhanced flow conditions.

Injection of cold water into a geothermal reservoir can trigger the chemical reactions. The ion content of the injected water is normally different from the original groundwater, affecting the equilibrium state of the chemical system. Also, the solubility of several minerals are temperature dependent, hence the cooling of the porous medium can itself trigger chemical reactions. As reported from field studies or simulations, porosity and permeability changes due to precipitation and dissolution of minerals as silica, quartz, anhydrite, gypsum and calcite can be observed [5, 6, 7, 10, 17, 18]. Modeling of the mineral precipitation and dissolution is important in order to

understand the processes and to better estimate to which extent the chemical reactions can affect the permeability of the porous medium.

When dealing with porosity changes, what happens at the pore scale is highly relevant. The pore geometry affects the reaction rates for the dissolution and precipitation process as the reactive surface area is changed, and the resulting permeability is affected by the pore geometry. To achieve expressions for both reaction rates and permeability that depend on the pore scale effects we start with a model at the pore scale, and derive the Darcy scale model by homogenization. We propose in this paper a pore scale model to investigate these matters. Pore scale models incorporating mineral precipitation and dissolution have been studied earlier in [12, 16] and the corresponding Darcy scale models have been investigated further in [3, 11]. These papers assume that the pore geometry is not changed by the chemical reactions, which is a valid assumption when the deposited or dissolved mineral layer is thin enough. Investigations honoring the porosity changes may be found in [4, 13, 14], where mineral precipitation and dissolution have been considered on either circular grains or in a thin strip. In these papers, the position of the interface between grain and void space is tracked, giving a problem with a free boundary. Similar models can also be obtained for biofilm growth [15], for drug release from collagen matrices [9], and on an evolving microstructure [8]. In the spirit of [14], we consider mineral precipitation and dissolution in a thin strip and take into account the effect of temperature on the chemical reactions and on the fluid flow, giving a more coupled system of equations. Temperature changes can initiate or accelerate the rate of chemical reactions due to changes in solubility of the minerals. Also, the fluid flow is affected by the temperature changes due to changes in the fluid density and viscosity. For geothermal systems, the temperature dependence can be of high importance [1].

The structure of this document is as follows: In Section 2 we present the geometry of the pore scale model, while in Section 3 the model equations together with boundary conditions are given. In Section 4 we take the model equations to dimensionless form, before we in Section 5 introduce asymptotic expansions and obtain effective equations. The paper ends with some concluding remarks on the resulting equations in Section 6.

2 Pore scale model

The thin strip is shown below in figure 1. The length of the strip is L , while the width is l , where l is much smaller than L . We assume symmetry around the x -axis, hence the upper half of the strip is a reflection of the lower half. The width of the mineral part is $d(x, t)$ where $0 \leq d(x, t) < l/2$, as a greater width would clog the pore channel and fluid flow would be impossible.

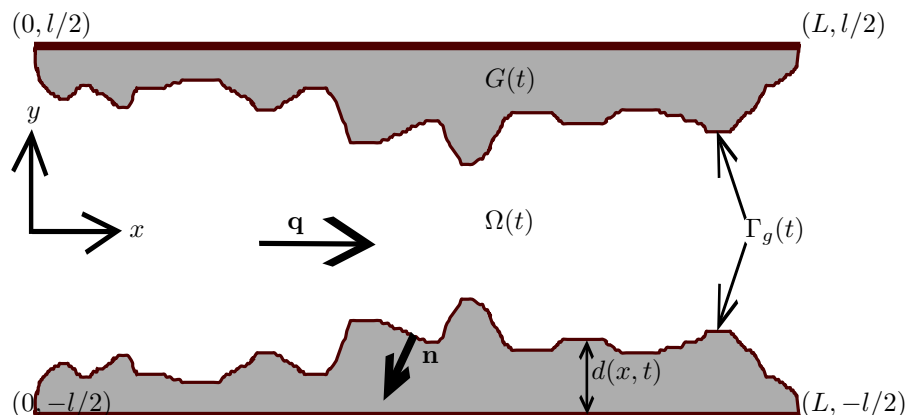


Figure 1: Model of thin strip.

The total domain Υ is the rectangle seen in the figure given by

$$\Upsilon = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq L, -l/2 \leq y \leq l/2\}.$$

The void space $\Omega(t)$ where fluid can flow is defined by

$$\Omega(t) = \{(x, y) \in R^2 | 0 \leq x \leq L, -(l/2 - d(x, t)) \leq y \leq (l/2 - d(x, t))\},$$

while the grain space $G(t)$ consisting of minerals is

$$G(t) = \{(x, y) \in R^2 | 0 \leq x \leq L, -l/2 \leq y \leq -(l/2 - d(x, t)) \vee (l/2 - d(x, t)) \leq y \leq l/2\}.$$

The interface $\Gamma_g(t)$ where mineral precipitation and dissolution can occur, is given by

$$\Gamma_g(t) = \{(x, y) \in R^2 | 0 < x < L, y = \pm(l/2 - d(x, t))\}.$$

The outward unit normal \mathbf{n} of the interface is (for the lower part) given by

$$\mathbf{n} = (\partial_x d, -1)^T / \sqrt{1 + (\partial_x d)^2}.$$

As the mineral width $d(x, t)$ changes with time, a point located at the interface $\Gamma_g(t)$ has a certain velocity. A point at the interface has coordinates $\mathbf{s}(t) = (x(t), -(l/2 - d(x, t)))$ and velocity $\mathbf{s}'(t) = (x'(t), \partial_x d(x(t), t)x'(t) + \partial_t d(x(t), t))$. Hence, the normal velocity of the lower boundary is

$$v_n = \mathbf{n} \cdot \mathbf{s}'(t) = -\partial_t d / \sqrt{1 + (\partial_x d)^2}. \quad (1)$$

The Rankine-Hugoniot condition guarantees conservation of quantities across a moving boundary:

$$\mathbf{n} \cdot [\mathbf{j}] = v_n [u] \quad (2)$$

where u is the preserved quantity (e.g. mass or energy) and \mathbf{j} is the flux of this quantity. The use of square brackets means the *jump* of the quantities, and is the difference between the quantities at each side of the interface. See figure 2 for illustration.

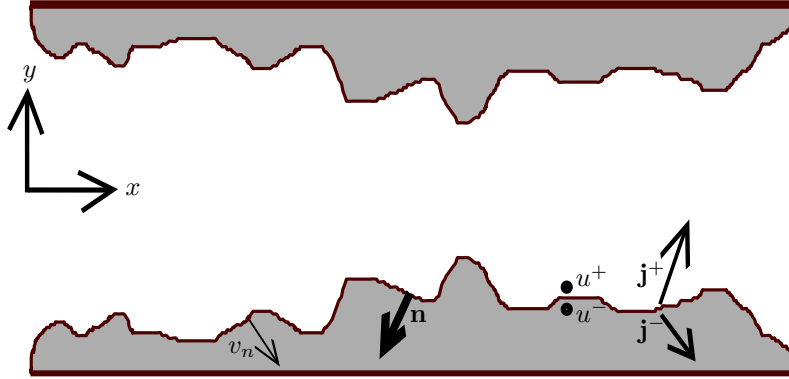


Figure 2: Illustration of the Rankine-Hugoniot jump condition for a moving boundary.

The Rankine-Hugoniot condition says that the normal component of the jump of the flux is equal to the normal velocity multiplied with the jump of the quantity across the boundary. The condition gives us a relationship between the preserved quantities and their fluxes across the moving boundary, hence giving boundary conditions becoming useful later.

3 Model equations

We assume conservation of ions, mass, momentum and energy. We also need one equation to describe how the chemical reactions affect the position of the interface $\Gamma_g(t)$.

