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Introduction

The origin of Lie theory is geometric and initialized with the view that the geometry of a space is determined by the group of its symmetries. Motivated by the study of differential equations, Sophus Lie (1842–1899) started to develop an analytic counterpart to Évariste Galois' (1811–1832) work on algebraic equations, and had the seminal idea to consider *infinitesimal* actions of local groups on manifolds. These *infinitesimal groups* could be studied by linearizing them, leading to the object that is known as a *Lie algebra* today. Being a linear object, the Lie algebra is more easily accessible than a group. Wilhelm Killing (1847–1923), who introduced Lie algebras independently, came up with a new approach for the study of these group actions: instead of classifying all group actions, one could also classify all (finite-dimensional complex) Lie algebras. Together with Friedrich Engel (1861–1941), he concluded that determining all *simple* Lie algebras was fundamental.

The finite dimensional *complex* simple Lie algebras consist of four infinite families A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$ and D_n $(n \geq 4)$, respectively, corresponding to the groups $\mathrm{SL}(n+1, \mathbb{C})$, $\mathrm{SO}(2n+1, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(2n, \mathbb{C})$, and five exceptional Lie algebras denoted by E_6, E_7, E_8, F_4 and G_2. A Lie algebra of one of these types is called a *classical Lie algebra*. The work by Claude Chevalley and Leonard Dickson shows that these types also exist over finite fields, i.e. for so-called *modular* Lie algebras. In the second half of the 20th century, the classification of finite-dimensional modular simple Lie algebras was completed for algebraically closed fields of characteristic greater than or equal to 5. It implies that such a simple modular Lie algebra in characteristic at least 5 is either classical, of Cartan type or Melikian. Hereby, the classification of the Lie algebras of Cartan type was the result of a long series of work, ending in papers of A. Premet and H. Strade, and subsumed in the books of H. Strade [Str04], [Str09],[Str13]. The Melikian Lie algebras are a single series of Lie algebras that occur in characteristic 5. The characteristics

2 and 3 seem very hard to characterize and many extraordinary examples have been found.

The progress in group theory influenced the work on the theory of Lie algebras and geometry. A mathematical milestone of the last century was the classification of all finite simple groups, finished in 1982, with the result that a finite simple group is either cyclic, alternating, a group of Lie type, or one of 26 sporadic examples.

The three different concepts of groups, Lie algebras and geometries are closely related and influenced the development of theory among each other in several ways. Where the connection between Lie algebras and group theory is intuitive considering the historic roots, the relationship between groups and geometries came into focus by the initial ideas of Fischer [Fis71] and Tits [Tit74]. Geometric methods found several applications in the theory of finite simple groups, leading to their final classification. This interaction gives a model for the further investigation of relations in the triad of geometries, groups and Lie algebras.

In this thesis, we consider the relationship between Lie algebras and geometries, more concretely, we take the path from the geometries to the Lie algebras, concentrating on classical modular Lie algebras. It is known that geometries related to buildings arise from classical Lie algebras (see e.g. [Coh12]). We will examine the converse: given a specific geometry related to a building, we will study to what extent a Lie algebra whose associated geometry is related to that building is unique.

The central objects in this approach are the *extremal elements* of a Lie algebra. Inside a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, a non-zero element x is called extremal if $[x, [x, \mathfrak{g}]]$ is contained in the 1-space spanned by x. Hereby, we exclude the special case where the space spanned by x is 0-dimensional, in which case x is called a *sandwich*.

The first time extremal elements occurred was in the article [Fau73] by J. Faulkner. He made use of inner ideals to identify shadows of buildings, where a 1-dimensional inner ideal is the 1-space spanned by an extremal element that is not a sandwich. Inner ideals were extensively studied by G. Benkart in her PhD thesis [Ben74] and the subsequent papers [Ben76] and [Ben77]. In [Che89], V. Chernousov used five extremal elements in a Lie algebra of type E_8 to settle the last open case of the Hasse principle conjecture. Recently, Extremal elements have been a topic of investigation in Eindhoven, with results published in several papers as [CIR08], [Di'p08],[i'pPR09] and [Roo11]. Similar presentations for other Lie algebras are given here as results of the work of A. Cohen, H. Cuypers, J. Draisma, G. Ivanyos, J. in 't panhuis, E. Postma and D. Roozemond, and most recently also K. Roberts and S. Shpectorov in [CRS14].

A definitive example of extremal elements are the long root elements of the classical Lie algebras. But extremal elements also occur in other classes of Lie algebras. By the result of A. Premet [Pre86b], we may assume that extremal elements or sandwiches exist in all simple Lie algebras, if the characteristic of the underlying field is at least 5. Moreover, A. Cohen, G. Ivanyos and D. Roozemond showed in [CIR08] that simple Lie algebras over algebraically closed fields are (with a single exception) generated by their extremal elements, provided that the characteristic is at least 5 and a non-sandwich extremal element is contained. Another result of their work is an elegant way to distinguish the classical simple Lie algebras from the Cartan type algebras (including the Melikian algebras), using their extremal elements: Either the space $[x, [x, \mathfrak{g}]]$ is 1-dimensional, in which case the Lie algebra is of classical type, or it is 0-dimensional, in which case x is a sandwich and the Lie algebra is of Cartan type.

The path from Lie algebras to geometries was introduced by A. Cohen and G. Ivanyos in [CI06], wherein they obtained a natural way to associate a geometry to a Lie algebra generated by extremal elements that are no sandwiches. The resulting geometric structure is a *root filtration space*, that is (under some mild restrictions) the shadow space of a spherical building. This construction was inspired by the geometric methods used in finite simple group theory. The resulting geometries have been classified, which raises the natural question: can the Lie algebra be recovered from the building in a canonical way? By the classification of spherical buildings, one can deduce that such a Lie algebra is in fact of a known classical type. In his PhD thesis [Rob12], K. Roberts already obtained this result for the A_n-case. Under the assumption that the Lie algebra contains no sandwiches and is spanned by its extremal elements, he identified a Lie algebra of type A_n from a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$ (see [Bou68] for notation).

This thesis addresses this reverse construction. We show under some weak assumptions on the underlying field that a simple Lie algebra that is generated by extremal elements that are not sandwiches and whose associated geometry is related to a spherical building of rank at least 3 is a classical Lie algebra.

The structure of this thesis. We start in the first chapter with some basic definitions and introduce our main object, namely, the classical linear Lie algebras $\mathfrak{gl}_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F}), \mathfrak{sp}_{\mathbb{F}}(V), \mathfrak{o}_n(\mathbb{F})$ and $(\mathfrak{s})\mathfrak{u}_n(\mathbb{F})$, where $\mathbb{F}$ denotes a field. In the second chapter, we proceed with the definition of extremal elements in Lie algebras and introduce the extremal form g on the Lie algebra $\mathfrak{g}$. Since we are mostly interested in the relations between extremal elements, we find and name five possible types of pairs of extremal elements $(x, y) \in E \times E$ that can occur. For the set of corresponding extremal points $\mathcal{E}(\mathfrak{g}) = \{\mathbb{F}x \mid x \text{ extremal}\}$, these relations have the following names: A pair of points can be *hyperbolic* (type $\mathcal{E}_2$, this is the case if the elements span an $\mathfrak{sl}_2$-subalgebra), *special* (type $\mathcal{E}_1$, where the elements do not commute but the extremal form of the pair is zero), *polar* (type $\mathcal{E}_0$, where the elements commute and do not belong to one of the following cases), *strongly commuting* (type $\mathcal{E}_{-1}$, in case that the elements commute, are not linearly dependent and $\mathbb{F}x + \mathbb{F}y \subseteq E \cup \{0\}$) or *equal* (type $\mathcal{E}_{-2}$, where the points are linearly dependent, so $\mathbb{F}x = \mathbb{F}y$). These relations also determine the corresponding geometry $\Gamma(\mathfrak{g})$ of a Lie algebra that we will examine in the following chapters. In particular, we consider the relation $\mathcal{E}_2$ where the extremal form of a pair of extremal elements is non-zero. Here, the two elements generate a subalgebra isomorphic to $\mathfrak{sl}_2$, so we also denote this relation by $\sim_{\mathfrak{sl}_2}$. We define a graph $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$, taking the extremal elements $\mathcal{E}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ as a vertex set, and the relation $\sim_{\mathfrak{sl}_2}$ naturally determines the edges. We call this graph the $\mathfrak{sl}_2$-graph of the Lie algebra. For simple Lie algebras, we can show our first result.

THEOREM (see 2.5.6). *If a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is simple with a non-trivial extremal form g, then $\mathcal{E}(\mathfrak{g})$ is connected with respect to the relation $\sim_{\mathfrak{sl}_2}$. In particular, the group $G = \langle \exp(x, t) \mid x \text{ extremal},\ t \in \mathbb{F}\rangle$ is transitive on the points in $\mathcal{E}(\mathfrak{g})$.*

To examine the extremal elements of classical Lie algebras in detail, we make use of the Chevalley basis in the third chapter. Via the construction of root systems and subsequently root elements, we obtain the Lie algebra as a span of long and short root elements. This allows us to classify *all* proper extremal elements and describe them explicitly. (Here a *proper* extremal element refers to a non-sandwich extremal element.)

THEOREM (see 3.4.11 and 3.4.12). *Let $\mathfrak{g}$ be a Chevalley Lie algebra over $\mathbb{F}$ with* $\operatorname{char}\mathbb{F} \neq 2$. *Then all proper extremal elements of $\mathfrak{g}$ are long root elements.*

The focus in the second half of the thesis is on the discrete geometric characterizations of the classical Lie algebras. In the fourth chapter, several geometric concepts that enable our results in the last two chapter are introduced. We start with fundamental concepts such as graphs, Coxeter groups and buildings. Proceeding with root shadow spaces and later root filtration spaces, we present the fundamental results of Cohen and Ivanyos in [CI06],[CI07]. Point-line spaces and, in particular, polar spaces will be used to apply the results of Cuypers in [Cuy94] for the symplectic Lie algebras. In the more general case in Chapter 5, we also work with polarized embeddings of point-line geometries and apply the main result of and Kasikova and Shult in [KS01]. We use the extremal geometry $\Gamma(\mathfrak{g})$ defined by the five relations that we introduced in the second chapter. We obtain the point-line space $(\mathcal{E}(\mathfrak{g}), \mathcal{F})$, where the extremal points form the point set and the lines are determined by the relation $\mathcal{E}_{-1}$, so the strongly commuting pairs. Using [KS01], we show that for two Lie algebras that are both spanned by their extremal elements and equipped with nondegenerate extremal forms, an isomorphic extremal geometry with an absolute universal embedding implies equivalence of the natural embeddings. By the classification of Cohen and Ivanyos, this holds in particular for Lie algebras with extremal geometries isomorphic to a root shadow space of type $\mathrm{BC}_{n,2}$, $\mathrm{D}_{n,2}$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, or $\mathrm{F}_{4,1}$, where $n \geq 3$. Using subalgebras isomorphic to $\mathfrak{sl}_2$, we prove that the Lie product for a fixed isomorphism type of the extremal geometry is unique (up to scalar multiples). Combining the previous results, we obtain our main conclusion.

THEOREM (see 5.4.1). *Let $\mathfrak{g}$ be a Lie algebra generated by its set of extremal elements and with trivial radical. If $\Gamma(\mathfrak{g})$ is nondegenerate and the natural embedding of the extremal geometry $\Gamma(\mathfrak{g})$ into the projective space on $\mathfrak{g}$ admits an absolute universal cover, then $\mathfrak{g}$ is uniquely determined (up to isomorphism) by $\Gamma(\mathfrak{g})$.*

In particular, this result applies to the Lie algebras of type BC_n ($n \geq 3$), D_n ($n \geq 4$), E_6, E_7, E_8, or F_4. Combined with results of [Rob12], it also characterizes Chevalley algebras of rank at least 3 and containing strongly commuting elements.

THEOREM (see 5.4.3). *Suppose $\mathfrak{g}$ is a Lie algebra and $\Gamma(\mathfrak{g})$ is isomorphic to $\Gamma(\mathfrak{ch})$ for some Lie algebra $\mathfrak{ch}$ of Chevalley type $X_n \neq \mathrm{C}_n$ where $n \geq 3$. Then*

$$\mathfrak{g}/\operatorname{Rad}(\mathfrak{g}) \cong \mathfrak{ch}/\operatorname{Rad}(\mathfrak{ch}).$$

It remains to consider the case where the set of lines in the extremal geometry as defined in chapter 5 is empty, in other words, there are no pairs of strongly commuting points. In this case, moreover, there are no pairs of special points leaving us with only hyperbolic and polar pairs. This holds in particular for the symplectic Lie algebras over a field of characteristic different from 2. In some other cases of this type, the problem of an empty line set in the extremal geometry as considered in the fifth chapter can be resolved by a quadratic extension of the underlying field, with the consequence that one can find strongly commuting pairs of extremal elements in the extended Lie algebra. We concentrate on the symplectic case and provide an alternative characterization for this type, using the $\mathfrak{sl}_2$-geometry as defined in the second chapter. For this purpose, we consider subalgebras spanned by a symplectic triple of extremal elements. An application of the main result of [Cuy94] shows that the partial linear space $\Gamma(\mathfrak{g})$ defined by the $\mathfrak{sl}_2$-relation is isomorphic to the geometry of hyperbolic lines of a symplectic geometry $HSp(V,f)$, where (V,f) denotes a symplectic space, with f as symplectic form. We find a projective space on the extremal elements with lines defined by $\mathfrak{sl}_2$-lines and polar lines. To complete the characterization, we introduce quadric Veroneseans and (universal) Veronesean embeddings, to apply the result of J. Schillewaert and H. Van Maldeghem in [SVM13]. We show that the projective embedding of the Lie algebra $\mathfrak{g}$ into $\mathbb{P}(\mathfrak{g})$ induces a universal Veronesean embedding of $\mathbb{P}(V)$, so that $\mathcal{E}(\mathfrak{g})$ is a quadric Veronesean. The Lie product is again unique (up to scalar multiples) on the Veronesean.
This leads to our final characterization of the symplectic Lie algebras by their geometries.

THEOREM (see 6.0.6). *Let $\mathfrak{g}$ be a simple Lie algebra of finite dimension over the field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ and generated by its set of extremal points $\mathcal{E}$ where*

$\mathcal{E}_{\pm 1}(\mathfrak{g}) = \emptyset$ *and for any* $(x, y), (y, z) \in \mathcal{E}_2(\mathfrak{g})$, *the subspace* $\langle x, y, z\rangle$ *embeds into a subalgebra isomorphic to* $\mathfrak{sp}_3(\mathbb{F})$ *or* $\mathfrak{psp}_3(\mathbb{F})$. *Then* $\mathfrak{g} \cong \mathfrak{sp}_n(\mathbb{F})$ *for some (even)* $n \geq 4$, *or* $\mathfrak{g} \cong (\mathfrak{p})\mathfrak{sl}_2(\mathbb{F})$.

Contents

CHAPTER 1

Lie algebras

This chapter introduces the basic terminology and notation that are essential for this thesis. Most of the proofs have been omitted and can be found in fundamental literature on Lie algebras, as e.g. [Hum78] or [Car72].

1.1. General theory

A vector space $\mathfrak{g}$ over a field $\mathbb{F}$ together with a binary operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

is a **Lie algebra** over $\mathbb{F}$ if the operation fulfills the following conditions:

(1) Bilinearity: for all $\alpha, \beta \in \mathbb{F}$ and for all $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}[\alpha x + \beta y, z] &= \alpha [x, z] + \beta [y, z], \\ [z, \alpha x + \beta y] &= \alpha [z, x] + \beta [z, y].\end{aligned}$$

(2) Alternation: for all $x \in \mathfrak{g}$, the identity $[x, x] = 0$ holds.

(3) Jacobi identity: all $x, y, z \in \mathfrak{g}$ fulfill

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

The operation $[\cdot,\cdot]$ is called the **Lie bracket** of the Lie algebra $\mathfrak{g}$.

We consider a first and fundamental example of a Lie algebra.
Let V be a (left) vector space over the (skew) field $\mathbb{K}$. We denote by $\mathrm{End}(V)$ the ring of all endomorphisms of V with the usual addition and composition as multiplication.
Now we can define an operation on $\mathrm{End}(V)$:

$$\text{for } x, y \in \mathrm{End}(V): \quad [x, y] := xy - yx,$$

the **bracket** or the **commutator** of x and y. This operation induces a Lie algebra structure $\mathfrak{gl}(V)$ on $\mathrm{End}(V)$ which is called the **general linear Lie algebra**. (Here $\mathrm{End}(V)$ is considered to be a vector space over a subfield $\mathbb{F}$ of $\mathbb{K}$.)

Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.
For $V = \mathbb{F}^n$, $n \in \mathbb{N}$ and $\mathbb{K} = \mathbb{F}$ a field, we have $\dim \mathfrak{gl}(V) = n^2$. In this case we denote $\mathfrak{gl}(V)$ by $\mathfrak{gl}_n(\mathbb{F})$, and identify $\mathrm{End}(V)$ with the algebra of all $n \times n$-matrices with entries in $\mathbb{F}$. This can be very useful for explicit calculations on linear Lie algebras.
An element of $\mathfrak{gl}(V)$ is called **finitary**, if its kernel has finite codimension. The finitary elements in $\mathfrak{gl}(V)$ form a subalgebra denoted by $\mathfrak{fgl}(V)$, the finitary general linear Lie algebra. Any subalgebra of $\mathfrak{fgl}(V)$ is called a **finitary linear Lie algebra**.
For two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a **Lie algebra homomorphism** if for all $x, y \in \mathfrak{g}_1$ we have $\varphi([x, y]) = [\varphi(x), \varphi(y)]$. If φ is also bijective, we call it a **Lie algebra isomorphism**. Note that a Lie algebra isomorphism is also an isomorphism in the usual sense, so a bijective homomorphism whose inverse is also an isomorphism.
A **Lie subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ where, for all $x, y \in \mathfrak{h}$, we have $[x, y] \in \mathfrak{h}$.
For $x \in \mathfrak{g}$ we define a linear map

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$$

by left multiplication by x, so:

$$\mathrm{ad}_x(y) = [x, y] \, .$$

The map ad_x is called the **adjoint** map of x.
The map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto \mathrm{ad}_x$ is a Lie algebra homomorphism. It is called the **adjoint** representation of $\mathfrak{g}$.

Definition and Example 1.1.1 (Special linear algebra). Let $n \in \mathbb{N}$, $n \geq 1$ and V be a vector space over the field $\mathbb{F}$ with $\dim V = n$. We denote by $\mathfrak{sl}(V)$ or $\mathfrak{sl}_n(\mathbb{F})$ the set of endomorphisms on V having trace zero. We denote the trace of an element $x \in \mathfrak{gl}$ by $\mathrm{Tr}(x)$. Because of $\mathrm{Tr}(xy - yx) = \mathrm{Tr}(xy) - \mathrm{Tr}(yx) = \mathrm{Tr}(xy) - \mathrm{Tr}(xy) = 0$ for $x, y \in \mathfrak{gl}_n$, we find that $\mathfrak{sl}_n(\mathbb{F})$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{F})$. It is called the **special linear algebra**.
If V is infinite dimensional, then we can define the trace function on finitary elements of $\mathfrak{gl}(V)$. In particular, we can define $\mathfrak{fsl}(V)$ to be the subalgebra of $\mathfrak{fgl}(V)$ consisting of finitary elements with trace 0.

DEFINITION AND EXAMPLE 1.1.2 (Heisenberg algebra). Consider the three-dimensional Lie subalgebra of $\mathfrak{gl}_3(\mathbb{F})$ generated by the matrices

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It satisfies the relations

$$[x, y] = z, \ [x, z] = 0, \ [y, z] = 0,$$

and is called **Heisenberg algebra**. The spanned vector space is the space of strictly upper-triangular 3×3 matrices over the underlying field $\mathbb{F}$.

The following theorem states that every finite dimensional Lie algebra is isomorphic to a linear Lie algebra. This result is due to I.D. Ado (1935) in the case where $\operatorname{char} \mathbb{F} = 0$. The restriction on the characteristic was removed later by Iwasawa and Harish-Chandra. Proofs for $\operatorname{char} \mathbb{F} = 0$ and p can be found e.g. in [Jac79], Chapter VI.

THEOREM 1.1.3. *Every finite dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(V)$ for some vector space V over the field $\mathbb{F}$.*

DEFINITION AND PROPOSITION 1.1.4. *An **ideal** I of a Lie algebra $\mathfrak{g}$ is a subspace where $[x, y] \in I$ for all $x \in \mathfrak{g}$ and $y \in I$.*
Suppose $\mathfrak{g}'$ is a second Lie algebra. Then the kernel of a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is an ideal of $\mathfrak{g}$, and the image is a subalgebra of $\mathfrak{g}'$.
*Conversely, for any ideal $I \subset \mathfrak{g}$ it holds that $\mathfrak{g}/I$ is a Lie algebra, called the **quotient algebra** of I in $\mathfrak{g}$ and I is the kernel of the quotient map $\mathfrak{g} \to \mathfrak{g}/I$.*

PROOF. If $x \in \ker \varphi$ and $a \in \mathfrak{g}$, then $\varphi[a, x] = [\varphi(a), \varphi(x)] = [\varphi(a), 0] = 0$. So $[a, x] \in \ker \varphi$.
Conversely, if $I \subset \mathfrak{g}$ is an ideal, $a \in \mathfrak{g}$, $x \in I$, then

$$[a + I, x + I] \subseteq [a, x] + [a, I] + [I, x] + [I, I] \subset [a, x] + I.$$

So the bracket is well defined in $\mathfrak{g}/I$ and $[\varphi(a), \varphi(x)] = \varphi[a, x]$. Moreover, as $a + I = I$ if $a \in I$, we find $I = \ker \varphi$. $\diamond$

The **center** of a Lie algebra $\mathfrak{g}$ is $Z(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, \mathfrak{g}] = 0\}$. The center is an ideal of $\mathfrak{g}$. It is the kernel of the adjoint representation.

We say that elements $x, y \in \mathfrak{g}$ **commute** if $[x, y] = 0$. So, the center of $\mathfrak{g}$ consists of those elements from $\mathfrak{g}$ that commute with all elements in $\mathfrak{g}$.
A Lie algebra is called **commutative** if any two elements commute.

EXAMPLE 1.1.5. The trace is a homomorphism from $\mathfrak{gl}_n(\mathbb{F})$ to $\mathbb{F}$ (where we consider $\mathbb{F}$ as the abelian Lie algebra over $\mathbb{F}$), since $\mathrm{Tr}[x,y]_{\mathfrak{gl}_n(\mathbb{F})} = \mathrm{Tr}(xy) - \mathrm{Tr}(yx) = 0 = [\mathrm{Tr}(x), \mathrm{Tr}(y)]_{\mathbb{F}}$. The subalgebra $\mathfrak{sl}_n(\mathbb{F})$ of $\mathfrak{gl}_n(\mathbb{F})$ is the kernel of Tr, so it is an ideal in $\mathfrak{gl}_n(\mathbb{F})$.

EXAMPLE 1.1.6. In a Heisenberg algebra $\mathfrak{g}$ over $\mathbb{F}$, there exists a 1-dimensional center, namely $Z(\mathfrak{g}) = \mathbb{F}z$ (using the notation of Example 1.1.2), which is an ideal in $\mathfrak{g}$.

DEFINITION AND PROPOSITION 1.1.7. *Let $\mathfrak{g}$ be a Lie algebra. Then $[\mathfrak{g}, \mathfrak{g}]$ is the subspace of $\mathfrak{g}$ spanned by all elements $[x, y]$ where $x, y \in \mathfrak{g}$.*
*The subspace $[\mathfrak{g}, \mathfrak{g}]$ is clearly an ideal of $\mathfrak{g}$. It is called the **commutator** subalgebra.*
In general, if $\mathfrak{i}$ is an ideal, then $[\mathfrak{g}, \mathfrak{i}]$, the subspace spanned by all elements of the form $[x, y]$ with $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$, is also an ideal of $\mathfrak{g}$.

A **simple** Lie algebra $\mathfrak{g}$ is a Lie algebra with $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and having no nontrivial ideals. In particular, in a simple Lie algebra one has $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. A Lie algebra that is a direct sum of simple Lie algebras is called **semisimple**.
For a Lie algebra $\mathfrak{g}$ we can define a sequence of ideals

$$\mathfrak{g}^0 := \mathfrak{g}, \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^2 := \left[\mathfrak{g}, \mathfrak{g}^1\right] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]], \mathfrak{g}^3 := \left[\mathfrak{g}, \mathfrak{g}^2\right], \dots$$

If there is a $n \in \mathbb{N}$ with $\mathfrak{g}^n = 0$, we call $\mathfrak{g}$ **nilpotent**.
For a Lie algebra $\mathfrak{g}$, we can also define the sequence of ideals

$$\mathfrak{g}^{(0)} := \mathfrak{g},\ \mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}],\ \mathfrak{g}^{(2)} := \left[\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}\right] = [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]],$$
$$\mathfrak{g}^3 := \left[\mathfrak{g}^{(2)}, \mathfrak{g}^{(2)}\right],\ \dots$$

If there is a $n \in \mathbb{N}$ with $\mathfrak{g}^{(n)} = 0$, we call $\mathfrak{g}$ **solvable**.

PROPOSITION 1.1.8. *Every nilpotent Lie algebra is solvable, but the converse is not true.*

PROOF. We show that for any nilpotent Lie algebra $\mathfrak{g}$, also $\mathfrak{g}^{k+1} \subseteq \mathfrak{g}^{(k)}$ holds for all $k \in \mathbb{N}$. We use induction on k. We have $\mathfrak{g}^{(0)} = \mathfrak{g}^0$. Assume $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^k$. Then

$$\mathfrak{g}^{(k+1)} \subseteq \left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}\right] \subseteq \left[\mathfrak{g}, \mathfrak{g}^k\right] = \mathfrak{g}^{k+1},$$

since $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}$ by definition.
As a counterexample for the converse, consider the two-dimensional (non-abelian) Lie algebra defined by $[a, b] = a$. It is solvable since $\mathfrak{g}^{(3)} = 0$, but not nilpotent since $\mathfrak{g}^k = \langle a \rangle$ for all $k \in \mathbb{N}$, $k \geq 2$. $\diamond$

PROPOSITION 1.1.9. *Let $\mathfrak{g}$ be a Lie algebra with solvable ideals I and J. Then also $I + J$ is a solvable ideal of $\mathfrak{g}$.*

PROOF. See [dG00, Prop. 2.3.1] or [Hum78, Section I.3] $\diamond$

DEFINITION 1.1.10. A **radical** of a finite dimensional Lie algebra $\mathfrak{g}$, denoted by $\mathrm{Rad}(\mathfrak{g})$, is a solvable ideal of $\mathfrak{g}$ of maximal possible dimension.

PROPOSITION 1.1.11. $\mathrm{Rad}(\mathfrak{g})$ *contains any solvable ideal of* $\mathfrak{g}$ *and is unique.*

PROOF. If I is a solvable ideal of $\mathfrak{g}$, then $I + \mathrm{Rad}\ (\mathfrak{g})$ is again a solvable ideal by 1.1.9. Since $\mathrm{Rad}\ (\mathfrak{g})$ is of maximal dimension, it follows $I + \mathrm{Rad}\ (\mathfrak{g}) = \mathrm{Rad}\ (\mathfrak{g})$ and $I \subset \mathrm{Rad}\ (\mathfrak{g})$. For uniqueness, if there are two distinct maximal solvable ideals of $\mathfrak{g}$, then like above, the sum is equal to both ideals, which is a contradiction. $\diamond$

1.2. Linear Lie algebras

Let V be a vector space. In the previous section, we have already seen some examples of linear Lie algebras, namely the general linear Lie algebra $\mathfrak{gl}(V)$ and the special linear Lie algebra $\mathfrak{sl}(V)$ or, in case V has infinite dimension, $\mathfrak{fsl}(V)$, all with the commutator as Lie bracket.
The special linear algebra is the first example of the classical Lie algebras that will be of importance in this work. There are four families of classical Lie algebras that we introduce in this section.
If V has finite dimension n, then the special linear Lie algebra $\mathfrak{sl}(V)$, considered as a subspace of the Lie algebra of $n \times n$-matrices, is spanned by the matrices

$$E_{i,j} \text{ with } i \neq j, i, j \in \{1, \ldots, n\}, \text{ and}$$

$$E_{i,i} - E_{i+1,i+1} \text{ with } i \in 1, \ldots, n-1,$$

where the $E_{i,j}$ denote the $n \times n$ matrix with a one in position (i,j) and 0 elsewhere. The dimension of $\mathfrak{sl}_n(\mathbb{F})$ is $n^2 - 1$.
Before introducing the other families of Lie algebras first a lemma:

LEMMA 1.2.1. *Let V be a vector space over the field $\mathbb{F}$, and suppose $h\colon V \times V \to \mathbb{F}$ is a map additive in both coordinates with $h(v,0) = 0 = h(0,v)$. If $S, T \in \mathfrak{gl}(V)$ satisfy $h(S(u),v) = -h(u,S(v))$ and $h(T(u),v) = -h(u,T(v))$ for all $u, v \in V$, then $h([S,T](u),v) = -h(u,[S,T](v))$.*

PROOF. Let $u, v \in V$. First, note that

$$\begin{aligned} 0 = h(v,0) =& h(v, w-w) = h(v,w) + h(v,-w) \\ \Rightarrow h(v,-w) =& - h(v,w). \end{aligned}$$

A similar argument leads to

$$h(-v,w) = -h(v,w).$$

Furthermore we have

$$\begin{aligned} h([S,T]u,v) = \;& h(STu - TSu, v) \\ =\;& h(S(Tu),v) - h(T(Su),v) \\ =\;& -h(Tu,Sv) + h(Su,Tv) \\ =\;& h(u,TSv) - h(u,STv) \\ =\;& h(u,TSv) - h(u,STv) \\ =\;& h(u,TSv - STv) \\ =\;& h(u,[T,S]v) \\ =\;& h(u,-[S,T]v) \\ =\;& -h(u,[S,T]v), \end{aligned}$$

and the lemma is proven. $\diamond$

The lemma shows that the property $h(Ru,v) = -h(u,Rv)$ of the bi-additive form h on V with $R \in \mathrm{End}\,(V)$ is preserved by the commutator on $\mathrm{End}(V)$. In particular, the set of all R satisfying $h(Ru,v) = -h(u,Rv)$ form a Lie subalgebra of $\mathfrak{gl}(V)$.
In the following, we consider various different types of bi-additive forms and the Lie subalgebras that they induce on $\mathfrak{gl}(V)$. In particular, we will consider so-called sesquilinear forms defined on vector spaces over skew fields.

DEFINITION 1.2.2. Let V be a left vector space over the skew field $\mathbb{K}$ and σ an anti-automorphism of $\mathbb{K}$ and $0 \neq \varepsilon \in \mathbb{K}$.

A map $h : V \times V \to \mathbb{K}$ is called a (reflexive) (σ, ε)-**sesquilinear form** on V if for all $\lambda, \mu \in \mathbb{K}$

(1) $h(v + w, u) = h(v, u) + h(w, u)$;
(2) $h(\lambda v, \mu w) = \lambda h(v, w)\mu^\sigma$;
(3) $h(v, w) = \varepsilon h(w, v)^\sigma$.

Notice that if h is nontrivial there are $v, w \in V$ with $h(v, w) = 1$. But then

$$1 = h(v, w) = \varepsilon h(w, v)^\sigma = \varepsilon(\varepsilon h(v, w)^\sigma)^\sigma = \varepsilon\varepsilon^\sigma.$$

So, $\varepsilon^\sigma = \varepsilon^{-1}$. Note that $h(w, v) = \varepsilon$ since $h(w, v) = \varepsilon h(v, w)^\sigma = \varepsilon 1^\sigma = \varepsilon$. Using this, we have for all $\lambda, \mu \in \mathbb{K}$:

$$\begin{aligned}\lambda\mu^\sigma =& \lambda h(v, w)\mu^\sigma \\ =& h(\lambda v, \mu w) \\ =& \varepsilon h(\mu w, \lambda v)^\sigma \\ =& \varepsilon(\mu h(w, v)\lambda^\sigma)^\sigma \\ =& \varepsilon\lambda^{\sigma^2} h(w, v)^\sigma \mu^\sigma \\ =& \varepsilon\lambda^{\sigma^2} \varepsilon^\sigma \mu^\sigma.\end{aligned}$$

This implies that $\lambda^{\sigma^2} = \varepsilon^{-1}\lambda\varepsilon$.

If σ is the identity, then $\mathbb{K}$ has to be a field and $\varepsilon = \pm 1$. In this case h is a **symmetric** ($\varepsilon = 1$) or **anti-symmetric** ($\varepsilon = -1$) bilinear form. If the form satisfies $h(v, v) = 0$ for all v, then we call it **alternating** or **symplectic**. Notice that an anti-symmetric bilinear form is symplectic if the characteristic of $\mathbb{K}$ is not 2.

Now we consider the case where σ is not the identity.

If $\alpha \neq 0$ and h is a (σ, ε)-sesquilinear form then αh is (τ, η)-sesquilinear, where $\tau(\lambda) = (\alpha^{-1}\lambda\alpha)^\sigma$ for all $\lambda \in \mathbb{K}$ and $\eta = \alpha\varepsilon\alpha^{-\sigma}$.

Indeed, we have

$$\begin{aligned}(\alpha h)(u, v) &= \alpha h(u, v) \\ &= \alpha\varepsilon h(v, u)^\sigma \\ &= \alpha\varepsilon h(v, u)^\sigma \alpha^\sigma \alpha^{-\sigma} \\ &= \alpha\varepsilon(\alpha h(v, u))^\sigma \alpha^{-\sigma} \\ &= \alpha\varepsilon\alpha^{-\sigma}\alpha^\sigma(\alpha h(v, u))^\sigma \alpha^{-\sigma}\end{aligned}$$

$$= \eta(\alpha^{-1}(\alpha h(v,u))\alpha)^{\sigma}$$
$$= \eta(\alpha h(v,u))^{\tau}.$$

Let $h \neq 0$ be a (σ, ε)-sesquilinear form with nontrivial σ.
Let $\beta \in \mathbb{K}$ such that $\alpha = \beta^{\sigma} - \varepsilon^{\sigma}\beta \neq 0$. (Clearly such an element β exists. For otherwise, $\beta^{\sigma} = \varepsilon^{\sigma}\beta$ for all $\beta \in \mathbb{K}$. In particular $1 = 1^{\sigma} = \varepsilon^{\sigma} \cdot 1$. So, $\varepsilon = 1$ and σ is the identity which contradicts our assumptions.) Then $\alpha^{\sigma} = \beta^{\sigma^2} - \beta^{\sigma}\varepsilon = \varepsilon^{\sigma}\beta\varepsilon - \beta^{\sigma}\varepsilon = -\alpha\varepsilon$.
So, αh is a (τ, η)-sesquilinear form, where $\eta = \alpha\varepsilon\alpha^{-\sigma} = \alpha\varepsilon(-\alpha\varepsilon)^{-1} = -1$. But then, as follows from the above, τ has order 2.
A sesquilinear form h on V, with

$$h(u,v) = (h(v,u))^{\sigma} \text{ for } u, v \in V$$

where σ is an anti-automorphism of order 2, is called a **Hermitian form** on V. If we have instead that

$$h(u,v) = -(h(v,u))^{\sigma} \text{ for } u, v \in V,$$

the form is called **skew-Hermitian**.
The elements $0 \neq v \in V$ with $h(v,v) = 0$ are called **singular** or **isotropic**. If $h(v,v) \neq 0$, then v is called **nonsingular** or **anisotropic**.
A pair of vectors v, w spanning a 2-dimensional subspace of V is called a **hyperbolic pair** of vectors if $h(v,v) = 0 = h(w,w)$ and $h(v,w) = 1$. The 2-space $\langle v, w\rangle$, where v, w is a hyperbolic pair, is called a hyperbolic 2-space.

The sesquilinear form h is called **nondegenerate** if $h(v,w) = 0$ for all $v \in V$, implies $w = 0$ and $h(v,w) = 0$ for all $w \in V$, implies $v = 0$.
Finally, the pair (V, h), where V is a vector space and h a sesquilinear (symplectic or (skew-)Hermitian) form on V is called a **sesquilinear (symplectic or (skew-) Hermitian) space**.

Notice that sesquilinear forms satisfy the conditions of Lemma 1.2.1, so we can use them to obtain subalgebras of $\mathfrak{gl}(V)$. We consider various types of forms.
First we consider a symplectic space (V, f). Clearly, if (V, f) is a nondegenerate symplectic space, then for any vector $0 \neq v \in V$ we can find a vector $w \in W$ with v, w forming a hyperbolic pair. For such v, w we find that the form f restricted to the space $\langle v, w\rangle^{\perp} := \{u \in V \mid f(\lambda v + \mu w, u) = 0 \text{ for all } \lambda, \mu \in \mathbb{F}\}$

is again nondegenerate. This implies that for finite dimensional V we can find a hyperbolic basis, i.e., a basis such that the symplectic form f on V is defined by the matrix

$$F = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

and $f(v,w) := v^t F w$. In particular, the dimension of V is even.

DEFINITION 1.2.3. Let (V, f) be a symplectic space over the field $\mathbb{F}$. Then the **symplectic Lie algebra** $\mathfrak{sp}(V, f)$ is the Lie subalgebra of $\mathfrak{gl}(V)$ that consists of all endomorphisms A in $\mathrm{End}(V)$ that satisfy

$$f\big(A(v), w\big) = -f\big(v, A(w)\big) \text{ for } v, w \in V.$$

If V is of finite dimension $n = 2m$ and f is the standard symplectic form with $f(v,w) = v^t F w$, as above, then the corresponding symplectic Lie algebra is denoted by $\mathfrak{sp}_n(\mathbb{F})$. The elements of $\mathfrak{sp}_n(\mathbb{F})$ can be represented by the matrices A satisfying $A^t F = -FA$. A basis of $\mathfrak{sp}_n(\mathbb{F})$ is given by the following matrices:

$$\begin{array}{ll} E_{i,m+i}, & 1 \le i \le m, \\ E_{m+i,i}, & 1 \le i \le m, \\ E_{i,j} - E_{m+j,m+i}, & 1 \le i, j \le m, \\ E_{i,m+j} + E_{j,m+i}, & 1 \le i < j \le m, \\ E_{m+i,j} + E_{m+j,i}, & 1 \le i < j \le m. \end{array}$$

In particular, the dimension of $\mathfrak{sp}_n(\mathbb{F})$ equals $2m^2 + m$.
The **finitary** symplectic Lie algebra $\mathfrak{fsp}(V)$ is the intersection of $\mathfrak{sp}(V)$ with $\mathfrak{fgl}(V)$.

DEFINITION 1.2.4. By $\mathfrak{psl}_n(\mathbb{F})$ (or $\mathfrak{psp}_n(\mathbb{F})$), we denote the Lie algebras obtained as a quotient of $\mathfrak{sl}_n(\mathbb{F})$ (or $\mathfrak{sp}_n(\mathbb{F})$, respectively) by its center. Notice that in most characteristics, the center is trivial and therefore in these cases we have $\mathfrak{psl}_n(\mathbb{F}) = \mathfrak{sl}_n(\mathbb{F})$ (and $\mathfrak{psp}_n(\mathbb{F}) = \mathfrak{sp}_n(\mathbb{F})$, respectively).

Next, consider the (skew)-Hermitian space (V, h), where V is a left vector space over a skew field $\mathbb{K}$ and h a (skew)-Hermitian form on V (relative to some σ).
The space (V, h) as well as the form h are called **anisotropic** if V does not contain singular vectors.
If (V, h) is a nondegenerate (skew)-Hermitian space containing a singular vector v, then we can find a second singular w such that v, w is a hyperbolic

pair. The subspace $\langle v, w\rangle^\perp := \{u \in V \mid h(\lambda v + \mu w, u) = 0 \text{ for all } \lambda, \mu \in \mathbb{K}\}$ is again nondegenerate. So we can decompose V into $V_1 \perp V_2$ where V_1 admits a hyperbolic basis and V_2 is anisotropic. This implies that, if V is finite dimensional, we can find a basis such that the form h is represented by the matrix H, i.e., $h(v, w) = v^t H w^\sigma$, where

$$H = \begin{pmatrix} 0 & I_k & 0 \\ \pm I_k & 0 & 0 \\ 0 & 0 & \Delta_m \end{pmatrix}.$$

Here Δ_m is a diagonal $m \times m$-matrix with on the diagonal entries $\lambda \in \mathbb{K}$ satisfying $\lambda^\sigma = \lambda$ in case h is Hermitian and $\lambda^\sigma = -\lambda$ in case h is skew-Hermitian.

Definition 1.2.5. Let (V, h) be a nontrivial (skew-) Hermitian space (relative to some σ) over a skew field $\mathbb{K}$. Then the **unitary Lie algebra** $\mathfrak{u}(V, h)$ consists of the endomorphisms T of V with

$$h(T(v), w) = -h(v, T(w)) \text{ for all } v, w \in V.$$

As a (skew-) Hermitian form satisfies the conditions of Lemma 1.2.1, this is a Lie algebra over any field $\mathbb{F}$ inside $\mathbb{K}$ which is fixed element-wise by σ. (Not over $\mathbb{K}$, since h is linear in the first, but not in the second variable.)
In case (V, h) is a finite dimensional nondegenerate (skew-) Hermitian space and h is represented by the matrix

$$H = \begin{pmatrix} 0 & I_k & 0 \\ \pm I_k & 0 & 0 \\ 0 & 0 & \lambda I_m \end{pmatrix}$$

as above, with $\lambda^\sigma = \pm\lambda$, we can identify $\mathfrak{u}(V, h)$ with the matrix algebra consisting of all matrices M satisfying $M^t H = -H M^\sigma$.
So, if $M = \begin{pmatrix} A & B & C \\ D & E & F \\ G & K & L \end{pmatrix}$, then

$$M^t H = \begin{pmatrix} A^t & D^t & G^t \\ B^t & E^t & K^t \\ C^t & F^t & L^t \end{pmatrix} \begin{pmatrix} 0 & I_k & 0 \\ \pm I_k & 0 & 0 \\ 0 & 0 & \lambda I_m \end{pmatrix} = \begin{pmatrix} \pm D^t & A^t & G^t\lambda \\ \pm E^t & B^t & K^t\lambda \\ \pm F^t & C^t & L^t\lambda \end{pmatrix}$$

and

$$-HM^\sigma = -\begin{pmatrix} 0 & I_k & 0 \\ \pm I_k & 0 & 0 \\ 0 & 0 & \lambda I_m \end{pmatrix} \begin{pmatrix} A & B & C \\ D & E & F \\ G & K & L \end{pmatrix}^\sigma = -\begin{pmatrix} D^\sigma & E^\sigma & F^\sigma \\ \pm A^\sigma & \pm B^\sigma & \pm C^\sigma \\ \lambda G^\sigma & \lambda K^\sigma & \lambda L^\sigma \end{pmatrix}.$$

One easily deduces, in case $\mathbb{K}$ is a field, that the dimension of $\mathfrak{u}(V,h)$ over $\mathbb{K}_\sigma = \{\mu \in \mathbb{K} \mid \mu^\sigma = \mu\}$ equals n^2 where $n = \dim(V) = 2k+m$.
Indeed, suppose h is skew-Hermitian, and μ an element from $\mathbb{K}$ not fixed by σ, then the following matrices form a basis for $\mathfrak{u}(V,h)$:

$$\begin{array}{ll}
E_{i,j} + E_{k+j,k+i}, & 1 \le i,j \le k, \\
\mu E_{i,j} + \mu^\sigma E_{k+j,k+i}, & 1 \le i,j \le k, \\
E_{k+i,j} + E_{k+j,i}, & 1 \le i < j \le k, \\
\mu E_{k+i,j} + \mu^\sigma E_{k+j,i}, & 1 \le i < j \le k, \\
E_{k+i,i}, & 1 \le i \le k, \\
E_{i,k+j} + E_{j,k+i}, & 1 \le i < j \le k, \\
\mu E_{i,k+j} + \mu^\sigma E_{j,k+i}, & 1 \le i < j \le k, \\
E_{i,k+i}, & 1 \le i \le k, \\
E_{2k+i,k+j} + \lambda E_{j,2k+i}, & 1 \le i \le m, \quad 1 \le j \le k, \\
\mu E_{2k+i,k+j} + \lambda\mu^\sigma E_{j,2k+i}, & 1 \le i \le m, \quad 1 \le j \le k, \\
E_{k+i,2k+j} + \lambda E_{2k+j,i}, & 1 \le i \le m, \quad 1 \le j \le k, \\
\mu E_{k+i,2k+j} + \lambda\mu^\sigma E_{2k+j,i}, & 1 \le i \le m, \quad 1 \le j \le k, \\
E_{2k+i,2k+j} - E_{2k+j,2k+i}, & 1 \le i < j \le m, \\
\mu E_{2k+i,2k+j} - \mu^\sigma E_{2k+j,2k+i}, & 1 \le i < j \le m, \\
\lambda E_{2k+i,2k+i}, & 1 \le i \le m.
\end{array}$$

In a similar way, a basis can be found in case h is Hermitian.

The **special unitary Lie algebra** $\mathfrak{su}(V,h)$ consists of those elements in $\mathfrak{u}(V,h)$ that are in $\mathfrak{sl}(V)$.
The **finitary** unitary and special unitary Lie algebras $\mathfrak{fu}(V,h)$ and $\mathfrak{fsu}(V,h)$ are the intersections of $\mathfrak{u}(V,h)$ with $\mathfrak{fgl}(V)$ and $\mathfrak{fsl}(V)$, respectively.
By $\mathfrak{psu}_n(\mathbb{F})$ (or $\mathfrak{pu}_n(\mathbb{F})$), we denote the Lie algebras obtained as a quotient of $\mathfrak{su}_n(\mathbb{F})$ (or $\mathfrak{u}_n(\mathbb{F})$, respectively) by its center.

Definition 1.2.6. Let V be a vector space over a field of characteristic $\neq 2$ and B be a nondegenerate symmetric bilinear form B on V. The **orthogonal Lie algebra** $\mathfrak{o}(V,B)$ consists of all $T \in \mathrm{End}(V)$ with the property

$$B(T(v),w) = -B(v,T(w)) \text{ for all } v,w \in V.$$

From Lemma 1.2.1 it follows that this property is invariant under the Lie bracket.

For $m \geq 2$ and $\dim V = n = 2m+1$ a nondegenerate symmetric bilinear (which means $B(v,w) = 0$ for all $w \in V$ implies $v = 0$) form can be represented (up to the choice of a basis and scalar) by the matrix

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix}.$$

Now $\mathfrak{o}_{2m+1}$ has dimension $m(2m+1)$ and consists of all endomorphisms T of V with $B(T(v),w) = -B(v,T(w))$. The following matrices form a basis for this Lie algebra:

$$\begin{aligned} &E_{i+1,j+1} - E_{m+j+1,m+i+1}, \quad 1 \leq i,j \leq m, \\ &E_{i+1,m+j+1} - E_{j+1,m+i+1}, \quad 1 \leq i < j \leq m, \\ &E_{m+i+1,j+1} - E_{m+j+1,i+1}, \quad 1 \leq i < j \leq m. \end{aligned}$$

For $m \geq 4$ and $\dim V = n = 2m$ we can represent the form B by the matrix

$$F = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

The dimension of the Lie algebra $\mathfrak{o}_{2n}$ defined by this form is $m(2m-1)$. It is spanned by the following matrices:

$$\begin{aligned} &E_{i,j} - E_{m+j,m+i}, \quad 1 \leq i,j \leq m, \\ &E_{i,m+j} - E_{j,m+i}, \quad 1 \leq i < j \leq m, \\ &E_{m+i,j} - E_{m+j,i}, \quad 1 \leq i < j \leq m. \end{aligned}$$

Note that the matrices in $\mathfrak{o}_{2m}$ are the skew-symmetric ones, in other words $\mathfrak{o}_n = \{X \in \mathrm{End}(V) | X + X^t = 0\}$.

The **special orthogonal Lie algebra** $\mathfrak{so}(V,h)$ consists of those elements in $\mathfrak{o}(V,h)$ that are in $\mathfrak{sl}(V)$.

The **finitary** orthogonal Lie algebra $\mathfrak{fo}(V,h)$ is the intersection of $\mathfrak{o}(V,h)$ with $\mathfrak{fgl}(V)$.

By $\mathfrak{pso}_n(\mathbb{F})$ (or $\mathfrak{po}_n(\mathbb{F})$), we denote the Lie algebras obtained as a quotient of $\mathfrak{so}_n(\mathbb{F})$ (or $\mathfrak{o}_n(\mathbb{F})$, respectively) by its center.

The general linear, special linear, symplectic, (special) unitary and orthogonal Lie algebras as described above are referred to as the **classical linear Lie algebras**.

CHAPTER 2

Extremal elements

In this chapter, we introduce extremal elements of Lie algebras, which are a basic structure for all Lie algebras considered in this work. Many details about Lie algebras spanned by extremal elements can be found in the fundamental paper [CSUW01]; the first three sections of this chapter follow their line. A very detailed and completely covering introduction of extremal elements can also be found in [Coh]. We give some properties and identities of extremal elements, and continue with some basic examples. Furthermore, we consider the low-dimensional cases of Lie algebras generated by two or three extremal elements. The chapter ends with the introduction of a geometric structure that can be defined on Lie algebras using the relations between extremal elements.

2.1. General theory

Definition 2.1.1. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F}$. A nonzero element $x \in \mathfrak{g}$ is called **extremal** if there is a map $g_x : \mathfrak{g} \to \mathbb{F}$ such that

$$\big[x, [x, y]\big] = 2g_x(y)x \tag{2.1}$$

and moreover

$$\big[[x, y], [x, z]\big] = g_x\big([y, z]\big)x + g_x(z)[x, y] - g_x(y)[x, z] \tag{2.2}$$

and

$$\big[x, [y, [x, z]]\big] = g_x\big([y, z]\big)x - g_x(z)[x, y] - g_x(y)[x, z] \tag{2.3}$$

for every $y, z \in \mathfrak{g}$.
The last two identities are called the **Premet identities**; see also Lemma 2.1.3 below.

As a consequence, it holds

$$\big[x, [x, \mathfrak{g}]\big] \subseteq \mathbb{F}x$$

for extremal $x \in \mathfrak{g}$, and for any $y \in \mathfrak{g}$, we have

$$\mathrm{ad}_x^3(y) = \big[x, [x, [x, y]]\big] = [x, \lambda x] = \lambda[x, x] = 0 \text{ for some } \lambda \in \mathbb{F}. \tag{2.4}$$

We say that x is ad-nilpotent of order at most 3.

The form g_x is called the **extremal form on** x. Note that the extremal form of $x \in \mathfrak{g}$ is denoted by f_x in most literature, but in order to distinguish between the extremal form and other forms on the Lie algebras as e.g. the symplectic form (see 1.2.2), we will denote it by g_x.

Note that in $\mathrm{char}(\mathbb{F}) \neq 2$, two elements x and $y \in \mathfrak{g}$ commute if and only if $[x, y] = [y, x]$.

We call an element $x \in \mathfrak{g}$ a **sandwich** if $\mathrm{ad}_x^2(y) = 0$ and $\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_x = 0$ for every $y \in \mathfrak{g}$. So, a sandwich is an element x for which g_x can be chosen to be identically zero. We introduce the convention that g_x is identically zero whenever x is a sandwich in $\mathfrak{g}$.

We denote the set of non-zero extremal elements of a Lie algebra by $E(\mathfrak{g})$ or, if $\mathfrak{g}$ is clear from the context, by E. Accordingly, we denote the set $\{\mathbb{F}x | x \in E(\mathfrak{g})\}$ of **extremal points** in the projective space on $\mathfrak{g}$ by $\mathcal{E}(\mathfrak{g})$ or $\mathcal{E}$.

LEMMA 2.1.2 ([CI06], Lemma 19). *For a Lie algebra $\mathfrak{g}$ and $x, y \in E(\mathfrak{g})$, we have $g_x(y) = g_y(x)$. Moreover, we have*

$$g_x([y, z]) = -g_y([x, z]) \tag{2.5}$$

for all $z \in \mathfrak{g}$.

PROOF. We start with the following observations: Let x, y commute. Then it follows from the identity (2.2) that $g_x(y)[x, z] = g_x([y, z])x$ for all $z \in \mathfrak{g}$. Assuming $g_x(y) \neq 0$, applying ad_x to both sides of the equation gives

$$g_x(y)\big[x, [x, z]\big] = \big[x, g_x(y)[x, z]\big] = \big[x, g_x\big([y, z]\big)x\big] = 0.$$

Using $[x, z] = \frac{g_x([y,z])}{g_x(y)} \cdot x$, that follows from the previous since $g_x(y) \neq 0$, we also deduce $\mathrm{ad}_x\mathrm{ad}_{y'}\mathrm{ad}_x(z) = \big[x, [y', [x, z]]\big] = 0$ for all $y', z \in \mathfrak{g}$, which implies that x is a sandwich, so $g_x(y) = 0$ by convention and we have a contradiction. So $g_x(y) = 0$.

Assume now $[x, y] \neq 0$. Consider the following equalities, that can be obtained from identity (2.3):

$$\big[x, [y, [x, [y, z]]]\big] + g_x(y)[x, [y, z]] = g_x([y, [y, z]])x - g_x([y, z])[x, y] \tag{2.6}$$

results from replacing z by $[y, z]$ in (2.3). Moreover

$$[y, [x, [y, [x, z]]]] + g_x(y)[y, [x, z]] = g_x([y, z])[x, z] - g_x(z)[y, [y, z]] \tag{2.7}$$

results also from (2.3) by applying ad_y to both sides. Now we exchange x and y in (2.7) and then subtract (2.6) and (2.7), and get

$$\big(g_x(y) - g_y(x)\big)\big[x,[y,z]\big] = -\big(g_x([y,z]) + g_y([x,z])\big)[x,y]. \tag{2.8}$$

This proves the required identities in the case where $[x,y]$ and $\big[x,[y,z]\big]$ are linearly independent. Furthermore, it suffices now to show $g_x(y) = g_y(x)$; the second equality follows since $[x,y] \neq 0$ by assumption.
If $\mathrm{char}(\mathbb{F}) \neq 2$, the Jacobi identity gives

$$[y,[x,[x,y]]] + [x,[[x,y],y]] + [[x,y],[x,y]] = 0,$$

which leads to $2g_x(y)[x,y] = 2g_y(x)[x,y]$. Since $[x,y] \neq 0$, this implies $g_x(y) = g_y(x)$. Actually, (2.8) implies the required identities if there is any $z' \in \mathfrak{g}$ such that $[x,[y,z']]$ and $[x,y]$ or, by interchanging x and y, $[y,[x,z']]$ and $[x,y]$ are linearly independent. So it remains to consider the case where $\mathrm{char}(\mathbb{F}) = 2$ and $[x,[y,\mathfrak{g}]] + [y,[x,\mathfrak{g}]] \subseteq \mathbb{F}[x,y]$. Applying ad_y, we see

$$[y,[x,[y,\mathfrak{g}]]] + [y,[y,[x,\mathfrak{g}]]] \subseteq \mathbb{F}[y,[x,y]] = 0.$$

Since $[y,[y,[x,\mathfrak{g}]]] = 2g_y([x,\mathfrak{g}])y = 0$, also $[y,[x,[y,\mathfrak{g}]]] = 0$. Using this in (2.3) we get

$$0 = [y,[x,[y,\mathfrak{g}]]] = g_y([x,\mathfrak{g}])y - g_y(\mathfrak{g})[y,x] - g_y(x)[y,\mathfrak{g}]$$

and we deduce $[y,\mathfrak{g}] \subseteq \mathbb{F}y + \mathbb{F}[x,y]$. Since $[y,[y,x]] = 0$ by (2.1), we know that $\mathbb{F}y + \mathbb{F}[x,y]$ is a commutative Lie subalgebra of $\mathfrak{g}$ and therefore also $[y,\mathfrak{g}]$ as its subspace. Using (2.2) and (2.3), we get $\big[y,[\mathfrak{g},[y,\mathfrak{g}]]\big] = \big[[y,\mathfrak{g}],[y,\mathfrak{g}]\big] = 0$ so $\mathrm{ad}_y\mathrm{ad}_u\mathrm{ad}_y = 0$ for all $u \in \mathfrak{g}$. Together with the fact that $\mathrm{ad}_y^2 = 0$, this implies that y is a sandwich and therefore by convention, g_y is zero. Similarly, we can deduce $[x,\mathfrak{g}] \subseteq \mathbb{F}x + \mathbb{F}[x,y]$ and with the same arguments as above, g_x is also zero. $\diamond$

LEMMA 2.1.3 (Premet). *If* $\mathrm{char}(\mathbb{F}) \neq 2$*, the equations* (2.2) *and* (2.3) *follow from* (2.1).

PROOF. We start with

$$\begin{aligned}
\big[[x,y],[x,z]\big] &= -\big[[x,z],[x,y]\big] \\
&= \big[x,[y,[x,z]]\big] + \big[y,[[x,z],x]\big] = \big[x,[y,[x,z]]\big] + \big[[x,[x,z]],y\big] \\
&= \big[x,[y,[x,z]]\big] + 2g_x(z)[x,y]. \qquad (2.9)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\big[[x,y],[x,z]\big] &= -\big[x,[z,[x,y]]\big] - \big[z,[[x,y],x]\big]\\ &= -\big[x,[z,[x,y]]\big] - \big[[x,[x,y]],z\big]\\ &= -\big[x,[z,[x,y]]\big] - 2g_x(y)[x,z]. \end{aligned} \tag{2.10}$$

Again using the Jacobi identity, we have

$$\begin{aligned}\big[x,[z,[x,y]]\big] &= -\big[x,[x,[y,z]]\big] - \big[x,[y,[z,x]]\big]\\ &= -2g_x\big([y,z]\big)x + \big[x,[y,[x,z]]\big]. \end{aligned} \tag{2.11}$$

Now replacing (2.11) in (2.10) and adding it to (2.9), we get twice (2.2), and if we subtract the two equation, we get twice (2.3). So if $\mathrm{char}(\mathbb{F}) \neq 2$, we deduce (2.2) and (2.3). $\diamond$

We can deduce the following three identities.

LEMMA 2.1.4. *Let $x, y \in E$ and $z \in \mathfrak{g}$. Then*

$$\big[[x,y],[x,[y,z]]\big] = 2g_y(z)g_x(y)x + g_x\big([y,z]\big)[x,y] - g_x(y)\big[x,[y,z]\big]. \tag{2.12}$$

and

$$\begin{aligned}\big[[x,y],[[x,y],z]\big] =& g_x(y)\big(2g_y(z)x - \big[x,[y,z]\big] + 2g_x(z)y - \big[y,[x,z]\big]\big)\\ &+ \big(g_x\big([y,z]\big) - g_y\big([x,z]\big)\big)[x,y]. \end{aligned} \tag{2.13}$$

Let now $x, y, z \in E$, then

$$\begin{aligned}\big[[x,[y,z]],[y,[x,z]]\big] = & -g_x\big([y,z]\big)g_y(z)x - g_x\big([y,z]\big)g_x(z)y\\ & -g_x\big([y,z]\big)g_x(y)z - 2g_x(z)g_y(z)[x,y]\\ & +2g_x(y)g_y(z)[x,z] - 2g_x(y)g_x(z)[y,z]. \end{aligned} \tag{2.14}$$

PROOF. For the first identity, we use (2.2) but replace z by $[y,z]$. This leads to

$$\begin{aligned}\big[[x,y],[x,[y,z]]\big] =& g_x\big(\big[y,[y,z]\big]\big)x + g_x\big([y,z]\big)[x,y] - g_x(y)\big[x,[y,z]\big]\\ =& g_x\big(2g_y(z)y\big)x + g_x\big([y,z]\big)[x,y] - g_x(y)\big[x,[y,z]\big]\\ =& 2g_x(y)g_y(z)x + g_x\big([y,z]\big)[x,y] - g_x(y)\big[x,[y,z]\big]. \end{aligned}$$

For the second equality, we use the Jacobi identity and then apply (2.12) twice.

$$\begin{aligned}\big[[x,y],[[x,y],z]\big] =& -\big[[x,y],[z,[x,y]]\big]\\ =& \big[[x,y],[x,[y,z]]\big] + \big[[x,y],[y,[z,x]]\big] \end{aligned}$$

$$
\begin{aligned}
&=\big[[x,y],[x,[y,z]]\big]+\big[[y,x],[y,[x,z]]\big]\\
&=2g_y(z)g_x(y)x+g_x\big([y,z]\big)[x,y]-g_x(y)\big[x,[y,z]\big]\\
&\quad+2g_x(z)g_y(x)y+g_y\big([x,z]\big)[y,x]-g_y(x)\big[y,[x,z]\big]\\
&=g_x(y)\big(2g_y(z)x-\big[x,[y,z]\big]+2g_x(z)y-\big[y,[x,z]\big]\big)\\
&\quad+\big(g_x\big([y,z]\big)-g_y\big([x,z]\big)\big)[x,y].
\end{aligned}
$$

Finally, with $x,y,z\in E$ now, the Jacobi identity gives

$$
\begin{aligned}
\big[[x,[y,z]],[y,[x,z]]\big]=&-\big[[y,[x,z]],[x,[y,z]]\big]\\
=&\big[x,[[y,z],[y,[x,z]]]\big]-\big[[y,z],[x,[y,[x,z]]]\big].
\end{aligned}
\tag{2.15}
$$

We split the equation (2.15) in two parts and consider them separately. Keep in mind that we have $g_x([y,z])=-g_y([x,z])$.

$$
\begin{aligned}
&\big[x,[[y,z],[y,[x,z]]]\big]\\
&\quad=\big[x,g_y\big([z,[x,z]]\big)y+g_y\big([x,z]\big)[y,z]-g_y(z)\big[y,[x,z]\big]\big]\\
&\quad=-2g_y(z)g_x(z)[x,y]-g_x\big([y,z]\big)\big[x,[y,z]\big]-g_y(z)\big[x,[y,[x,z]]\big]\\
&\quad=-2g_y(z)g_x(z)[x,y]-g_x\big([y,z]\big)\big[x,[y,z]\big]-g_y(z)g_x\big([y,z]\big)x\\
&\qquad+g_y(z)g_x(y)[x,z]+g_y(z)g_x(z)[x,y].
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
&-\big[[y,z],[x,[y,[x,z]]]\big]\\
&\quad=-\big[[y,z],g_x\big([y,z]\big)x-g_x(y)[x,z]-g_x(z)[x,y]\big]\\
&\quad=g_x\big([y,z]\big)\big[x,[y,z]\big]-g_x(y)g_x\big([y,z]\big)z+g_x(y)g_y(z)[x,z]\\
&\qquad-g_x(y)g_x(z)[y,z]-g_x(z)g_x\big([y,z]\big)y\\
&\qquad-g_x(y)g_x(z)[y,z]-g_x(z)g_y(z)[x,y].
\end{aligned}
$$

Adding the two terms, we find (2.14)

$$
\begin{aligned}
&\big[[x,[y,z]],[y,[x,z]]\big]\\
&\quad=-g_x\big([y,z]\big)g_y(z)x-g_x\big([y,z]\big)g_x(z)y-g_x\big([y,z]\big)g_x(y)z\\
&\qquad-2g_x(z)g_y(z)[x,y]+2g_x(y)g_y(z)[x,z]-2g_x(y)g_x(z)[y,z]. \qquad \diamond
\end{aligned}
$$

DEFINITION 2.1.5. For $x, y \in E$ extremal elements we define

$$(x,y) \in \begin{cases} E_{-2}, & \Longleftrightarrow \mathbb{F}x = \mathbb{F}y, \\ E_{-1}, & \Longleftrightarrow [x,y] = 0, (x,y) \notin E_{-2} \text{ and } \mathbb{F}x + \mathbb{F}y \subseteq E \cup \{0\} \ , \\ E_0, & \Longleftrightarrow [x,y] = 0 \text{ and } (x,y) \notin E_{-2} \cup E_{-1}, \\ E_1, & \Longleftrightarrow [x,y] \neq 0 \text{ and } g_x(y) = 0, \\ E_2, & \Longleftrightarrow g_x(y) \neq 0. \end{cases}$$

For the corresponding extremal points $\langle x \rangle, \langle y \rangle$, we define

$$(\langle x \rangle, \langle y \rangle) \in \mathcal{E}_i \Longleftrightarrow (x,y) \in E_i.$$

For two distinct extremal points $\langle x \rangle, \langle y \rangle$ we say that the pair

$$(\langle x \rangle, \langle y \rangle) \text{ is } \begin{cases} \textbf{hyperbolic} \ , & \text{if } \ i = 2, \\ \textbf{special} \ , & \text{if } \ i = 1, \\ \textbf{polar} \ , & \text{if } \ i = 0, \\ \textbf{strongly commuting} \ , & \text{if } \ i = -1, \\ \textbf{commuting} \ , & \text{if } \ i \leq 0. \end{cases}$$

By abuse of notation, we will in the following often just write x for $\langle x \rangle$ if it is clear that an extremal point is meant.
Let (x, y) be a hyperbolic pair. If $z \in \mathcal{E}$ makes (x, y, z) a hyperbolic path of length two, i.e. $x \neq z$ and $(y, z) \in \mathcal{E}_2$, then we call (x, y, z) a **symplectic triple** if $(x, z) \in \mathcal{E}_{\leq 0}$. By abuse of notation, we sometimes also call the triple of extremal elements (x, y, z) where $x, y, z \in E$ a symplectic triple.

Note that for a symplectic triple (x, y, z) as defined above:

$$g_x(y) \neq 0 \neq g_y(z) \text{ and}$$
$$[x, z] = 0.$$

2.2. The exponential map

Since extremal elements are ad-nilpotent of order at most 3 (see 2.4), we can define the **exponential map** for an extremal element $x \in E$, any $y \in \mathfrak{g}$ and some $\lambda \in \mathbb{F}$:

$$\exp(x, \lambda)y = y + \lambda \cdot [x, y] + \lambda^2 g_x(y)x. \tag{2.16}$$

PROPOSITION 2.2.1. *Let $x \in E(\mathfrak{g})$ with extremal form $g_x : \mathfrak{g} \to \mathbb{F}$. Then* $\exp(x, \lambda)$ *is an endomorphism on $\mathfrak{g}$ for every $\lambda \in \mathbb{F}$, and* $\exp(x, \lambda + \mu) = \exp(x, \lambda)\exp(x, \mu)$ *for all $\lambda, \mu \in \mathbb{F}$.*

PROOF. Before we start with the proof of the statement, we provide two equations that we use in the proof. We have

$$\big[[x,[x,y]],[x,[x,z]]\big] = 4g_x(y)g_x(z)[x,x] = 0,$$

and

$$\begin{aligned}
[\mathrm{ad}_x(y), \mathrm{ad}_x^2(z)] + [\mathrm{ad}_x^2(y), \mathrm{ad}_x(z)] =& \big[[x,y],[x,[x,z]]\big] + \big[[x,[x,y]],[x,z]\big] \\
=& 2g_x(z)\big[[x,y],x\big] + \big[2g_x(y)x,[x,z]\big] \\
=& -2g_x(z)\big(\big[x,[x,y]\big]\big) + 4g_x(y)g_x(z)x \\
=& -4g_x(z)g_x(y)x + 4g_x(y)g_x(z)x = 0.
\end{aligned}$$

Now let $x, y, z \in \mathfrak{g}$ and $\lambda \in \mathbb{F}$. We have

$$\begin{aligned}
&[\exp(x,\lambda)(y), \exp(x,\lambda)(z)] \\
&\quad = [y + \lambda \cdot [x,y] + \lambda^2 \cdot g_x(y)x, z + \lambda \cdot [x,z] + \lambda^2 \cdot g_x(z)x] \\
&\quad = [y,z] + [y, \lambda \cdot [x,z]] + [y, \lambda^2 \cdot [x,z]] + [\lambda \cdot [x,y], z] \\
&\qquad + [\lambda \cdot [x,y], \lambda \cdot [x,z]] + [\lambda \cdot [x,y], \lambda^2 \cdot [x,z]] \\
&\qquad + \big[\lambda^2 \cdot g_x(y)x, z\big] + \big[\lambda^2 \cdot g_x(y)x, \lambda \cdot [x,z]\big] + \big[\lambda^2 \cdot g_x(y)x, \lambda^2 \cdot g_x(z)x\big] \\
&\quad = [y,z] + \lambda \cdot [y,[x,z]] + \lambda \cdot [[x,y],z] + \lambda^2 \cdot [[x,y],[x,z]] \\
&\qquad + \lambda^2 \cdot [y, g_x(z)x] + \lambda^2 \cdot [g_x(y)x, z] \\
&\quad = [y,z] + \lambda \cdot [x,[y,z]] + \lambda^2 \cdot g_x([y,z])x \\
&\quad = \exp(x,\lambda)\big([y,z]\big).
\end{aligned}$$

This proves $\exp(x, \lambda)$ to be an automorphism. To show $\exp(x, \lambda + \mu) = \exp(x, \lambda)\exp(x, \mu)$ for all $x \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{F}$, note that $g_x(x) = g_x([x, y]) = 0$ for every $y \in \mathfrak{g}$. Now it follows that

$$\begin{aligned}
&\exp(x,\lambda)\exp(x,\mu)y \\
&\quad = \exp(x,\lambda)\big(y + \mu[x,y] + \mu^2 g_x(y)x\big) \\
&\quad = y + \mu[x,y] + \mu^2 g_x(y)x + \lambda\big[x, y + \mu[x,y] + \mu^2 g_x(y)x\big] \\
&\qquad + \lambda^2 g_x\big(y + \mu[x,y] + \mu^2 g_x(y)x\big)x \\
&\quad = y + \mu[x,y] + \mu^2 g_x(y)x + \lambda[x,y] + 2\lambda\mu g_x(y)x + \lambda\mu^2 g_x(y)[x,x]
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda^2 g_x(y)x + \lambda^2\mu g_x\big([x,y]\big)x + \lambda^2\mu^2 g_x(y)g_x(x)x \\
&= y + \lambda[x,y] + \mu[x,y] + \lambda^2 g_x(y)x + 2\lambda\mu g_x(y)x \\
&= y + (\lambda+\mu)[x,y] + (\lambda+\mu)^2 g_x(y)x \\
&= \exp(x,\lambda+\mu)y.
\end{aligned}
$$

◇

LEMMA 2.2.2. *For $x \in E$ the set $U_x = \{\exp(x,\lambda) | \lambda \in \mathbb{F}\}$ is a subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *isomorphic to the additive group of* $\mathbb{F}$*.*

PROOF. We have seen in the previous proposition that the map $\lambda \mapsto \exp(x,\lambda)$, $\lambda \in \mathbb{F}$, is a homomorphism. Therefore, the map $\mathbb{F} \to U_x$ given by $\lambda \mapsto \exp(x,\lambda)$ is an isomorphism of groups. Let $y \in E$, $\lambda,\mu \in \mathbb{F}$:

$$
\begin{aligned}
&\exp(x,\lambda)\exp(x,\mu)y \\
&\quad = y + \mu[x,y] + \mu^2 g_x(y)x + \lambda[x,y] + \lambda\mu[x,[x,y]] + \lambda\mu^2 g_x(y)[x,x] \\
&\qquad + \lambda^2 g_x(y)x + \lambda^2\mu g_x([x,y])x + \lambda^2\mu^2 g_x(y)g_x(x)x \\
&\quad = y + (\lambda+\mu)[x,y] + (\lambda+\mu)^2 g_x(y)x \\
&\quad = \exp(x,\lambda+\mu)y.
\end{aligned}
$$

We see that $\exp(x,-\lambda)$ is the inverse of $\exp(x,\lambda)$, so both are automorphisms of $\mathfrak{g}$. ◇

2.3. The extremal form

Suppose $\mathfrak{g}$ is a Lie algebra over the field $\mathbb{F}$ generated by its extremal elements. The purpose of this section is to show the existence of a bilinear form g on $\mathfrak{g}$ with the property that $g(x,y) = g_x(y)$ for any two extremal elements $x,y \in \mathfrak{g}$. We start with the following observation:

LEMMA 2.3.1 ([CSUW01, Lemma 2.5]). *If a Lie algebra is generated by extremal elements, it is linearly spanned by the set E of all its extremal elements.*

PROOF. Clearly, $\mathfrak{g}$ is spanned by brackets of extremal elements. We prove by induction on its length that every such bracket is a linear combination of extremal elements. If the length is 1, there is nothing to show. Assume now that for $n \in \mathbb{N}$, it is already shown that all elements of length n are linearly spanned by E. Let $z \in E$ be an element with bracketing of length $n+1$, and let $x,y \in E$ be two elements such that $[x,y]$ is the innermost bracket of z, that is $z = \big[\cdot,\big[\cdot,\big[\ldots\big[\cdot,[x,y]\big]\ldots\big]\big]\big]$. Consider $v = \exp(x,1)y = y + [x,y] + g_x(y)x$.