3.1 Conservation of ions

There are two active ions in the fluid, with molar concentrations u^1 and u^2 . They satisfy the convection-diffusion equation in the void space:

$$\partial_t u^i = \nabla \cdot (D \nabla u^i - \mathbf{q} u^i) \text{ for } (x, y) \in \Omega(t). \quad (3)$$

In the above equation, D is the diffusion coefficient which we assume to be constant, and \mathbf{q} is the fluid velocity. At the mineral interface, one ion from each species can together form a mineral molecule, or going the opposite way, one mineral can dissolve into two ions: $u^1 + u^2 \leftrightarrow C$, where C is the mineral. Hence the ions can appear on both sides of the interface $\Gamma_g(t)$: either as ion in the void space, or as part of a mineral in the grain space. The Rankine-Hugoniot condition (2) for conserving ions across the moving interface is

$$\mathbf{n} \cdot (D \nabla u^i - \mathbf{q} u^i) = v_n (\rho_C - u^i) \text{ on } \Gamma_g(t) \quad (4)$$

where ρ_C is the molar density of the formed solid. As the minerals in the grain space do not move within the grain space $G(t)$, they have zero flux. Hence the difference in flux only has contribution from the ion flux. On the right-hand side, the difference $(\rho_C - u^i)$ appears as one mineral molecule contains one ion of u^i . We assume that our two ions have initially the same concentration. As the same number of ions disappear or are produced through the reaction, the two ions will always have the same concentration. Hence, $u^1 = u^2 = u$.

3.2 How reactions affect $d(x, t)$

At the boundary $\Gamma_g(t)$, minerals can precipitate and dissolve. When a mineral molecule dissolves and releases two ions into the fluid, the boundary changes position at the same time as the mineral molecule is no longer a part of the grain space $G(t)$, see figure 3 below.

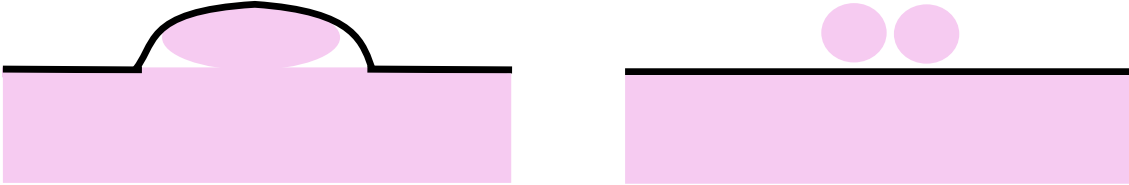


Figure 3: Mineral molecule dissolving into the fluid creating two ions. The black line shows the position of the interface $\Gamma_g(t)$.

Oppositely, when two ions come together and form a mineral molecule, they attach themselves at the boundary between void and grain space and become a part of the grain space. Reading figure 3 from right to left, illustrates this situation.

As the position of the interface $\Gamma_g(t)$ changes with precipitation and dissolution, we can quantify this change using the width of $G(t)$; $d(x, t)$, and the normal velocity of the interface v_n . To quantify the extent of dissolution and precipitation taking place, we introduce precipitation and dissolution *rates*. As we defined the normal vector \mathbf{n} pointing into the grain space (out of the void space) as seen in figure 1, the normal velocity of the interface is positive when dissolution occurs, and negative when precipitation occurs. Hence, the normal velocity is proportional to the local difference between the dissolution and precipitation rates and is given by

$$\rho_C v_n = -(f_p - f_d) \text{ on } \Gamma_g(t), \quad (5)$$

where f_p and f_d are the precipitation and dissolution rates for the reaction. If we think of the precipitation and dissolution rates as fluxes for the mineral, the above equation can be interpreted

¹If the reaction was $n_1 u^1 + n_2 u^2 \leftrightarrow C$ as considered by van Noorden[14], the difference $(n_i \rho_C - u^i)$ would appear as one mineral molecule then contains n_i ions.

as a result from the Rankine-Hugoniot condition (2), where the minerals are to be preserved. The left-hand side corresponds to the jump of the preserved quantity. As minerals only appear in the grain space $G(t)$, the molar density ρ_C appears. The right-hand side of the above equation is the net flux of minerals through the boundary.

We assume the precipitation rate to increase with ion concentration and with temperature. This is described through a kinetic rate depending on ion concentration with an Arrhenius factor:

$$f_p(T_f, u) = k_0 e^{-E/RT_f} \frac{(\gamma u)^2}{K_m(T_f)}, \quad (6)$$

where k_0 is a positive rate constant, E is the activation energy, R is the gas constant and T_f is fluid temperature. Further, γ is the activity coefficient of the ions and $K_m(T_f)$ is the equilibrium constant for the mineral. As the two ions have equal concentration u , the expression in the numerator is squared. The equilibrium constant is called a constant as it does not depend on ion or mineral concentration, but it may depend on fluid temperature. We assume dissolution to take place as long as there are minerals present; that is, as long as $d(x, t) > 0$. We further assume that the dissolution happens faster at higher temperatures, hence

$$f_d(T_f, d) = k_0 e^{-E/RT_f} w(d(x, t)), \quad (7)$$

where the function $w(d)$ is where $w(d)$ is given by

$$w(d) = \begin{cases} 0 & \text{if } d < 0 \\ \min\left(\frac{(\gamma u)^2}{K_m(T_f)}, 1\right) & \text{if } d = 0 \\ 1 & \text{if } d > 0. \end{cases}$$

The reason for defining the rates this way is to incorporate equilibrium states of the reaction properly. At equilibrium, the ion concentration u does not change, meaning the precipitation and dissolution rates are equal. When there are minerals present, meaning $d(x, t) > 0$, then $f_p - f_d = k_0 \exp(-E/RT_f) \left(\frac{(\gamma u)^2}{K_m(T_f)} - 1 \right) = 0$ at equilibrium, resulting in $K_m(T_f) = (\gamma u_{eq})^2$, which is how the equilibrium constant is defined. At equilibrium, the precipitation rate and dissolution rate have the same magnitude, hence the position of the interface is not changed. When there are no minerals left, the dissolution rate can be either the same magnitude or smaller than the precipitation rate, corresponding to the system either being in equilibrium or supersaturated with ions. Away from equilibrium, the magnitude of the reaction rates is such that the equilibrium is approached: If the fluid is supersaturated with ions, the precipitation rate is larger than the dissolution rate, resulting in a net precipitation of ions. If the fluid is undersaturated with ions, the precipitation rate is smaller than the dissolution rate, as long as there are more minerals left.

Collecting the equations (5), (6) and (7) and combining with the expression for v_n , (1), yields

$$\rho_C \partial_t d(x, t) = k_0 e^{-E/RT_f} \left(\frac{(\gamma u)^2}{K_m(T_f)} - w(d(x, t)) \right) \sqrt{1 + (\partial_x d(x, t))^2} \text{ on } \Gamma_g(t). \quad (8)$$

This equation describes how the reaction rates affect the aperture through the derivatives of the mineral width $d(x, t)$.