By Lemma 2.2.1, v is also extremal. But now we can express z as a sum of elements of length n:

$$\begin{aligned} z =&[\cdot,[\cdot,[\dots[\cdot,v-y-g_x(y)x]\dots]]] \\ =&[\cdot,[\cdot,[\dots[\cdot,v]\dots]]]-[\cdot,[\cdot,[\dots[\cdot,y]\dots]]]-g_x(y)[\cdot,[\cdot,[\dots[\cdot,x]\dots]]]. \diamond \end{aligned}$$

THEOREM 2.3.2 ([CSUW01, Theorem 2.6]). *Let $\mathfrak{g}$ be generated by $E(\mathfrak{g})$. Then there is a unique bilinear symmetric form $g : \mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$ such that the linear form g_x coincides with $y\mapsto g(x,y)$ for each $x\in E$. Moreover, this form is associative in the sense that $g(x,[y,z]) = g([x,y],z)$ for all $x,y,z\in\mathfrak{g}$.*

PROOF. From the previous result, we deduce that there exists a basis of $\mathfrak{g}$, say $\{u_i|i\in I\}$, consisting of extremal elements from $E(\mathfrak{g})$. Notice that if $x\in E$ and $\lambda\in\mathbb{F}$, then $\big[\lambda x,[\lambda x,y]\big] = 2\lambda g_x(y)\lambda x = 2\lambda g(x,y)\lambda x$ for all $y\in\mathfrak{g}$, so λx is extremal with $g(\lambda x,y) = \lambda g(x,y)$.
We define the extremal form g_x for an element $x\in\mathfrak{g}$ with $x=\sum_{i\in I}\lambda_i u_i$ by $g_x=\sum_{i\in I}\lambda_i g_{u_i}$. Notice from the above that this is $\sum_{i\in I} g_{\lambda_i u_i}$.
Now we choose an element $y\in\mathfrak{g}$ with $y=\sum_{i\in I}u_i$, and suppose that there is a second way to write it as a sum of extremal elements, say $y=\sum_{i\in I}v_i$, where $v_i\in E$. So for $z\in E$, we have (using the result from 2.1.2):

$$\begin{aligned} g_y(z) =&\sum_{i\in I} g_{u_i}(z) = \sum_{i\in I} g_z(u_i) = g_z\big(\sum_{i\in I}u_i\big) \\ =&g_z\big(\sum_{i\in I}v_i\big) = \sum_{i\in I} g_z(v_i) = \sum_{i\in I} g_{v_i}(z). \end{aligned}$$

Since E spans $\mathfrak{g}$ by 2.3.1, we conclude $\sum_{i\in I} g_{u_i} = \sum_{i\in I} g_{v_i}$, so g_x is a well defined linear functional. Consequently, $g(x,y) = g_x(y)$ defines a bilinear form, which is symmetric by 2.1.2 and 2.3.1.
To show that g is associative, take $x,y,z\in E$. Exchanging x and y in (2.12) gives
(2.17)

$$\begin{aligned} \big[[x,y],[y,[x,z]]\big] =&-\big[[y,x],[y,[x,z]]\big] \\ =&-2g(x,z)g(y,z)y+g(y,[x,z])[x,y]+g(y,x)\big[y,[x,z]\big]. \end{aligned}$$

On the other hand, we get by the Jacobi identity and with 2.1.2:

(2.18)

$$
\begin{aligned}
\big[[x,y],[y,[x,z]]\big] =& -\big[y,[[x,z],[x,y]]\big] - \big[[x,z],[[x,y],y]\big] \\
=& \big[y, g(x,[y,z])x + g(x,z)[x,y] - g(x,y)[x,z]\big] \\
& - 2g(y,x)\big[[x,z],y\big] \\
=& - g(x,[y,z])[x,y] + g(x,z)\big[y,[x,y]\big] \\
& - g(x,y)\big[y,[x,z]\big] + 2g(y,x)\big[y,[x,z]\big] \\
=& - g(x,[y,z])[x,y] - 2g(x,z)g_y, x)y + g(y,x)\big[y,[x,z]\big].
\end{aligned}
$$

Suppose now that $[x,y] \neq 0$. Then, comparing the coefficients in the equations (2.17) and (2.18) leads to

$$g\big(y,[x,z]\big) = -g\big(x,[y,z]\big). \tag{2.19}$$

But then, by symmetry of g, we have

$$g\big(x,[z,y]\big) = -g\big(x,[y,z]\big) = g\big(y,[x,z]\big) = g\big([x,z],y\big).$$

So we have

$$g\big(x,[z,y]\big) = g\big([x,z],y]\big) \text{ whenever } [x,y] \neq 0. \tag{2.20}$$

Similarly, we get

$$g\big(x,[y,z]\big) = g\big([x,y],z]\big) \text{ whenever } [x,z] \neq 0 \text{ and} \tag{2.21}$$

$$g\big(y,[x,z]\big) = g\big([y,x],z]\big) \text{ whenever } [y,z] \neq 0. \tag{2.22}$$

So it remains to consider the case $[x,y] = 0$. Obviously, in this case it holds $g([x,y],z) = 0$, so our goal is to show that also $g(x,[y,z]) = 0$. This is clear if $[y,z] = 0$. Assuming $[y,z] \neq 0$, we can either have $[x,z] \neq 0$, in which case the required equality follows from (2.21), or $[x,z] = 0$. In the latter case, we can apply (2.5) and get $g(x,[y,z]) = -g(y,[x,z]) = 0$ if $[x,z] = 0$. ◇

Definition 2.3.3. For a Lie algebra $\mathfrak{g}$ generated by extremal elements, a form $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ as in Proposition 2.3.2 is called an **extremal form** of $\mathfrak{g}$. Notice that g is unique (for an explicit reasoning, see e.g. [CI06]).

Definition 2.3.4. The **radical** of the extremal form g on $\mathfrak{g}$ is

$$\operatorname{rad}(g) := \{u \in \mathfrak{g} | g_u(z) = 0 \ \forall \ z \in \mathfrak{g}\}.$$

LEMMA 2.3.5. *If B is a symmetric or antisymmetric bilinear form on $\mathfrak{g}$ such that*

$$B(x,[y,z]) = B([x,y],z)$$

for all $x, y, z \in \mathfrak{g}$, then $\mathrm{rad}(B)$ *is an ideal of* $\mathfrak{g}$.

PROOF. We know that $\mathrm{rad}(B)$ is a linear subspace of $\mathfrak{g}$. Let $x \in \mathrm{rad}(B)$ and $y \in \mathfrak{g}$. Then $B([x,y],z) = B(x,[y,z]) = 0$ and moreover $B([y,x],z) = \varepsilon B(z,[y,x]) = \varepsilon B([z,y],x) = 0$ for each $z \in \mathfrak{g}$ and $\varepsilon = 1$ if B is symmetric and $\varepsilon = -1$ if B is antisymmetric. It follows $[x,y],[y,x] \in \mathrm{rad}(B)$. ⋄

DEFINITION 2.3.6. For $\dim V < \infty$ and $x, y \in \mathfrak{g}$, the bilinear form

$$\kappa_{\mathfrak{g}}(x,y) = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$$

is called the **Killing form** of $\mathfrak{g}$.

LEMMA 2.3.7. *In the corresponding Lie algebra* $\mathfrak{g}$, $\mathrm{rad}(g)$ *and* $\mathrm{rad}(\kappa)$ *are ideals.*

PROOF. By Proposition 2.3.2, g is associative, so can apply Lemma 2.3.5. The symmetry of κ follows from the property $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$, $X, Y \in \mathrm{End}(\mathfrak{g})$ of the trace function. This identity also induces the associativity of κ:

$$\begin{aligned}
\kappa([x,y],z) =& \mathrm{Tr}([\mathrm{ad}_x, \mathrm{ad}_y] \circ \mathrm{ad}_z) = \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_z - \mathrm{ad}_y\mathrm{ad}_x\mathrm{ad}_z)\\
=& \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_z) - \mathrm{Tr}(\mathrm{ad}_y\mathrm{ad}_x\mathrm{ad}_z)\\
=& \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_z) - \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_z\mathrm{ad}_y)\\
=& \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_z - \mathrm{ad}_x\mathrm{ad}_z\mathrm{ad}_y)\\
=& \mathrm{Tr}(\mathrm{ad}_x \circ [\mathrm{ad}_y, \mathrm{ad}_z])\\
=& \kappa(x,[y,z]).
\end{aligned}$$

⋄

In the following, we give a few facts about the radicals of the forms g and κ and the radical of the Lie algebra. The proofs for these statements can be found in [CSUW01], section 9. Note that they are just proved there for $\mathrm{char}(\mathbb{F}) \neq 2$. The first statement is about the relation between the radicals of the extremal and the Killing-form:

LEMMA 2.3.8 ([CSUW01], 9.3 and 9.5). *In general,* $\mathrm{rad}(g) \subseteq \mathrm{rad}(\kappa)$. *If moreover* $\mathrm{char}(\mathbb{F}) = 0$, *then* $\mathrm{rad}(g) = \mathrm{rad}(\kappa)$.

The following gives the relation between the radical of the extremal form and the radical of the Lie algebra $\mathfrak{g}$ as defined in 1.1.10:

LEMMA 2.3.9 ([CSUW01], 9.8 and 9.12). *In general,* $\mathrm{Rad}(\mathfrak{g}) \subseteq \mathrm{rad}(g)$. *Moreover, if* $\mathrm{char}(\mathbb{F}) \neq 2$ *or* 3, *then* $\mathrm{Rad}(\mathfrak{g}) = \mathrm{rad}(g)$.

The following results give a correspondence between the semisimplicity of the Lie algebra and $\mathrm{rad}(g)$:

PROPOSITION 2.3.10 ([CSUW01] 9.14). *We have* $\mathrm{rad}(g) = 0$ *if and only if* $\mathfrak{g}$ *is a direct sum of simple ideals.*

This implies:

COROLLARY 2.3.11 ([CSUW01] 9.15). *We have that* $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ *is a direct sum of simple ideals if and only if* $\mathrm{Rad}(\mathfrak{g}) = \mathrm{rad}(g)$.

2.4. Classical linear Lie algebras, tensors and extremal elements

In this section we describe the classical linear Lie algebras as defined in the previous section with the help of tensors. This description turns out to be a useful model in various computations later on.
Let V be a left vector space over the skew field $\mathbb{K}$ and W^* a subspace of V^*, the dual of V, which is a right vector space over $\mathbb{K}^{opp}$, the opposite of $\mathbb{K}$. Let $\mathbb{F}$ be the center of $\mathbb{K}$.

DEFINITION 2.4.1. On $\mathfrak{g} := V \otimes W^*$, we define an $\mathbb{F}$-bilinear product $[\cdot,\cdot]$ by

$$[v \otimes \phi, w \otimes \psi] := \phi(w)(v \otimes \psi) - \psi(v)(w \otimes \phi)$$

with $v, w \in V$ and $\phi, \psi \in W^*$.

PROPOSITION 2.4.2. *Equipped with the product defined above,* $\mathfrak{g}$ *is a Lie algebra over* $\mathbb{F}$.

PROOF. We have $[\ ,\ ]$ defined to be $\mathbb{F}$-bilinear and since

$$[v \otimes \phi, v \otimes \phi] = \phi(v)(v \otimes \phi) - \phi(v)(v \otimes \phi) = 0,$$

the bracket is also alternating. It remains to show the Jacobi identity. For this it suffices to show that for all $v, w, u \in V$ and $\psi, \phi, \chi \in V^*$ we have

$$[w \otimes \chi, [v \otimes \phi, u \otimes \psi]] + [v \otimes \phi, [u \otimes \psi, w \otimes \chi]] + [u \otimes \psi, [w \otimes \chi, v \otimes \phi]] = 0.$$

Consider the first summand:

$$[w\otimes\chi[v \otimes \phi, u \otimes \psi]]$$

$$\begin{aligned}&=[w\otimes\chi,\phi(u)v\otimes\psi)-\psi(v)u\otimes\phi)]\\&=\phi(u)\chi(v)w\otimes\psi-\psi(w)\phi(u)v\otimes\chi-\psi(v)\chi(u)w\otimes\phi+\phi(w)\psi(v)u\otimes\chi.\end{aligned}$$

We get the second and the third summand just by rotation of the variables:

$$\begin{aligned}&[v\otimes\phi,[u\otimes\psi,w\otimes\chi]]\\&\quad=\psi(w)\phi(u)v\otimes\chi-\chi(v)\psi(w)u\otimes\phi-\chi(u)\chi(w)v\otimes\psi+\psi(v)\chi(u)w\otimes\phi.\end{aligned}$$

and

$$\begin{aligned}&[u\otimes\psi,[w\otimes\chi,v\otimes\phi]]\\&\quad=\chi(v)\psi(w)u\otimes\phi-\phi(u)\chi(v)w\otimes\psi-\phi(w)\phi(v)u\otimes\chi+\chi(u)\phi(w)v\otimes\psi.\end{aligned}$$

So, indeed,

$$[w\otimes\chi,[v\otimes\phi,u\otimes\psi]]+[v\otimes\phi,[u\otimes\psi,w\otimes\chi]]+[u\otimes\psi,[w\otimes\chi,v\otimes\phi]]=0,$$

which proves the Jacobi identity. ◇

So we have a Lie algebra $\mathfrak{g}$ on the vector space V, W^*, which we denote $\mathfrak{g}(V, W^*)$.
We call a nontrivial pure tensor $v\otimes\phi\in V\otimes V^*$ **singular** if $\phi(v)=0$. If $\phi(v)\neq 0$, it is called **nonsingular**.

Proposition 2.4.3. *Let g be an $\mathbb{F}$-bilinear form on $V\otimes W^*$ given by*

$$g(v\otimes\phi,w\otimes\psi)=-\psi(v)\phi(w)$$

for $v\otimes\phi, w\otimes\psi\in V\otimes V^*$.
Then for all singular pure tensors $v\otimes\phi$ and tensors $w\otimes\psi$ and $u\otimes\chi\in V\otimes V^*$ we have

$$\begin{aligned}\big[v\otimes\phi,[v\otimes\phi,w\otimes\psi]\big]&=2g(v\otimes\phi,w\otimes\psi)v\otimes\phi,\\ \big[[v\otimes\phi,w\otimes\psi],[v\otimes\phi,u\otimes\chi]\big]&=g(v\otimes\phi,\psi(u)w\otimes\chi-\chi(w)u\otimes\psi)v\otimes\phi\\&\quad+g(v\otimes\phi,u\otimes\chi)[v\otimes\phi,w\otimes\psi]\\&\quad-g(v\otimes\phi,w\otimes\psi)[v\otimes\phi,u\otimes\chi],\end{aligned}$$

and

$$\begin{aligned}\big[v\otimes\phi,[w\otimes\psi,[v\otimes\phi,u\otimes\chi]\big]&=g(v\otimes\phi,w\otimes\chi\psi(u)-u\otimes\psi\chi(w))v\otimes\phi\\&\quad-g(v\otimes\phi,u\otimes\chi)[v\otimes\phi,w\otimes\psi]\\&\quad-g(v\otimes\phi,w\otimes\psi)[v\otimes\phi,u\otimes\chi].\end{aligned}$$

PROOF. Let $v \otimes \phi$ be a singular pure tensor. Using the definition of the bracket from 2.4.1, we have

$$\begin{aligned}\big[v \otimes \phi, [v \otimes \phi, w \otimes \psi]\big] =& \big[v \otimes \phi, \phi(w) v \otimes \psi - \psi(v) w \otimes \phi\big] \\ =& -\psi(v)\phi(w) v \otimes \phi - \psi(v)\phi(w) v \otimes \phi \\ =& -2\psi(v)\phi(w) v \otimes \phi \\ =& 2g(v \otimes \phi, w \otimes \psi) v \otimes \phi.\end{aligned}$$

Furthermore we find

$$\begin{aligned}&\big[[v \otimes \phi, w \otimes \psi], [v \otimes \phi, u \otimes \chi]\big] \\ &\quad = [\phi(w) v \otimes \psi - \psi(v) w \otimes \phi, \phi(u) v \otimes \chi - \chi(v) u \otimes \phi] \\ &\quad = \phi(u)\psi(v)\phi(w) v \otimes \chi - \phi(w)\chi(v)\phi(u) v \otimes \psi \\ &\quad - \chi(v)\psi(u)\phi(w) v \otimes \phi + \phi(w)\phi(v)\chi(v) u \otimes \psi \\ &\quad - \phi(u)\phi(v)\psi(v) w \otimes \chi + \psi(v)\chi(w)\phi(u) v \otimes \phi \\ &\quad + \chi(v)\phi(u)\psi(v) w \otimes \phi - \psi(v)\phi(w)\chi(v) u \otimes \phi \\ &\quad = (-\chi(v)\psi(u)\phi(w) + \psi(v)\chi(w)\phi(u)) v \otimes \phi \\ &\quad - \phi(w)\chi(v)\phi(u) v \otimes \psi + \phi(u)\psi(v)\phi(w) v \otimes \chi \\ &\quad + \chi(v)\phi(u)\psi(v) w \otimes \phi - \psi(v)\phi(w)\chi(v) u \otimes \phi \\ &\quad = (-\chi(v)\psi(u)\phi(w) + \psi(v)\chi(w)\phi(u)) v \otimes \phi \\ &\quad + (-\phi(u)\chi(v))(\phi(w) v \otimes \psi - \psi(v) w \otimes \phi) \\ &\quad - (-\phi(w)\psi(v))(\phi(u) v \otimes \chi - \chi(v) u \otimes \phi) \\ &\quad = g(v \otimes \phi, \psi(u) w \otimes \chi - \chi(w) u \otimes \psi) v \otimes \phi \\ &\quad + g(v \otimes \phi, u \otimes \chi)[v \otimes \phi, w \otimes \psi] \\ &\quad - g(v \otimes \phi, w \otimes \psi)[v \otimes \phi, u \otimes \chi].\end{aligned}$$

Finally,

$$\begin{aligned}&\big[v \otimes \phi, [w \otimes \psi, [v \otimes \phi, u \otimes \chi]\big] \\ &\quad = [v \otimes \phi, [w \otimes \psi, \phi(u) v \otimes \chi - \chi(v) u \otimes \phi]] \\ &\quad = [v \otimes \phi, \phi(u)(\psi(v) w \otimes \chi - \chi(w) v \otimes \psi) \\ &\quad - \chi(v)(\psi(u) w \otimes \phi - \phi(w) u \otimes \psi)] \\ &\quad = \phi(u)\phi(w)\psi(v) v \otimes \chi - \phi(u)\psi(v)\chi(v) w \otimes \phi\end{aligned}$$

$$
\begin{aligned}
&+ \chi(w)\phi(u)\psi(v)v \otimes \phi - \chi(v)\psi(u)\phi(w)v \otimes \phi\\
&- \chi(v)\phi(w)\psi(v)u \otimes \phi + \chi(v)\phi(w)\phi(u)v \otimes \psi\\
&= (-\chi(v)\psi(u)\phi(w) + \psi(v)\chi(w)\phi(u))v \otimes \phi\\
&+ \phi(u)\phi(w)\chi(v)v \otimes \psi + \phi(u)\phi(w)\psi(v)v \otimes \chi\\
&- \psi(v)\chi(v)\phi(u)w \otimes \phi - \psi(v)\chi(v)\phi(w)u \otimes \phi\\
&= (-\chi(v)\psi(u)\phi(w) + \psi(v)\chi(w)\phi(u))v \otimes \phi\\
&- (-\phi(u)\chi(v))(\phi(w)v \otimes \psi - \psi(v)w \otimes \phi)\\
&- (-\phi(w)\psi(v))(\phi(u)v \otimes \chi - \chi(v)u \otimes \phi)\\
&= g(v \otimes \phi, w \otimes \chi\psi(u) - u \otimes \psi\chi(w))v \otimes \phi\\
&- g(v \otimes \phi, u \otimes \chi)[v \otimes \phi, w \otimes \psi]\\
&- g(v \otimes \phi, w \otimes \psi)[v \otimes \phi, u \otimes \chi]. \qquad \diamond
\end{aligned}
$$

Corollary 2.4.4. *Let* $\mathbb{K} = \mathbb{F}$ *be a field. Then the pure tensors* $v \otimes \phi \in V \otimes V^*$ *are extremal.*
Moreover,

$$\exp(v \otimes \phi, \lambda)(w \otimes \psi) = \big(w + \lambda\phi(w)v\big) \otimes \big(\psi - \lambda\psi(v)\phi\big) \tag{2.23}$$

for any $\lambda \in \mathbb{F}$.

Proof. This is a direct consequence of the above proposition. As it is sufficient to check the extremal identities only for pure tensors $w \otimes \psi$ and $u \otimes \chi$, we find that the singular pure tensors are extremal with associated extremal form g.
Using the above, we also find

$$
\begin{aligned}
&\exp(v \otimes \phi, \lambda)(w \otimes \psi)\\
&\qquad = w \otimes \psi + \lambda[v \otimes \phi, w \otimes \psi] + \lambda^2 g(v \otimes \phi, w \otimes \psi)v \otimes \phi\\
&\qquad = w \otimes \psi + \lambda\big(\phi(w)v \otimes \psi - \psi(v)w \otimes \phi\big) - \lambda^2\phi(w)\psi(v)v \otimes \phi\\
&\qquad = \big(w + \lambda\phi(w)v\big) \otimes \big(\psi - \lambda\psi(v)\phi\big). \qquad \diamond
\end{aligned}
$$

Notice that in case $\mathbb{F} \neq \mathbb{K}$ and the characteristic is not 2, the elements $v \otimes \phi$, for which there is an element $w \otimes \psi$ with $\phi(w)\psi(v) \neq 0$, are not extremal. Indeed, in this situation we find $\big[v \otimes \phi, [v \otimes \phi, \mathfrak{g}]\big]$ to contain $\mathbb{K}v \otimes \phi$, which is bigger than $\mathbb{F}v \otimes \phi$.

2.4.1. General linear Lie algebras. As before, let V be a left vector space over the skew field $\mathbb{K}$ with center $\mathbb{F}$ and suppose W^* is a subspace of V^*.

As in the case where $\mathbb{K}$ is a field, we can denote by $\mathfrak{gl}(V)$ the algebra of $\mathbb{K}$-linear maps from V to V. Equipped with the standard Lie commutator, i.e., for all $T, S \in \mathfrak{gl}(V)$ we have $[T,S] = TS - ST$, we find $\mathfrak{gl}(V)$ to be a Lie algebra over $\mathbb{F}$.

The tensor product $V \otimes W^*$ is isomorphic to a subspace of $\mathfrak{gl}(V)$ via the map

$$\Phi : V \otimes W^* \longrightarrow \mathfrak{gl}(V)$$
$$v \otimes \phi \mapsto t_{v,\phi},$$

where

$$t_{v,\phi} : V \to V$$
$$\text{with } t_{v,\phi}(w) = \phi(w) \cdot v.$$

PROPOSITION 2.4.5. *The map Φ is a homomorphism of Lie algebras.*

PROOF. To show that the Lie product is preserved, we have to prove that

$$\Phi\big([v \otimes \phi, w \otimes \psi]_{\mathfrak{g}}\big) = \big[\Phi(v \otimes \phi), \Phi(w \otimes \psi)\big]_{\mathfrak{gl}(V)}$$

where $v, w \in V$, $\phi, \psi \in V^*$ and $[\ ,\ ]_{\mathfrak{g}}$ and $[\ ,\ ]_{\mathfrak{gl}(V)}$ denote the Lie products on $\mathfrak{g}$ and $\mathfrak{gl}(V)$, respectively. Starting with the left side, we have

$$\begin{aligned}\Phi\big([v \otimes \phi, w \otimes \psi]_{\mathfrak{g}}\big) =& \Phi\big(\phi(w)v \otimes \psi - \psi(v)w \otimes \phi\big)\\ =& t_{\phi(w)v,\psi} - t_{\psi(v)w,\phi}.\end{aligned}$$

Applying this to an element $u \in V$, we get

$$\begin{aligned}\big(t_{\phi(w)v,\psi} - t_{\psi(v)w,\phi}\big)(u) =& \psi(u)\phi(w)v - \phi(u)\psi(v)w\\ =& t_{v,\phi}\big(\psi(u)w\big) - t_{w,\psi}\big(\phi(u)v\big)\\ =& t_{v,\phi}\big(t_{w,\psi}(u)\big) - t_{w,\psi}\big(t_{v,\phi}(u)\big)\\ =& \big(t_{v,\phi}t_{w,\psi} - t_{w,\psi}t_{v,\phi}\big)(u)\\ =& \big[t_{v,\phi}, t_{w,\psi}\big]_{\mathfrak{gl}(V)}(u).\end{aligned}$$ ◇

If V is finite dimensional over $\mathbb{F}$ and $W^* = V^*$, it is well known that Φ is onto and hence is an isomorphism. If V is infinite dimensional this is not the case. However, we do have the following.

PROPOSITION 2.4.6. *Suppose* $\mathbb{K} = \mathbb{F}$ *is a field. Then the map* Φ *induces a Lie algebra isomorphism of* $\mathfrak{g}(V \otimes V^*)$ *into* $\mathfrak{fgl}(V)$.

PROOF. Clearly Φ is a linear map, mapping elements of $V \otimes V^*$, as these are finite sums of pure tensors, to finitary linear maps.
If V is finite dimensional, then it is well known that Φ is a bijection. In particular, then Φ is injective. Now, if V is infinite dimensional, and $x \in V \otimes V^*$ is an element of the kernel, then it can be written as a finite sum of pure tensors. So, there is a finite dimensional subspace V_0 of V such that these pure tensors are inside $V_0 \otimes V_0^*$. But that implies that x is in the kernel of the map Φ restricted to $V_0 \otimes V_0^*$ and hence $x = 0$. Thus also Φ is injective.
A similar argument proves that Φ is surjective. Indeed, if g is an element from $\mathfrak{fgl}(V)$, then we can decompose V as $V = V_1 \oplus V_2$ such that V_1 is finite dimensional and contains $g(V)$ (so the image of g) and V_2 is contained in the kernel of g. But then g is already contained in $\Phi(V_1 \otimes V_1^*)$. $\diamond$

The subalgebra of $\mathfrak{g}(V, W^*)$ generated by the elements $v \otimes \phi$ with $v \in V$, $\phi \in W^*$ and $\phi(v) = 0$ will be denoted by $\mathfrak{g}_0(V, W^*)$.
The elements $t_{v,\phi}$, which are images of singular pure tensors, are called **infinitesimal transvections**. We call $\langle v \rangle$ the **center** and $\langle \phi \rangle$ the **axis** of the infinitesimal transvection.
The elements $t_{v,\phi}$, which are images of nonsingular pure tensors, are called **infinitesimal reflections**.
For an element t in $\mathfrak{gl}(V)$, we can define an action on V^* by

$$t^*(\psi)(v) := -\psi(tv),$$

where $v \in V$ and $\psi \in V^*$. For t_1, t_2 two elements in $\mathfrak{gl}(V)$ and $w \in V$, $\phi \in V^*$, we find

$$\begin{aligned}
([t_1, t_2])^*(\phi)(w) &= -\phi([t_1, t_2](w)) \\
&= -\phi(t_1 t_2 - t_2 t_1)(w)) \\
&= \phi((t_2 t_1 - t_1 t_2) w) \\
&= \phi(t_2 t_1 w) - \phi(t_1 t_2 w) \\
&= -t_2^* \phi(t_1 w) + t_1^* \phi(t_2 w) \\
&= (t_1^* t_2^* \phi)(w) - t_2^* t_1^* \phi(w) \\
&= ([t_1^*, t_2^*](\phi))(w).
\end{aligned}$$

So we have defined a Lie algebra action of $\mathfrak{gl}(V)$ on V^*, called the **dual action**. This of course extends to an action $t^{\otimes}$ of t on $V \otimes V^*$ by

$$\begin{aligned} t^{\otimes}\colon V \otimes V^* &\to V \otimes V^* \\ v \otimes \phi &\mapsto tv \otimes t^*\phi. \end{aligned}$$

As follows from Proposition 2.4.3, the infinitesimal transvections are extremal if $\mathbb{K}$ is a field. Moreover, the exponent of an infinitesimal transvection is a **transvection**, i.e, a linear transformation of the form $T_{v,\phi} := 1 + t_{v,\phi}$ where $\phi(v) = 0$ and v, ϕ nonzero. A **transvection group** is a group of the form $\{1 + \lambda t_{v,\phi} | \lambda \in \mathbb{F}\}$. The actions of a transvection on V^* and $V \otimes V^*$ are then given by

$$\begin{aligned} T^*_{v,\phi} : V^* \longrightarrow & V^* \\ \psi \mapsto & \psi + t^*_{v,\phi}\psi = \psi - \phi\psi(v) \end{aligned}$$

and

$$\begin{aligned} T^{\otimes}_{v,\phi} : V \otimes V^* \longrightarrow & V \otimes V^* \\ (w \otimes \psi) \mapsto & \big(w + t_{v,\phi}(w)\big) \otimes \big(\psi + t^*_{v,\phi}(\psi)\big) \\ & = \big(w + \phi(w)v\big) \otimes \big(\psi - \phi\psi(v)\big). \end{aligned}$$

This defines an action of the group generated by all transvections on $V \otimes V^*$, respecting the Lie product.
In case the characteristic of $\mathbb{F}$ is not 2, we can also associate an invertible linear map to an infinitesimal reflection $t_{v,\phi}$, namely the **reflection** $R_{v,\phi} := 1 - t_{v,\phi}$. Such reflections are studied in [CCS99].

2.4.2. Special linear Lie algebras. Let V be a vector space over the field $\mathbb{F}$. If V is of finite dimension, then $\mathfrak{sl}(V)$ is the subalgebra of $\mathfrak{gl}(V)$ of elements of trace 0. This subalgebra is the Φ-image of the subspace $\mathfrak{g}_0(V, V^*)$ of $V \otimes V^*$ generated by the elements

$$v \otimes \phi \in V \otimes V^* \text{ with } \phi(v) = 0.$$

Indeed, each such element is mapped to an element in $\mathfrak{sl}(V)$. Moreover, if $v_1, \ldots, v_n$ is a basis of V with dual basis $\phi_1, \ldots, \phi_n$, it is readily seen that the elements $v_i \otimes \phi_j$ and $v_k + v_{k+1} \otimes \phi_k - \phi_{k+1}$ with $1 \leq i \neq j \leq n$ and $1 \leq k \leq n-1$ form an independent set. Thus, the image of Φ is of dimension at least $n^2 - 1$ and hence $\mathfrak{sl}(V)$.

As it is well known, the algebra $\mathfrak{sl}(V)$ and hence $\mathfrak{g}_0(V, V^*)$ is, up to its center, simple.

If V, however, is infinite dimensional, we encounter more simple Lie algebras in the following way.

Let W^* be a subspace of V^*. Then we can consider $\mathfrak{fgl}(V, W^*)$ (and $\mathfrak{fsl}(V, W^*)$) to be the subalgebras of $\mathfrak{fgl}(V)$ generated by the elements $t_{v,\phi}$ with $v \in V$ and $\phi \in W^*$ (and $\phi(v) = 0$), i.e., the Φ-image of $\mathfrak{g}(V, W^*)$ (or $\mathfrak{g}_0(V, W^*)$, respectively).

Let U be a subspace of the annihilator $\mathrm{Ann}_V(W^*) := \{u \in V \mid \psi(u) = 0 \text{ for all } \psi \in W^*\}$. Then for $u \otimes \psi$ with $u \in U$ we have $[v \otimes \phi, u \otimes \psi] = -\psi(v) u \otimes \phi$. Thus, if $\{0\} \neq U \neq V$ we find $\langle u \otimes \psi \mid u \in U, \psi \in W^* \rangle$ to be a proper ideal in $\mathfrak{g}(V, W^*)$. However, if $\mathrm{Ann}_V(W^*) = 0$, then the algebra $\mathfrak{fsl}(V, W^*)$ for infinite dimensional V is also simple.

2.4.3. Symplectic Lie algebras. Now suppose (V, f) is a symplectic space, then the (finitary) symplectic Lie algebra $\mathfrak{fsp}(V, f)$ is the image under Φ of the subalgebra spanned by the elements $v \otimes f(v, \cdot) \in \mathfrak{g}$. This follows from the following two results:

LEMMA 2.4.7. *Let $f : V \times V \to \mathbb{F}$ be a symplectic form. Then with $t_v := \Phi(v \otimes f(v, \cdot))$ as defined before, we have $f(t_v(w), u) = -f(w, t_v(u))$ for all $u, v, w \in V$.*

PROOF. Let $u, v, w \in V$. Then by definition of Φ, we have

$$\begin{aligned} f(t_v(w), u) =& f(v f(v, w), u) = f(v, w) f(v, u) = -f(w, v) f(v, u) \\ =& - f(w, f(v, u) v) = -f(w, t_v(u)). \end{aligned} \qquad \diamond$$

This lemma implies that the elements t_v with $0 \neq v \in V$ are in $\mathfrak{sp}(V, f)$. That these elements generate the finitary part of this algebra follows from the next result.

PROPOSITION 2.4.8. *Let (V, f) be a nondegenerate symplectic space over the field $\mathbb{F}$. Then the finitary symplectic Lie algebra $\mathfrak{fsp}(V)$ is generated by its extremal elements t_v, with $0 \neq v \in V$.*

Proof. Let $\mathfrak{g} = \langle t_v \mid v \in V \setminus \{0\}\rangle$. By the above lemma we have $\mathfrak{g} \subseteq \mathfrak{fsp}(V)$. We will prove equality.
First assume that V has finite dimension $n = 2m$.
Let $v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m}$ be a hyperbolic basis for V with $f(v_i, v_j) = 1$ for $1 \leq i \leq m$ and $j = m + i$ and 0 otherwise. Then consider the elements t_{v_i} together with the elements $t_{v_i+v_j}$ where $i < j$ are in $\{1, \ldots, 2m\}$. These elements are linearly independent. Indeed, suppose

$$\sum_{i=1}^{2m} \lambda_i t_{v_i} + \sum_{1 \leq i < j \leq 2m} \lambda_{ij} t_{v_i+v_j} = 0.$$

Then evaluating this element in v_l, with $1 \leq l \leq m$ yields

$$\lambda_{l+m} v_{l+m} + \sum_{k<l+m} \lambda_{k(l+m)}(v_k + v_{l+m}) + \sum_{k>l+m} \lambda_{(l+m)k}(v_k + v_{l+m}) = 0.$$

We deduce that $\lambda_{k(l+m)} = 0$ and $\lambda_{(l+m)k'} = 0$ for $k < l + m$ and $k' > l + m$, as well as $\lambda_{l+m} = 0$.
Evaluating the above map element in v_l, where $m < l \leq 2m$, yields $\lambda_{k(l-m)} = 0$ and $\lambda_{(l-m)k'} = 0$ for $k < l - m$ and $k' > l - m$, as well as $\lambda_{l-m} = 0$.
Thus indeed, the elements t_{v_i} together with the elements $t_{v_i+v_j}$ where $i < j$ are in $\{1, \ldots, 2m\}$ form an independent set. This implies that the subspace $\mathfrak{g}$ has dimension $m(2m + 1)$ and hence equals $\mathfrak{sp}(V)$.
Now assume that V is infinite dimensional and $x \in \mathfrak{fsp}(V)$. Then there is a nondegenerate finite dimensional subspace V_0 of V such that $x \in \mathfrak{sp}(V_0)$. So, by the above, $x \in \mathfrak{g}$. This proves the proposition. $\diamond$

The elements t_v are called **symplectic infinitesimal transvections**. Notice that exponentiation of symplectic infinitesimal transvections leads to so called **symplectic transvections**. on V.

2.4.4. Unitary Lie Algebras. Let V be a (left) vector space over the skew field $\mathbb{K}$. Let h be a skew-Hermitian form on V with respect to some anti-automorphism σ of $\mathbb{K}$. We consider elements of the form $v \otimes h(\cdot, v) \in \mathfrak{g}(V, V^*)$.