3.3 Conservation of mass

The mass of the fluid is conserved when it flows through the pore. As the fluid consists mainly of water, the fluid molar density ρ_f is not affected by the chemical reactions, but depends on temperature. As the temperature can change, the fluid density cannot be assumed constant. Hence, the mass conservation equation is

$$\partial_t \rho_f + \nabla \cdot (\rho_f \mathbf{q}) = 0 \text{ for } (x, y) \in \Omega(t). \quad (9)$$

Note that in some geothermal systems, the fluid density is also assumed to depend on pressure due to the large pressure changes that can occur through injection and production of fluid. As we are mainly interested in the temperature dependence, we neglect the pressure dependence and only allow for the fluid density to change with temperature. At the boundary, ions can leave the fluid and become part of the grain space in stead. Hence, there is a mass flux through the boundary $\Gamma_g(t)$. The Rankine-Hugoniot boundary condition applied to mass is

$$\mathbf{n} \cdot (-\rho_f \mathbf{q}) = v_n(2\rho_C - \rho_f) \text{ on } \Gamma_g(t). \quad (10)$$

The preserved quantity is mass, here represented as the molar density ρ_f . As one molecule of minerals, represented with ρ_C , contains *two* ions; one of u^1 and one of u^2 , the term $2\rho_C$ appears^{II}. Note that if $\rho_f = 2\rho_C$, the normal component of the velocity is zero at the interface, meaning that the chemical reactions do not cause volume change. Also note that our way of modeling the fluid density assumes that the density is not changed by the reactions; we assume that the ion concentrations are so low compared to the molar density of water, that the density of the fluid is not affected through the reactions. This is a valid assumption as long as the ion concentrations do not vary significantly.

3.4 Conservation of momentum

During the fluid flow, we also assume conservation of momentum, which can be expressed as

$$\partial_t(\rho_f \mathbf{q}) + \nabla \cdot (\rho_f \mathbf{q} \mathbf{q}) = \mathbf{b},$$

where \mathbf{b} is body force. In our case, body forces come from viscous stress, implying $\mathbf{b} = \nabla \cdot \sigma$ where σ is Cauchy's stress tensor. We assume the fluid is Newtonian, and we further assume that the stress tensor is a linear function of the strain rates, that the fluid is isotropic, and that the body forces are such that the fluid is at rest at hydrostatic pressure. Then, $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$ where p is pressure and $\tau_{ij} = \mu(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{3} \frac{\partial q_k}{\partial x_k} \delta_{ij})$. Hence the conservation of momentum becomes

$$\partial_t(\rho_f \mathbf{q}) + \nabla \cdot (\rho_f \mathbf{q} \mathbf{q}) = -\nabla p + \nabla \cdot \left(\mu(\nabla \mathbf{q} + (\nabla \mathbf{q})^T) \right) - \frac{2}{3} \nabla(\mu \nabla \cdot \mathbf{q}) \text{ for } (x, y) \in \Omega(t). \quad (11)$$

No-slip conditions are assumed at the boundary, meaning that \mathbf{q} has no tangential component at $\Gamma_g(t)$, but is parallel to the normal vector \mathbf{n} . Combining with the boundary condition appearing from conservation of mass, equation (10), the new boundary condition becomes

$$\mathbf{q} = \frac{\rho_f - 2\rho_C}{\rho_f} v_n \mathbf{n} \text{ on } \Gamma_g(t). \quad (12)$$

3.5 Conservation of energy

For now, we separate between two temperatures; the temperature in the fluid T_f and temperature in the grain T_g . Fluid temperature is only defined in the void space $\Omega(t)$ and grain temperature is defined in the grain space $G(t)$. The reason for separating between two temperatures is to easier emphasize how the energy is transferred differently in the two domains, but the separation has no physical meaning.

As the fluid is flowing, energy transfer can happen through both diffusion and convection, hence

$$\partial_t(\rho_f c_f T_f) = \nabla \cdot (k_f \nabla T_f - \rho_f c_f \mathbf{q} T_f) \text{ in } \Omega(t). \quad (13)$$

In the grain space flow is not possible, hence

$$\partial_t(\rho_C c T_g) = \nabla \cdot (k_g \nabla T_g) \text{ in } G(t). \quad (14)$$

^{II}If the reaction was $n_1 u^1 + n_2 u^2 \leftrightarrow C$, then the right-hand side in the above equation would be $v_n((n_1 + n_2)\rho_C - \rho_f)$

In the above equations, c_f and c are specific heats, and k_f and k_g are heat conductivities, of fluid and mineral respectively. The Rankine-Hugoniot condition for conservation of energy across the moving interface is

$$\mathbf{n} \cdot (k_f \nabla T_f - \rho_f c_f \mathbf{q} T_f - k_g \nabla T_g) = v_n (\rho_C c T_g - \rho_f c_f T_f) \text{ on } \Gamma_g(t), \quad (15)$$

and we also assume temperature continuity at the interface:

$$T_g = T_f \text{ on } \Gamma_g(t). \quad (16)$$

For the lower and upper part of $G(t)$, we assume homogeneous Neumann boundary conditions:

$$\partial_y T_g = 0 \text{ for } 0 \leq x \leq L, y = \pm l/2. \quad (17)$$

4 Dimensionless form

To achieve non-dimensional quantities, we introduce t_{ref} , $x_{ref} = L$, u_{ref} , $q_{ref} = L/t_{ref}$, $p_{ref} = L^3 u_{ref}/t_{ref}^2 l^{III}$, $\mu_{ref} = l^2 p_{ref}/L q_{ref}^{IV}$, T_{ref} and let $\epsilon = l/L$. Non-dimensional variables are denoted with a hat and are defined as

$$\begin{aligned} \hat{t} &= t/t_{ref} & \hat{x} &= x/L & \hat{y} &= y/L & \hat{u} &= u/u_{ref} & \hat{d}^\epsilon &= d/l \\ \hat{\mathbf{q}}^\epsilon &= \mathbf{q}/q_{ref} & \hat{p}^\epsilon &= p/p_{ref} & \hat{\rho}_f &= \rho_f/c_{ref} & \hat{\rho} &= \rho_C/c_{ref} \\ \hat{k} &= k_0 t_{ref}/u_{ref} l & \hat{D} &= D t_{ref}/L^2 & \hat{\mu} &= \mu/\mu_{ref} & \hat{T}^\epsilon &= T/T_{ref} \end{aligned}$$

We emphasize dependence on the small variable ϵ by denoting our main variables with ϵ as a superscript. Since we will only use non-dimensional variables in the following, we skip the hat.

Using non-dimensional variables, the total domain is defined by

$$\Upsilon^\epsilon = \{(x, y) \in R^2 | 0 \leq x \leq 1, -\epsilon/2 \leq y \leq \epsilon/2\}.$$

The void space is now given by

$$\Omega^\epsilon(t) = \{(x, y) \in R^2 | 0 \leq x \leq 1, -\epsilon(1/2 - d^\epsilon(x, t)) \leq y \leq \epsilon(1/2 - d^\epsilon(x, t))\},$$

while the grain space is defined as

$$G^\epsilon(t) = \{(x, y) \in R^2 | 0 \leq x \leq 1, -\epsilon/2 \leq y \leq -\epsilon(1/2 - d^\epsilon) \vee \epsilon(1/2 - d^\epsilon) \leq y \leq \epsilon/2\}.$$

The interface between the void and grain space is now

$$\Gamma^\epsilon(t) = \{(x, y) \in R^2 | 0 \leq x \leq 1, y = \pm \epsilon(1/2 - d^\epsilon(x, t))\},$$

while the outward unit normal for the lower part of the interface is given by

$$\mathbf{n}^\epsilon = (\epsilon \partial_x d^\epsilon, -1)^T / \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2}. \quad (18)$$

Inserting the dimensionless variables into the model equations give the following set of equations and boundary conditions. We insert the normal velocity v_n , equation (1), when necessary.

The convection-diffusion equation (3) describing the ion concentration, the model equation becomes

$$\partial_t u^\epsilon = \nabla \cdot (D \nabla u^\epsilon - \mathbf{q}^\epsilon u^\epsilon) \text{ in } \Omega^\epsilon(t), \quad (19)$$

^{III}This choice of p_{ref} is in order to make the left-hand side of equation (11) disappear in the asymptotic expansion and achieve Darcy's law.

^{IV}This is the same choice made by [14] in order to balance viscous forces and pressure in equation (11) to achieve Darcy's law.

with the boundary condition (4) now written as

$$\mathbf{n}^\epsilon \cdot (D\nabla u^\epsilon - \mathbf{q}^\epsilon u^\epsilon) = -\epsilon \partial_t d^\epsilon (\rho - u^\epsilon) / \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} \text{ on } \Gamma^\epsilon(t). \quad (20)$$

Note that an underlying assumption is that the non-dimensional diffusion coefficient D is not depending on ϵ , hence diffusion and convection occur at the same time scale.

The non-dimensional version of the equation (8) describing how the width $d(x, t)$ is changed by chemical reactions, is

$$\rho \partial_t d^\epsilon = (f_p(T_f^\epsilon, u^\epsilon) - f_d(T_f^\epsilon, d^\epsilon)) \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} \text{ on } \Gamma^\epsilon(t). \quad (21)$$

The reaction rates are non-dimensional and are given by

$$f_p(T_f^\epsilon, u^\epsilon) = k e^{-\alpha/T_f^\epsilon} \frac{(\gamma u^\epsilon)^2}{K_m(T_f^\epsilon)} \text{ and } f_d(T_f^\epsilon, d^\epsilon) = k e^{-\alpha/T_f^\epsilon} w(d^\epsilon(x, t)), \quad (22)$$

where $\alpha = E/RT_{ref}$ is non-dimensional. Also note that $K_m(T_f^\epsilon)$ is non-dimensional as it has been scaled with u_{ref}^2 . The activity coefficient is by definition non-dimensional.

The mass conservation equation (9) transforms into

$$\partial_t \rho_f + \nabla \cdot (\rho_f \mathbf{q}^\epsilon) = 0 \text{ in } \Omega^\epsilon(t). \quad (23)$$

The corresponding Rankine-Hugoniot boundary equation (10) has the non-dimensional form

$$\mathbf{q}^\epsilon \cdot \mathbf{n}^\epsilon = -\epsilon \frac{\rho_f - 2\rho}{\rho_f} \partial_t d^\epsilon / \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} \text{ on } \Gamma^\epsilon(t). \quad (24)$$

The momentum balance equation (11) becomes

$$\begin{aligned} \epsilon \left(\partial_t (\rho_f \mathbf{q}^\epsilon) + \nabla \cdot (\rho_f \mathbf{q}^\epsilon \mathbf{q}^\epsilon) \right) &= -\nabla p^\epsilon \\ &+ \epsilon^2 \left(\nabla \cdot \left(\mu (\nabla \mathbf{q}^\epsilon + (\nabla \mathbf{q}^\epsilon)^T) \right) - \frac{2}{3} \nabla (\mu \nabla \cdot \mathbf{q}^\epsilon) \right) \text{ in } \Omega^\epsilon(t), \end{aligned} \quad (25)$$

while the boundary condition (12) is

$$\mathbf{q}^\epsilon = -\epsilon \frac{\rho_f - 2\rho}{\rho_f} \partial_t d^\epsilon \mathbf{n}^\epsilon / \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} \text{ on } \Gamma^\epsilon(t). \quad (26)$$

The non-dimensional form of the energy conservation equations (13) and (14) is

$$\partial_t (\rho_f T_f^\epsilon) + \nabla \cdot (\rho_f \mathbf{q}^\epsilon T_f^\epsilon) = \frac{1}{Pe} \nabla^2 T_f^\epsilon \text{ in } \Omega^\epsilon(t) \quad (27)$$

and

$$\partial_t (\varsigma \rho T_g^\epsilon) = \frac{1}{Pe} \kappa \nabla^2 T_g^\epsilon \text{ in } G^\epsilon(t), \quad (28)$$

where $Pe = L^2 u_{ref} c_f / k_f t_{ref}$ is the Péclet number, and $\varsigma = c/c_f$ and $\kappa = k_g/k_f$. These three constants are all assumed to not depend on ϵ , but are of order 1. The boundary condition (15) is written

$$\mathbf{n}^\epsilon \cdot \left(\frac{1}{Pe} \nabla T_f^\epsilon - \rho_f \mathbf{q}^\epsilon T_f^\epsilon - \frac{1}{Pe} \kappa \nabla T_g^\epsilon \right) = -\epsilon (\varsigma \rho T_g^\epsilon - \rho_f T_f^\epsilon) \partial_t d^\epsilon / \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} \text{ on } \Gamma^\epsilon(t), \quad (29)$$

and the continuity condition (16) is

$$T_g^\epsilon = T_f^\epsilon \text{ on } \Gamma^\epsilon(t). \quad (30)$$

The boundary condition (17) for T_g is now

$$\partial_y T_g^\epsilon = 0 \text{ for } 0 \leq x \leq 1, y = \pm \epsilon/2. \quad (31)$$

5 Asymptotic expansion

We perform a formal asymptotic expansion for the variables depending on ϵ , namely u^ϵ , d^ϵ , \mathbf{q}^ϵ , p^ϵ , T_f^ϵ and T_g^ϵ . For all expecting d^ϵ we assume

$$u^\epsilon(x, y, t) = u_0(x, \frac{y}{\epsilon}, t) + \epsilon u_1(x, \frac{y}{\epsilon}, t) + O(\epsilon^2).$$

where $u_0(x, z, t)$ is the function describing the first order behavior of the variable u^ϵ . The following term $\epsilon u_1(x, y, t)$ is less important when ϵ is small. As the velocity \mathbf{q}^ϵ is a vector function, we assume the above expansion for both the horizontal component $q^{(1)}$ and the vertical component $q^{(2)}$. Due to the scaling of the second variable, $u_i(x, z, t)$ is defined in the domain

$$\tilde{\Omega}(t) = \{(x, z) | 0 \leq x \leq 1, -(1/2 - d^\epsilon) \leq z \leq (1/2 - d^\epsilon)\}.$$

The exception is T_{gi} which is defined in

$$\tilde{G}(t) = \{(x, y) \in R^2 | 0 \leq x \leq 1, -1/2 \leq z \leq -(1/2 - d^\epsilon) \vee (1/2 - d^\epsilon) \leq z \leq 1/2\}.$$

The width of the grain space, d^ϵ , does not depend on y and has the expansion

$$d^\epsilon(x, t) = d_0(x, t) + \epsilon d_1(x, t) + O(\epsilon^2).$$

The component functions $d_i(x, t)$ are defined for $0 \leq x \leq 1$.

We further assume that fluid density ρ_f and viscosity μ to depend linearly on fluid temperature T_f^ϵ :

$$\rho_f(T_f^\epsilon) = \rho_0 - \beta_{\rho_f} T_f^\epsilon = \rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1} + O(\epsilon^2)); \quad (32)$$

$$\mu(T_f^\epsilon) = \mu_0 + \beta_\mu T_f^\epsilon = \mu_0 + \beta_\mu (T_{f0} + \epsilon T_{f1} + O(\epsilon^2)), \quad (33)$$

where ρ_0 and μ_0 are reference values of the density and viscosity, and β_{ρ_f} and β_μ are positive constants. Other forms for the density and viscosity can be considered straightforwardly.

Below we follow the ideas in [14]. The goal is to derive an upscaled effective model describing the thin strip with vanishing width, obtaining a one-dimensional model still honoring the changes in aperture.