Lemma 2.4.9. *With $t_v := \Phi(v \otimes h(\cdot, v))$ we have $h(t_v(w), u) = -h(w, t_v(u))$ for all $u, v \in V$.*

Proof. Let $u, w \in V$. Then, as h is skew-Hermitian we get

$$h(t_v(w), u) = h(h(w, v)v, u) = h(w, v)h(v, u)$$

$$\begin{aligned} &= h(w, (h(v,u)v)^\sigma) = h(w, -h(u,v)v) \\ &= -h(w, t_v(u)). \end{aligned}$$ ◇

The elements t_v as defined above, with $v \neq 0$ a singular vector, are called **unitary infinitesimal transvections**. If v is nonsingular, we call t_v a **unitary infinitesimal reflection**.
For two unitary infinitesimal transvections t_v and t_w we have

$$[t_v, [t_v, t_w]] = -2h(v,w)h(w,v)t_v.$$

So, if we consider $\mathfrak{u}(V,h)$ as an algebra over a field $\mathbb{F}$ inside $\mathbb{K}_\sigma$, then for t_v to be an extremal element we should have $h(v,w)h(w,v) \in \mathbb{F}$ for all t_w. In this case, an extremal form g can be defined on $\mathfrak{u}(V,h)$ via

$$g(t_v, t_w) = -h(v,w)h(w,v).$$

Proposition 2.4.10. *Let h be a nondegenerate skew-Hermitian form on the vectors space V over the field $\mathbb{K}$ with respect to the field automorphism σ of order 2. The Lie algebra $\mathfrak{fu}(V,h)$ over $\mathbb{K}_\sigma$ is generated by its elements t_v, where $v \in V$.*

Proof. We first consider the case where $\dim(V) = n < \infty$. In this case it is well known that $\mathfrak{u}(V,h)$ has dimension n^2, see 1.2.5.
Let $v_1, \ldots v_n$ be a basis such that the matrix of the form h with respect to this basis equals $H = \begin{pmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & \lambda I_m \end{pmatrix}$, where $2m + k = n$, and $\lambda \in \mathbb{F}$ with $\lambda^\sigma = -\lambda$.
Now consider the elements t_{v_i}, $t_{v_i+v_j}$ and $t_{v_i+\mu v_j}$, where $1 \leq i < j \leq n$ and $\mu \in \mathbb{F}$ a fixed element with $\mu^\sigma \neq \mu$. (If the characteristic of $\mathbb{F}$ is different from 2, we can take $\mu = \lambda$.)
As in the symplectic case, we easily verify that these n^2 elements form an independent set in $\mathfrak{u}(V,h)$. So, as the dimension of $\mathfrak{u}(V,h)$ equals n^2, we have shown that $\mathfrak{u}(V,h)$ is generated by its infinitesimal transvections and reflections.
If V has infinite dimension, the result follows like in the linear and symplectic case. ◇

In general, a unitary space does not necessarily contain singular vectors. But when it does, $\mathfrak{fsu}(V,h)$ is generated by its infinitesimal transvections. Indeed,

in view of the above result and the fact that $\mathfrak{su}(V,h)$ has codimension 1 in $\mathfrak{u}(V,h)$, it suffices to prove that $\mathfrak{u}(V,h)$ can be generated by all its infinitesimal transvections (if they exist) and a unique reflection.
This is clearly true in the case where (V,h) is a hyperbolic 2-space over the field $\mathbb{K}$. Indeed, if v_1, v_2 is a hyperbolic basis of V, then $t_{v_1}, t_{v_2}, t_{v_1+v_2}$ generate $\mathfrak{su}(V,h)$ and together with any infinitesimal reflection they generate $\mathfrak{u}(V,h)$.
So, to prove in general that $\mathfrak{fu}(V,h)$ can be generated by its infinitesimal transvections together with one reflection, it suffices to prove connectedness of the graph Γ on the nonsingular points of V, where two such points are adjacent if and only if they span a hyperbolic line (i.e., 2-space).
We will prove this to be true for skew-Hermitian forms h.

Proposition 2.4.11. *Suppose (V,h) is a nondegenerate skew-Hermitian space over the field $\mathbb{K}$, containing isotropic 1-spaces. Then the graph Γ, as defined above, is connected.*

Proof. A 2-space W of V can be singular, hyperbolic, tangent or anisotropic. Here singular means that it only consists of singular vectors, tangent means it contains a unique singular 1-space (which is in the radical of $h|_W$) and all other 1-spaces are nonsingular, and anisotropic means that all 1-spaces are nonsingular.
As V contains an isotropic vector $v \neq 0$, there is also a vector $w \in V$ with $h(v,w) \neq 0$. Then the 2-space $L = \langle v, w\rangle$ is hyperbolic. Without loss of generality we can assume that v, w is a hyperbolic basis of L.
Let u be a nonsingular vector in V. We will prove that $\langle u\rangle$ is in the same connected component of Γ as some nonsingular point on $\langle v, w\rangle$.
First assume that u is perpendicular to both v and w, i.e, $h(u,v) = h(u,w) = 0$. Let $u' = u + \lambda v$ be a nonsingular vector on $\langle u, v\rangle$ not in $\langle u\rangle$. Then $h(u', w) \neq 0$, and $\langle u', w\rangle$ is hyperbolic. In particular, we find a singular vectors $w' = u + \lambda v + \mu w$ in $\langle u', w\rangle$, which are not scalar multiples of w. For each such w' the 2-space $\langle u, w'\rangle$ is hyperbolic.
Then, for $\mu \in \mathbb{K}$ satisfying $\mu - \mu^\sigma = h(u,u)$ we have $h(u+v+\mu w, u+v+\mu w) = h(u,u) + \mu^\sigma - \mu = 0$ and $h(v + \mu w, v + \mu w) = \mu^\sigma - \mu = -h(u,u) \neq 0$. Such μ exists, as $\{\mu - \mu^\sigma \in \mathbb{K} \mid \mu \in \mathbb{K}\} = \{\mu\mu^\sigma \mid \mu \in \mathbb{K}\}$, see [BC13]. Moreover, as $h(u, u+v+\mu w) = h(u,u) \neq 0$, we find the 2-space $\langle u, v+\mu w\rangle$ to be hyperbolic. This implies that $\langle u\rangle$ is adjacent to $\langle v + \mu w\rangle$.

Now assume that $h(u,v) \neq 0$ but $h(u,w)=0$. After scaling v (and w), we can assume that $h(u,u)-h(u,v) \neq 0$ Let $\mu \in \mathbb{K}$ with $h(u,u)+(\mu-h(u,v))^\sigma-(\mu-h(u,v))=0$. Then $h(u+v+\mu w, u+v+\mu w) = h(u,u)+h(u,v)-h(v,u)+\mu^\sigma-\mu = h(u,u)+(\mu-h(u,v))^\sigma-(\mu-h(u,v))=0$. Moreover, $h(u,u+v+\mu w) = h(u,u)-h(u,v) \neq 0$. So, $\langle u, u+v+\mu w\rangle$ is hyperbolic and meets $\langle v,w\rangle$ in $v+\mu w$, which is nonsingular, as $h(v+\mu w, v+\mu w) = \mu-\mu^\sigma = h(u,u)$.
Again, we find that $\langle u\rangle$ is adjacent to a nonsingular point on $\langle v,w\rangle$.
Finally assume that $h(u,v) \neq 0$ and $h(u,w) \neq 0$. Let $u' \in \langle u,v\rangle$ be perpendicular to w. If u' is nonsingular, then $\langle u\rangle$ is adjacent to $\langle u'\rangle$, and the latter is, by the above adjacent to some nonsingular point in $\langle v,w\rangle$. Thus, assume u' is singular. Then $\langle u',w\rangle$ is singular, and as u is perpendicular to at most one point on $\langle u',w\rangle$, all lines on $\langle u\rangle$, except for maybe one, are hyperbolic. This clearly implies that there is at least one hyperbolic line on $\langle u\rangle$ meeting $\langle v,w\rangle$ in a nonsingular point. This finishes the proof. $\diamond$

As explained above, this result implies the following.

Proposition 2.4.12. *Suppose (V,h) is a nondegenerate unitary space over a field $\mathbb{K}$ containing isotropic points. Then $\mathfrak{fsu}(V,h)$ is generated by its infinitesimal transvections.*

2.4.5. Orthogonal Lie algebras. For the **orthogonal Lie algebras**, we use the following form:

Lemma 2.4.13. *Let $B: V\times V \to \mathbb{F}$ be a symmetric bilinear form. Then with*

$$S = \Phi(v \otimes B(w,\cdot) - w \otimes B(v,\cdot))$$

we have $B(S(x),y) = -B(x,S(y))$ for $u,v,x,y \in V$.

Proof. Let $u,v,x,y \in V$, then we have:

$$\begin{array}{rcl} B(S(x),y) = & B(vB(w,x)-wB(v,x),y) \\ = & B(w,x)B(v,y)-B(v,x)B(w,y) \\ = & B(x,B(v,y)w)-B(x,B(w,y)v) \\ = & B(x,B(v,y)w-B(w,y)v) \\ = & B(x,-S(y)) \\ = & -B(x,S(y)). \end{array}.$$

$\diamond$

The condition for the element to be in $\mathfrak{sl}_n(\mathbb{F})$ is $B(w,v)+B(v,w)=0$. So, if $\mathrm{char}(\mathbb{F}) \neq 2$ we have $B(v,w)=0$. We obtain the full orthogonal Lie algebra this way as the image of Φ.
To consider the two orthogonal types of Lie algebras and their extremal elements, we need a new type of linear transformations. Let therefore in the following the vector space V be of dimension $2n$ or $2n-1$, where $n \in \mathbb{N}$.

DEFINITION 2.4.14. Let B denote the orthogonal bilinear form as in 2.4.13, and $u, v \in V$ two linearly independent vectors such that $B(u,u) = B(u,v) = B(v,u) = B(v,v) = 0$. Then,

$$T_{u,v} : V \to V, \quad x \mapsto x + B(u,x)v - B(v,x)u \tag{2.24}$$

is called the **Siegel transvection** with respect to $u, v \in V$. In this case, the map $t_{u,v} = T_{u,v} - 1$ is called **infinitesimal Siegel transvection**. The group $\langle 1 + s \cdot t_{u,v} | s \in \mathbb{F}\rangle$ is the **Siegel transvection group**.

LEMMA 2.4.15. *The Lie algebra $\mathfrak{so}(V)$ is spanned by infinitesimal Siegel transvections.*

PROOF. This can be proven as we did before, for example in the symplectic case. For a proof, see [Pos07, Lemma 2.39]. ◇

LEMMA 2.4.16. *Infinitesimal Siegel transvections are extremal elements in $\mathfrak{so}(V)$ with*

$$g(t_{u,v}, t_{v,x}) = (B(u,w)B(v,x) - B(u,x)B(v,w)) \tag{2.25}$$

for $u, v, w, x \in V$ as in 2.4.14. Moreover, we have

$$\exp(t_{u,v}, s)t_{w,x} = t_{w+st_{u,v}(w), w+st_{u,v}(x)}. \tag{2.26}$$

PROOF. For the extremality, let $a \in V$ be any vector. Then

$$\begin{aligned}
&[t_{u,v}, t_{w,x}](a) \\
&\quad = \big(t_{u,v}t_{w,x} - t_{w,x}t_{u,v}\big)(a) \\
&\quad = t_{u,v}\big(B(w,a)x - B(x,a)w\big) - t_{w,x}\big(B(u,a)v - B(v,a)u\big) \\
&\quad = B(w,a)t_{u,v}(x) - B(x,a)t_{u,v}(w) - B(u,a)t_{w,x}(v) + B(v,a)t_{w,x}(u) \\
&\quad = B(w,a)\big(B(u,x)v - B(v,x)u\big) - B(x,a)\big(B(u,w)v - B(v,w)u\big) \\
&\qquad - B(u,a)\big(B(w,v)x - B(x,v)w\big) + B(v,a)\big(B(w,u)x - B(x,u)w\big)
\end{aligned}$$

$$
\begin{aligned}
&= B(u,x)\big(B(w,a)v - B(v,a)w\big) + B(v,x)\big(B(u,a)w - B(w,a)u\big) \\
&\quad + B(u,w)\big(B(v,a)x - B(x,a)v\big) + B(v,w)\big(B(x,a)u - B(u,a)x\big) \\
&= B(u,x)t_{w,v}(a) + B(v,x)t_{u,w}(a) + B(u,w)t_{v,x}(a) - B(v,w)t_{u,x}(a) \\
&= \big(B(u,x)t_{w,v} + B(v,x)t_{u,w} + B(u,w)t_{v,x} - B(v,w)t_{u,x}\big)(a)
\end{aligned}
$$

using the fact that $t_{x,u} = -t_{u,x}$.
With this and with $B(u,u) = B(u,v) = B(v,v) = 0$ and $t_{u,u} = t_{v,v} = 0$, we can deduce

$$
\begin{aligned}
&\big[t_{u,v},[t_{u,v},t_{w,x}]\big] \\
&\quad = \big[t_{u,v}, B(u,x)t_{w,v} - B(v,w)t_{u,x} + B(v,x)t_{u,w} + B(u,w)t_{v,x}\big] \\
&\quad = B(u,x)\big[t_{u,v},t_{w,v}\big] - B(v,w)\big[t_{u,v},t_{u,x}\big] \\
&\qquad + B(v,x)\big[t_{u,v},t_{u,w}\big] + B(u,w)\big[t_{u,v},t_{v,x}\big] \\
&\quad = B(u,x)\big(B(u,v)t_{w,v} - B(v,w)t_{u,v} + B(v,v)t_{u,w} + B(u,w)t_{v,v}\big) \\
&\qquad - B(v,w)\big(B(u,x)t_{u,v} - B(v,u)t_{u,x} + B(v,x)t_{u,u} + B(u,u)t_{v,x}\big) \\
&\qquad + B(v,x)\big(B(u,w)t_{u,v} - B(v,u)t_{u,w} + B(v,w)t_{u,u} + B(u,u)t_{v,w}\big) \\
&\qquad + B(u,w)\big(B(u,x)t_{v,v} - B(v,v)t_{u,x} + B(v,x)t_{u,v} + B(u,v)t_{v,x}\big) \\
&\quad = - B(u,x)B(v,w)t_{u,v} - B(v,w)B(u,x)t_{u,v} \\
&\qquad + B(v,x)B(u,w)t_{u,v} + B(u,w)B(v,x)t_{u,v} \\
&\quad = 2\big(B(u,v)B(v,x) - B(u,x)B(v,w)\big)t_{u,v},
\end{aligned}
$$

so $g(t_{u,v},t_{v,x}) = (B(u,w)B(v,x) - B(u,x)B(v,w))$.
Using these equalities, we get for any $s \in \mathbb{F}$ and any $a \in V$:

$$
\begin{aligned}
&\big(\exp(t_{u,v},s)t_{w,x}\big)(a) \\
&\quad = t_{w,x}(a) + s\big(B(u,x)t_{w,v}(a) - B(v,w)t_{u,x}(a) + B(v,x)t_{u,w}(a) \\
&\qquad + B(u,w)t_{v,x}(a)\big) + s^2 t_{u,v}(a)\big(B(u,w)B(v,x) - B(u,x)B(v,w)\big) \\
&\quad = t_{w,x}(a) + t_{s(B(u,w)v - B(v,w)u),x}(a) + t_{w,s(B(u,x)v-B(v,x)u)}(a) \\
&\qquad + s^2 t_{B(u,w)v-B(v,w)u,B(u,x)v-B(v,x)u}(a) \\
&\quad = t_{w,x}(a) + t_{st_{u,v}(w),x}(a) + t_{w,st_{u,v}(x)}(a) + t_{st_{u,v}(w),st_{u,v}(x)}(a) \\
&\quad = t_{w+st_{u,v}(w),x+st_{u,v}(x)}(a).
\end{aligned}
$$

◇

2.5. The $\mathfrak{sl}_2$-relation

As in the previous sections, let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F}$ generated by its set E of extremal elements. Let g be an extremal form on $\mathfrak{g}$.
We first analyse what subalgebras of $\mathfrak{g}$ two extremal elements elements in E generate.

PROPOSITION 2.5.1. *Let $x, y \in E$ and $\langle x, y\rangle$ the subalgebra of $\mathfrak{g}$ generated by x and y. Then exactly one of the following assertions holds:*

(1) *$g(x, y) = 0$ and $\langle x, y\rangle = \mathbb{F}x + \mathbb{F}y$ is abelian.*
(2) *$g(x, y) = 0$, and $\langle x, y\rangle = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$, where $z = [x, y] \neq 0$, and E contains all elements of $\langle x, y\rangle \setminus \langle z\rangle$.*
(3) *$g(x, y) \neq 0$, the subalgebra $\langle x, y\rangle$ equals $\mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ and is isomorphic to $\mathfrak{sl}_2$. The set E contains all elements that are mapped by this isomorphism onto infinitesimal transvections of $\mathfrak{sl}_2$.*

PROOF. We define $[x, y] := z$ and distinguish three cases:

(1) $z = 0$, $g(x, y) = 0$.
We know from 2.3.1 that $\mathfrak{g}$ is spanned by x and y, so $\mathfrak{g} = \mathbb{F}x + \mathbb{F}y$, and $\mathfrak{g}$ is abelian.
(2) $z \neq 0$ and the extremal form $g(x, y) = 0$.
Clearly $\mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ is closed under multiplication with x and y. So, $\langle x, y\rangle = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$. Now, for all $\lambda \in \mathbb{F}$ we find $\exp(x, \lambda)(y) = y + \lambda \cdot [x, y] + \lambda^2 g_x(y)x = y + \lambda \cdot z$ and $\exp(y, \lambda)(x) = x + \lambda \cdot [x, y] + \lambda^2 g_y(x)x = x + \lambda \cdot z$ to be extremal.
(3) $g(x, y) \neq 0$.
As $\mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ is closed under multiplication with x and y, we do have $\langle x, y\rangle = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$. Without loss of generality we can assume $g(x, y) = 1$. Now consider $\mathfrak{g}(V)$, where V is a 2-dimensional vector space over $\mathbb{F}$ with basis v_1, v_2 and dual basis ϕ_1, ϕ_2 . Let $\hat{x} = v_1 \otimes \phi_2$ and $\hat{y} = v_2 \otimes \phi_1$. Then, with $\hat{z} = [\hat{x}, \hat{y}]$ we find that the structure constants of x, y, z and $\hat{x}, \hat{y}, \hat{z}$ are the same. So, we have $\langle x, y\rangle \cong \mathfrak{g}_0(V) \cong \mathfrak{sl}_2$. Under this isomorphism we find that the element $\exp(x, s)y$ is mapped to $t_{v_2+sv_1,\phi_1-s\phi_2}$. This implies that all elements that are mapped to infinitesimal transvections are in E. $\diamond$

REMARK 2.5.2. Notice that in the above proposition $E \cap \langle x, y\rangle$ may contain more elements than those indicated. Indeed, all non-zero elements of $\langle x, y\rangle$

might be extremal in the first and second case. However, in case $\langle x, y\rangle \cong \mathfrak{sl}_2$ and the characteristic of $\mathbb{F}$ is not 2, there are no other extremal elements in $\langle x, y\rangle$.

DEFINITION 2.5.3. On the set of extremal points $\mathcal{E}$ of $\mathfrak{g}$, we define the relation

$$x \sim_{\mathfrak{sl}_2} y :\iff g_{x_1}(y_1) \neq 0 \iff g(x_1, y_1) \neq 0$$

for some extremal elements $x_1 \in \langle x\rangle$ and $y_1 \in \langle y\rangle$ with $x, y \in \mathcal{E}$. This is, in case $\mathbb{F}$ is not of characteristic 2, equivalent with saying that $\langle x_1, y_1\rangle \cong \mathfrak{sl}_2$. This relation defines a graph structure on $\mathfrak{g}$ by taking the point set $\mathcal{E}$ as the set of vertices and define two points $x, y \in \mathcal{E}$ as adjacent if and only if $x \sim_{\mathfrak{sl}_2} y$. We denote the graph $(\mathcal{E}, \sim_{\mathfrak{sl}_2})$ by $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ or, if $\mathfrak{g}$ is clear from the context, just by $\Gamma_{\mathfrak{sl}_2}$.

In this section, we relate properties of the graph $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ to properties of $\mathfrak{g}$.

By abuse of notation, in the following we will not distinguish between the extremal element $x \in E$ and the corresponding extremal point $\langle x\rangle$ in $\mathcal{E}$, and denote both by x, if it is clear from the context what x refers to.

LEMMA 2.5.4. *Let $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ have at least two connected components Γ_1 and Γ_2 with corresponding point sets $\mathcal{E}_1$ and $\mathcal{E}_2$. Then, we have*

$$[x, y] \in \mathrm{rad}(g)$$

for all $x \in \mathcal{E}_1$ and $y \in \mathcal{E}_2$.

PROOF. Let $z = [x, y] \neq 0$. Assume that $z \notin \mathrm{rad}(g)$, so let $u \in \mathcal{E}$ be such that $g(u, z) = g_u(z) \neq 0$. We consider the 2-dimensional space $\langle x, z\rangle$. By Proposition 2.5.1 all 1-spaces in $\langle x, z\rangle$ except for possibly $\langle z\rangle$ are extremal points. If there are two extremal elements x_1, x_2 in $\langle x, z\rangle$ with $g_u(x_1) = 0 = g_u(x_2)$, then by linearity of g, also $g_u(z) = 0$, so all extremal points except for maybe one in $\langle x, z\rangle$ must be in $\mathfrak{sl}_2$-connection with u. So choose such an element $x_3 \notin \langle z\rangle$ with $g(x_3, u) \neq 0$. Now fix an element $a \in \exp(y)$ with $x_3^a = x$ and an element $b \in \exp(x)$ with $y_3^b = y$. Then u^{ab} is connected to $x_3^{ab} = x^b = x$ and to $y_3^{ab} = y_3^b = y$ proving x and y to be in the same connected component of $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$. This is a contradiction. So indeed $z \in \mathrm{rad}(g)$. ◇

This leads to the following consequences:

COROLLARY 2.5.5. *Let E_0 be a subset of extremal elements in $\mathfrak{g}$ and $\mathcal{E}_0$ the corresponding set of extremal points. Further assume $\langle E_0 \rangle = \mathfrak{g}$ and $\mathcal{E}_0$ is a connected component of the $\mathfrak{sl}_2$-graph of $\mathfrak{g}$. If $E_1 := E \setminus E_0$ is non-empty, it consists of sandwich elements in $\mathfrak{g}$ and $\langle E_1 \rangle$ is an ideal of $\mathfrak{g}$ contained in* $\mathrm{rad}(g)$.

PROOF. For all $x \in E_0$ and $y \in E_1$ we have $g(x, y) = g(y, x) = 0$. Now since $\mathfrak{g} = \langle x | x \in E_0 \rangle$, we have $g(y, z) = 0$ for all $z \in \mathfrak{g}$. So, $y \in \mathrm{rad}(g)$. But then,

$$[y, [y, z]] = 2g_y(z)y = 0$$

and by (2.3)

$$[y, [x, [y, z]]] = g_y([x, z])y - g_y(z)[y, x] - g_y(x)[y, z] = 0$$

for all $z \in \mathfrak{g}$. This implies that y is a sandwich.
Finally notice that for all $x \in E_0$ and $y \in E_1$ we have $[x, y] = \exp(x)y - y \in \langle E_1 \rangle$. So, $\langle E_1 \rangle$ is an ideal contained in $\mathrm{rad}(g)$.

◇

COROLLARY 2.5.6. *If $\mathfrak{g}$ is simple and the bilinear form g is not trivial, then $\mathcal{E}$ is connected with respect to the relation $\sim_{\mathfrak{sl}_2}$. In particular, then the group $G = \langle \exp(x, t) \mid x \text{ extremal}, \ t \in \mathbb{F} \rangle$ is transitive on the points in $\mathcal{E}$.*

PROOF. Suppose $\mathfrak{g}$ is simple and the bilinear form g is not trivial. Then, by Lemma 2.3.5 we find that $\mathrm{rad}(g) = 0$. But then Lemma 2.5.4 implies that $\mathcal{E}$ is connected with respect to the relation $\sim_{\mathfrak{sl}_2}$.
Now let $x, y \in E$ with $(x, y) \in E_2$, and without loss of generality, assume $g_x(y) = g_y(x) = 1$. Then $\exp(x, 1)y = x + [x, y] + y = y - [y, x] + x = \exp(y, -1)x$. So, $\exp(y, 1)\exp(x, 1)y = x$ and the elements $\langle x \rangle, \langle y \rangle \in \mathcal{E}$ are in the same orbit under the automorphism group. Now, the connectedness of $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ implies that $\mathcal{E}$ is one G-orbit. ◇

THEOREM 2.5.7. *Suppose* $\mathrm{rad}(g) = 0$ *and the characteristic of* $\mathbb{F}$ *is not* 2. *Then $\mathfrak{g}$ is a direct sum of simple Lie subalgebras.*

PROOF. By Lemma 2.5.4 we find that $\mathfrak{g}$ can be written as the direct sum of Lie subalgebras, each generated by a connected class of extremal elements. Let $\mathfrak{g}_1$ be such summand generated by its extremal elements in the set $\mathcal{E}_1$ and suppose $\mathcal{I}$ is a nonzero ideal of $\mathfrak{g}_1$. Let $0 \neq i \in \mathcal{I}$. Then, as $\mathrm{rad}(g) = 0$, there is an element $x \in \mathfrak{g}_1$ with $g_x(i) \neq 0$. Moreover, as $\mathfrak{g}_1$ is generated by $\mathcal{E}_1$, we can assume this x to be an element in $\mathcal{E}_1$. So, since the characteristic of $\mathbb{F}$ is not 2, we have $x \in [x,[x,i]] \subseteq \mathcal{I}$. But then also each element $y \in \mathcal{E}_1$ with $g_x(y) \neq 0$ is in $\mathcal{I}$ and by connectedness of $\mathcal{E}_1$ with respect to $\sim_{\mathfrak{sl}_2}$, we find $\mathcal{I} = \mathfrak{g}_1$. $\diamond$

CHAPTER 3

Chevalley algebras

In this chapter, we consider Lie algebras with a Chevalley basis, which has the property that all structure constants are integers. We give concrete multiplication tables for this type of Lie algebras. This enables us to find their extremal elements, as defined in the previous chapter, and the extremal form, in a very concrete way.
We start with the introduction of the basic geometric concept of root systems, which leads to the definition of the well-known Dynkin diagrams. So we find the geometric motivation for the different types of Chevalley algebras, that we introduce in the second section, and see some examples. Finally, we finish the chapter with the proof of our result about the extremal elements in Chevalley algebras: They are, with some exceptions, exactly the long root elements with respect to the underlying root system of the Lie algebra. The descriptions in the first two sections of this chapter are based on Carter [Car72], Buekenhout-Cohen [BC13] and Roozemond [Roo10].

3.1. Root systems

Let V be a Euclidean space of finite dimension $n \in \mathbb{N}$ and let (v, w) denote the value of the inner product on V, for $v, w \in V$. A **reflection** in V is an invertible linear transformation that leaves some hyperplane fixed (pointwise) and sends any vector orthogonal to that hyperplane to its negative. Obviously, every reflection preserves the inner product on V, so it is **orthogonal**.

Definition 3.1.1. For a nonzero vector $\alpha \in V$, the reflection σ_α with root α is given by the **reflecting hyperplane**

$$H_\alpha = \{\beta \in V \mid (\beta, \alpha) = 0\}.$$

So, we see that nonzero vectors proportional to α yield the same reflection. An explicit formula for the reflection is:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

Since the number $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ occurs frequently, we denote it abbreviatory by $\langle\beta,\alpha\rangle$. Note that this is only linear in the first variable.
We define the **dual vector**

$$\alpha^* := \frac{2\alpha}{(\alpha,\alpha)}.$$

DEFINITION 3.1.2. A subset Φ of V is a **root system** if the following conditions hold:

(1) Φ is finite, does not contain zero and spans V.
(2) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.
(3) If $\alpha, \beta \in \Phi$, then $\langle\beta,\alpha\rangle \in \mathbb{Z}$.
(4) if $\alpha, t\alpha \in \Phi$ with $t \in \mathbb{R}$, then $t = \pm 1$.

The elements of a root system are called **roots**, and the rank of Φ is defined to be $\dim V$ and is denoted by $\operatorname{rk}\Phi$. For a root system Φ, its **dual** $\Phi^* = \{\alpha^* | \alpha \in \Phi\}$, is also a root system. A **set of fundamental roots** or **set of simple roots** is a subset $\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi$ that is a basis of V relative to which each $\alpha \in \Phi$ has a unique expression $\alpha = \sum s_i\alpha_i$ with s_i integers and all either nonnegative or nonpositive. Such a set of roots always exists; this is proven e.g. in [Car72, 2.1.3]. Those roots where all s_i are nonnegative are called **positive roots** and form the subset Φ^+ of Φ and those with all s_i nonpositive are **negative roots**, forming Φ^-. So of course, Φ^+ and Φ^- depend on the choice of Δ.
A root system Φ is **irreducible** if it cannot be partitioned into the union of two proper subsets $\Phi = \Phi_1 \cup \Phi_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

The **length** of a root $\alpha \in \Phi$ is its length in V. Since there are at most two different lengths of roots occurring in an irreducible root system (see e.g. [Hum78, 10.4 Lemma C]), it makes sense to divide the root system into **long roots** and **short roots**, denoted by Φ^{long} and Φ^{short}. By convention, if all roots have the same length, we call them all long roots. The **height** of a root $\alpha = \sum s_i\alpha_i$ relative to Δ is $\operatorname{ht}(\alpha) := \sum s_i$.
Two root systems Φ and Φ' corresponding to the Euclidian spaces V and V' are **isomorphic** if there exists a vector space isomorphism $\phi : V \to V'$ sending Φ to Φ' s.t. $\langle\phi(\beta), \phi(\alpha)\rangle = \langle\beta,\alpha\rangle$ for each pair of roots $\alpha, \beta \in \Phi$.

Let Φ be a root system of the vector space V. The **Weyl group** $W(\Phi)$ is the group generated by the reflections σ_α with $\alpha \in \Phi$. Because of the first two properties of root systems, $W(\Phi)$ permutes the elements of Φ and acts faithful on Φ, so we can identify $W(\Phi)$ with a subgroup of the symmetric group on Φ, and see that $W(\Phi)$ is finite.

In order to construct root systems, we find a few additional properties that can be deduced from the previous ones. For two independent roots $\alpha, \beta \in \Phi$, there always exist integers $p, q \geq 0$ such that $i\alpha + \beta \in \Phi$ for $-p \leq i \leq q$, but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ are no roots. We call the sequence

$$-p\alpha + \beta, \dots, \beta, \dots, q\alpha + \beta$$

the **α-chain through** β. The reflection σ_α from the Weyl group inverts each α-chain of roots. In particular it transforms $-p\alpha + \beta$ into $q\alpha + \beta$, so $-p\alpha + \beta, q\alpha + \beta$ are mirror images in the hyperplane orthogonal to α. Hence $((-p\alpha + \beta) + (q\alpha + \beta), \alpha) = 0$, which leads to

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = p - q.$$

If $\alpha_i, \alpha_j \in \Delta$ are two distinct fundamental roots, then $-\alpha_i + \alpha_j$ is not a root, so α_j is the first member of the α_i chain through α_j. With the previous relation and $p = 0$, we see that $(\alpha_i, \alpha_j) \leq 0$, so the angle $\theta_{i,j}$ between α_i, α_j is obtuse. There are just a few possibilities for the value of this angle. Since $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ and $2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ are both integers, we have

$$\frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} = 4\cos^2\theta_{i,j}.$$

Since $0 \leq \cos^2\theta_{i,j} \leq 1$, we have $4\cos^2\theta_{i,j} = 0, 1, 2, 3$ or 4, and together with the fact that $\theta_{i,j}$ is obtuse we have $\theta_{i,j} \in \{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi\}$. We can exclude the possibility $\theta_{i,j} = \pi$ since α_i, α_j are linearly independent. So we have

$$n_{i,j} := 4\cos^2\theta_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = 0, 1, 2, 3$$

is a product of two non-positive integers. The possible factorizations that arise in the different cases are the following:

(1) If $n_{i,j} = 1$, the factorization must be $1 = (-1)(-1)$, so $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$ and the roots α_i and α_j have the same length.
(2) If $n_{i,j} = 2$, the factorization must be $2 = (-1)(-2)$ and thus one of α_i, α_j is $\sqrt{2}$ times longer than the other.
(3) If $n_{i,j} = 3$, then the factorization is $3 = (-1)(-3)$ and one of α_i, α_j is $\sqrt{3}$ times longer than the other.

(4) If $n_{i,j} = 0$, no information about the relative length of the roots can be obtained.

To simplify handling roots of different lengths in the following, in those cases where two different root lengths occur we define the shorter one to have length 1 and the longer one to have length $\sqrt{2}, \sqrt{3}$, respectively, if not mentioned otherwise explicitly.
Regarding the defining properties of a root system, it is immediate that there is (up to isomorphism) just one root system of rank one. The irreducible root systems of higher rank are classified; we give them explicitly in the following. We consider in detail the irreducible root systems of rank two afterwards.
In order to describe the irreducible root systems it is convenient to use an orthonormal basis of the vector space containing the roots.

EXAMPLE 3.1.3. (1) Type A_n. Let $e_0, e_1, \ldots, e_n$ be an orthonormal basis of a Euclidian space of dimension $n+1$, and let V be the subspace of vectors

$$\sum_{i=0}^{n} \lambda_i e_i \quad \text{with} \quad \sum_{i=0}^{n} \lambda_i = 0.$$

The vectors $e_0 - e_1, e_1 - e_2, \ldots, e_{n-1} - e_n$ form a fundamental system of type A_n and the full system of roots with this fundamental system is given by

$$\Phi = \{e_i - e_j \mid i, j = 0, 1, \ldots, n, i \neq j\}.$$

For Examples 2, 3 and 4, assume that $e_1, e_2, \ldots, e_n$ is an orthonormal basis of a Euclidian space V of dimension n.

(2) Type B_n. The vectors $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n$ form a fundamental system of type B_n. The full root system is given by

$$\Phi = \{\pm e_i \pm e_j, \pm e_i \mid i, j = 1, \ldots, n, i \neq j\}.$$

(3) Type C_n. The vectors $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n$ form a fundamental system of type C_n. The full root system is given by

$$\Phi = \{\pm e_i \pm e_j, \pm 2e_i \mid i, j = 1, \ldots, n, i \neq j\}.$$

Note that in this case, we define the short roots to have length $\sqrt{2}$ and long ones length 2.

(4) Type D_n. The vectors $e_1-e_2, e_2-e_3, \ldots, e_{n-2}-e_{n-1}, e_{n-1}-e_n, e_{n-1}+e_n$ form a fundamental system of type D_n. The full root system is given by

$$\Phi = \{\pm e_i \pm e_j \mid i,j = 1,\ldots,n, i \neq j\}.$$

(5) Type E_8. A fundamental system of type E_8 is given by $e_1-e_2, e_2-e_3, e_3-e_4, e_4-e_5, e_5-e_6, e_6-e_7, e_6+e_7, -\frac{1}{2}\sum_{i=1}^{8} e_i$. The full root system is

$$\begin{aligned}\Phi = \; & \Big\{ \pm e_i \pm e_j \;\Big|\; i,j = 1,\ldots,8, i \neq j \Big\} \\ & \cup \Big\{ \frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i \;\Big|\; \varepsilon_i = \pm 1, \prod_{i=1}^{8} \varepsilon_i = 1 \Big\}.\end{aligned}$$

The systems E_7 and E_6 are easily obtainable as subsystems of E_8:

(6) Type E_7. Let $e_1, e_2, \ldots, e_8$ be as in the E_8 case. A fundamental system of type E_7 is given by $e_2-e_3, e_3-e_4, e_4-e_5, e_5-e_6, e_6-e_7, e_6+e_7, -\frac{1}{2}\sum_{i=1}^{8} e_i$.
The vectors lie in the subspace of elements $\sum_{i=1}^{8} \lambda_i e_i$ with $\lambda_1 = \lambda_8$. The full root system is

$$\begin{aligned}\Phi = \; & \Big\{ \pm (e_1+e_8), \pm e_i \pm e_j \;\Big|\; i,j = 2,\ldots,7, i \neq j \Big\} \\ & \cup \Big\{ \frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i \;\Big|\; \varepsilon_1 = \varepsilon_8 = 1, \varepsilon_i = \pm 1, \prod_{i=1}^{8} \varepsilon_i = 1 \Big\} \\ & \cup \Big\{ -\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i \;\Big|\; \varepsilon_1 = \varepsilon_8 = 1, \varepsilon_i = \pm 1, \prod_{i=1}^{8} \varepsilon_i = 1 \Big\}.\end{aligned}$$

(7) Type E_6. Let $e_1, e_2, \ldots, e_8$ be as in the E_8 case. A fundamental system of type E_6 is given by $e_3-e_4, e_4-e_5, e_5-e_6, e_6-e_7, e_6+e_7, -\frac{1}{2}\sum_{i=1}^{8} e_i$.
The vectors lie in the 6-dimensional subspace of elements $\sum_{i=1}^{8} \lambda_i e_i$ with $\lambda_1 = \lambda_2 = \lambda_8$. The full root system is

$$\begin{aligned}\Phi = \; & \Big\{ \pm e_i \pm e_j \;\Big|\; i,j = 3,\ldots,7, i \neq j \Big\} \\ & \cup \Big\{ \frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i \;\Big|\; \varepsilon_1 = \varepsilon_2 = \varepsilon_8 = 1, \varepsilon_i = \pm 1, \prod_{i=1}^{8} \varepsilon_i = 1 \Big\}\end{aligned}$$

$$\cup \Big\{ -\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i \;\Big|\; \varepsilon_1 = \varepsilon_2 = \varepsilon_8 = 1, \varepsilon_i = \pm 1, \prod_{i=1}^{8} \varepsilon_i = 1 \Big\}.$$

(8) Type F_4. Let e_1, e_2, e_3, e_4 be an orthonormal basis for V. A fundamental system of vectors for type F_4 consists of $e_1 - e_2$, $e_2 - e_3$, e_3, $\frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$.
The full root system is

$$\begin{aligned}\Phi = \;& \Big\{ \pm e_i \pm e_j, \pm e_i, \;\Big|\; i, j = 1, 2, 3, 4, i \neq j \Big\} \\ & \cup \Big\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \Big\}.\end{aligned}$$

(9) Type G_2. Let e_1, e_2, e_3 be an orthonormal basis for V. A fundamental system of vectors for type G_2 is $\{e_1 - e_2, -2e_1 + e_2 + e_3\}$. The full root system is

$$\begin{aligned}\Phi = \Big\{ & \pm (e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \pm(2e_1 - e_2 - e_3), \\ & \pm (-e_1 + 2e_2 - e_3), \pm(-e_1 - e_2 + 2e_3) \Big\}.\end{aligned}$$

Explicitly, we can illustrate the previously defined data regarding the irreducible root systems of rank two.