5.1 Conservation of ions

We substitute the expansion for u^ϵ into the convection-diffusion equation (19). Note that due to the scaling of the second variable, the factor $1/\epsilon$ appears in the derivatives of z .

$$\begin{aligned} \partial_t(u_0 + \epsilon u_1) &= (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot \left(D(\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j})(u_0 + \epsilon u_1) - (\mathbf{q}_0 + \epsilon \mathbf{q}_1)(u_0 + \epsilon u_1) \right) + O(1); \\ \partial_t u_0 &= \frac{1}{\epsilon^2} D \partial_{zz} u_0 + \frac{1}{\epsilon} (D \partial_{zz} u_1 - \partial_z (q_0^{(2)} u_0)) + O(1) \text{ in } \tilde{\Omega}(t). \end{aligned}$$

The boundary equation (20) is written

$$\begin{aligned} (\epsilon \partial_x d_0 \mathbf{i} - \mathbf{j}) \cdot \left(D(\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j})(u_0 + \epsilon u_1) - (\mathbf{q}_0 + \epsilon \mathbf{q}^\epsilon)(u_0 + \epsilon u_1) \right) + O(\epsilon) \\ = -\epsilon \partial_t (d_0 + \epsilon d_1) (\rho - (u_0 + \epsilon u_1)); \\ -\frac{1}{\epsilon} D \partial_z u_0 + (-D \partial_z u_1 + q_0^{(2)} u_0) = O(\epsilon) \text{ on } \tilde{\Gamma}(t). \end{aligned}$$

We collect the terms of lowest orders in ϵ as these dominate the equations when ϵ approaches zero. For the two above equations, the lowest order terms are

$$\begin{aligned} \partial_{zz} u_0 &= 0 \text{ in } \tilde{\Omega}(t); \\ \partial_z u_0 &= 0 \text{ on } \tilde{\Gamma}(t), \end{aligned}$$

implying that u_0 cannot depend on z , hence $u_0 = u_0(x, t)$.

We integrate the original convection-diffusion equation (19) over y , with integration limits $-\epsilon(1/2 - d^\epsilon) \leq y \leq \epsilon(1/2 - d^\epsilon)$:

$$\int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \partial_t u^\epsilon dy = \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \left(\partial_x (D \partial_x u^\epsilon - q^{\epsilon(1)} u^\epsilon) + \partial_y (D \partial_y u^\epsilon - q^{\epsilon(2)} u^\epsilon) \right) dy.$$

We interchange the order of integration and differentiation, taking into account that our integration limits depend on x and t through d^ϵ . For the second term in the integral on the right-hand side, we can evaluate the integral as differentiation and integration of the variable y are opposite operations. Hence,

$$\begin{aligned} \partial_t \left(\frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} u^\epsilon dy \right) + 2 \partial_t d^\epsilon u^\epsilon|_{y=-\epsilon(1/2-d^\epsilon)} &= \partial_x \left(\frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} (D \partial_x u^\epsilon - q^{\epsilon(1)} u^\epsilon) dy \right) \\ &+ 2 \partial_x d^\epsilon (D \partial_x u^\epsilon - q^{\epsilon(1)} u^\epsilon)|_{y=-\epsilon(1/2-d^\epsilon)} - \frac{2}{\epsilon} (D \partial_y u^\epsilon - q^{\epsilon(2)} u^\epsilon)|_{y=-\epsilon(1/2-d^\epsilon)}. \end{aligned}$$

We apply the unit normal vector $\mathbf{n}^\epsilon = (\epsilon \partial_x d^\epsilon, -1)^T / \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2}$, hence the last two terms in the above equation can be written as

$$\frac{2}{\epsilon} \mathbf{n}^\epsilon \cdot (D \nabla u^\epsilon - \mathbf{q}^\epsilon u^\epsilon) \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} |_{y=-\epsilon(1/2-d^\epsilon)}.$$

Using the boundary condition (20), this expression can be written

$$-2 \partial_t d^\epsilon (\rho - u^\epsilon) |_{y=-\epsilon(1/2-d^\epsilon)}.$$

We insert the asymptotic expansion for u^ϵ and \mathbf{q}^ϵ . At the same time, we make a substitution in the remaining integrals, letting $z = y/\epsilon$. The lowest order terms are

$$\begin{aligned} &\partial_t \left(\int_{-(1/2-d_0)}^{(1/2-d_0)} u_0 dz \right) + 2 \partial_t d_0 u_0 |_{z=-(1/2-d_0)} \\ &= \partial_x \left(\int_{-(1/2-d_0)}^{1/2-d_0} (D \partial_x u_0 - q_0^{(1)} u_0) dz \right) - 2 \partial_t d_0 (\rho - u_0) |_{z=-(1/2-d_0)}. \end{aligned}$$

Since u_0 does not depend on z , the integrals can easily be evaluated. We define the transmissivity $\bar{q}(x, t) = \int_{-(1/2-d_0)}^{1/2-d_0} q_0^{(1)}(x, z, t) dz$. Hence, the above equation can be written

$$\partial_t ((1 - 2d_0)u_0 + 2\rho d_0) = \partial_x (D(1 - 2d_0)\partial_x u_0 - \bar{q}u_0) \text{ for } 0 \leq x \leq 1.$$

5.2 Conservation of energy part I

As for the concentration, the lowest order expansions of the temperatures T_f and T_g do not depend on z . Inserting the asymptotic expansions for T_f^ϵ , T_g^ϵ and \mathbf{q}^ϵ into (27) and (28) provides

$$\begin{aligned} &\partial_t \left((\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) (T_{f0} + \epsilon T_{f1}) \right) \\ &+ (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot \left((\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) (\mathbf{q}_0 + \epsilon \mathbf{q}_1) (T_{f0} + \epsilon T_{f1}) \right) \\ &= \frac{1}{P_e} (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) (T_{f0} + \epsilon T_{f1}) + O(1) \text{ in } \tilde{\Omega}(t), \end{aligned}$$

and

$$\partial_t \left(\varsigma \rho (T_{g0} + \epsilon T_{g1}) \right) = \frac{1}{P_e} \kappa (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) (T_{g0} + \epsilon T_{g1}) + O(1) \text{ in } \tilde{G}(t).$$

The lowest order term of these two equations both arise from the diffusive terms;

$$\partial_{zz}T_{f0} = 0 \text{ in } \tilde{\Omega}(t)$$

and

$$\partial_{zz}T_{g0} = 0 \text{ in } \tilde{G}(t).$$

The boundary condition (29) combined with the asymptotic expansions and the density equation (32) provides

$$\begin{aligned} & (\epsilon\partial_x(d_0 + \epsilon d_1)\mathbf{i} - \mathbf{1j}) \cdot \left(\frac{1}{Pe}(\partial_x\mathbf{i} + \frac{1}{\epsilon}\partial_z\mathbf{j})(T_{f0} + \epsilon T_{f1}) \right. \\ & \left. - (\rho_0 - \beta_{\rho_f}(T_{f0} + \epsilon T_{f1}))(\mathbf{q}_0 + \epsilon\mathbf{q}_1)(T_{f0} + \epsilon T_{f1}) - \frac{1}{Pe}\kappa(\partial_x\mathbf{i} + \frac{1}{\epsilon}\partial_z\mathbf{j})(T_{g0} + \epsilon T_{g1}) \right) \\ & = -\epsilon\partial_t(d_0 + \epsilon d_1) \left(\varsigma\rho(T_{g0} + \epsilon T_{g1}) - (\rho_0 - \beta_{\rho_f}(T_{f0} + \epsilon T_{f1}))(T_{f0} + \epsilon T_{f1}) \right) + O(\epsilon^2). \end{aligned}$$

Here, the dominating terms as ϵ decreases are

$$\partial_zT_{f0} - \kappa\partial_zT_{g0} = 0 \text{ on } \tilde{\Gamma}(t).$$

The lowest order of the boundary conditions (30) and (31) are

$$T_{f0} = T_{g0} \text{ on } \tilde{\Gamma}(t), \quad \partial_zT_{g0} = 0 \text{ at } z = \pm 1/2.$$

The only possible solution of the two above equations with these boundary conditions is that T_{f0} and T_{g0} do not depend on z , hence

$$T_{f0} = T_{f0}(x, t) \text{ and } T_{g0} = T_{g0}(x, t). \quad (34)$$

Since the continuity condition assures that T_{g0} and T_{f0} are equal for all x , we introduce $T_0 = T_{g0} = T_{f0}$.

5.3 The d^ϵ equation

In order to derive an effective equation for how the width d^ϵ is affected by the reactions, we need to regularize the dissolution rate to ensure a Lipschitz continuous function. We define $w_\delta(d^\epsilon)$ such that

$$w_\delta(d^\epsilon) = \begin{cases} 0 & \text{if } d^\epsilon < 0 \\ d^\epsilon/\delta & \text{if } 0 \leq d^\epsilon \leq \delta \\ 1 & \text{if } d^\epsilon > \delta. \end{cases}$$

We insert the asymptotic expansions for d^ϵ , u^ϵ and T_f^ϵ into equation (21), obtaining

$$\rho\partial_t(d_0 + \epsilon d_1) = (f_p(T_{f0} + \epsilon T_{f1}, u_0 + \epsilon u_1) - f_d(T_{f0} + \epsilon T_{f1}, d_0 + \epsilon d_1))\sqrt{1 + (\epsilon\partial_x(d_0 + \epsilon d_1))^2} + O(\epsilon^2),$$

where $w(d^\epsilon)$ in f_d in equation (22) is to be replaced with the regularized function defined here. Since f_p and f_d are both Lipschitz, the lowest order expansion is

$$\rho\partial_t d_0 = f_p(T_{f0}, u_0) - f_d(T_{f0}, d_0) \text{ for } 0 \leq x \leq 1.$$

If we now let δ approach zero, we obtain our original expression for $w(d_0)$.

5.4 Conservation of mass

We integrate the mass conservation equation (23) across a thin section in the void space with width δx ; the integration area is given by $Y = \{(x, y) \in R^2 | x_1 \leq x \leq x_1 + \delta x, -\epsilon(1/2 - d^\epsilon) \leq y \leq \epsilon(1/2 - d^\epsilon)\}$:

$$0 = \int_Y \partial_t \rho_f dV + \int_Y \nabla \cdot (\rho_f \mathbf{q}^\epsilon) dV.$$

We apply Gauss' theorem to the second integral and divide the whole equation by δx at the same time:

$$0 = \frac{1}{\delta x} \int_Y \partial_t \rho_f dV + \frac{1}{\delta x} \int_{\partial Y} \rho_f \mathbf{q}^\epsilon \cdot \mathbf{n} ds.$$

The figure 4 indicates how Y and its boundary ∂Y together with outward unit normals \mathbf{n} look like.

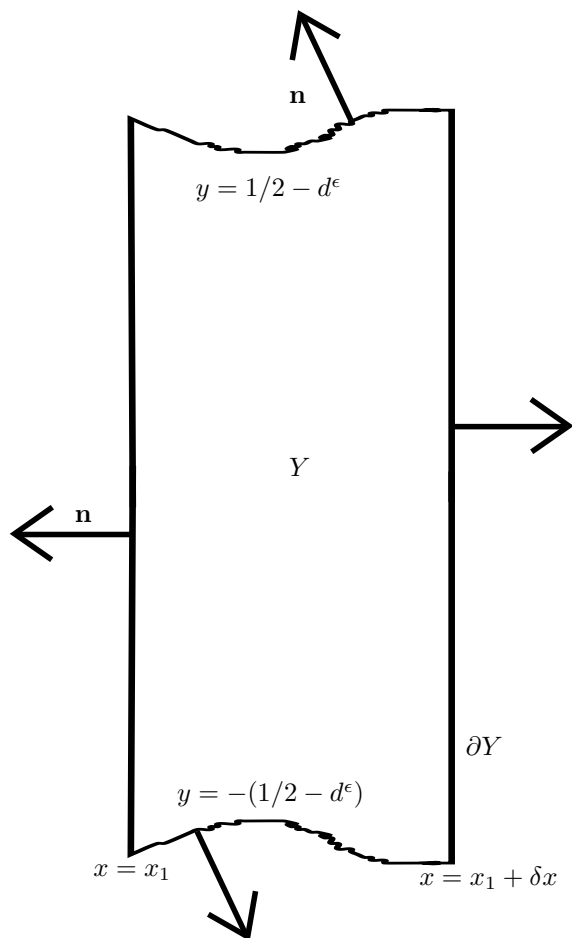


Figure 4: Thin section of the void space.

We insert the limits in our two integrals accordingly. The boundary ∂Y consists of four parts; the right, left, upper and lower. Due to the symmetry around the x -axis the upper and lower

integrals are equal, hence

$$\begin{aligned}
0 &= \frac{1}{\delta x} \int_{x_1}^{x_1+\delta x} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \partial_t \rho_f dy dx + \frac{1}{\delta x} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f q^{\epsilon(1)} dy|_{x=x_1+\delta x} \\
&\quad - \frac{1}{\delta x} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f q^{\epsilon(1)} dy|_{x=x_1} + \frac{2}{\delta x} \int_{x_1}^{x_1+\delta x} \rho_f \mathbf{q}^\epsilon \cdot \mathbf{n}^\epsilon \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} dx|_{y=-\epsilon(1/2-d^\epsilon)}.
\end{aligned}$$

In the last term, we can rewrite the integrand using the boundary condition (24). We also insert the expression for the density (32) together with asymptotic expansions for \mathbf{q}^ϵ and T_f^ϵ . We make a change in variables, letting $z = y/\epsilon$:

$$\begin{aligned}
O(\epsilon^2) &= \frac{\epsilon}{\delta x} \int_{x_1}^{x_1+\delta x} \int_{-(1/2-(d_0+\epsilon d_1))}^{1/2-(d_0+\epsilon d_1)} \partial_t (\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) dz dx \\
&\quad + \frac{\epsilon}{\delta x} \int_{-(1/2-(d_0+\epsilon d_1))}^{1/2-(d_0+\epsilon d_1)} (\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) (q_0^{(1)} + \epsilon q_1^{(1)}) dz|_{x=x_1+\delta x} \\
&\quad - \frac{\epsilon}{\delta x} \int_{-(1/2-(d_0+\epsilon d_1))}^{1/2-(d_0+\epsilon d_1)} (\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) (q_0^{(1)} + \epsilon q_1^{(1)}) dz|_{x=x_1} \\
&\quad - \frac{2\epsilon}{\delta x} \int_{x_1}^{x_1+\delta x} \left((\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) - 2\rho \right) \partial_t (d_0 + \epsilon d_1) dx|_{z=-(1/2-(d_0+\epsilon d_1))}.
\end{aligned}$$

We collect the lowest order terms, hence

$$\begin{aligned}
0 &= \frac{1}{\delta x} \int_{x_1}^{x_1+\delta x} \int_{-(1/2-d_0)}^{1/2-d_0} \partial_t \rho_{f0} dz dx \\
&\quad + \frac{1}{\delta x} \int_{-(1/2-d_0)}^{1/2-d_0} \rho_{f0} q_0^{(1)} dz|_{x=x_1+\delta x} - \frac{1}{\delta x} \int_{-(1/2-d_0)}^{1/2-d_0} \rho_{f0} q_0^{(1)} dz|_{x=x_1} \\
&\quad - \frac{2}{\delta x} \int_{x_1}^{x_1+\delta x} (\rho_{f0} - 2\rho) \partial_t d_0 dx.
\end{aligned}$$