Example 3.1.4. The four possible irreducible root systems of rank two are shown in Figure 1. Table 1 gives the previously defined data of root diagrams for the cases A_2, B_2, C_2 and G_2. Hereby, we use a more familiar notation for the roots in the rank two cases, where the root system is spanned by two roots named α and β. The exact subscription to the notation previously given for the cases of higher rank is:

- A_2: $\alpha = e_0 - e_1, \beta = e_1 - e_2$.
- B_2: $\alpha = e_1 - e_2, \beta = e_2$.
- C_2: $\alpha = e_1 - e_2, \beta = 2e_2$.
- G_2: $\alpha = e_1 - e_2, \beta = 2e_1 + e_2 + e_3$.

Definition 3.1.5. Let Φ be a root system, $W(\Phi)$ (or abbreviatory W) the corresponding Weyl group and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a set of fundamental roots. A **Coxeter system** is a pair (W, S), where $S := \{\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_n}\}$ a set of reflections on the roots in Φ.
The **Cartan matrix** of Φ follows from the Coxeter system by taking the $n \times n$-matrix with entry $\langle \alpha_i, \alpha_j^* \rangle$ at position (i, j). There is a relation between

	A_2	B_2	G_2
Φ	$\pm\alpha, \pm\beta, \pm\alpha+\beta$	$\pm\alpha, \pm\beta, \pm\alpha+\beta,$ $\pm\alpha+2\beta$	$\pm\alpha, \pm\beta, \pm\alpha+\beta,$ $\pm 2\alpha+\beta, \pm 3\alpha+\beta,$ $\pm 3\alpha+2\beta$
Δ	α, β	α, β	α, β
Φ^{long}	Φ	$\pm\alpha, \pm\alpha+2\beta$	$\pm\beta, \pm 3\alpha+\beta,$ $\pm 3\alpha+2\beta$
Φ^{short}	$\emptyset$	$\pm\beta,\ \pm(\alpha+\beta)$	$\pm\alpha, \pm\alpha+\beta,$ $\pm 2\alpha+\beta$
height	$\text{ht}(\pm\alpha) = \pm 1$ $\text{ht}(\pm\beta) = \pm 1,$ $\text{ht}(\pm(\alpha+\beta)) = \pm 2$	$\text{ht}(\pm\alpha) = \pm 1$ $\text{ht}(\pm\beta) = \pm 1$ $\text{ht}(\pm(\alpha+\beta)) = \pm 2,$ $\text{ht}(\pm(\alpha+2\beta)) = \pm 3$	$\text{ht}(\pm\alpha) = \pm 1$ $\text{ht}(\pm\beta) = \pm 1$ $\text{ht}(\pm(\alpha+\beta)) = \pm 2$ $\text{ht}(\pm(2\alpha+\beta)) = \pm 3$ $\text{ht}(\pm(3\alpha+\beta)) = \pm 4$ $\text{ht}(\pm(3\alpha+2\beta)) = \pm 5$

TABLE 1. Root data for A_2, B_2, C_2

the Cartan matrix C and the Coxeter system: Let m_{ij} be the order of $\sigma_{\alpha_i}\sigma_{\alpha_j}$, then

$$\cos\Big(\frac{\pi}{m_{ij}}\Big)^2 = \frac{\langle\alpha_i, \alpha_j^*\rangle\langle\alpha_j, \alpha_i^*\rangle}{4}.$$

The $n \times n$ matrix with entries (m_{ij}) is called the **Coxeter matrix**. The next step is the construction of the **Coxeter diagram** Π: It is a graph with one vertex for each root in Δ, so these vertices can be numbered as $1, \dots, n$. The edges of the graphs are given by the pairs $\{i, j\}$ with $m_{ij} > 2$ and have the label m_{ij}. The **Dynkin diagram** is the Coxeter diagram containing an additional information about the root lengths: In case $\langle\alpha_i, \alpha_j^*\rangle < \langle\alpha_j, \alpha_i^*\rangle$, the edge $\{i, j\}$ is replaced by an directed edge (i, j) in the Dynkin diagram. The arrow points from the vertex of the longer root to the vertex of the shorter one.

Notice that, if a root system is reducible, the Dynkin diagram is disconnected and vice versa. The Dynkin diagrams of the irreducible root systems are given in Figure 2.

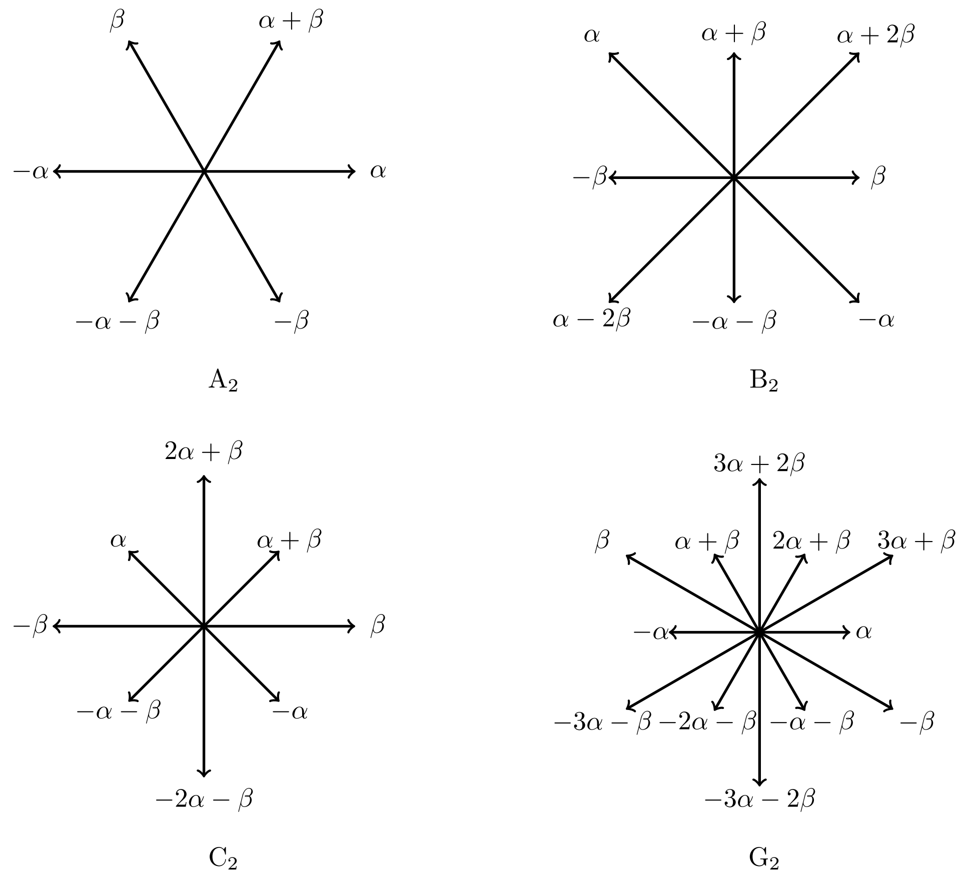

FIGURE 1. Irreducible root systems in dimension two

DEFINITION 3.1.6. A vector $w \in V$ is called a **weight** if $\langle w, \alpha^* \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$. The weights form the **weight lattice** Λ. We denote by Λ_Φ the sublattice (of finite index) spanned by Φ. If $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a set of fundamental roots of Φ, then Λ has a corresponding basis of **fundamental weights** $\{\lambda_1, \ldots, \lambda_n\}$ such that $\langle \lambda_i, \alpha_j^* \rangle = \delta_{ij}$. The quotient Λ/Λ_Φ is called the **fundamental group** of the root system Φ.
The fundamental groups corresponding to the irreducible root systems are the following (see [Hum78], section 13): $\mathbb{Z}/(n+1)\mathbb{Z}$ for A_n, $\mathbb{Z}/2\mathbb{Z}$ for $\mathrm{B}_n, \mathrm{C}_n, \mathrm{E}_7$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for D_n, n even and $\mathbb{Z}/4\mathbb{Z}$ for D_n if n odd, $\mathbb{Z}/3\mathbb{Z}$ for E_6 and trivial for $\mathrm{E}_8, \mathrm{F}_4$ and G_2.

In order to define Chevalley Lie algebras, we will make use of the more general concept of root data.

DEFINITION 3.1.7. A **root datum** is a quadruple $R = (X, Y, \Phi, \Phi^*)$ where

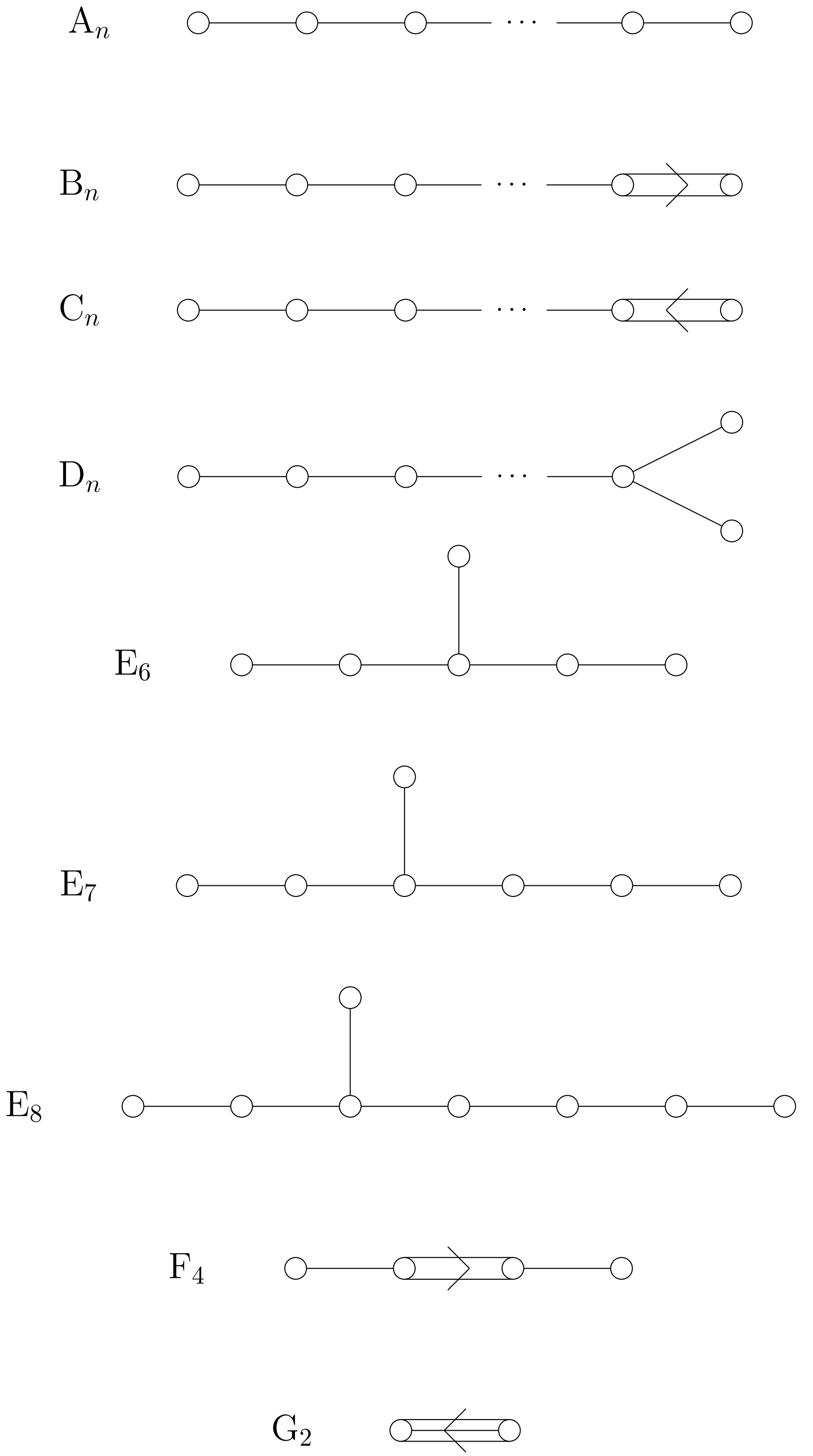

FIGURE 2. Dynkin diagrams

(1) X and Y are dual free $\mathbb{Z}$-modules of finite rank.
(2) $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$ is a bilinear pairing putting X and Y into duality.
(3) Φ is a finite subset of X and Φ^* is a finite subset of Y.
(4) There is a one-to-one correspondence $^* : \Phi \to \Phi^*$.

For $\alpha \in \Phi$, we define the linear map $s_\alpha : X \to X$ for $\alpha \in \Phi$ by

$$s_\alpha(x) = x - \langle x, \alpha^* \rangle \alpha$$

and similarly the linear map $s^*_\alpha : Y \to Y$ by

$$s_\alpha(y) = y - \langle \alpha, y \rangle \alpha^*.$$

We assume moreover

(1) $\langle \alpha, \alpha^* \rangle = 2$ for all $\alpha \in \Phi$.
(2) Φ is closed under the action of s_α.
(3) Φ^* is closed under the action of s^*_α.
(4) If $\alpha, t\alpha \in \Phi$ with $t \in \mathbb{R}$, then $t = \pm 1$.

Then the elements of Φ are the **roots** and the elements of Φ^* the **coroots** of the root datum. The group W generated by all s_α for $\alpha \in \Phi$ is the **Weyl group** of the root datum.

The connection between root systems and root data is clear: Denote by $\langle \Phi \rangle_X$ the submodule of X generated by $\Phi \neq \emptyset$ and let $V = \langle \Phi \rangle_X \otimes \mathbb{R}$, then obviously Φ is a root system in V. Similarly, Φ^* is a root system in $\langle \Phi^* \rangle_Y \otimes \mathbb{R}$.

One can also construct a root datum from a root system in the following way: Let Φ be a root system in some Euclidian space V with inner product $(\cdot, \cdot)$. We defined earlier $\alpha^* = \frac{2\alpha}{(\alpha,\alpha)}$ and $\Phi^* = \{\alpha^* | \alpha \in \Phi\}$. We take $X = \mathbb{Z}\Phi$ and $Y = \{y \in V | (x, y) \in \mathbb{Z} \text{ for all } x \in X\}$, and define $\langle x, y^* \rangle = (x, y^*)$ for $x \in X$ and $y \in Y$. This makes $R = (X, Y, \Phi, \Phi^*)$ a root datum.

Example 3.1.8. Let $\Phi = \{\pm\alpha \pm \beta, \pm(\alpha + \beta), \pm(\alpha + 2\beta)\}$ be a root system of type B_2 in $\mathbb{R}^2$. A possible choice is $\alpha = (-1, 1)$ and $\beta = (1, 0)$. Then $\alpha^* = (-1, 1)$ and $\beta^* = (2, 0)$, and the vectors $(1, 0)$ and $(0, 1)$ form a basis of $\mathbb{Z}\Phi$ and the vectors $(-1, 1)$ and $(1, 1)$ form a basis for $\mathbb{Z}\Phi^*$. With $X = Y = \mathbb{Z}\Phi$, we have a root datum $R = (X, Y, \Phi, \Phi^*)$.

Definition 3.1.9. The **rank** of a root datum is the dimension of $X \otimes \mathbb{R}$ which is equal to that of $Y \otimes \mathbb{R}$; the **semisimple rank** is the dimension of $\mathbb{Z}\Phi \otimes \mathbb{R}$, and the root datum is called **semisimple** if the rank and the semisimple rank are equal. The root datum is called **irreducible** if Φ is irreducible.

A root datum $R = (X, Y, \Phi, \Phi^*)$ is **isomorphic** to another root datum $R' = (X', Y', \Phi', \Phi^{*'})$ if there are isomorphisms between X and X' and between Y and Y' both denoted by φ, such that their restrictions to Φ and Φ^* are isomorphisms of root systems, and fulfill $\langle \varphi x, \varphi y \rangle = \langle x, y \rangle$ for all $x \in \Phi, y \in \Phi^*$. We defined a weight vector w in $X \otimes \mathbb{R}$ to be any vector such that $\langle w, \alpha^* \rangle \in \mathbb{Z}$ for all $\alpha \in \mathbb{Z}$. The weights form a weight lattice and the fundamental group is the quotient of this lattice by the root lattice $\mathbb{Z}\Phi$. This fundamental group determines the possible semisimple root data with a given root system Φ via the quotient $X/\mathbb{Z}\Phi$. For our work, it will be of some importance that there can be more than one possible root datum corresponding to a root system Φ, depending on the choice of the weight lattice. To consider this, we introduce the **isogeny type** of a root datum. If $X/\mathbb{Z}\Phi$ is the trivial group, we say that R is of **adjoint** isogeny type, or the **adjoint root datum** of type Φ. If $X/\mathbb{Z}\Phi$ is on the other hand the full fundamental group, R is said to be of **simply connected** isogeny type or the **simply connected root datum** of type Φ. If neither of these hold, R is said to be of **intermediate** isogeny type, but note this can only occur for root systems of type A_n (if (n+1) is not prime) and D_n.

To distinguish the different root data, we denote the irreducible adjoint root datum of type X_n by $\mathrm{X}_n^{\mathrm{ad}}$ and the corresponding simply connected root datum by $\mathrm{X}_n^{\mathrm{sc}}$. Intermediate root data of type A_n are denoted by $\mathrm{A}_n^{(k)}$, where $k|n+1$, and intermediate root data of type D_n will be denoted by $\mathrm{D}^{(1)}$ if n is odd, and by $\mathrm{D}_n^{(1)}, \mathrm{D}_n^{(n-1)}$ and $\mathrm{D}_n^{(n)}$ if n is even.

The following computational rules and examples are taken from [Roo10].
Let n be the rank of R and l be the semisimple rank. Fix $X = Y = \mathbb{Z}^n$ and define $\langle x, y \rangle = xy^T \in \mathbb{Z}$ for x a row vector and y a transposed row, so a column vector. Let A be the integral $l \times n$-matrix containing the simple roots as row vectors. We call A the **root matrix** of R. Similarly, let B be the $l \times n$-matrix containing the simple coroots in the corresponding order; B is the **coroot matrix** of R. Then the Cartan matrix C is equal to AB^T and $\mathbb{Z}\Phi = \mathbb{Z}A$ and $\mathbb{Z}\Phi^* = \mathbb{Z}B$. For $\alpha \in \Phi$ we define c^α to be the $\mathbb{Z}$-valued size l row vector satisfying $\alpha = c^\alpha A$.
In the following, we will mostly deal with semisimple root data, where $l = n$. Here, the definition of the adjoint isogeny type implies that for the adjoint isogeny type we may take A to be the $n \times n$ identity matrix and $B = C^T$.

Vice versa, for the simply connected root datum we may take $A = C$ and B as the identity.

EXAMPLE 3.1.10 (Rank one root data). We classify the semisimple root data of rank one. There is only one root system of rank one, namely A_1 with the roots α and $-\alpha$, but nevertheless there are two non-isomorphic semisimple root data of rank one, $\mathrm{A}_1^{\mathrm{ad}}$ and $\mathrm{A}_1^{\mathrm{sc}}$. They can be obtained as follows: Fix the root lattice $X = \mathbb{Z}$ and the coroot lattice $Y = \mathbb{Z}$, so the pairing is just simple multiplication: $\langle x, y\rangle = xy$. The Cartan matrix C is equal to $(\langle \alpha, \alpha^*\rangle) = (2)$. So for A and B we are looking for integral 1×1 matrices such that $AB^T = C$. So obviously $A = (1), B = (2)$ and $A = (2), B = (1)$ are two possible choices, where the first one is the adjoint and the second the simply connected case. The choices are nonisomorphic since the determinants of the root matrices A differ.

EXAMPLE 3.1.11 (Rank two root data). In the 2-dimensional case, we have a couple of cases to distinguish. Recall that there were 5 possible root systems $\mathrm{A}_1\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_2 \cong \mathrm{C}_2$ and G_2, giving rise to different Cartan matrices. We also know that the fundamental group of A_n is $\mathbb{Z}/(n+1)\mathbb{Z}$, the fundamental group of B_n and C_2 is $\mathbb{Z}/2\mathbb{Z}$ and the fundamental group of G_2 is trivial. Using the same computational procedures as before, we obtain the choices for the root and the coroot matrices that can be found in Table 2. They are unique up to multiplication with elements of $\mathrm{SL}(2,\mathbb{Z})$: if $m \in \mathrm{SL}(2,\mathbb{Z})$, then $AB^T = (Am)(Bm^{-T})^T, \det(A) = \det(Am)$ and $\det(B) = \det(Bm)$.

3.2. Definition of Chevalley algebras

The following definition is according to [Coh], chapter 7.

DEFINITION 3.2.1. Let (X, Y, Φ, Φ^*) be a root datum of rank n, with bilinear pairing $\langle ., .\rangle : X \times Y \to \mathbf{Z}$. Consider the free $\mathbb{Z}$-module

$$\mathfrak{g}_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}x_\alpha$$

where x_α formal basis elements (complementary to Y). As a $\mathbb{Z}$-module, it is of rank $n + |\Phi|$. We define on $\mathfrak{g}_{\mathbb{Z}}$ a bilinear map

$$[\cdot,\cdot] : \mathfrak{g}_{\mathbb{Z}} \times \mathfrak{g}_{\mathbb{Z}} \to \mathfrak{g}_{\mathbb{Z}}$$

	Cartan matrix	Root matrix	Coroot matrix
$A_1^{ad}A_1^{ad}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
$A_1^{ad}A_1^{sc}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$A_1^{sc}A_1^{sc}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
A_2^{ad}	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
A_2^{sc}	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$B_2^{ad} = C_2^{sc}$	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$
$B_2^{sc} = C_2^{ad}$	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
G_2	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

TABLE 2. Root data of rank 2

by the following rules:

$$\begin{aligned} [y, z] &= 0 \\ [y, x_\beta] &= \langle \beta, y \rangle x_\beta \\ [x_\alpha, x_\beta] &= \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ \alpha^* & \text{if } \beta = -\alpha, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha, \beta \in \Phi$, $y, z \in Y$. The numbers $N_{\alpha,\beta}$ are integral **structure constants** chosen to be $\pm(p_{\alpha,\beta} + 1)$, where $p_{\alpha,\beta}$ is the biggest number such that $-p_{\alpha,\beta}\alpha + \beta$ is a root. With respect to the root datum (X, Y, Φ, Φ^*), these relations define a $\mathbb{Z}$-algebra on $\mathfrak{g}_\mathbb{Z}$ that is called a **Chevalley algebra**. The formal basis elements $x_\alpha, \alpha \in \Phi$ together with a basis of Y form a **Chevalley basis** of $\mathfrak{g}_\mathbb{Z}$. If $\mathfrak{g}_\mathbb{Z}$ is moreover a Lie algebra, we call it an **integral Chevalley Lie algebra**.

If a Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{F}$ can be obtained by tensoring with a field $\mathbb{F}$, we call $\mathfrak{g}$ a **Chevalley Lie algebra** and denote it by $\mathfrak{g}_{\mathbb{F}}$.

Actually, one can find some necessary and sufficient restrictions on the structure constants $N_{\alpha,\beta} \in \mathbb{Z}$ for the Chevalley algebra $\mathfrak{g}_{\mathbb{Z}}$ to be a Lie ring.

LEMMA 3.2.2 ([Coh],7.1.2). *Using the previous notation, the following conditions are necessary and sufficient for the bracket $[\cdot,\cdot]$ to define a Lie ring on $\mathfrak{g}_{\mathbb{Z}}$ (note that it is not a Lie algebra since $\mathfrak{g}_{\mathbb{Z}}$ is no vector space, but a $\mathbb{Z}$-module).*

$$N_{\beta,\alpha} = - N_{\alpha,\beta}; \tag{3.1}$$

$$N_{\alpha,\beta} = 0 \textit{ if } \alpha + \beta \notin \Phi; \tag{3.2}$$

$$(\alpha,\alpha) N_{\alpha,\beta} = (\gamma,\gamma) N_{\beta,\gamma} \quad \textit{if } \alpha,\beta,\gamma \in \Phi \textit{ are without opposite pairs and } \alpha+\beta+\gamma = 0; \tag{3.3}$$

$$\frac{\langle \beta, \alpha^* \rangle}{(\beta,\beta)} = \frac{N_{\alpha,\beta} N_{-\alpha,-\beta}}{(\beta+\alpha,\beta+\alpha)} - \frac{N_{-\alpha,\beta} N_{\alpha,-\beta}}{(\beta-\alpha,\beta-\alpha)} \quad \textit{if } \alpha,\beta \in \Phi \textit{ are linearly independent roots}; \tag{3.4}$$

$$\frac{N_{\alpha,\beta} N_{\gamma,\delta}}{(\alpha+\beta,\alpha+\beta)} + \frac{N_{\beta,\gamma} N_{\alpha,\delta}}{(\beta+\gamma,\beta+\gamma)} + \frac{N_{\gamma,\alpha} N_{\beta,\delta}}{(\gamma+\alpha,\gamma+\alpha)} = 0 \quad \textit{if } \alpha,\beta,\gamma,\delta \in \Phi \textit{ are without opposite pairs and } \alpha+\beta+\gamma+\delta = 0. \tag{3.5}$$

The number of possible choices for the structure constants is restricted by these conditions and parameterized by so-called extraspecial pairs. To define them we equip Φ with a total ordering $\prec$, that we choose in such a way that $0 \prec \alpha$ for all $\alpha \in \Phi^+$, respecting the height as defined above. This means $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ implies $\alpha \prec \beta$.

DEFINITION 3.2.3. Having chosen a total ordering $\prec$ on the root system Φ, an ordered pair of roots (α, β) with $\alpha, \beta \in \Phi$ is called **special** (with respect to the ordering $\prec$) if $\alpha + \beta \in \Phi$ and $0 \prec \alpha \prec \beta$. A special pair of roots is called **extraspecial** (with respect to the ordering $\prec$) if for all special pairs (α', β') for which $\alpha + \beta = \alpha' + \beta'$ we have $\alpha \preceq \alpha'$.

So we can conclude that every root in Φ^+ that is a sum of two roots in Φ^+ is the sum of exactly one extraspecial pair.

EXAMPLES 3.2.4.

(1) The A_2 case

For a Lie algebra of type A_2 and with fundamental roots α, β as in 3.1.4, we have the (long) root elements x_α, x_β, $x_{\alpha+\beta}$, $x_{-\alpha}$, $x_{-\beta}$ and $x_{-\alpha-\beta}$. The only extraspecial pair with respect to the ordering $\prec$ by height is (α, β). In the following table we specify the structure constants $N_{\alpha,\beta}$ of the brackets in the A_2 case. Here, one can choose $\delta_1 \in \{-1, 1\}$

	α	β	$\alpha+\beta$	$-\alpha$	$-\beta$	$-\alpha-\beta$
α	0	δ_1	0	0	0	$-\delta_1$
β	$-\delta_1$	0	0	0	0	δ_1
$\alpha+\beta$	0	0	0	$-\delta_1$	δ_1	0
$-\alpha$	0	0	δ_1	0	$-\delta_1$	0
$-\beta$	0	0	$-\delta_1$	δ_1	0	0
$-\alpha-\beta$	δ_1	$-\delta_1$	0	0	0	0

From now on and for the remaining thesis, we choose $\delta_1 = +1$ and use this for all computations in A_2.

(2) The B_2 case.

In the Lie algebra of type B_2 and with the choice of the fundamental roots α, β as in 3.1.4, the positive roots are denoted by $\alpha, \beta, \alpha+\beta$ and $\alpha+2\beta$. The extraspecial pairs are (β, α) and $(\beta, \alpha+\beta)$. The following table gives the structure constants for a Chevalley basis, with $\delta_1, \delta_2 \in \{1, -1\}$:

	α	β	$\alpha+\beta$	$\alpha+2\beta$	$-\alpha$	$-\beta$	$-\alpha-\beta$	$-\alpha-2\beta$
α	0	$-\delta_1$	0	0	0	0	δ_1	0
β	δ_1	0	$2\delta_2$	0	0	0	$-2\delta_1$	$-\delta_2$
$\alpha+\beta$	0	$-2\delta_2$	0	0	δ_1	$-2\delta_1$	0	δ_2
$\alpha+2\beta$	0	0	0	0	0	$-\delta_2$	δ_2	0
$-\alpha$	0	0	$-\delta_1$	0	0	δ_1	0	0
$-\beta$	0	0	$2\delta_1$	δ_2	$-\delta_1$	0	$-2\delta_2$	0
$-\alpha-\beta$	$-\delta_1$	$2\delta_1$	0	$-\delta_2$	0	$2\delta_2$	0	0
$-\alpha-2\beta$	0	δ_2	$-\delta_2$	0	0	0	0	0

From now on and for the remaining thesis, we choose $\delta_1 = \delta_2 = +1$ and use this for all computations in B_2.

Using the notation of the roots in the orthonormal basis as given in 3.1.4, so $\alpha = e_1 - e_2$ and $\beta = e_2$, we can easily deduce the structure constants for the C_2 root system. Since $C_2 = B_2$ via scaling by $\sqrt{2}$ and a 45 degree rotation, the tables of structure constants in these two cases are equal. Using the previous notation, the long roots in the C_n case are $\pm 2e_i$ with $i \in \{1, \ldots, n\}$ and the short roots are $\pm(e_i \pm e_j)$ with $i \neq j$ and $i, j \in \{1, \ldots, n\}$. The (non-unique) correspondence between B_2 and C_2 is therefore

$$\alpha = 2e_1, \quad \beta = -e_1 + e_2 \Rightarrow \alpha + \beta = e_1 + e_2, \quad \alpha + 2\beta = 2e_2.$$

(3) The G_2 case
Again with the notations from 3.1.4, we have the positive roots $\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta$ and $3\alpha + 2\beta$. As before, we have $\delta_i \in \{+1, -1\}$ for $i = 1, 2, 3, 4$.
Table 3 gives the structure constants $N_{\alpha,\beta}$ for this root system.
The extraspecial pairs are $(\alpha, \beta), (\alpha, \alpha + \beta), (\alpha, 2\alpha + \beta)$ and $(\beta, 3\alpha + \beta)$.
From now on and for the remaining thesis, we choose $\delta_1 = \delta_2 = \delta_3 = \delta_4 = +1$ and use this for all computations in G_2.

Note that the matrices obtained from the tables of the structure constants are skew-symmetric by definition, so any opposite choice of the signs δ_i (for i in the index set of the signs), leads either to the same or to the negative (e.g. a scalar multiple) of the same Lie product and therefore to an isomorphic Lie algebra.

3.3. Independence of the basis

Previously, we have seen that we can consider Chevalley algebras over an arbitrary field $\mathbb{F}$ by tensoring: $\mathfrak{g}_{\mathbb{F}} = \mathbb{F} \otimes \mathfrak{g}_{\mathbb{Z}}$. When we work with Chevalley Lie algebras, we call two Chevalley bases to have the same **fixed type** if the multiplication tables with respect to these bases are isomorphic, so they have the same structure constants.
One of the main results about Chevalley algebras is the following (see e.g. [Car72]).

THEOREM 3.3.1. *The automorphism group* $\mathrm{Aut}(\mathfrak{g})$ *of a Chevalley Lie algebras* $\mathfrak{g}$ *acts transitively on the set of Chevalley bases of* $\mathfrak{g}$ *of a given fixed type.*

In particular, there is an automorphism in $\mathrm{Aut}(\mathfrak{g})$ that transforms any Chevalley basis of $\mathfrak{g}$ into $\{h_\alpha, \alpha \in \Delta; \pm x_\alpha, \alpha \in \Phi\}$, where by h_α, we denote the elements of the Cartan subalgebra (we will keep this notation in the following). This allows us in the following to pick a fixed Chevalley basis and root system for $\mathfrak{g}$.

TABLE 3. Multiplication table of G_2

	α	β	$\alpha+\beta$	$2\alpha+\beta$	$3\alpha+\beta$	$3\alpha+2\beta$	$-\alpha$	$-\beta$	$-\alpha-\beta$	$-2\alpha-\beta$	$-3\alpha-\beta$	$-3\alpha-2\beta$
α	0	δ_1	$2\delta_2$	$3\delta_3$	0	0	0	0	$-3\delta_1$	$-2\delta_2$	$-\delta_3$	0
β	$-\delta_1$	0	0	0	δ_4	0	0	0	δ_1	0	0	$-\delta_4$
$\alpha+\beta$	$-2\delta_2$	0	0	$-3\delta_1\delta_3\delta_4$	0	0	$-3\delta_1$	δ_1	0	δ_2	0	$\delta_1\delta_3\delta_4$
$2\alpha+\beta$	$-3\delta_3$	0	$3\delta_1\delta_3\delta_4$	0	0	0	$-2\delta_2$	0	$2\delta_2$	0	δ_3	$-\delta_1\delta_3\delta_4$
$3\alpha+\beta$	0	$-\delta_4$	0	0	0	0	$-\delta_1$	0	0	δ_3	0	δ_4
$3\alpha+2\beta$	0	0	0	0	0	0	0	$-\delta_4$	$\delta_1\delta_3\delta_4$	$-\delta_1\delta_3\delta_4$	δ_4	0
$-\alpha$	0	0	$3\delta_1$	$2\delta_2$	δ_3	0	0	$-\delta_1$	$-2\delta_2$	$-3\delta_3$	0	0
$-\beta$	0	0	$-\delta_1$	0	0	δ_4	δ_1	0	0	0	$-\delta_4$	0
$-\alpha-\beta$	$3\delta_1$	$-\delta_1$	0	$-2\delta_2$	0	$-\delta_1\delta_3\delta_4$	$2\delta_2$	0	0	$3\delta_1\delta_3\delta_4$	0	0
$-2\alpha-\beta$	$2\delta_2$	0	$-2\delta_2$	0	$-\delta_3$	$\delta_1\delta_3\delta_4$	$3\delta_3$	0	$-\delta_1\delta_3\delta_4$	0	0	0
$-3\alpha-\beta$	δ_3	0	0	$-\delta_3$	0	$-\delta_4$	0	δ_4	0	0	0	0
$-3\alpha-2\beta$	0	δ_4	$-\delta_1\delta_3\delta_4$	$\delta_1\delta_3\delta_4$	$-\delta_4$	0	0	0	0	0	0	0

It also implies that the following definition is independent of the choice of the basis and the root system Φ.

DEFINITION 3.3.2. Let $\mathrm{Aut}(\mathfrak{g})$ be the automorphism group of a Chevalley Lie algebra $\mathfrak{g}$. Then elements of the form x_α^g, with $\alpha \in \Phi$ and $g \in \mathrm{Aut}(\mathfrak{g})$, are called **root elements**. If α is a long root (a short root, respectively), x_α^g is in particular a **long root element** (a **short root element**, respectively).

3.4. Extremal elements in Chevalley algebras

In this section, we analyse the structure of extremal elements in a Chevalley Lie algebra $\mathfrak{g}$ and its $\mathfrak{sl}_2$-graph. We will first see that all long root elements are extremal. Actually, we will prove that in most cases the extremal elements of Chevalley algebras are *exactly* the long root elements. There are exceptions that will be considered in the end of the chapter.

DEFINITION 3.4.1. For a Chevalley Lie algebra $\mathfrak{g}_\mathbb{Z} = Y \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}x_\alpha$, we denote $\mathfrak{g}_\Phi = \langle x_\alpha | \alpha \in \Phi\rangle$, the linear span of the formal generators x_α. By $h_\alpha := [x_\alpha, x_{-\alpha}]$, we denote the elements of the Cartan subalgebra.

PROPOSITION 3.4.2. *Long root elements x_α with $\alpha \in \Phi^{long}$ in $\mathfrak{g}_\Phi$ fulfill*

$$[[x_\alpha, [x_\alpha, y]] = cx_\alpha$$

for all $y \in \mathfrak{g}_\Phi$, where $c \in \mathbb{F}$ depends on y.

PROOF. Let $\alpha \in \Phi$ be a long root, $\beta \in \Phi$ an arbitrary root, and x_α, x_β be the corresponding elements in $\mathfrak{g}$. We verify the identity for any y in the Chevalley basis. We distinguish the following cases:
If $y \in Y$, we have $[x_\alpha, [x_\alpha, y]] = \langle\alpha, y\rangle[x_\alpha, x_\alpha] = 0$.
For $y = x_\beta$ with $\beta \in \Phi$, we have two cases:

(1) $\beta = -\alpha$: Applying 3.2.1 , we have

$$[x_\alpha, [x_\alpha, x_\beta]] = [x_\alpha, [x_\alpha, x_{-\alpha}]] = [x_\alpha, h_\alpha] = cx_\alpha$$

for $c = -\langle\alpha, h_\alpha\rangle \in \mathbb{F}$.