In the above equation, $\rho_{f0} = \rho_0 - \beta_{T_f} T_{f0}$. To evaluate the integrals in z , we can take advantage of ρ_{f0} not depending on z as T_{f0} is independent of z . Hence,

$$\begin{aligned}
0 &= \frac{1}{\delta x} \int_{x_1}^{x_1+\delta x} (1 - 2d_0) \partial_t \rho_{f0} dx \\
&\quad + \frac{1}{\delta x} (\rho_{f0} \bar{q}|_{x=x_1+\delta x} - \rho_{f0} \bar{q}|_{x=x_1}) \\
&\quad - \frac{2}{\delta x} \int_{x_1}^{x_1+\delta x} (\rho_{f0} - 2\rho) \partial_t d_0 dx.
\end{aligned}$$

Next step is to let δx approach zero. Using the fundamental theorem in calculus on the two integrals, and the definition of the derivative in the middle expression, we obtain

$$0 = (1 - 2d_0) \partial_t \rho_{f0} + \partial_x (\rho_{f0} \bar{q}) - 2(\rho_{f0} - 2\rho) \partial_t d_0,$$

which can be rewritten as

$$\partial_t \left((1 - 2d_0) \rho_{f0} + 2d_0 2\rho \right) + \partial_x (\rho_{f0} \bar{q}) = 0 \text{ for } 0 \leq x \leq 1.$$

We can show that the vertical component of \mathbf{q}_0 , is zero by inserting the asymptotic expansions directly into the mass conservation equation (23):

$$\partial_t (\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) + (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot ((\rho_0 - \beta_{\rho_f} (T_{f0} + \epsilon T_{f1})) (\mathbf{q}_0 + \epsilon \mathbf{q}_1)) = O(\epsilon).$$

The lowest order term is

$$\partial_z(\rho_{f0}q_0^{(2)}) = 0.$$

As ρ_{f0} does not depend on z , this means that $\partial_z q_0^{(2)} = 0$. The lowest order term of the boundary condition (24) gives $q_0^{(2)} = 0$ at $z = \pm(1/2 - d_0)$, which means that

$$q_0^{(2)} \equiv 0 \text{ in } \tilde{\Omega}(t).$$

5.5 Conservation of momentum

We insert the asymptotic expansion into the equation for conservation of momentum (25):

$$\begin{aligned} & \epsilon \left(\partial_t \left((\rho_0 - \beta_{\rho_f}(T_{f0} + \epsilon T_{f1}))(\mathbf{q}_0 + \epsilon \mathbf{q}_1) \right) \right. \\ & \quad \left. + (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot \left((\rho_0 - \beta_{\rho_f}(T_{f0} + \epsilon T_{f1}))(\mathbf{q}_0 + \epsilon \mathbf{q}_1)(\mathbf{q}_0 + \epsilon \mathbf{q}_1) \right) \right) \\ & = - (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j})(p_0 + \epsilon p_1) \\ & \quad + \epsilon^2 \left((\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot \left((\mu_0 + \beta_{\mu}(T_{f0} + \epsilon T_{f1})) \left((\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j})(\mathbf{q}_0 + \epsilon \mathbf{q}_1) \right. \right. \right. \\ & \quad \left. \left. \left. + ((\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j})(\mathbf{q}_0 + \epsilon \mathbf{q}_1))^T \right) \right) \right) \\ & \quad \left. - \frac{2}{3} (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \left((\mu_0 + \beta_{\mu}(T_{f0} + \epsilon T_{f1})) (\partial_x \mathbf{i} + \frac{1}{\epsilon} \partial_z \mathbf{j}) \cdot (\mathbf{q}_0 + \epsilon \mathbf{q}_1) \right) \right) + O(\epsilon). \end{aligned}$$

The lowest order term arise from the pressure gradient;

$$\partial_z p_0 = 0,$$

implying that p_0 does not depend on z , hence $p_0 = p_0(x, t)$. The second-lowest order terms are

$$\begin{aligned} \partial_z(\rho_{f0} \mathbf{q}_0 \mathbf{q}_0) \cdot \mathbf{j} &= -\partial_x p_0 \mathbf{i} - \partial_z p_1 \mathbf{j} + \partial_z \left(\mathbf{j} \cdot (\mu_{f0} \partial_z (\mathbf{j} \mathbf{q}_0)) \right) \\ & \quad + \partial_z \left(\mathbf{j} \cdot (\mu_{f0} \partial_z (\mathbf{j} \mathbf{q}_0))^T \right) - \frac{2}{3} \partial_z (\mu_{f0} \partial_z (\mathbf{j} \cdot \mathbf{q}_0)) \mathbf{j}. \end{aligned}$$

The horizontal component of the above equation is

$$-\partial_x p_0 + \partial_z (\mu_{f0} \partial_z q_0^{(1)}) = 0.$$

As $\mu_{f0} = \mu_0 + \beta_{\mu} T_{f0}$ only depends on T_{f0} , which is independent of z , this equation reads as

$$\mu_{f0} \partial_{zz} q_0^{(1)} = \partial_x p_0. \quad (35)$$

The lowest order approximation of the no-slip boundary condition (26) results in

$$q_0^{(1)} = 0 \text{ at } z = \pm(1/2 - d_0).$$

Integrating the above equation (35) twice with respect to z and applying this boundary condition, gives

$$q_0^{(1)} = \frac{1}{2\mu_{f0}} \partial_x p_0 (z^2 - (1/2 - d_0)^2).$$

We then integrate this equation from $z = -(1/2 - d_0)$ to $z = 1/2 - d_0$ to obtain \bar{q} :

$$\bar{q} = -\frac{(1 - 2d_0)^3}{12\mu_{f0}} \partial_x p_0.$$

Note that this expression is similar to Darcy's law. In a fracture, the permeability is known to be proportional to the aperture width, and we obtain a cubic relationship as we integrate the velocity across the height of our strip.

5.6 Conservation of energy part II

Previously we showed that the lowest order temperature approximations T_{f0} and T_{g0} do not depend on z and are equal for all x . We seek one upscaled equation containing information from (27) and (28). We integrate both equations over their respective domains in y and sum the integrals.

$$\begin{aligned}
& \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \partial_t(\varsigma\rho T_g^\epsilon) dy + \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \partial_t(\rho_f T_f^\epsilon) dy \\
& + \int_{\epsilon(1/2-d^\epsilon)}^{\epsilon/2} \partial_t(\varsigma\rho T_g^\epsilon) dy + \int_{\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \nabla \cdot (\rho_f \mathbf{Q}^\epsilon T^\epsilon) dy \\
& = \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \kappa \nabla^2 T_g^\epsilon dy + \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \nabla^2 T_f^\epsilon dy + \int_{\epsilon(1/2-d^\epsilon)}^{\epsilon/2} \frac{1}{Pe} \kappa \nabla^2 T_g^\epsilon dy.
\end{aligned}$$

We change the order of differentiation and integration, taking into account that d^ϵ depends on both t and x . For the integrals of derivatives with respect to y , we can calculate the integrals. We make use of the symmetry in y , hence,

$$\begin{aligned}
& \partial_t \left(\frac{1}{\epsilon} \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \varsigma \rho T_g^\epsilon dy \right) - \partial_t d^\epsilon \varsigma \rho T_g^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} \\
& + \partial_t \left(\frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f T_f^\epsilon dy \right) + 2\partial_t d^\epsilon \rho_f T_f^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} \\
& + \partial_t \left(\frac{1}{\epsilon} \int_{\epsilon(1/2-d^\epsilon)}^{\epsilon/2} \varsigma \rho T_g^\epsilon dy \right) - \partial_t d^\epsilon \varsigma \rho T_g^\epsilon |_{y=\epsilon(1/2-d^\epsilon)} \\
& + \partial_x \left(\frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f q^{\epsilon(1)} T_f^\epsilon dy \right) + 2\partial_x d^\epsilon \rho_f q^{\epsilon(1)} T_f^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} - \frac{2}{\epsilon} \rho_f q^{\epsilon(2)} T_f^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} \\
& = \partial_x \left(\frac{1}{\epsilon} \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \kappa \partial_x T_g^\epsilon dy \right) - \partial_x d^\epsilon \frac{1}{Pe} \kappa \partial_x T_g^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} + \frac{1}{\epsilon} \frac{1}{Pe} \kappa \partial_y T_g^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} \\
& + \partial_x \left(\frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \partial_x T_f^\epsilon dy \right) + \frac{2}{Pe} \partial_x d^\epsilon \partial_x T_f^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} - \frac{2}{\epsilon} \frac{1}{Pe} \partial_y T_f^\epsilon |_{y=-\epsilon(1/2-d^\epsilon)} \\
& + \partial_x \left(\frac{1}{\epsilon} \int_{\epsilon(1/2-d^\epsilon)}^{\epsilon/2} \frac{1}{Pe} \kappa \partial_x T_g^\epsilon dy \right) - \partial_x d^\epsilon \frac{1}{Pe} \kappa \partial_x T_g^\epsilon |_{y=\epsilon(1/2-d^\epsilon)} + \frac{1}{\epsilon} \frac{1}{Pe} \kappa \partial_y T_g^\epsilon |_{y=\epsilon(1/2-d^\epsilon)}.
\end{aligned}$$

The boundary condition (31) implies that two of the boundary-evaluation terms disappear. The rest of the boundary-evaluation terms are at $y = -\epsilon(1/2 - d^\epsilon)$ due to the symmetry across the x -axis. For these terms, we make use of the normal vector $\mathbf{n}^\epsilon = (\epsilon \partial_x d^\epsilon, -1) / \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2}$, which we see fit nicely with the structure found in the boundary evaluation terms arising from the spatial derivatives. Hence,

$$\begin{aligned}
& \partial_t \left(\frac{2}{\epsilon} \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \varsigma \rho T_g^\epsilon dy + \frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f T_f^\epsilon dy \right) + 2\partial_t d^\epsilon (\rho_f T_f^\epsilon - \varsigma \rho T_g^\epsilon) |_{y=-\epsilon(1/2-d^\epsilon)} \\
& + \partial_x \left(\frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f q^{\epsilon(1)} T_f^\epsilon dy \right) + \frac{2}{\epsilon} \mathbf{n}^\epsilon \cdot (\rho_f \mathbf{Q}^\epsilon T_f^\epsilon) \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} |_{y=-\epsilon(1/2-d^\epsilon)} \\
& = \partial_x \left(\frac{2}{\epsilon} \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \kappa \partial_x T_g^\epsilon dy + \frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \partial_x T_f^\epsilon dy \right) \\
& - \frac{2}{\epsilon} \mathbf{n}^\epsilon \cdot \left(\frac{1}{Pe} \kappa \nabla T_g^\epsilon \right) \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} |_{y=-\epsilon(1/2-d^\epsilon)} + \frac{2}{\epsilon} \mathbf{n}^\epsilon \cdot \left(\frac{1}{Pe} \nabla T_f^\epsilon \right) \sqrt{1 + (\epsilon \partial_x d^\epsilon)^2} |_{y=-\epsilon(1/2-d^\epsilon)}.
\end{aligned}$$

The Rankine-Hugoniot boundary condition (29), together with the equation for d^ϵ , equation (21), imply that the boundary terms cancel each other out, and we are left with

$$\begin{aligned} \partial_t \left(\frac{2}{\epsilon} \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \varsigma \rho T_g^\epsilon dy + \frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f T_f^\epsilon dy \right) + \partial_x \left(\frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \rho_f q^{\epsilon(1)} T_f^\epsilon dy \right) \\ = \partial_x \left(\frac{2}{\epsilon} \int_{-\epsilon/2}^{-\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \kappa \partial_x T_g^\epsilon dy + \frac{1}{\epsilon} \int_{-\epsilon(1/2-d^\epsilon)}^{\epsilon(1/2-d^\epsilon)} \frac{1}{Pe} \partial_x T_f^\epsilon dy \right). \end{aligned}$$

We insert asymptotic expansions and substitute $z = y/\epsilon$. The lowest order expansion is

$$\begin{aligned} \partial_t \left(2 \int_{-1/2}^{-(1/2-d_0)} \varsigma \rho T_0 dy + \int_{-(1/2-d_0)}^{1/2-d_0} \rho_{f0} T_0 dy \right) + \partial_x \left(\int_{-(1/2-d_0)}^{1/2-d_0} \rho_{f0} q_0^{(1)} T_0 dy \right) \\ = \partial_x \left(2 \int_{-1/2}^{-(1/2-d_0)} \frac{1}{Pe} \kappa \partial_x T_0 dy + \int_{-(1/2-d_0)}^{1/2-d_0} \frac{1}{Pe} \partial_x T_0 dy \right), \end{aligned}$$

where we have used $T_f = T_g = T_0$. As T_0 does not depend on z , integrating up is easy. Hence,

$$\partial_t \left((1-2d_0)\rho_{f0}T_0 + 2d_0\varsigma\rho T_0 \right) + \partial_x(\rho_{f0}\bar{q}T_0) = \frac{1}{Pe} \partial_x \left((1-2d_0)\partial_x T_0 + 2d_0\kappa\partial_x T_0 \right) \text{ for } 0 \leq x \leq 1.$$

6 Resulting model equations

To summarize, we have derived an upscaled model for mineral precipitation and dissolution in a thin strip honoring changes in aperture, with fluid flow and heat transport. The model includes five unknowns: $u_0(x, t)$, $d_0(x, t)$, $\bar{q}(x, t)$, $p_0(x, t)$ and $T_0(x, t)$. We can note that all our main variables only depend on x and t , hence the thin strip problem has reduced to a one-dimensional problem which is as expected. We have five equations to describe our main variables:

$$\partial_t \left((1-2d_0)u_0 + 2d_0\rho \right) = \partial_x \left((1-2d_0)D\partial_x u_0 - \bar{q}u_0 \right)$$

$$\rho \partial_t d_0 = f_p(T_0, u_0) - f_d(T_0, d_0)$$

$$\partial_t \left((1-2d_0)\rho_{f0} + 2d_02\rho \right) + \partial_x(\rho_{f0}\bar{q}) = 0$$

$$\bar{q} = -\frac{(1-2d_0)^3}{12\mu_{f0}} \partial_x p_0$$

$$\partial_t \left((1-2d_0)\rho_{f0}T_0 + 2d_0\varsigma\rho T_0 \right) + \partial_x(\rho_{f0}\bar{q}T_0) = \frac{1}{Pe} \partial_x \left((1-2d_0)\partial_x T_0 + 2d_0\kappa\partial_x T_0 \right)$$

All equations are valid for $0 \leq x \leq 1$. Recall that the fluid density ρ_{f0} and viscosity μ_{f0} are not constant, but depend linearly on the temperature T_0 .

The equations follow a certain pattern; terms associated with the void space appear in combination with the factor $(1-2d_0)$, which is the width of the void space. Terms associated with the grains have the factor $2d_0$, which is the width of the grain space. This means we obtain upscaled equations taking into account the changing aperture through the derivatives of d_0 . Our findings are consistent with the findings of van Noorden [14], where our results include the effect from temperature dependence through fluid properties ρ_{f0} and μ_{f0} and the reaction rates.

The thin strip problem is relevant for a geothermal setting as our thin strip may represent a fracture. In geothermal reservoirs where the rock is highly fractured and has otherwise low permeability; such as in e.g. granite reservoirs, the fluid flow is mainly through the fractures. The model presented here can describe how the efficient equations for flow, heat transport and solute transport are affected as minerals precipitate and dissolve inside the fractures through the production period of the geothermal reservoir.

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