(2) $\beta \neq -\alpha$: Here, we use 3.2.1 again. Assuming that $\alpha + \beta$ is a root, we have

$$[x_\alpha, [x_\alpha, x_\beta]] = [x_\alpha, N_{\alpha,\beta}x_{\alpha+\beta}] = 0$$

because $2\alpha+\beta \notin \Phi$, where $N_{\alpha,\beta} = \pm(r+1)$ and r the biggest number such that $-r\alpha + \beta$ is a root. If $\alpha + \beta$ is no root, then $N_{\alpha,\beta} = 0$, so $[x_\alpha, [x_\alpha, x_\beta]] = 0$ holds again.

So in all cases, we see that $[x_\alpha, [x_\alpha, y]]$ is either zero or a multiple of x_α. Since $\mathfrak{g}_\Phi$ is spanned by the elements x_β and the elements of Y, the result is proven. $\diamond$

Proposition 3.4.3. *The Lie algebra $\mathfrak{g}_\Phi$ as defined in 3.4.1 is generated by its long root elements.*

Proof. We defined $\mathfrak{g}_\Phi$ to be generated by all elements x_α with $\alpha \in \Phi$, and we can distinguish between short and long roots in Φ by 3.3.2.
So let x_β with $\beta \in \Phi$ be a short root element. We prove: x_β is the sum of at most three long root elements.

(1) A_n-case: In this case, all roots are long by definition.
(2) B_n-case: If $\mathfrak{g}_\Phi$ of B_n-type, there exists a Lie subalgebra $\mathfrak{g}_1$ of type B_2 with $x_\beta \in \mathfrak{g}_1$. We know that for every Lie algebra of type B_2, we can identify β with a short positive root such that the positive roots of $\mathfrak{g}_1$ are $\alpha, \beta, \alpha+\beta$ and $2\beta+\alpha$. Hereby, β and $\alpha+\beta$ are the short roots and α and $2\beta+\alpha$ are the long ones. With respect to the given Chevalley basis, we see:

$$\underbrace{\exp(x_{-\alpha-\beta}, 1)(x_{\alpha+2\beta})}_{long} = \underbrace{x_{\alpha+2\beta}}_{long} - \underbrace{x_\beta}_{short} - \underbrace{x_{-\alpha}}_{long},$$

so

$$x_\beta = x_{\alpha+2\beta} - x_{-\alpha} - \exp(x_{-\alpha-\beta}, 1)(x_{\alpha+2\beta})$$

is a sum of three long root elements.
(3) C_n-case: Parallel to the B_n-case, we can also here find a Lie subalgebra $\mathfrak{g}_1$ of type C_2 with $x_\beta \in \mathfrak{g}_1$. Now we use the correspondence between B_2 and C_2 mentioned in 3.2.4, so the long roots are $\alpha = 2\varepsilon_1$ and $\alpha + 2\beta = 2\varepsilon_2$ and the short roots are $\beta = -\varepsilon_1 + \varepsilon_2$ and $\alpha + \beta = \varepsilon_1 + \varepsilon_2$,

Using this, we can compute:

$$\underbrace{\exp(x_{\alpha+\beta}, 1)(x_{-\alpha})}_{long} = \underbrace{x_{-\alpha}}_{long} + \underbrace{x_\beta}_{short} - \underbrace{x_{\alpha+2\beta}}_{long},$$

so

$$x_\beta = x_{\alpha+2\beta} - x_{-\alpha} + \exp(x_{\alpha+\beta}, 1)(x_{-\alpha})$$

is a sum of three long root elements.

(4) G_2-case: In a Lie algebra of type G_2, we have the long positive roots $\beta, 3\alpha+2\beta$ and $3\alpha+\beta$, and the short positive roots $\alpha, \alpha+\beta$ and $2\alpha+\beta$. As before, we identify the short root element x_α and get (assumed that $\mathrm{char}(\mathbb{F}) \neq 2$):

$$\underbrace{\exp(x_{-\alpha-\beta}, 1)(x_{3\alpha+2\beta})}_{long} + \underbrace{\exp(x_{-\alpha-\beta}, -1)(x_{3\alpha+2\beta})}_{long}$$
$$= 2\underbrace{x_{3\alpha+2\beta}}_{long} + 2\underbrace{x_\alpha}_{short},$$

so

$$x_\alpha = \frac{1}{2}\big(\exp(x_{-\alpha-\beta}, 1)(x_{3\beta+2\alpha}) + \exp(x_{-\alpha-\beta}, -1)(x_{3\beta+2\alpha}) - 2x_{3\beta+2\alpha}\big)$$

is the sum of three long root elements.

In case that $\mathrm{char}\,\mathbb{F} = 2$, we use the following observation:

$$\underbrace{\exp(x_{-2\alpha-\beta}, 1)(x_{3\alpha+\beta})}_{long} = x_{3\alpha+\beta} + x_\alpha + 2x_{-\alpha-\beta}$$
$$= \underbrace{x_{3\alpha+\beta}}_{long} + \underbrace{x_\alpha}_{short},$$

and consequently

$$x_\alpha = x_{3\alpha+\beta} + \exp(x_{-2\alpha-\beta}, 1)(x_{3\alpha+\beta}),$$

so we can express x_α as the sum of two long root elements.

(5) In the cases $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ and D_n, all roots are long by definition, so there is nothing to show.

(6) In the F_4-case, every short root element lies in a Lie subalgebra of type B_2, so we can solve this case by referring to the B_2 case above.

$\diamond$

To consider the extremal elements in Chevalley Lie algebras, we need the corresponding extremal form g as defined in 2.3. We will see that the following definition gives a suitable choice for g.

DEFINITION 3.4.4. For $\mathfrak{g}_\Phi$ with root system Φ, define l to be the length of a long root in Φ. For $\alpha, \beta \in \Phi$ roots, denote by $\theta_{\alpha,\beta}$ the angle between α, β. We define the following form on $\mathfrak{g}_\Phi$:

$$g(x_\alpha, x_\beta) = 0 \text{ if } \alpha \neq -\beta.$$

$$g(x_\beta, x_{-\beta}) = 1 \text{ if } \beta \text{ a long root.}$$

$$g(x_\alpha, x_{-\alpha}) = l^2 \text{ if } \alpha \text{ a short root.}$$

$$g(x_\alpha, h_\beta) = 0.$$

$$g(h_\alpha, h_\beta) = \begin{cases} 2\|\alpha\|\|\beta\|\cos(\theta_{\alpha,\beta}) \text{ if } \beta \neq \alpha \text{ of different length.} \\ 2\cos(\theta_{\alpha,\beta}) \text{ for } \alpha \neq \beta \text{ both long.} \\ 2l^2\cos(\theta_{\alpha,\beta}) \text{ for } \alpha \neq \beta \text{ both short.} \\ 2 \text{ if } \alpha = \beta \text{ and } \alpha \text{ both long.} \\ 2l^2 \text{ if } \alpha = \beta \text{ and } \alpha \text{ both short.} \end{cases}$$

PROPOSITION 3.4.5. *The previous choice of the form g on $\mathfrak{g}_\Phi$ over a field $\mathbb{F}$ defines an extremal form as defined in 2.3, and the long root elements in $\mathfrak{g}_\Phi$ are extremal with respect to g.*

PROOF. As we have seen in 2.1.3, the extremality of an element x_α follows if just $[x_\alpha, [x_\alpha, y]] = g(x_\alpha, y)x_\alpha$ for all $y \in \mathfrak{g}$ and g an extremal form in the sense of 2.3 if $\mathrm{char}(\mathbb{F}) \neq 2$. In this case, moreover the given choice for g can be deduced from the relations defining a Chevalley algebra. So assume first that indeed $\mathrm{char}(\mathbb{F}) \neq 2$. We consider the form evaluated on the various elements of the Chevalley basis.
For the first case, consider $\alpha, \beta \in \Phi$ any roots and h_γ be any element in Y. Then

$$\begin{aligned} g([h_\gamma, x_\alpha], x_\beta) =& g(\langle \alpha, \gamma \rangle x_\alpha, x_\beta) \\ =& \langle \alpha, \gamma \rangle g(x_\alpha, x_\beta). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g([h_\gamma, x_\alpha], x_\beta) =& g(x_\alpha, -[h_\gamma, x_\beta]) \\ =& g(x_\alpha, -\langle \beta, \gamma \rangle x_\beta) \\ =& -\langle \beta, \gamma \rangle g(x_\alpha, x_\beta). \end{aligned}$$

Combining these two equalities, we get $\langle\alpha,\gamma\rangle g(x_\alpha,x_\beta) = -\langle\beta,\gamma\rangle g(x_\alpha,x_\beta)$, so $(\langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle)g(x_\alpha,x_\beta) = \langle\alpha+\beta,\gamma\rangle g(x_\alpha,x_\beta) = 0$. But if $\alpha+\beta \neq 0$, we can always choose an $\gamma \in \Phi$ such that $\langle\alpha+\beta,\gamma\rangle \neq 0$, so $g(x_\alpha,x_\beta) = 0$.

In the second case, we assume $\beta \in \Phi$ to be a long root. Then by Definition 2.1, we have

$$[x_\beta,[x_\beta,x_\alpha]] = 2g(x_\beta,x_\alpha)x_\beta$$

for any $\alpha \in \Phi$. So choose $\alpha = -\beta$:

$$[x_\beta,[x_\beta,x_{-\beta}]] = [x_\beta,h_\beta] = \langle\beta,\beta\rangle x_\beta = 2x_\beta$$

since $\langle\beta,\beta\rangle = \frac{2(\beta,\beta)}{(\beta,\beta)}$. So since $\mathrm{char}(\mathbb{F}) \neq 2$, this implies $g(x_\beta,x_{-\beta}) = 1$.

Now assume $\alpha \in \Phi$ a short root, and let $\beta \in \Phi$ be any long root. Now

$$\begin{aligned} g(h_\alpha,h_\beta) =& g([x_\alpha,x_{-\alpha}],[x_\beta,x_{-\beta}]) \\ =& g(x_\alpha,[x_{-\alpha},[x_\beta,x_{-\beta}]]) \\ =& g(x_\alpha,[x_{-\alpha},h_\beta]) \\ =& g(x_\alpha,-\langle-\alpha,\beta\rangle x_{-\alpha}) \\ =& \langle\alpha,\beta\rangle g(x_\alpha,x_{-\alpha}). \end{aligned} \tag{3.6}$$

On the other hand, the same expression can be transformed as follows:

$$\begin{aligned} g(h_\alpha,h_\beta) =& g([x_\alpha,x_{-\alpha}],[x_\beta,x_{-\beta}]) \\ =& g([[x_\alpha,x_{-\alpha}],x_\beta],x_{-\beta}) \\ =& g([h_\alpha,x_\beta],x_{-\beta}) \\ =& g(\langle\beta,\alpha\rangle x_\beta,x_{-\beta}) \\ =& \langle\beta,\alpha\rangle g(x_\beta,x_{-\beta}). \end{aligned} \tag{3.7}$$

Since we know from the first case that $g(x_\beta,x_{-\beta}) = 1$, this leads to

$$\langle\alpha,\beta\rangle g(x_\alpha,x_{-\alpha}) = \langle\beta,\alpha\rangle.$$

Finally, we have

$$\begin{aligned} g(x_\alpha,x_{-\alpha}) =& \frac{\langle\beta,\alpha\rangle}{\langle\alpha,\beta\rangle} \\ =& \frac{(\beta,\alpha)}{(\alpha,\alpha)}\frac{(\beta,\beta)}{(\alpha,\beta)} \\ =& \frac{(\beta,\beta)}{(\alpha,\alpha)} = l^2 \end{aligned}$$

since α is a short root with normalized length 1.
For the next case, we first consider the case where $\beta = \alpha$, so we compute $g(x_\alpha, h_\alpha)$. We have

$$\begin{aligned} g(x_\alpha, h_\alpha) =& g(x_\alpha, [x_\alpha, x_{-\alpha}]) \\ =& g([x_\alpha, x_\alpha], x_{-\alpha}) \\ =& g(0, x_{-\alpha}) = 0. \end{aligned}$$

Now consider $g(x_\alpha, h_\beta)$ for $\alpha \neq \beta$. We have

$$\begin{aligned} g(x_\alpha, h_\beta) =& g(x_\alpha, [x_\beta, x_{-\beta}]) \\ =& g([x_\alpha, x_\beta], x_{-\beta}). \end{aligned}$$

Now the following cases can occur: First $[x_\alpha, x_\beta] = 0$; then of course it holds $g(0, x_{-\beta}) = 0$. Secondly, if $\alpha = -\beta$, we get $-g(h_\beta, x_\beta) = 0$ by the previous considerations. Or finally, if we have $[x_\alpha, x_\beta] = N_{\alpha,\beta} x_{\alpha+\beta}$, we get $N_{\alpha,\beta} g(x_{\alpha+\beta}, x_\beta) = 0$ using previous cases, since $\alpha + \beta \neq -\beta$.
For the last case, we distinguish $g(h_\alpha, h_\alpha)$ and $g(h_\alpha, h_\beta)$ with $\alpha \neq \beta$. If $\alpha \neq \beta$ and exactly one of them is a short root, then either $g(x_\beta, x_{-\beta}) = 1$ or $g(x_\alpha, x_{-\alpha}) = 1$. W.l.o.g., assume that β is long, and α is short, so $\|\alpha\| = 1$. Then (3.7) gives

$$\begin{aligned} g(h_\alpha, h_\beta) =& \langle \beta, \alpha \rangle \\ =& \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \\ =& \frac{2\|\beta\|\|\alpha\| \cos(\theta_{\alpha,\beta})}{\|\alpha\|^2} \\ =& 2\|\beta\| \cos(\theta_{\alpha,\beta}). \end{aligned}$$

If $\alpha \neq \beta$ both long, then $g(x_\beta, x_{-\beta}) = 1 = g(x_\alpha, x_{-\alpha})$, so (3.6) as well as (3.7) lead to

$$\begin{aligned} g(h_\alpha, h_\beta) =& \langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle \\ =& \frac{2\|\beta\|\|\alpha\| \cos(\theta_{\alpha,\beta})}{\|\alpha\|^2} = \frac{2\|\alpha\|\|\beta\| \cos(\theta_{\alpha,\beta})}{\|\beta\|^2} \\ =& 2\cos(\theta_{\alpha,\beta}). \end{aligned}$$

If $\alpha \neq \beta$ both short roots, we have

$$g(h_\alpha, h_\beta) = \langle \beta, \alpha \rangle g(x_\beta, x_{-\beta}) = \langle \alpha, \beta \rangle g(x_\alpha, x_{-\alpha})$$

$$
\begin{aligned}
&=\frac{2\|\beta\|\|\alpha\|\cos(\theta_{\alpha,\beta})}{\|\alpha\|^2}l^2 = \frac{2\|\alpha\|\|\beta\|\cos(\theta_{\alpha,\beta})}{\|\beta\|^2}l^2 \\
&=2l^2\cos(\theta_{\alpha,\beta}).
\end{aligned}
$$

For $g(h_\alpha, h_\alpha)$, we have

$$
g(h_\alpha, h_\alpha) = \langle \alpha, \alpha \rangle g(x_\alpha, x_{-\alpha}) = \begin{cases} 2 \text{ if } \alpha \text{ long}, \\ 2l^2 \text{ if } \alpha \text{ short}, \end{cases}
$$

by the previous results.
Obviously, since all coefficients appearing in these computations are in $\mathbb{Z}$ by definition of Chevalley Lie algebras, the previously defined Lie bracket $[\cdot, \cdot]$ together with the given definition of g satisfies the Premet identities for all characteristics $\neq 2$. So they are also true in a Chevalley Lie algebra $\mathbb{F} \otimes \mathfrak{g}_{\mathbb{Z}}$ as constructed in [Car72] (see section 3.3) where $\operatorname{char}(\mathbb{F}) = 2$. So g also defines an extremal form in this case, and the long root elements are extremal with respect to this form. $\diamond$

The previous results enable us to compute the radical $\operatorname{rad}(g)$ for $\mathfrak{g}_\Phi$.

TABLE 4. The cases with nontrivial radical of the extremal form

type	$\operatorname{char}(\mathbb{F}) =: p$	$\dim(\mathfrak{g}/\operatorname{rad}(g))$	generators of $\operatorname{rad}(g)$
A_n	$p \mid (n+1)$	n^2+2n-1	$h_{e_0-e_1} + 2h_{e_1-e_2} + \dots nh_{e_{n-1}-e_n}$
B_n, n even	$p=2$	$2n^2-n-2$	all x_α with $\alpha \in \Phi$ short, $h_{e_1-e_2} + h_{e_3-e_4} + \dots + h_{e_{n-1}-e_n}$
B_n, n odd	$p=2$	$2n^2-n-1$	all x_α with $\alpha \in \Phi$ short
C_n	$p=2$	$2n$	all x_α with $\alpha \in \Phi$ short, all h_α with $\alpha \in \Phi$
D_n, n even	$p=2$	$2n^2-n-2$	$h_{e_1-e_2} + h_{e_3-e_4} + \dots + h_{e_{n-1}+e_n}$, $h_{e_{n-1}-e_n} + h_{e_{n-1}+e_n}$
D_n, n odd	$p=2$	$2n^2-n-1$	$h_{e_{n-1}-e_n} + h_{e_{n-1}+e_n}$
E_6	$p=3$	77	$h_{e_3-e_4} - h_{e_4-e_5} + h_{e_6+e_7} - h_{e_m}$ where $e_m := -\frac{1}{2}\sum_{i=1}^{8} e_i$
E_7	$p=2$	132	$h_{e_2-e_3} + h_{e_4-e_5} + h_{e_6-e_7}$
F_4	$p=2$	26	all x_α with $\alpha \in \Phi$ short
G_2	$p=3$	7	all x_α, h_α with $\alpha \in \Phi$ short

PROPOSITION 3.4.6. *Let* $\mathfrak{g}_\Phi$ *be as defined in 3.4.1. Then the radical of the extremal form* g *as defined in 3.4.4 is trivial on* $\mathfrak{g}_\Phi$*, except for the cases listed in Table 4 on page 66 (note that the dimensions in Table 4 are vector space dimensions).*

PROOF. The proof is a straightforward computation of the radicals of the extremal form; the concrete values of g on the Cartan subalgebra of the Chevalley algebras can be found in the appendix on page 125.

◇

PROPOSITION 3.4.7. *In general,*

$$\mathfrak{g}_\Phi = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}, \tag{3.8}$$

where $\mathfrak{g}$ *is the Chevalley Lie algebra corresponding to the root system* Φ *as defined in 3.2.1, and* $\mathfrak{g}_\Phi$ *as defined in 3.4.1. Moreover, we have* $\mathfrak{g}_\Phi = \mathfrak{g}$*, or we are in one of the following cases: The root datum is of adjoint or intermediate isogeny type (as defined in 3.1.9) and the underlying field* $\mathbb{F}$ *is of characteristic* 2 *for* Φ *of type* $\mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n, \mathrm{E}_7$*, characteristic* 3 *for* E_6 *or characteristic* p *with* $p \mid (n+1)$ *or* $p^2 \mid (n+1)$ *for* Φ *of type* A_n*.*

PROOF. If $\mathrm{char}(\mathbb{F}) \neq 2$, we have $x_\alpha \in \langle [h_\alpha, x_\alpha] \rangle \subseteq [\mathfrak{g}, \mathfrak{g}]$, and $h_\alpha \in \langle [x_\alpha, x_{-\alpha}] \rangle \subseteq [\mathfrak{g}, \mathfrak{g}]$, the inclusion $\mathfrak{g}_\Phi \subseteq [\mathfrak{g}, \mathfrak{g}]$ in 3.8 is obvious. If $\mathrm{char}(\mathbb{F}) = 2$, then we find that $x_\alpha = [x_\beta, x_\gamma]$ for some roots $\beta, \gamma \in \Phi$ such that γ is in the α-chain through β with the property $\gamma + \beta = \alpha$. The other direction $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_\Phi$ is also clear regarding the bracket rules of Chevalley Lie algebras given in 3.2.1.
The second inclusion holds for any Lie algebra $\mathfrak{g}$, so it just remains to consider in which cases there is indeed an equality.

◇

In [Hog82], the exact structure and codimension of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ is described. We find that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ exactly in the cases of root data of adjoint and intermediate isogeny type and characteristic 2 for Φ of type $\mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n, \mathrm{E}_7$, characteristic 3 for E_6 and characteristic p with $p \mid (n+1)$ or $p^2 \mid (n+1)$ for Φ of type A_n. So just in these cases, we have to distinguish furthermore between $\mathfrak{g}_\Phi = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}$.

COROLLARY 3.4.8. *In all cases where all roots are of one length, we have* $\mathrm{rad}(g) = Z(\mathfrak{g}_\Phi)$*.*

PROOF. It is clear that $Z(\mathfrak{g}_\Phi) \subseteq \mathrm{rad}(g)$. So just the opposite inclusion has to be shown. There we obviously just have to consider the cases where $\mathrm{rad}(g)$ is nontrivial. So we go case by case.

- A_n, $\mathrm{char}(\mathbb{F})|(n+1)$:
Let $x \in \langle h_{e_0-e_1}+2h_{e_1-e_2}+\dots+nh_{e_{n-1}-e_n}\rangle$, without loss of generality we can assume $x = h_{e_0-e_1}+2h_{e_1-e_2}+\dots+nh_{e_{n-1}-e_n}$. It is sufficient to consider $[x, x_{e_i-e_j}]$ for $e_i - e_j \in \Phi$, $i \neq j$ arbitrary, and without loss of generality assume $j > i$.

$$\begin{aligned}[x, x_{e_i-e_j}] = &\left[h_{e_0-e_1}+2h_{e_1-e_2}+\dots nh_{e_{n-1}-e_n}, x_{e_i-e_j}\right]\\ =&\Big(\langle e_0-e_1, e_i-e_j\rangle + 2\langle e_1-e_2, e_i-e_j\rangle\\ &+\dots+n\langle e_{n-1}-e_n, e_i-e_j\rangle\Big)x_{e_i-e_j}\\ =&\langle e_0+e_1+\dots+e_{n-1}-ne_n, e_i-e_j\rangle x_{e_i-e_j}.\end{aligned}$$

Now we distinguish two cases. Either $j \neq n$, and the previous expression is equal to $(1\cdot 1 - 1\cdot 1)x_{e_i-e_j} = 0$, or $j = n$ ($i \neq n$ follows from $j > i$), and we have $(1\cdot 1 + (-n)(-1))x_{e_i-e_j} = (n+1)\cdot x_{e_i-e_j} = 0$ since $\mathrm{char}(\mathbb{F}) \mid n+1$. Hence $x \in Z(\mathfrak{g}_\Phi)$.

- D_n, $\mathrm{char}(\mathbb{F}) = 2$: Here, without loss of generality let $x = h_{e_{n-1}-e_n} + h_{e_{n-a}+e_n}$, and $x_{\pm e_i \pm e_j} \in \mathfrak{g}_\Phi$, $i \neq j$ arbitrary. Now

$$\begin{aligned}[x, x_{\pm e_i\pm e_j}] = &\left[h_{e_{n-1}-e_n}+h_{e_{n-a}+e_n}, x_{\pm e_i \pm e_j}\right]\\ =&\langle e_{n-1}-e_n+e_{n-1}+e_n, \pm e_i \pm e_j\rangle x_{\pm e_i\pm e_j}\\ =&\langle 2e_{n-1}, \pm e_i \pm e_j\rangle x_{\pm e_i \pm e_j}\\ =&0\cdot x_{\pm e_i\pm e_j} = 0.\end{aligned}$$

So $x \in Z(\mathfrak{g}_\Phi)$.

- E_6, $\mathrm{char}(\mathbb{F}) = 3$. W.l.o.g., let $x = h_{e_3-e_4} - h_{e_4-e_5} + h_{e_6+e_7} - h_{-\frac{1}{2}\sum_{i=1}^8 e_i}$ be the element from the radical and $y \in \mathfrak{g}$ be arbitrary. We consider separately $y_1 = x_{\pm e_k \pm e_j}, j \neq k, j, k \in \{1, \dots, 8\}$, and $y_{\pm 2} = x_{\pm(\frac{1}{2}\sum_{j=1}^8 \varepsilon_k e_k)}$, where $\varepsilon_i = \pm 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_8 = 1, \prod_{i=1}^8 \varepsilon_i = 1$.

$$[x, y_1] = \left[h_{e_3-e_4} - h_{e_4-e_5} + h_{e_6+e_7} - h_{-\frac{1}{2}\sum_{i=1}^8 e_i}, x_{\pm e_k \pm e_j}\right]$$

$$
\begin{aligned}
&=\langle e_3 - e_4 - e_4 + e_5 + e_6 + e_7 + \frac{1}{2}\sum_{i=1}^{8} e_i, \pm e_k \pm e_j\rangle x_{\pm e_k \pm e_j}\\
&=\langle e_k + e_j + \frac{1}{2}(e_k + e_j), \pm e_k \pm e_j\rangle x_{\pm e_k \pm e_j}\\
&=\left(\pm\frac{3}{2} \pm \frac{3}{2}\right) x_{\pm e_k \pm e_j}\\
&=0 \cdot x_{\pm e_k \pm e_j} = 0.
\end{aligned}
$$

Now for $y_{\pm 2}$:

$$
\begin{aligned}
&[x, y_{\pm 2}]\\
&= \left[h_{e_3-e_4} - h_{e_4-e_5} + h_{e_6+e_7} - h_{-\frac{1}{2}\sum_{i=1}^{8} e_i}, x_{\pm\left(\frac{1}{2}\sum_{j=1}^{8}\varepsilon_k e_k\right)}\right]\\
&=\Big\langle e_3 - e_4 - e_4 + e_5 + e_6 + e_7 + \frac{1}{2}\sum_{i=1}^{8} e_i, \pm\left(\frac{1}{2}\sum_{j=1}^{8}\varepsilon_k e_k\right)\Big\rangle x_{\pm\left(\frac{1}{2}\sum_{j=1}^{8}\varepsilon_k e_k\right)}\\
&= \pm\Big(\frac{1}{4}\langle e_1 + e_2 + e_8, e_1 + e_2 + e_8\rangle\\
&\quad + \frac{3}{4}\langle e_3 + \cdots + e_7, \varepsilon_3 e_3 + \cdots + \varepsilon_7 e_7\rangle\Big) x_{\pm\left(\frac{1}{2}\sum_{j=1}^{8}\varepsilon_k e_k\right)}\\
&= \pm 6 \cdot \frac{3}{4} x_{\pm\left(\frac{1}{2}\sum_{j=1}^{8}\varepsilon_k e_k\right)} = 0.
\end{aligned}
$$

◇

PROPOSITION 3.4.9. *The extremal points $\langle x_\alpha\rangle$, with $\alpha \in \Phi$ a long root, are contained in a single connected component of the graph $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g}_\Phi)$.*

PROOF. Let α, β be long roots and $\langle x_\alpha\rangle$ and $\langle x_\beta\rangle$ be extremal points. If $\langle \alpha, \beta\rangle = \pm 2$, then $\langle x_\alpha\rangle$ and $\langle x_\beta\rangle$ are the same or adjacent.
If $\langle \alpha, \beta\rangle = -1$, so $(\alpha, \beta) \leq 0$, then $\alpha + \beta$ is also a root. This implies that $[x_\alpha, x_\beta] = N_{\alpha,\beta} x_{\alpha+\beta}$ with $N_{\alpha,\beta} \neq 0$. By 2.5.1, it follows that x_α and x_β span a Lie algebra of Heisenberg type (if they generate a $\mathfrak{sl}_2$, where $\langle \alpha, \beta\rangle = \pm 2$ holds, we are in the previous case). Now 2.5.1 also implies that all elements in $\langle x_\alpha, x_\beta\rangle$ except for $x_{\alpha+\beta}$ are extremal, so especially $x_\alpha + x_\beta$. This implies $g(x_{-\alpha}, x_\alpha + x_\beta) = g(x_{-\alpha}, x_\alpha) + g(x_{-\alpha}, x_\beta) = 1$ as well as $g(x_{-\beta}, x_\alpha + x_\beta) = 1$. We get the chain $x_\alpha \sim_{\mathfrak{sl}_2} x_{-\alpha} \sim_{\mathfrak{sl}_2} x_\alpha + x_\beta \sim_{\mathfrak{sl}_2} x_{-\beta} \sim_{\mathfrak{sl}_2} x_\beta$, which proves that x_α and x_β are in the same connected component of the $\mathfrak{sl}_2$-graph.
If $\langle \alpha, \beta\rangle = 1$, the same argument as in the previous case is applicable just replacing $\alpha + \beta$ by $\alpha - \beta$.
It remains to consider the case where $\langle \alpha, \beta\rangle = 0$, so the roots α and β are orthogonal to each other. If Φ is not of type C_n, we can find a long root γ

such that $\langle \alpha, \gamma \rangle \neq 0$ and $\langle \beta, \gamma \rangle \neq 0$ and then apply the above, to conclude that both $\langle x_\alpha \rangle$ and $\langle x_\beta \rangle$ are in the connected component of $\Gamma_{\mathfrak{sl}_2}$ containing $\langle x_\gamma \rangle$.
If Φ is of type C_n, the long roots are $\pm 2\varepsilon_i$ and the short roots are $\pm(\varepsilon_i \pm \varepsilon_j)$, where $1<i \leq j<n$. Note that in this case, we have the root lengths $\sqrt{2}$ and 2. We will show that all elements $x_{\pm 2\varepsilon_i}$ are in the same $\mathfrak{sl}_2$-component.
Obviously, pairwise the elements $x_{2\varepsilon_i}$ and $x_{-2\varepsilon_i}$ span an $\mathfrak{sl}_2$-subalgebra. We need to prove that all these $\mathfrak{sl}_2$-subalgebras are connected. By way of example, we consider the pairs $x_{2\varepsilon_1}, x_{-2\varepsilon_1}$ and $x_{2\varepsilon_2}, x_{-2\varepsilon_2}$. Obviously, the result is then also true for any other two pairs $x_{2\varepsilon_i}, x_{-2\varepsilon_i}$ and $x_{2\varepsilon_{i+1}}, x_{-2\varepsilon_{i+1}}$.
Define

$$d = \exp(x_{-\varepsilon_1-\varepsilon_2})(x_{2\varepsilon_1}) = x_{2\varepsilon_1} - x_{\varepsilon_1-\varepsilon_2} - x_{-2\varepsilon_2},$$

which is a long root element since $x_{2\varepsilon_1}$ is one.
Bilinearity of g gives

$$\begin{aligned} g(x_{-2\varepsilon_1}, d) =& g(x_{-2\varepsilon_1}, x_{2\varepsilon_1}) - g(x_{-2\varepsilon_1}, x_{\varepsilon_1-\varepsilon_2}) - g(x_{-2\varepsilon_1}, x_{-2\varepsilon_2}) \\ =& 1 + 0 + 0 \neq 0. \end{aligned}$$

Moreover

$$\begin{aligned} g(x_{2\varepsilon_2}, d) =& g(x_{2\varepsilon_2}, x_{2\varepsilon_1}) - g(x_{2\varepsilon_2}, x_{\varepsilon_1-\varepsilon_2}) - g(x_{2\varepsilon_2}, x_{-2\varepsilon_2}) \\ =& 0 + 0 + 1 \neq 0. \end{aligned}$$

So we have a chain $\langle x_{2\varepsilon_1} \rangle \sim_{\mathfrak{sl}_2} \langle x_{-2\varepsilon_1} \rangle \sim_{\mathfrak{sl}_2} \langle d \rangle \sim_{\mathfrak{sl}_2} \langle x_{2\varepsilon_2} \rangle \sim_{\mathfrak{sl}_2} \langle x_{-2\varepsilon_2} \rangle$. $\diamond$

PROPOSITION 3.4.10. *Let* $\bar{\mathfrak{g}} := \mathfrak{g}_\Phi / \mathrm{rad}(g)$. *Then* $\Gamma_{\mathfrak{sl}_2}(\bar{\mathfrak{g}})$ *is connected, and if* $\mathrm{char}(\mathbb{F}) \neq 2$*, then* $\bar{\mathfrak{g}}$ *is simple.*

PROOF. Assume that $x \in \bar{\mathfrak{g}}$ is extremal, but not in the same connected component as some element $x_\alpha \in \bar{\mathfrak{g}}$ with $\alpha \in \Phi^{long}$. Since $\mathfrak{g}_\Phi$ is generated by its long root elements, that are in one connected component by 3.4.9, it follows by 2.5.4 that $[x, \bar{\mathfrak{g}}] = 0$ and $x \in Z(\bar{\mathfrak{g}}) = \{0\}$. So $\Gamma_{\mathfrak{sl}_2}(\bar{\mathfrak{g}})$ is connected, and therefore $\bar{\mathfrak{g}}$ is simple, applying 2.5.4 and 2.5.7. $\diamond$

If $\mathrm{char}\,\mathbb{F} \neq 2$, we will determine all extremal elements in $\mathfrak{g}_\Phi$.
We assume $\mathrm{char}(\mathbb{F}) \neq 2$. Using the previous result, we conclude that each extremal element in $\bar{\mathfrak{g}}$ is in the $\mathrm{Aut}(\mathfrak{g}_\Phi)$-orbit of an element $t(x_\alpha + \mathrm{rad}(g))$ for some scalar $t \in \mathbb{F}$, where $\alpha \in \Phi$ a long root.
Considering an extremal element $x \in \mathfrak{g}_\Phi$, we know that also $x + \mathrm{rad}(g) \in \bar{\mathfrak{g}}$ is extremal. We have seen that if $\mathrm{char}(\mathbb{F}) \neq 2$, we have $Z(\mathfrak{g}_\Phi) = \mathrm{rad}(g)$ in all

cases except for G_2 in characteristic 3. This case, we will consider later. So instead of $x + \mathrm{rad}(g)$ we can write $x + Z(\mathfrak{g}_\Phi)$.
But any extremal element in $\bar{\mathfrak{g}}$, so in particular also $x + Z(\mathfrak{g}_\Phi)$, is in the orbit of $t(x_\alpha + Z(\mathfrak{g}_\Phi))$, $t \in \mathbb{F}$. W.l.o.g., we can assume $x + Z(\mathfrak{g}_\Phi) = x_\alpha + Z(\mathfrak{g}_\Phi)$. So $x - x_\alpha \in Z(\mathfrak{g}_\Phi)$. But this implies that $x = x_\alpha + z$ for some $z \in Z(\mathfrak{g}_\Phi)$.
Now we can choose some $y \in \mathfrak{g}_\Phi$ such that

$$\begin{aligned}[x_\alpha + z, [x_\alpha + z, y]] =&[x_\alpha + z, [x_\alpha, y]]\\ =&[x_\alpha, [x_\alpha, y]]\\ =&cx_\alpha\end{aligned}$$

for some $0 \neq c \in \mathbb{F}$ (which must exist, since $x_\alpha + z$ is a sandwich otherwise). But since $x = x_\alpha + z$ was supposed to be extremal, there also must be a $c' \in \mathbb{F}$ such that $cx_\alpha = c'(x_\alpha + z)$. So $z = 0$, and all extremal elements in $\mathfrak{g}_\Phi$ are long root elements.

COROLLARY 3.4.11. *Let $\mathfrak{g}_\Phi$ as before and $\mathrm{char}(\mathbb{F}) \neq 2$, and if $\mathrm{char}(\mathbb{F}) = 3$ assume that Φ is not of type G_2. Then all (non-sandwich) extremal elements of $\mathfrak{g}_\Phi$ are long root elements.*

Finally, we consider the exceptional case of a Chevalley Lie algebra of type G_2 in characteristic 3.

PROPOSITION 3.4.12. *For $\mathfrak{g}_\Phi$ with root system Φ of type G_2 over a field $\mathbb{F}$ of characteristic 3, the short root elements are in $\mathrm{rad}(g)$ and $\mathfrak{g}_\Phi/\mathrm{rad}(g)$ is simple. The extremal elements in $\mathfrak{g}_\Phi$ are the long root elements.*

PROOF. We have already seen that the short root elements are in $\mathrm{rad}(g)$ in 3.4.6. Moreover, $\mathfrak{g}_\Phi/\mathrm{rad}(g)$ is of type A_2 (modulo center). Let $x \in \mathfrak{g}_\Phi$ be extremal. Then $x = x_\beta + r$, where $r \in \langle x_\alpha, h_\alpha | \alpha \in \Phi^{short}\rangle = \mathrm{rad}(g)$, and $\beta \in \Phi^{long}$. We show that r must be zero here. W.l.o.g. we can assume that the long root is indeed the one denoted by β in 3.2.4, just by symmetry of the root system. We have

$$\begin{aligned}r =&c_\alpha x_\alpha + c_{-\alpha}x_{-\alpha} + c_{\alpha+\beta}x_{\alpha+\beta} + c_{-\alpha-\beta}x_{-\alpha-\beta}\\ &+ c_{2\alpha+\beta}x_{2\alpha+\beta} + c_{-2\alpha-\beta}x_{-2\alpha-\beta} + d_\alpha h_\alpha + d_{\alpha+\beta}h_{\alpha+\beta} + d_{2\alpha+\beta}h_{2\alpha+\beta},\end{aligned}$$

where $c_\alpha, c_{-\alpha}, c_{\alpha+\beta}, c_{-\alpha-\beta}, c_{2\alpha+\beta}, c_{-2\alpha-\beta}, d_\alpha, d_{\alpha+\beta}, d_{2\alpha+\beta} \in \mathbb{F}$.

If $x_\beta + r$ is extremal, the equality

$$[x_\beta + r, [x_\beta + r, y]] = k(x_\beta + r)$$

must hold for all $y \in \mathfrak{g}_\Phi$ and some $k \in \mathbb{F}$ depending on y. A tedious, but straightforward computation filling in $x_{-\beta}, x_\alpha, x_{-\alpha-\beta}, x_{3\alpha+\beta}$ in place of y and comparing the coefficients on both sides of the resulting equation leads to $c_\alpha = c_{-\alpha} = c_{\alpha+\beta} = c_{-\alpha-\beta} = c_{2\alpha+\beta} = c_{-2\alpha-\beta} = d_\alpha = d_{\alpha+\beta} = d_{2\alpha+\beta} = 0$, so indeed $x_\beta + r$ is extremal only if $r = 0$, as required. $\diamond$

REMARK 3.4.13. Note that in a Lie algebra $\mathfrak{g}_\Phi$ of type A_2 over a field $\mathbb{F}$ of characteristic 3, there is a non-trivial center $Z = \operatorname{rad}(g)$ containing sandwich elements, see 3.4.6 and 3.4.8. As we have seen before, $\bar{\mathfrak{g}} := \mathfrak{g}_\Phi / \operatorname{rad}(g)$ is isomorphic to $\bar{\mathfrak{g}}' := \mathfrak{g}'_\Phi / \operatorname{rad}(g')$, where $\mathfrak{g}'_\Phi$ is of type G_2 (and g' the corresponding extremal form). So, the extremal elements of $\bar{\mathfrak{g}}'$ and $\bar{\mathfrak{g}}$ are the same and come from $\bar{\mathfrak{g}}'$. With the previous result, it follows that they are the long root elements.

CHAPTER 4

Buildings and geometries

In this chapter, we lay the groundwork for the subsequent chapters. It is a collection of definitions and results that will, all together, enable us to give a geometric characterization of the Lie algebras considered in the following two chapters. In the first section, the basic concepts of graphs, Coxeter systems and buildings are introduced, based on the fundamental book of R. Weiss [Wei03]. We use it to define root shadow spaces, the subject of an important result of A. Cohen and G. Ivanyos about the extremal geometry of Lie algebras and central in Chapter 5.
In the second section, we define point-line spaces and consider their properties, followed by the central result of [Cuy94]. We use this in Chapter 6 for the special consideration of the symplectic Lie algebras.
Section 3 prepares the use of the main results of [CI06] and [CI07], giving the definition and some examples of root filtration spaces.
Finally in section 4, we consider embeddings of point-line spaces into projective spaces and deduce some helpful properties; we close with the main statement of [KS01].

4.1. Buildings

We have already been concerned with graphs in chapter 2, but we start here by giving their formal definitions to implement the notation for subsequent introduction of buildings and root shadow spaces. We follow [Wei03].

Definition 4.1.1. A **graph** Γ is a pair (V, E) of two sets V, E where the elements of V are called **vertices** and the elements of E are pairs (v, w) of vertices $v, w \in V$ and are called **edges**. If an edge (v, w) exists in E, we say that $v, w \in V$ are **joined by an edge** or that they are **adjacent**, and write $v \sim w$.
A **subgraph** $\Gamma' \subseteq \Gamma = (V, E)$ is a pair (V', E'), where $V' \subseteq V$ and $E' \subseteq E$, and is moreover a **induced subgraph** if for all $v, w \in V'$ with $(v, w) \in E$

also $(v, w) \in E'$. A **path** in Γ is a sequence $v_1, v_2, \ldots, v_n$ $(n \in \mathbb{N})$ of elements of V with the property that $v_1 \sim v_2 \sim \cdots \sim v_n$. The number of edges going through a vertex $v \in V$ is called the **valency** of v; it is the cardinality of the set $\{(x, y) \in E | x = v \text{ or } y = v\}$.

Consider an **index set** I, where we usually choose $I = \{1, \ldots, n\}$, and whose elements we call **colours**. Then an **edge-coloured graph** $\Gamma = (V, E)$ is a graph where there is an element $i \in I$ assigned to each edge $(v, w) \in E$, in which case we write $v \sim_i w$, and say that v and w are i**-adjacent**. Considering a subset $J \subseteq I$, a connected component of the graph obtained from Γ by deleting all edges in E labelled with a colour in $I \setminus J$ is called a J**-residue** of Γ. In the special case where $J = \{j\}$, we call a J-residue of just one colour a j**-panel** of Γ. The cardinality of J is called the **rank** and the cardinality of $I \setminus J$ the **corank** of a J-residue.

An **isomorphism** of two edge-coloured graphs $\Gamma = (V, E), \Gamma' = (V', E')$ with the same index set I is a pair of bijections (ϕ, σ) such that $\phi : V \to V'$ is a bijection and $\sigma : I \to I$ such that the vertices v and w in V are i-adjacent if and only if the vertices $\phi(v)$ and $\phi(w)$ in V' are $\sigma(i)$-adjacent, in symbols

$$v \sim_i w \Leftrightarrow \phi(v) \sim_{\sigma(i)} \phi(w).$$

If hereby σ is the identity, we call the isomorphism **special**.

A **chamber system** Δ is an edge-coloured graph $\Gamma = (V, E)$ with index set I, where the elements of V are called **chambers** and for all $i \in I$, the i-panels in Γ are complete graphs with at least two vertices. A chamber system is called **thick** if each panel has at least three chambers and **thin** if each panel has exactly two chambers. A **gallery** is a path in a chamber system and the **distance** of two chambers $v, w \in V$, denoted by $\operatorname{dist}(v, w)$, is the length of a minimal gallery between v and w. A subset of chambers X is called **convex** if every minimal gallery of points $v, w \in X$ is also contained in X, and the **diameter** of X is defined as $\operatorname{diam}(X) := \sup\{\operatorname{dist}(v, w) | v, w \in X\}$. A chamber system which is an edge-coloured induced subgraph of a chamber system Δ and preserves the colours of Δ is a **subchamber system**.

In the previous chapter, we already introduced Weyl groups, which are a special, namely the finite, case of Coxeter groups. Recall the definition of **Coxeter systems** (W, S), **Coxeter Diagrams** Π and the **Coxeter matrix**

(m_{ij}) in 3.1.5. The Coxeter group is a generalization of the Weyl group introduced there, omitting the finiteness assumption.

Definition 4.1.2. Let $I = \{1, \dots, n\}$ be an index set and m_{ij} a Coxeter matrix with entries $m_{ij} \geq 2$ and $m_{ii} = 1$ for all $i, j \in I$. Then a **Coxeter group** W is a group having a set of generators $\{r_i | i \in I\}$ indexed by I such that W is defined by the relations

$$W = \langle r_i | (r_i r_j)^{m_{ij}} = 1 \text{ for all } i, j \in I, m_{ij} \neq \infty \rangle.$$

In particular, $r_i^2 = 1$ for all $i \in I$.

Since Coxeter group and Coxeter diagram determine each other (up to automorphism of the diagram), we consider them as a pair and say that W is the Coxeter groups of type Π and (W, S) is the Coxeter System of type Π.
Let W_J for $J \subset I$ be a subgroup of W generated by $S_J = \{r_j | j \in J\}$. We define Π_J to be the subgraph of Π obtained by deleting the vertices $I \setminus J$, and we get the Coxeter system (W_J, S_J) of type Π_J (for a proof, see [Bou68, Ch. iv, §1.8 Thm. 2]).

Definition 4.1.3. For a Coxeter system (W, S) of type Π, we define the **Coxeter chamber system** Σ_Π, having as chambers the elements of W and two chambers x and y are i-adjacent if and only if $x r_i = y$ for $r_i \in S$.

The group of special automorphisms of a Coxeter chamber system Σ_Π is denoted by $\mathrm{Aut}^\circ(\Sigma_\Pi)$. Notice that left multiplication by an arbitrary element of W is a special automorphism of Σ_Π, and moreover $\mathrm{Aut}^\circ(\Sigma_\Pi) \cong W$ (for a proof, see e.g. [Wei03, 2.8]). Using this identification, a **reflection** is a nontrivial element $s \in W$ of order two, i.e. s interchanges two chambers of an edge. The set of edges that is fixed by a reflection s is called the **wall** of s and denoted by M_s. The complimentary set $\Gamma \setminus M_s$ of the wall of s in the graph has two connected components, called **half-apartments**.
Equipped with all this nomenclature, we are now able to introduce those types of graphs that we are actually concerned about in this thesis.

Definition 4.1.4 (according to [Wei09, Thm 29.35]). Let W be a Coxeter group of type Π and let I be the vertex set of Π. A **building** of type Π with index set I is a chamber system Δ with index set I with a collection of subchamber systems A called **apartments** such that

(1) Each Σ in A is isomorphic to the Coxeter chamber system Σ_Π.
(2) Each pair of chambers x, y is contained in a common apartment.
(3) For each pair of chambers x, y and each pair of apartments Σ, Σ' containing both x and y , there exists a special isomorphism from Σ to Σ' that fixes x and y.
(4) For each chamber x and each pair of apartments Σ, Σ' that contain x and each panel P such that $P \cap \Sigma$ and $P \cap \Sigma'$ are nonempty, there exists a special isomorphism that fixes x and sends $P \cap \Sigma$ to $P \cap \Sigma'$.

A building is called **spherical** if its apartments have finite diameter, **thick** (resp. **thin**) if the underlying chamber system is thick (resp. thin), **irreducible** if the corresponding diagram Π is connected and **reducible** if Π is not connected. The **rank** of a building is the cardinality of the index set I.

REMARK 4.1.5. Let Σ be a Coxeter chamber system of type Π. Then Σ is a thin building of type Π whose collection of apartments is $\{\Sigma\}$. The properties of a building follow easily since there is only one apartment in the building.

The following result is well known, a proof can be found in [Wei03, Chapter 12].

THEOREM 4.1.6. *Let Δ be a thick irreducible spherical building of type Π and rank at least 3. Then Π is A_n for $n \geq 2$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4 or G_2.*

In the following, we specify the type of a building Δ if it is known according to the previous theorem and say that Δ is a **building of type** X_n.
The following definition is useful for the construction of an example.

DEFINITION 4.1.7. The **Witt index** of a quadratic form κ of a vector space V over a field $\mathbb{F}$ is the maximum dimension of a linear subspace of V on which κ vanishes.

EXAMPLE 4.1.8. Let (V, f) be a pair of a vector space V over the field $\mathbb{F}$ and a form f on V. We consider all subspaces V_i of V that are singular with respect to f, i.e. $f|_{V_i} = 0$. The maximal chain of such singular subspaces can be written as a flag geometry on subspaces V_i of V with the incidence relation of inclusion. This means that a chain of inclusions

$$V_1 \subset V_2 \subset V_3 \subset \ldots V_{n-1} \subset V_n$$

is represented by

$$\underset{V_1}{\circ}\!-\!\!-\!\!-\underset{V_2}{\circ}\!-\!\!-\!\!-\underset{V_3}{\circ}\!-\!\!-\cdots-\!\!-\underset{V_{n-1}}{\circ}\!-\!\!-\!\!-\underset{V_n}{\circ}$$

A_n Here, we choose the form f on V just to be trivial, so $f \equiv 0$. Therefore, the singular subspaces w.r.t. f are simply *all* subspaces of V. The obtained flag complex is a building of type A_n.
Conversely, a spherical building of type A_n with $n \geq 3$ is isomorphic to the flag complex of an n-dimensional projective space over $\mathbb{F}$.

$\mathrm{B}_n = \mathrm{C}_n$ Now, let f be non-degenerate quadratic form of Witt index $n \geq 2$ (if V is of odd dimension $2n+1$) or a symplectic form (if V is of even dimension $2n$). The singular subspaces w.r.t. these forms give rise to a building of type $\mathrm{B}_n = \mathrm{C}_n$, that we therefore denote in the following by BC_n (we obtain the Coxeter diagram, with is the Dynkin diagram without arrows, so there is no distinction between B_n and C_n).

D_n Assume that V is a vector space of even dimension $2n$ and f is a non-degenerate quadratic form of Witt index n. The flag complex of the singular subspaces w.r.t. f is a building of type D_n. We denote this incidence system by $\Gamma(V)$. Assume moreover that each residue of $\Gamma(V)$ of type n has size 2.
Then the corresponding dual polar graph, which is the graph having maximal singular subspaces as vertices that are connected by an edge if and only if they have a common geometric hyperplane, is the disjoint union of two cocliques C_1 and C_2 (see [BC13, Lemma 7.8.4]). The **oriflamme geometry** $\Delta(V)$ of V is the incidence system over $\{1, \ldots, n\}$ whose elements of type $i = 1, \ldots, n-2$ are of the same type and with the same incidence as in $\Gamma(V)$. The elements of type $n-1$ (respectively, n) of $\Delta(V)$ are the members of C_1 (respectively, C_2). Elements in $C_1 \cup C_2$ are incident in $\Delta(V)$ if and only if their corresponding maximal subspaces of V have a common geometric hyperplane. Now a spherical building of type D_n with $n \geq 4$ is the chamber system of the oriflamme geometry (see [BC13] for details).

To construct the apartments for these buildings in this example, we introduce **frames** for the vector space V. A frame for (V, f) is a hyperbolic basis $\{v_i\}_{i \in I}$ (with respect to the form f) determined up to scalar multiplication. So a frame determines a set of one-dimensional singular subspaces $L_i := \{\mathbb{F}v_i\}_{i\in I}$, and any subset consisting of k of these subspaces generates a k-dimensional singular subspace. An ordered frame $L_1, \dots, L_n$ defines a complete flag

$$U_i := L_1 \oplus \cdots \oplus L_i.$$

Any reordering of the spaces $\{L_i\}_{i\in I}$ also gives a frame. So the subspaces obtained as sums of the L_i's form the apartments of the building.

The new point of view on our geometry uses points and lines instead of vertices and edges. We deepen this in the following section. This enables us to give examples of spherical buildings and root shadow spaces of the Coxeter families introduced in the previous chapter.

4.2. Point-line spaces

We introduce a new geometric structure of central importance.

Definition 4.2.1. A **point-line space** $(\mathcal{P}, \mathcal{L})$ is a pair of a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines, where each element of $\mathcal{L}$ is a subset of $\mathcal{P}$ of size at least two.
It is called a **partial linear space** if any two points are on at most one line, and a **linear space** if any two points are on exactly one line.
Let $(\mathcal{P}, \mathcal{L})$ be a point-line space. The **collinearity graph** of $(\mathcal{P}, \mathcal{L})$ is the graph where two (possibly coinciding) points in $\mathcal{P}$ are connected if and only if there is a line in $\mathcal{L}$ containing both of these points.
Two points p, q of $(\mathcal{P}, \mathcal{L})$ are called **collinear** if they are adjacent in the collinearity graph, and the line through them is denoted by pq in this case. Notice that in this definition, a point is not collinear with itself. The set of points that is collinear with a point $p \in \mathcal{P}$ is denoted by $p^{\sim}$.
The point-line space $(\mathcal{P}, \mathcal{L})$ is **connected point-line space** if and only if the collinearity graph is connected.
A **subspace** of $\mathcal{P}$ is a subset $\mathcal{P}'$ of $\mathcal{P}$ such that whenever p and q are two collinear points of $\mathcal{P}'$ are on a line $l \in \mathcal{L}$, then l is fully contained in $\mathcal{P}'$. So if $\mathcal{P}'$ is a subspace of $(\mathcal{P}, \mathcal{L})$ then $\mathcal{P}'$ together with the set of lines in $\mathcal{L}$ that meet $\mathcal{P}'$ in at least 2 points forms a partial linear space. It is clear that the intersection of any collection of subspaces is again a subspace, and we define for any subset $\mathcal{X}$ of $\mathcal{P}$ the subspace **generated by** $\mathcal{X}$ to be the intersection of all subspaces containing $\mathcal{X}$ and denote it by $\langle \mathcal{X} \rangle$.

If any two points of a subspace of a space are collinear, we call it a **singular subspace**, and the **singular rank** of the space is the supremum of all ranks of maximal singular subspaces.
If we have a point-line space $(\mathcal{P}, \mathcal{L})$ and if $n \in \mathbb{N}$ is the minimal number of generating elements of $(\mathcal{P}, \mathcal{L})$, then n is the **generating rank** of $(\mathcal{P}, \mathcal{L})$.

We now relate some partial linear spaces to buildings. For an irreducible building Δ of type X_n, we denote by Φ the root system of the corresponding Dynkin diagram, as defined in 3.1.2. Since the buildings of type B_n and C_n are equal, we agree to take the roots from B_n. We can choose a root α of maximal length form a set of fundamental roots $\{\alpha_i\}$ with $i \in I$, and define the subset $j \subseteq I$ such that J consists of all $i \in I$ with $\langle -\alpha, \alpha_i \rangle \neq 0$.
In the Dynkin diagram of type Π whose vertices are numbered by $\{1, \ldots, n\}$, we add a new node numbered 0 and connect this new node with the vertices carrying a number of the set J as defined above. The **root nodes** of X_n are exactly these nodes in J.
Equipped with all this notation, we can construct a point-line space on a building of type X_n.

DEFINITION 4.2.2. Let Δ be an irreducible spherical building of type X_n with root nodes in the set J. We construct a point-line geometry $(\mathcal{E}, \mathcal{F})$ on Δ, where there points are called ***J*-shadows**, defined to be the $(I \setminus J)$-residues. The lines are the sets of all J-shadows that contain chambers form a given j-panel, called the ***j*-lines**, for $j \in J$. The point-line space $(\mathcal{E}, \mathcal{F})$ is the **root shadow space** of type $X_{n,J}$, or $X_{n,j}$, if $J = \{j\}$.

EXAMPLE 4.2.3. We pick up the cases considered in Example 4.1.8 again.

A_n We have seen that any building of type A_n is associated to the flag complex of an n-dimensional projective space and vice versa. A building of type A_n is the only irreducible case where the set J of root nodes has more than one element, namely $J = \{1, n\}$. This implies that, regarding the flag of singular subspaces as in 4.1.8, the points of the corresponding root shadow space can be identified with incident point-hyperplane pairs of a projective geometry of rank n. The lines are of two different types, namely the sets of incident point-hyperplane pairs (p, H) where p runs over a projective line, and H is fixed, or dually, where p is fixed and H runs through the set of hyperplanes containing a fixed codimension 2 subspace. See also example 4.3.2.

BC_n Examples of root shadow space of type $\mathrm{BC}_{n,1}$ can be obtained from a vector space V equipped with a (nondegenerate) sesquilinear form f whose singular subspaces give rise to a building of type BC as in 4.1.8. The points and lines are the singular 1 and 2-dimensional subspaces of V with respect to the form f. In these partial linear spaces a point p is collinear to one or all points of a line.

Examples of root shadow spaces of type $\mathrm{BC}_{n,2}$ are the point-line space where the points are the singular 2-spaces. A line of the root shadow space is then the sets of all singular 2-spaces on a singular point 1-space and contained in a singular plane 3-space. We consider the resulting geometric structure somewhat closer in 4.3.3.

D_n Here again, we have the root node $J = \{2\}$. Starting with the oriflame geometry of singular subspaces of an orthogonal geometry of type D_n, root shadow spaces of type $\mathrm{D}_{n,i}$ can be obtained in a similar way as those of type $\mathrm{BC}_{n,i}$ for $i \in \{1, 2\}$ as described above, see 4.3.3.

As we have seen above, buildings give rise to partial linear spaces. These partial linear spaces have been used to give geometric characterizations of buildings, just in terms of points and lines. With discuss some of these results.

A **projective plane** is a point-line space with the following properties:

(1) For any two distinct points, there is exactly one line containing both of them.
(2) For any two distinct lines, there is exactly one intersecting point.
(3) There are three distinct points such that no line contains more than two of them.

If all lines in a projective plane have the same number r of points, r is said to be the **order** of the projective plane.

Clearly, the points and lines from a projective space $\mathbb{P}(V)$ of some vector space V form a partial linear space in which any two intersecting lines generate a subspace isomorphic to a projective plane. One of the earliest and most famous results on partial linear spaces, the Veblen and Young Theorem, characterizes projective spaces, and hence buildings of type A_n, by this property:

THEOREM 4.2.4 (Veblen and Young). *Let* $(\mathcal{P}, \mathcal{L})$ *be a connected partial linear space such that*

(1) *all lines contain at least* 3 *points;*
(2) *any two intersecting lines generate a subspace isomorphic to a projective plane;*
(3) *there are two lines in* $\mathcal{L}$ *that do not intersect.*

Then $(\mathcal{P}, \mathcal{L})$ *is isomorphic to the partial linear spaces of* 1- *and* 2-*dimensional subspaces of some vector space* V.

A **polar space** is a partial linear space $(\mathcal{P}, \mathcal{L})$ satisfying the so-called 'one-or-all' or Buekenhout-Shult axiom:

A point p is collinear with one or all points of a line ℓ.

If p, q are points of a polar space, then by $p \perp q$ we denote that $p = q$ or p and q are collinear. By $p^\perp$ we denote the set of all points collinear to p. A polar space is called **nondegenerate** if $p^\perp \neq \mathcal{P}$ for all points $p \in \mathcal{P}$.

As we have seen in example 4.2.3, buildings of type BC and D related to sesquilinear and quadratic forms give rise to polar spaces. More generally, given a vector space equipped with a sesquilinear form f, one can construct a polar space whose points are the singular 1-spaces and whose lines are the singular 2-spaces of V.
Building upon work of Veldkamp and Tits, Buekenhout and Shult [BS74] showed that under some weak restrictions, the converse is also true. See also the work of Johnson [Joh90] and Cuypers, Johnson, and Pasini [CJP93].

THEOREM 4.2.5. *Let* $(\mathcal{P}, \mathcal{L})$ *be a nondegenerate polar space such that*

(1) *all lines have at least* 3 *points;*
(2) *there exist two nonintersectiong lines* l, m *such that* $p \perp q$ *for all* $p \in l, q \in m$.

Then $(\mathcal{P}, \mathcal{L})$ *is isomorphic to the polar space of* 1- *and* 2-*dimensional singular subspaces of vector space* V *with respect to a sequilinear or pseudoquadratic form on* V.

Here, a pseudoquadratic form generalizes the concept of a quadratic form.

The results of Buekenhout and Shult characterize geometries on singular subspaces with respect to sesquilinear forms. The following result, due to Cuypers [Cuy94], provides a characterization of symplectic spaces in terms of nonsingular 2-spaces.
Before we state the result, we give some definitions.

Definition 4.2.6. Let V be a vector space equipped with a nontrivial symplectic form f. Then denote by $\mathcal{P}$ the set of 1-dimensional subspaces of V outside the radical of f. A hyperbolic line of V is the set of 1-spaces of a 2-space of V on which f is nondegenerate. By $HSp(V, f)$ we denote the partial linear space $(\mathcal{P}, \mathcal{L})$, where $\mathcal{L}$ is the set of all hyperbolic lines of V. We call $HSp(V, f)$ the **geometry of hyperbolic lines** of (V, f).

Definition 4.2.7. A projective plane from which a single line and all points on that line are removed is called an **affine plane**. A **dual affine plane**, also called **symplectic plane**, is a projective plane from which a single point and all lines through this point are removed.
A (dual) affine plane corresponding to a projective plane of order r is also of order r.

It is straightforward to check that inside the geometry of hyperbolic lines of a symplectic space two intersecting lines generate a symplectic plane. But, there are more examples of partial linear spaces with this property. Indeed, if one considers a projective space $\mathbb{P}$ and removes from it all the points and lines that are in or meet a fixed codimension 2 space nontrivially, then what is left is again a partial linear space in which any two intersecting lines generate a symplectic plane. In case the projective space $\mathbb{P}$ is the projective space of a vector space V over a commutative field, this does not provide new examples. However, if the underlying field is not commmuative it does.
Now we can state Cuypers' result:

Theorem 4.2.8 ([Cuy94], Thm. 1.1). *Let $(\mathscr{P}, \mathscr{L})$ be a connected partial linear space such that*

(1) *all lines contain at least* 4 *points;*
(2) *any pair of intersecting lines is contained in a subspace isomorphic to a symplectic plane;*

(3) *there are two lines in* $\mathcal{L}$ *that do not intersect.*

Then $(\mathscr{P}, \mathscr{L})$ *is isomorphic to the geometry of hyperbolic lines of a symplectic* (V, f) *or to the space of points and lines of a projective space* $\mathbb{P}(V)$*, where* V *is of a vector space over a noncommutative field, missing a codimension* 2 *subspace.*

4.3. Root filtration spaces

In the following section, we introduce an additional structure on partial linear spaces. This leads to the main results of [CI06], [CI07]. We follow their notation.

DEFINITION 4.3.1. Let $(\mathcal{E}, \mathcal{F})$ be a partial linear space. For $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ a quintuple of symmetric relations partitioning $\mathcal{E}\times\mathcal{E}$, we call $(\mathcal{E}, \mathcal{F})$ a **root filtration space** with filtration $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ if the following properties are satisfied, where we write $\mathcal{E}_{\leq i}$ for $\cup_{j\leq i}\mathcal{E}_j$.

(A) The relation $\mathcal{E}_{-2}$ is equality on $\mathcal{E}$.
(B) The relation $\mathcal{E}_{-1}$ is collinearity of distinct points of $\mathcal{E}$.
(C) There is a map $\mathcal{E}_1 \to \mathcal{E}$, denoted by $(u, v) \mapsto [u, v]$, such that, if $(u, v) \in \mathcal{E}_1$ and $x \in \mathcal{E}_i(u) \cap \mathcal{E}_j(v)$, then $[u, v] \in \mathcal{E}_{\leq i+j}(x)$.
(D) For each $(x, y) \in \mathcal{E}_2$, we have $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$.
(E) For each $x \in \mathcal{E}$, the subsets $\mathcal{E}_{\leq -1}(x)$ and $\mathcal{E}_{\leq 0}(x)$ are subspaces of $(\mathcal{E}, \mathcal{F})$.
(F) For each $x \in \mathcal{E}$, the subset $\mathcal{E}_{\leq 1}(x)$ is a geometric hyperplane of $(\mathcal{E}, \mathcal{F})$.

We call a pair $(x, y) \in \mathcal{E}_i$ **hyperbolic** if $i = 2$, **special** if $i = 1$, **polar** if $i = 0$, **collinear** if $i = -1$ (that means that only distinct points are considered to be collinear) and **commuting** if $i \leq 0$.
According to previous definitions, the **collinearity graph** of $(\mathcal{E}, \mathcal{F})$ is the graph whose vertices are the points in $\mathcal{E}$ and two vertices x, y are joined by an edge if and only if they are contained in a common line, so if $(x, y) \in \mathcal{E}_{-1}$, so we can denote it by $(\mathcal{E}, \mathcal{E}_{-1})$. Two points joined by an edge inside $(\mathcal{E}, \mathcal{E}_{-1})$ are called **neighbours**.
If $(\mathcal{E}, \mathcal{F})$ satisfies additionally the following two conditions, it is called a **non-degenerate** root filtration space.

(G) For each $x \in \mathcal{E}$, the set $\mathcal{E}_2(x)$ is not empty.
(H) The collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.

We describe in detail the filtrations of the root shadow spaces of type $\mathrm{A}_{n,\{1,n\}}$ and $\mathrm{BC}_{n,2}$, $\mathrm{D}_{n,2}$ from the examples 4.2.3.

EXAMPLE 4.3.2. We consider the space $\mathcal{E} = \{(p,H)|p \in H\}$ of point-hyperplane pairs of a projective space, where collinearity of $(p,H),(q,K) \in \mathcal{E}$ is given if $p = q$ or $H = K$. The lines in this space are given as follows: Let $(p,H),(q,K)$ be collinear, then the line through them consists of all points (r,M) with $r \in \langle p,q\rangle$, the line on p and q in the underlying projective space. This is the root shadow space of type $\mathrm{A}_{n,\{1,n\}}$, as introduced in 4.2.3, provided the dimension of the underlying vector space is $n+1$. We show that it is also a root filtrations space. The relations on pairs of points $x := (p,H), y := (q,K)$ are defined as follows:

(-2) $x \sim_{-2} y \Leftrightarrow p = q, H = K$
(-1) $x \sim_{-1} y \Leftrightarrow p = q$ or $H = K$ but not $(p,H) = (q,K)$
(0) $x \sim_0 y \Leftrightarrow p \in K, q \in H,$ but $H \neq K, p \neq q$
(1) $x \sim_1 y \Leftrightarrow q \in H$ but $p \notin K$ or $p \in K$ but $q \notin H$
(2) $x \sim_2 y \Leftrightarrow p \notin K, q \notin H$.

The properties (A) and (B) of 4.3.1 are fulfilled by construction of the space. For property (C), assume that $(x,y) \in \mathcal{E}_1$, and furthermore w.l.o.g. assume that $q \in H$ but $p \notin K$. We define $[x,y] := (q,H) \in \mathcal{E}$. Now consider some $z \in \mathcal{E}_i(x) \cap \mathcal{E}_j(y)$, say $z = (r,L) \in \mathcal{E}$. We have to show that $(q,H) \in \mathcal{E}_{i+j}(z)$. Therefore, we consider the possible cases for $i,j \in \{-2,\dots,2\}$ that can occur, and w.l.o.g. assume $i \leq j$. If $i = -2$, then $j = -2$ gives the trivial case with $x = y = z$ and there is nothing to show. The only other possible case if $i = -2$ and therefore $x = z$ is $j = 1$, since $(x,y) \in \mathcal{E}_1$. But $z = x$ implies that $(q,H) = [x,y] \sim_{-1} z = (p,H)$, so indeed $[x,y] \in \mathcal{E}_{\leq -1}(z)$. If $i = -1 = j$, we have by definition of x that $r = p$ of $L = H$ and by definition of y that $r = q$ or $L = K$. The only combination of these assumptions that does not lead to a contradiction is $L = H$ and $r = q$, which implies that $(q,H) \in \mathcal{E}_{-2}(r,L)$. Now consider $i = -1$, so $r = p$ or $L = H$, and $j = 0$ implying $r \in K, q \in L$ but $L \neq K$ and $q \neq r$. The required $(q,L) \in \mathcal{E}_{\leq -1}(r,L)$ is equivalent to $q = r$ or $H = L$, and the latter is obviously fulfilled. The next possible combination $i = j = 0$ implies that $r \in H, p \in L$ but $H \neq L, p \neq r$ as well as $r \in K, q \in L$ but $K \neq L, r \neq q$. So in particular $q \in L, r \in H, H \neq L, r \neq q$, which is equivalent to $(q,H) \in \mathcal{E}_0(r,L)$. Next, let $= -1$ and $j = 1$. The first one implies $r = p$ or $L = H$ and the second $r \in K$ but $q \notin L$ or

$q \in L$ but $r \notin K$. The only combinations of these cases that do not lead to a contradiction are $r = p, r \in K, q \notin L$ and $r = p, q \in L, r \notin K$, and both imply that $(q, H) \in \mathcal{E}_{\leq 0}(r, L)$. The last (nontrivial) case to consider is $i = 0, j = 1$. We have $p \in L, r \in H, H \neq L, r \neq p$ and either $q \in L, r \notin K$ or $q \notin L, r \in K$. Since the $\mathcal{E}_2$-case requires $r \notin H$, it cannot occur and we have $(q, H) \in \mathcal{E}_{\leq 1}(r, L)$.

Considering property (D), assume $(x, y) \in \mathcal{E}_2$. Then by definition $p \notin K$ and $q \notin H$. Now suppose that $z := (s, M) \in \mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) \neq \emptyset$. By $z \in \mathcal{E}_{\leq 0}(x)$, it follows $s \in H$ and $p \in M$. By $(s, M) \in \mathcal{E}_{\leq -1}(y)$, it follows either $q = s$ or $M = K$. In the first case, we have by assumption that $s = q \notin H$, a contradiction. In the second case, it follows $p \in M = K$, also a contradiction. So it follows $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$. Property (E) is obviously fulfilled, since $\mathcal{E}_{\leq -1}(x)$ is the set of all hyperplanes containing a distinct point p if $x = (p, H)$, and $\mathcal{E}_{\leq 0}(x)$ is the intersection of all hyperplanes containing p. The subset $\mathcal{E}_{\leq 1}(x)$ is the set of all hyperplanes having nonempty intersection with H for $x = (p, H)$. This is a geometric hyperplane of $\mathcal{E}$ and (F) holds.

Example 4.3.3. Here, we consider the root shadow spaces of type $\mathrm{BC}_{n,2}$ and $\mathrm{D}_{n,2}$, as described in 4.2.3. Notice that the root shadow spaces of type $\mathrm{BC}_{n,1}$ are the polar spaces themselves. Let $(\mathcal{P}, \mathcal{E})$ be a nondegenerate polar space. Recall that we defined in 4.2.3 the point-line space $(\mathcal{E}, \mathcal{F})$ by taking the lines of the polar space as our points (so the points here are the lines in the $\mathrm{BC}_{n,1}$ type) and $\mathcal{F}$ to be the set of all lines through a point p in a singular plane π. The collinearity relation for two elements $l, m \in \mathcal{E}$ is that l, m must span a singular plane. We define the following relations:

(-2) $l \sim_{-2} m \Leftrightarrow l = m$.
(-1) $l \sim_{-1} m \Leftrightarrow l, m$ span a singular plane.
(0) $l \sim_0 m \Leftrightarrow l, m$ either span a singular subspace not contained in a plane or l, m intersect but do not span a singular plane.
(1) $l \sim_1 m \Leftrightarrow$ there is a unique line n such that the span of n and l and the span of n and m are singular planes. Define $n := [l, m]$ in this case.
(2) $l \sim_2 m$ if none of the previous cases occurs.

This defines a root filtration space on $(\mathcal{E}, \mathcal{F})$, as one can deduce following the same steps as in 4.3.2.

The following Lemma follows from several technical results in [CI06]. A proof can be found in [Rob12], Lemma 4.2.8.

LEMMA 4.3.4. *Let $(\mathcal{E}, \mathcal{F})$ be a nondegenerate root filtration space. Then its defining relations can be characterized by the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ in the following way.*

(-2) $(x, y) \in \mathcal{E}_{-2}$ *if and only if* $x = y$.
(-1) $(x, y) \in \mathcal{E}_{-1}$ *if and only if x and y are distinct collinear points.*
(0) $(x, y) \in \mathcal{E}_0$ *if and only if x and y have at least two common neighbours.*
(1) $(x, y) \in \mathcal{E}_1$ *if and only if x and y have a unique common neighbour.*
(2) $(x, y) \in \mathcal{E}_2$ *if and only if x and y have no common neighbours.*

Furthermore, pairs of points in $\mathcal{E}_{-2}, \mathcal{E}_{-1}, \mathcal{E}_0 \cup \mathcal{E}_1$ and $\mathcal{E}_2$ have a distance between them in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of $0, 1, 2$ and 3, respectively.

We have seen that the root shadow spaces of type $\mathrm{A}_{n,\{1,n\}}$ and $\mathrm{BC}_{n,2}$ are root filtration spaces. The following theorem gives the general statement.

THEOREM 4.3.5 ([CI07, Thm. 36]). *Suppose that X_n is an irreducible Dynkin diagram, $n \geq 2$. Then the root shadow space Γ of type $X_{n,J}$ (where J denotes the set of root nodes for X_n, according to 4.2.2), is either a non-degenerate polar space, namely in case that $X_{n,J}$ is of type $\mathrm{C}_{n,1}$, or a non-degenerate root filtrations space for all other types. If the latter is the case, then Γ is a root filtration space with respect to the relations from 4.3.4.*

PROOF. The first part is Thm. 36 from [CI07], and the second statement follows from the remarks on page 1438 in [CI07]. $\diamond$

The following is the main result from [CI07] and gives us the opposite assignment.

THEOREM 4.3.6 ([CI07, Thm. 1]). *Let $\Gamma = (\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space. If the singular rank of Γ is finite, then Γ is a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$ $(n \geq 2)$,$\mathrm{BC}_{n,2}$ $(n \geq 3)$,$\mathrm{D}_{n,2}$ $(n \geq 4)$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$ or $\mathrm{G}_{2,2}$.*

4.4. Polarized embeddings

The following leads to the main result of [KS01] that will be given in 4.4.6; we use it in the next chapter. We follow the notation in [KS01].

DEFINITION 4.4.1. Let V be a vector space over the field $\mathbb{F}$. The **projective space** $\mathbb{P}(V)$ of V is the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, where the projective points $\mathcal{P}$ are the 1-dimensional subspaces and the projective lines $\mathcal{L}$ are the 2-dimensional subspaces of V, with the natural incidence.
A **projective embedding** of Γ over $\mathbb{F}$ is an injective map e from $\mathcal{P}$ to a set of points that span $\mathbb{P}(V)$, such that the image of the point-shadow of each line comprises all projective points of a projective line. Note that this induces an injection from $\mathcal{L}$ into the line set of $\mathbb{P}(V)$.
Let now $e : \Gamma \to \mathbb{P}(V)$ be such an embedding and $t : V \to W$ be a surjective semilinear transformation, with the property that $K := \ker(t)$ intersects any span $\langle p, q\rangle$ for any pair $(p, q) \in \mathcal{P} \times \mathcal{P}$ trivially. Then e can be carried onwards to the coset of K, and we obtain an embedding $e' : \Gamma \to \mathbb{P}(W)$. We call e' the **morphic image** of e, or we say that e' is **derived** from e or e **covers** e'. In particular, $e'(p) := t(e(p)) \in W$ is a 1-space in W for all $p \in \mathcal{P}$.
If all embeddings e' of Γ can be obtained in such a way from e, we call e **absolute** or **absolute universal**.

Let now $\Gamma = (\mathcal{E}, \mathcal{F})$ be a nondegenerate root filtration space.
Let $\psi : \Gamma \to \mathbb{P}$ be an arbitrary projective embedding of Γ. We call ψ **polarized** if and only if $\psi(\mathcal{E}_{\leq 1}(x))$ is contained in a hyperplane of $\mathbb{P}$ for all $x \in \mathcal{E}$.
The radical R_ψ of a polarized embedding ψ is the intersection

$$R_\psi := \bigcap_{x \in \mathcal{E}} \langle \psi(\mathcal{E}_{\leq 1}(x))\rangle.$$

Here $\langle \psi(\mathcal{E}_{\leq 1}(x))\rangle$ denotes the subspace of $\mathbb{P}$ generated by $\psi(\mathcal{E}_{\leq 1}(x))$.

LEMMA 4.4.2. *Let $\psi : \Gamma \to \mathbb{P}$ be a projective embedding covering of a polarized embedding ϕ. Then ψ is polarized.*
Moreover the kernel of the projection of ψ to ϕ is contained in the radical of ψ.

PROOF. The first statement is trivial.
Now suppose the kernel K of the projection τ of ψ to ϕ is not contained in the radical of ψ. Then there is an element $x \in \mathcal{E}$ such that $\langle \mathcal{E}_{\leq 1}(\psi(x))\rangle$ does not contain K. But that implies that the image under τ of the hyperplane $\langle \mathcal{E}_{\leq 1}(\psi(x))\rangle$ of $\mathbb{P}$ is the full space $\mathbb{P}(\mathfrak{g})$. This contradicts that ϕ is polarized. $\diamond$

PROPOSITION 4.4.3. *Let ψ be a cover of a polarized embedding ϕ of Γ. If the radical of ϕ trivial, then ϕ is isomorphic to ψ modulo its radical R_ψ.*

PROOF. The projection τ of ψ onto ϕ maps the radical of ψ into the radical of ϕ. However, since the radical of ϕ is trivial, we find the kernel of τ to be the radical R_ψ. $\diamond$

THEOREM 4.4.4. *Suppose* Γ *admits an absolute universal embedding and a polarized embedding* ϕ *with trivial radical.*
Then any polarized embedding ψ *of* Γ *covers* ϕ.

PROOF. Let χ be the absolute universal embedding of Γ. By Lemma 4.4.2, χ is polarized and both ψ and ϕ are isomorphic to the quotient of χ by a subspace K_ψ and K_ϕ, respectively, of its radical R_χ.
Since the radical of ϕ is trivial, we find K_ϕ to be equal to R_χ. But this implies that $K_\psi \subseteq K_\phi$ and ψ clearly covers ϕ. $\diamond$

THEOREM 4.4.5. *Suppose* $(\mathcal{P}, \mathcal{L})$ *is a point-line geometry admitting an absolute universal embedding* ψ. *If* ϕ *is a polarized embedding of* $(\mathcal{P}, \mathcal{L})$*, then*

$$\phi/R_\phi \cong \psi/R_\psi.$$

PROOF. Lemma 4.4.2 shows that $\phi \cong \psi/R$ for some $R \subseteq R_\psi$. The radical of ψ/R is $R_\psi/R \cong R_\phi$, and we get

$$\phi/R_\phi \cong (\psi/R)/R_\phi \cong (\psi/R)/(R_\psi/R) \cong \psi/R_\psi.$$

$\diamond$

We close this section with the following result of A. Kasikova and E. Shult:

THEOREM 4.4.6 ([KS01]). *Let* $\Gamma = (\mathcal{E}, \mathcal{F})$ *be a root filtration space of type* $\mathrm{BC}_{n,2}$, $\mathrm{D}_{n,2}$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$ *or* $\mathrm{F}_{4,1}$. *Then* Γ *admits an absolute universal embedding.*

PROOF. Kasikova and Shult prove the existence for each of these cases in [KS01]. The case $\mathrm{BC}_{n,2}$ for $n \geq 4$ can be found in [KS01, 4.8] and $\mathrm{C}_{3,2}$ in 4.7, $\mathrm{D}_{n,2}$ for $n \geq 5$ is covered in 4.5 and the special case $\mathrm{D}_{4,2}$ is treated in 4.1, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$ and $\mathrm{E}_{8,8}$ in 4.11, and $\mathrm{F}_{4,1}$ in 4.9.

$\diamond$

The results in this section on polarized embeddings have also been obtained by R. Blok, see [Blo11].

CHAPTER 5

From the geometry to the Lie algebra

In this chapter we use the structures introduced in the previous chapter for a geometric characterization of Lie algebras generated by extremal elements. So, we start with a Lie algebra $\mathfrak{g}$ that is generated by its extremal elements and equipped with an extremal form g as defined in 2.3.3.

5.1. The extremal geometry

In order to assign a geometry to the given Lie algebra $\mathfrak{g}$, we follow the method explained in [Coh12], and firstly construct a point-line-geometry out of the extremal elements $E(\mathfrak{g})$.

In 2.1.5, we introduced names for the five possible relations on a pair (x, y) of extremal elements. Recall that we named these relations as follows:

(-2) $(x, y) \in E_{-2}$ if and only if x and y are linearly dependent;

(-1) $(x, y) \in E_{-1}$ if and only if x and y are linearly independent, $[x, y] = 0$, and $\lambda x + \mu y \in E$ for all $(\lambda, \mu) \in \mathbb{F}^2, (\lambda, \mu) \neq (0, 0)$;

(0) $(x, y) \in E_0$ if and only if $[x, y] = 0$ and (x, y) is not in $E_{-2} \cup E_{-1}$;

(1) $(x, y) \in E_1$ if and only if $[x, y] \neq 0$, but $g_x(y) = 0$;

(2) $(x, y) \in E_2$ if and only if $g_x(y) \neq 0$.

Note that $g_x(y) = 0$ whenever $(x, y) \in E_{\leq 1}$ and $[x, y] \neq 0$ for all $(x, y) \in E_{\geq 1}$. Moreover, as follows from Lemma 24 [CI06] and Lemma 5.3.5 that we state later in this chapter, the sum $x + y$ of two commuting linearly independent extremal elements x and y is extremal if and only if $(x, y) \in E_{-1}$.

As in 2.1.5, the symmetric relations $\{\mathcal{E}_i\}_{i=-2}^{2}$ correspond to $\{E_i\}_{i=-2}^{2}$ in a natural way via $(\mathbb{F}x, \mathbb{F}y) \in \mathcal{E}_i$ if and only if $(x, y) \in E_i$ for $i \in \{-2, \ldots, 2\}$. The five relations $\{\mathcal{E}_i\}_{i=-2}^{2}$ on $\mathcal{E}$ are disjoint where $\mathcal{E}_{-1}$ is collinearity and $\mathcal{E}_{-2}$ is equality.

Definition 5.1.1. Let $\mathcal{E}$ be the set of projective extremal points of the Lie algebra $\mathfrak{g}$ and let $\mathcal{F}$ be the set of projective lines $\mathbb{F}x + \mathbb{F}y$ for $(x, y) \in \mathcal{E}_{-1}$. Hereby, we identify a 2-space with the set of 1-spaces it contains. Then the point-line space $(\mathcal{E}, \mathcal{F})$ together with the previously defined relations $\mathcal{E}_i$, $i \in \{-2, \ldots, 2\}$ on $\mathcal{E}$ define the **extremal geometry** of $\mathfrak{g}$. We usually denote it by $\Gamma(\mathfrak{g})$.

So the unique line in $\mathcal{F}$ containing two incident points $\mathbb{F}x$ and $\mathbb{F}y$ is $\mathbb{F}x + \mathbb{F}y$, which makes $(\mathcal{E}, \mathcal{F})$ a partial linear space.
In the previous chapter, we already have seen extremal geometries of classical Lie algebras, namely in terms of root filtration spaces. The relations $\mathcal{E}_{-2 \leq i \leq 2}$ of root filtration spaces are exactly the relations of the extremal geometry, as defined above. The extremal elements of the classical families of Lie algebras were computed in section 2.4. In 4.3.2, we discussed the case of a root shadow space $\mathrm{A}_{n,\{1,n\}}$, where the point-hyperplane pairs of a polar space define the points in the extremal geometry. The root shadow space for Lie algebras of the families $\mathrm{B}_n, \mathrm{C}_n$ and D_n were considered in 4.3.3. Hereby, the root shadow space of type $\mathrm{BC}_{n,1}$ is a polar space, so in this case we find $\mathcal{E}_{-1} = \mathcal{E}_1 = \emptyset$. The other root shadow spaces and therefore the (connected components of the) extremal geometries are *nondegenerate* root filtration spaces.
We will use the following fundamental results of Cohen and Ivanyos (see [CI06] and [CI07]) in the next section.

THEOREM 5.1.2 ([CI06], Theorem 28). *Suppose that $\mathfrak{g}$ is a Lie algebra, generated by its extremal elements $\mathcal{E}(\mathfrak{g})$ and with extremal form g, where the radical of g is trivial, i.e. $Rad(g) = 0$. Then the extremal geometry $(\mathcal{E}, \mathcal{F})$ of $\mathfrak{g}$ is a root filtration space with filtration $\{\mathcal{E}_i\}_{i=-2}^{2}$ as defined above. Let $\mathcal{B}_i$ be the connected components of $(\mathcal{E}, \mathcal{E}_2)$ and let $\mathfrak{g}_i$ be the Lie subalgebra generated by $\mathcal{B}_i$ of $\mathfrak{g}$. Then each $\mathcal{B}_i$ is a nondegenerate root filtration space or a root filtration space without lines, $\mathfrak{g}$ is the direct sum of Lie subalgebras $\mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ whenever $i \neq j$. In particular, $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$.*

By the above result we are able to use the classification of root filtration spaces as discussed in 4.3.6 and find the following.

THEOREM 5.1.3 ([CI07], Theorem 1). *A connected compontent of the extremal geometry $(\mathcal{E}, \mathcal{F})$ of a finite dimensional Lie algebra $\mathfrak{g}$, generated by its set of extremal elements and equipped wth a nondegenerate extremal form g, is isomorphic to a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$,$\mathrm{BC}_{n,2}$,$\mathrm{D}_{n,2}$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$ or $\mathrm{G}_{2,2}$ or consists of a single point.*

PROOF. The results follows from Theorem 5.1.2 and Theorem 4.3.6 together with the observation that extremal points in different components $\mathcal{B}_1$ and $\mathcal{B}_2$ of $(\mathcal{E}, \mathcal{E}_2)$ are not collinear, so that these components are unions of connected subspaces of the extremal geometry. Indeed, suppose $x \in \mathcal{B}_1$ and $y \in \mathcal{B}_2$ are collinear, then, as g is nondegenerate, there is a $z \in \mathcal{E}_2(x)$. Then it follows necessarily that $z \notin \mathcal{E}_2(y)$. But, as $\mathcal{E}_{\leq 1}(y)$ is a geometric hyperplane, see 4.3.1(F), that implies that $z \in \mathcal{E}_2(v)$ for each extremal point v on the line through x and y different from y. In particular, all these points v are in $\mathcal{B}_1$. Similarly we can prove that all points v on the line through x and y but different from x are in $\mathcal{B}_2$. But, as the line through x and y contains at least three points, we find that there is a point in the intersection of $\mathcal{B}_1$ and $\mathcal{B}_2$. This contradiction proves that points from different components of $(\mathcal{E}, \mathcal{E}_2)$ are never collinear in $(\mathcal{E}, \mathbb{F})$. $\diamond$

Note that the labeling of the Coxeter diagrams follows [Bou68].

5.2. The embedding

We fix the properties that we assume for Lie algebras in this chapter. If not mentioned otherwise, any Lie algebra in the remainder of this chapter is supposed to fulfill these conditions.

SETTING 5.2.1. By $\mathfrak{g}$ we denote a Lie algebra generated by its set E of extremal elements and with nondegenerate extremal form g. By $\Gamma = (\mathcal{E}, \mathcal{F})$, we denote the extremal geometry of $\mathfrak{g}$. We assume Γ to be nondegenerate and connected so in particular $\mathcal{E}_{-1} \neq \emptyset$.

Considering $\mathfrak{g}$ as a vector space, it carries a natural **projective geometry** via the natural incidence geometry of all proper subspaces of $\mathfrak{g}$. Hereby, the 1-subspaces of $\mathfrak{g}$ are the projective points and the 2-subspaces are the projective lines. We denote this point-line geometry by $\mathbb{P}(\mathfrak{g})$ and call it the **projective space** on $\mathfrak{g}$.

The *natural* projective embedding of the extremal geometry $\Gamma = (\mathcal{E}, \mathcal{F})$ into $\mathbb{P}(\mathfrak{g})$ is defined to be the injection

$$\phi : \mathcal{E} \hookrightarrow \text{ projective points of } \mathbb{P}(\mathfrak{g}),$$

so for $x \in \mathcal{E}$, we have

$$\phi(x) = x.$$

For any line $l \in \mathcal{F}$, the restriction of ϕ to all points of l is the full set $\phi(l)$ of points of some projective line and, as the extremal points in $\mathcal{E}$ linearly span $\mathfrak{g}$ (see 2.3.1), the set $\phi(\mathcal{E})$ spans $\mathbb{P}(\mathfrak{g})$. We find

$$\begin{aligned} &\phi : \Gamma \hookrightarrow \mathbb{P}(\mathfrak{g}) \\ &p \in \mathcal{E} \mapsto \text{1-spaces} = \text{points}, \\ &l \in \mathcal{F} \mapsto \text{2-spaces} = \text{lines}. \end{aligned}$$

LEMMA 5.2.2. *The embedding ϕ is polarized.*

PROOF. For each $x \in \mathcal{E}$ we find $\phi\big(\mathcal{E}_{\leq 1}(x)\big)$ to be contained in the hyperplane $\{y \in \mathfrak{g} \mid g(x, y) = 0\}$. $\diamond$

THEOREM 5.2.3. *Suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras as in the setting 5.2.1, each of them generated by its set of nondegenerate extremal elements and equipped with a nondegenerate extremal form. Assume their corresponding extremal geometries Γ_1 and Γ_2 are isomorphic to each other and admit an absolute universal embedding. Then their natural embeddings are equivalent.*

PROOF. We can apply the results of section 4.4 and find by 4.4.4 that the natural embeddings ϕ_1 and ϕ_2 are equivalent, provided their radicals are trivial.
Since, for $i = 1, 2$, the radical R_i of embedding ϕ_i is the intersection of all the subspaces $\langle \mathcal{E}_{\leq 1}(x) \rangle$ where x runs through the set of extremal points of $\mathfrak{g}_i$, we find these radicals to be contained in the radical of the extremal form g_i of $\mathfrak{g}_i$. As the radical of the forms g_1 and g_2 are trivial by assumptions, the radicals of the embeddings are also trivial. $\diamond$

In view of the Theorems 5.1.2 and 5.1.3 and using the classification given in 5.1.3, we obtain the following.

COROLLARY 5.2.4. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras as in 5.2.1. Assume the corresponding extremal geometries Γ_1 and Γ_2 are isomorphic to each other and to a connected root shadow space of type* $\mathrm{BC}_{n,2}$, $\mathrm{D}_{n,2}$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, *or* $\mathrm{F}_{4,1}$, *where $n \geq 3$. Then their natural embeddings are equivalent.*

PROOF. As stated in 4.4.6, Kasikova and Shult [KS01] show that Γ_i, with $i = 1, 2$ admits an absolute universal embedding. So Theorem 5.2.3 applies. $\diamond$

REMARK 5.2.5. For root shadow spaces of type $A_{n,\{1,n\}}$ and of type $G_{2,2}$ it is not known whether they admit an absolute universal embedding.
The results of Völklein [Völ89] imply that the natural embeddings of the extremal geometries of type $A_{n,\{1,n\}}$ and of type $G_{2,2}$ of the Chevalley Lie algebras of type A_n and G_2 do have a universal cover.
Blok and Pasini [BP03] obtain some partial results on embeddings of the geometries of type $A_{n,\{1,n\}}$ under some extra conditions on the underlying field. Van Maldeghem and Thas [TVM04] show that the natural embedding of finite dual Cayley hexagons in the Chevalley Lie algebra is, up to isomorphism, the unique embedding of the hexagon in dimension ≥ 14.

In the next section we will prove that given an embedding of the extremal geometry of a Lie algebra there is, up to a scalar multiple, at most one Lie bracket corresponding to it. This implies that the Lie structures $\mathfrak{g}_1$ and $\mathfrak{g}_2$, in the cases considered in 5.2.4 are isomorphic.

5.3. Uniqueness of the Lie product

Let $\mathfrak{g}$ be a Lie algebra generated by its set of extremal elements E with respect to a nondegenerate extremal form g. As before let $\Gamma = (\mathcal{E}, \mathcal{F})$ be the extremal geometry of $\mathfrak{g}$. In the previous section we have seen that the natural embedding of Γ into the projective space $\mathbb{P}(\mathfrak{g})$ is uniquely determined (up to isomorphism), if Γ admits an absolute universal embedding. Our goal is to prove that not only the embedding of the extremal geometry is uniquely determined, but that also the Lie product is determined up to scalar multiples.
Our strategy to show this uniqueness of the Lie product (up to scalar multiples) for a given embedded extremal geometry is to prove uniqueness first on small subspaces and then extend the result to the full projective space by building it up by small subalgebras. We consider these subalgebras in small dimension in detail in the following.
So, in this section we assume the following.

SETTING 5.3.1. Let $\mathfrak{g}$ be a Lie algebra as in 5.2.1, with Γ naturally embedded into the projective space $\mathbb{P}(\mathfrak{g})$.
Let $[\cdot,\cdot]$ denote the Lie product on $\mathfrak{g}$. We consider a second Lie product $[\cdot,\cdot]_1$ defining a Lie algebra $\mathfrak{g}_1$ on the vector space underlying $\mathfrak{g}$ with extremal form g^1 and also Γ as extremal geometry.

We want to show that $[\cdot,\cdot]_1 = \lambda[\cdot,\cdot]$ for some fixed $\lambda \in \mathbb{F}^*$. Notice that the relations $\mathcal{E}_i$ with $-2 \leq i \leq 2$ are determined by Γ (see 5.1.1). So elements $x, y \in \mathcal{E}$ are in relation $\mathcal{E}_i$ in $\mathfrak{g}$ if and only if they are in relation $\mathcal{E}_i$ in $\mathfrak{g}_1$.

LEMMA 5.3.2. *Let $(x, y) \in E_{\leq 1}$, then there is a $\lambda \in \mathbb{F}^*$ such that $[x, y]_1 = \lambda[x, y]$.*

PROOF. If $(x, y) \in E_{\leq 0}$, then $[x, y]_1 = 0 = [x, y]$.
If $(x, y) \in E_1$, then both $[x, y]$ and $[x, y]_1$ span the unique point in $\mathcal{E}$ collinear to both $\langle x\rangle$ and $\langle y\rangle$. So indeed, there is a $\lambda \in \mathbb{F}^*$ with $[x, y]_1 = \lambda[x, y]$. $\diamond$

Now we concentrate on the subalgebra of $\mathfrak{g}$ generated by a hyperbolic pair. Such a subalgebra is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.
We examine $\mathfrak{sl}_2(\mathbb{F})$ a bit closer.

EXAMPLE 5.3.3. Recall the definition of classical linear Lie algebras given in section 2.4. The Lie product of a Lie algebra structure on $V \otimes V^*$, where $v, w \in V, \phi, \psi \in V^*$, is given by

$$[v \otimes \phi, w \otimes \psi] = \phi(w)(v \otimes \psi) - \psi(v)(w \otimes \phi)$$

as defined in 2.4.1 and the extremal elements with respect to the extremal form $g(v\otimes\phi, w\otimes\psi) = -\psi(v)\phi(w)$, introduced in 2.4.3, are the pure tensors (we proved this in 2.4.4 and 2.5.1). We consider the case where V is 2-dimensional and denote the corresponding Lie algebra by $\mathfrak{g}$. Then $\mathfrak{g}$ is isomorphic to the special linear Lie algebra $\mathfrak{sl}_2(\mathbb{F})$.
Any hyperbolic pair of elements $(x, y) \in \mathcal{E}_2(\mathfrak{g})$ generates the algebra $\mathfrak{g}$. Let

$$x := e_1 \otimes \phi_2 \text{ and } y := e_2 \otimes \phi_1,$$

where $\{e_1, e_2\}$ denotes the standard basis of the underlying 2-dimensional vector space V and $\{\phi_1, \phi_2\}$ a dual basis such that $\phi_i(e_i) = 1$ and $\phi_i(e_j) = 0$ for $i, j \in \{1, 2\}, i \neq j$. Written as matrices, we have

$$e_1 \otimes \phi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_2 \otimes \phi_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[e_1 \otimes \phi_2, e_2 \otimes \phi_1] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so the span is given by matrices of the form

$$m = \begin{pmatrix} c & a \\ b & -c \end{pmatrix},$$

with $a, b, c \in \mathbb{F}$, and the (nonzero) pure tensors correspond to the nonzero elements where $ab + c^2 = 0$, which are exactly those matrices of rank 1. We see that

$$g(x, y) = g(e_1 \otimes \phi_2, e_2 \otimes \phi_1) = -\phi_1(e_1)\phi_2(e_2) = -1.$$

So these extremal elements form a quadric which can be described by $ab = g(x, y)c^2$.

Below, we will see that the extremal elements are *exactly* the matrices of rank one. To consider the case of a Lie subalgebra generated by a hyperbolic pair where $\operatorname{char} \mathbb{F} = 2$, we use the following results of [CI06].

LEMMA 5.3.4 ([CI06, Lemma 21]). *Let $x \in E(\mathfrak{g}), y \in \mathfrak{g}$ and $g_x(y) \neq 0$. Then $x, y, [x, y]$ are linearly independent. If $y \in E(\mathfrak{g})$, then the subalgebra generated by x and y is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.*

LEMMA 5.3.5 ([CI06, Lemma 27]). *Let $\mathfrak{g}$ be a Lie algebra generated by its set of extremal elements E and let x, y be linearly independent extremal elements with x not a sandwich in $\mathfrak{g}$. If $\lambda x + \mu y \in E$ for some $\lambda, \mu \in \mathbb{F}^*$, then $(x, y) \in E_{-1}$.*

PROPOSITION 5.3.6. *Let (x, y) be a hyperbolic pair of $\mathfrak{g}$. Then the subalgebra of $\mathfrak{g}$ generated by x and y is 3-dimensional. If $\operatorname{char} \mathbb{F} \neq 2$, the extremal points inside this subalgebra form a quadric. If $\operatorname{char} \mathbb{F} = 2$, the extremal elements can be found in the union of a quadric and the center of the subalgebra.*

PROOF. By 5.3.4, the Lie subalgebra $\mathfrak{h}$ generated by x and y is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. So we identify $\mathfrak{h}$ with $\mathfrak{sl}_2(\mathbb{F})$ as in Example 5.3.3. In particular, we can identify

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Suppose an element

$$z = ax + by + c[x, y] = \begin{pmatrix} c & b \\ a & -c \end{pmatrix}$$

is extremal for some $a, b, c \in \mathbb{F}$. With $\exp(x,t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $t \in \mathbb{F}$, the element

$$z' := \exp(x,t) \cdot z \cdot \exp(x,-t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & b \\ a & -c \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} c+at & -t(c+at)+b-ct \\ a & -ta-c \end{pmatrix}$$

is also extremal. Now suppose that $a \neq 0$. If we choose $t = a^{-1}c$, so $c+ta = 0$, and we obtain

$$z' = \begin{pmatrix} 0 & b-ct \\ a & 0 \end{pmatrix}.$$

It follows $z' = a \cdot y + (b - tc) \cdot x$. By Lemma 5.3.5, this is just possible if $b - ct = 0$ (since $a \neq 0$ and $(x, y) \in E_2$). Then we have $b = ct = a^{-1}c^2$, and therefore

$$c^2 - ab = c^2 - aa^{-1}c^2 = 0,$$

so z is of rank one.

If we suppose $b \neq 0$, a similar argument with $\exp(y,t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ leads to same result that z is of rank one.

It remains to consider the case where $a = b = 0$. Here, we have $z = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$, so $z = c \cdot [x, y]$ and therefore not extremal if $c \neq 0$ and $\operatorname{char} \mathbb{F} \neq 2$. If $\operatorname{char} \mathbb{F} = 2$, the element z clearly lies in the center Z of $\mathfrak{g}$. This completes the proof.

◇

LEMMA 5.3.7. *Let (x, y) be a hyperbolic pair generating a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then there is a $\lambda \in \mathbb{F}^*$ such that for all $v, w \in \mathfrak{h}$ we have $[v, w]_1 = \lambda[v, w]$.*

PROOF. Without loss of generality suppose that $g(x, y) = 1$. Inside both $\mathfrak{g}$ and $\mathfrak{g}_1$ the elements x and y generate a subalgebra isomorphic to $\mathfrak{sl}_2$.
Inside $\Gamma(\mathfrak{g})$ we take two distinct lines l_1 and l_2 on $\langle x \rangle$ with $[l_1, l_2] = \langle x \rangle$. Notice that such lines exist. For $i = 1, 2$, fix a point $\langle x_i \rangle$ on l_i which is at distance 2 from $\langle y \rangle$. Let $y_i := [y, x_i]$. Then for each point $\langle z_1 \rangle$ on the line through $\langle x_1 \rangle$ and $\langle y_1 \rangle$, there is a point $\langle z_2 \rangle$ on the line through $\langle x_2 \rangle$ and $\langle y_2 \rangle$ which is in relation $\mathcal{E}_1$ with $\langle z_1 \rangle$. This follows from the observation that the

group $\langle \mathrm{Exp}(x), \mathrm{Exp}(y)\rangle$ leaves the lines $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ invariant and is transitive on the points of these lines.

We claim that both in $\mathfrak{g}_1$ and $\mathfrak{g}_2$ the unique common neighbour $\langle z\rangle = \langle [z_1, z_2]\rangle$ of $\langle z_1\rangle$ and $\langle z_2\rangle$ is inside the subalgebra generated by x and y.

Indeed, within $\mathrm{Aut}(\mathfrak{g})$ we find that the elements of $\mathrm{Exp}(\langle x\rangle)$ fix, for $i = 1, 2$, the point $\langle x_i\rangle$ as well as the line spanned by x_i and y_i. Moreover, $\mathrm{Exp}(\langle x\rangle)$ acts transitively on the points of this line different from $\langle x_i\rangle$. Thus there is an element $g \in \mathrm{Exp}(\langle x\rangle)$ that maps $\langle y_1\rangle$ to $\langle z_1\rangle$. As g leaves the line spanned by x_2 and y_2 invariant, it maps $\langle y_2\rangle$ to the unique point on this line which is at distance 2 from $\langle z_1\rangle$, the point $\langle z_2\rangle$. But then $\langle y\rangle$ is mapped to $\langle z\rangle$ by the element g, as $\langle z\rangle$ is the unique common neighbor of $\langle z_1\rangle$ and $\langle z_2\rangle$. This clearly implies that $\langle z\rangle$ is inside the subalgebra of $\mathfrak{g}$ generated by x and y. In particular, x, y and z linearly span the subalgebra of $\mathfrak{g}$ generated by x and y. But similarly, these three elements are also contained in the subalgebra of $\mathfrak{g}_1$ generated by x and y and span this subalgebra. So, these subalgebras have to coincide as linear subspaces.

The above proves more. Indeed, it shows that the point $\langle z\rangle$ is in the $\mathrm{Exp}(\langle x\rangle)$-orbit of $\langle y\rangle$ both with respect to $[\cdot,\cdot]$ and to $[\cdot,\cdot]_1$. Actually, these two orbits have to be equal.

In $\mathfrak{g}$ this orbit, together with $\langle x\rangle$, consists of all 1-spaces spanned by elements

$$ax + by + c[x, y],$$

where $a, b, c \in \mathbb{F}$ satisfy $ab = c^2$.

Now, suppose

$$[x, y]_1 = \alpha x + \beta y + \gamma[x, y],$$

for some fixed $\alpha, \beta, \gamma \in \mathbb{F}$. Note that $[x, y]_1 \neq 0 \neq [x, y]$, since (x, y) is a hyperbolic pair with respect to both Lie products by the general assumptions in 5.3.1. Note moreover that $\gamma \neq 0$, since otherwise the result of 5.3.4 leads to a contradiction.

The images of y under elements from $\mathrm{Exp}(\langle x\rangle)$, but now with respect to $[\cdot,\cdot]_1$, are of the form

$$y + \lambda[x, y]_1 + \lambda^2 g^1(x, y)x = (\lambda^2 g^1(x, y) + \lambda\beta)x + (1 + \lambda\alpha)y + \lambda^2\gamma^2[x, y]$$

where $\lambda \in \mathbb{F}$. These elements are also extremal in $\mathfrak{g}$ and hence satisfy the equation

$$(1 + \lambda\alpha)(\lambda^2 g^1(x, y) + \lambda\beta) = \lambda^2\gamma^2.$$

This implies that the qubic equation

$$(1+\alpha X)(g^1(x,y)X^2+\beta X)=\gamma^2X^2$$

has $|\mathbb{F}|$ zeros. So, if $|\mathbb{F}|>3$, this means

$$\alpha=\beta=0, \text{ and } \gamma^2=g^1(x,y),$$

and we deduce

$$[x,y]_1=\gamma[x,y].$$

But this implies that $[\ ,\ ]_1$ is a scalar multiple of $[\ ,\ ]$, and completes the proof. In case that $|\mathbb{F}|=2$, the above equation for $\lambda=1$ reads as follwos:

$$(1+\alpha)(1+\beta)=\gamma^2.$$

Now if $\alpha=1$ or $\beta=1$, it follows that $\gamma^2=0$ and so $\gamma=0$, which is a contradiction. So also here, $\alpha=\beta=0$ must hold.
In case that $|\mathbb{F}|=3$, there are more cases to consider for $\alpha\neq 0$ or $\beta\neq 0$, all leading either to no possible solution for γ or to $\gamma=0$, which is a contradiction. So, we conclude that the Lie product is unique up to scalar multiples. $\diamond$

LEMMA 5.3.8. *Let $x\in\mathcal{E}$ and $l\in\mathcal{F}$. Suppose $y_1,y_2\in\mathcal{E}$ span l. Then we can find an element $\lambda\in\mathbb{F}^*$ with $[x,y_i]_1=\lambda[x,y_i]$ for both $i=1$ and $i=2$.*

PROOF. Under the given conditions, Lemma 5.3.2 and Lemma 5.3.7 imply that there exist λ_i for $i=1,2$ with $[x,y_i]_1=\lambda_i[x,y_i]$. If $[x,y_1]=0$ or $[x,y_2]=0$, then clearly we can take λ_1 and λ_2 to be equal. So assume $[x,y_1]\neq 0\neq[x,y_2]$ and let $y_3:=-(y_1+y_2)$ such that there exists λ_3 with $[x,y_3]_1=\lambda_3[x,y_3]$. Suppose $\lambda_1\neq\lambda_2$. We find

$$0=[x,y_1+y_2+y_3]=[x,y_1]+[x,y_2]+[x,y_3]$$

and

$$\begin{aligned}0=&[x,y_1+y_2+y_3]_1=[x,y_1]_1+[x,y_2]_1+[x,y_3]_1\\ =&\lambda_1[x,y_1]+\lambda_2[x,y_2]+\lambda_3[x,y_3].\end{aligned}$$

But this implies that

$$(\lambda_1-\lambda_2)[x,y_1]+(\lambda_3-\lambda_2)[x,y_3]=0$$

and hence

$$0=[x,(\lambda_1-\lambda_2)y_1+(\lambda_3-\lambda_2)y_3]$$

$$=[x, (\lambda_1 - \lambda_2)y_1 + (\lambda_3 - \lambda_2)(-y_1 - y_2)]$$
$$=[x, (\lambda_1 - \lambda_3)y_1 + (\lambda_2 - \lambda_3)y_2].$$

With the definition $z := (\lambda_1 - \lambda_3)y_1 + (\lambda_2 - \lambda_3)y_2 \neq 0$, we have $[x, z] = 0$ and we find $z \in \mathcal{E}_{\leq 0}(x)$ and hence $[x, z]_1 = 0$ by Lemma 5.3.2. Since $[x, y_1] \neq 0 \neq [x, y_2]$, the element z is not a multiple of y_1 or y_2, and hence there are $\mu_1, \mu_2 \in \mathbb{F}^*$ with $y_2 = \mu_1 y_1 + \mu_2 z$. Now we find

$$\begin{aligned}[x, y_2]_1 =& [x, \mu_1 y_1 + \mu_2 z]_1 = [x, \mu_1 y_1]_1 + [x, \mu_2 z]_1 \\ =& \lambda_1 \mu_1 [x, y_1] + 0 = \lambda_1 \mu_1 [x, y_1]\end{aligned}$$

and

$$[x, y_2] = [x, \mu_1 y_1 + \mu_2 z] = \mu_1 [x, y_1] + 0 = \mu_1 [x, y_1].$$

So,

$$[x, y_2]_1 = \lambda_1 [x, y_2].$$

This contradicts λ_1 to be different from λ_2. $\diamond$

We are now ready to prove the main objective of this section.

THEOREM 5.3.9. *Let $\mathfrak{g}$ be a Lie algebra generated by its set of extremal elements E, equipped with the Lie product denoted by $[\cdot, \cdot]$ and a nondegenerate extremal form g. Assume that there is a second Lie product $[\cdot, \cdot]_1$ defined on the underlying vector space, with corresponding nondegenerate extremal form g^1, giving rise to the same extremal geometry Γ. Then, there is a $\lambda \in \mathbb{F}^*$ with $[x, y]_1 = \lambda [x, y]$ for all $x, y \in \mathfrak{g}$.*

PROOF. Fix a hyperbolic pair (x, y). Then, by Lemma 5.3.7, there is a $\lambda \in \mathbb{F}^*$ with $[x, y]_1 = \lambda [x, y]$. We prove that this element λ is the one we are looking for. We begin with the proof that for all $z \in E$ we have $[x, z]_1 = \lambda [x, z]$. Suppose $z \in E$ different from y. If $[x, z] = 0$, then $z \in E_{\leq 0}(y)$, hence also $[x, z]_1 = 0$, and $[x, z]_1 = \lambda [x, z]$.
If $z \in E_2(x)$, then, as Γ has diameter 3, we can find elements z_1 and z_2 in $E_{\geq 1}(x)$ such that $\langle y, z_1 \rangle$, $\langle z_1, z_2 \rangle$, and $\langle z_2, z \rangle$ are in $\mathcal{F}$. (Notice that we allow these subspaces to be equal to each other.) Now we can apply the above Lemma 5.3.8 to each of these lines and eventually find that $[x, z]_1 = \lambda [x, z]$.
Finally consider the case where $z \in E_1(x)$. We construct an element $z' \in E_2(x) \cap E_{-1}(z)$. To show that such an element exists, we use two technical

results that can be found in [CI07]. Since $(\langle x\rangle, \langle z\rangle) \in \mathcal{E}_1$, the points $\langle x\rangle$ and $\langle z\rangle$ have a unique common neighbour $\langle [x,z]\rangle \in \mathcal{E}_{-1}(\langle x\rangle) \cap \mathcal{E}_{-1}(\langle z\rangle)$. Now by [CI07, Lemma 3(ii)], there exists a $z' \in E$ such that the point $\langle z'\rangle$ is in $\mathcal{E}_{-1}(z) \cap \mathcal{E}_1([x,z])$. Applying [CI07, Lemma 2(v)], it follows that $(x,z') \in E_2$. So actually $z' \in E_2(x) \cap E_{-1}(z)$ exists. By the above we have $[x,z']_1 = \lambda[x,z']$ and Lemma 5.3.8 implies now that $[x,z]_1 = \lambda[x,z]$.
Since we started with a fixed $x \in E$, it remains to show that the scalar factor λ is independent of x. The graph $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ (as defined in 3.4) corresponds to the graph $(\mathcal{E}, \mathcal{E}_2)$. Since both g and g^1 are nondegenerate, we can apply 2.5.4, so $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ is a connected graph. This means that every point in $\mathcal{E}$ is contained in at least one hyperbolic pair and all points are in the one single connected component of $(\mathcal{E}, \mathcal{E}_2)$. So starting with the hyperbolic pair (x,y) on x, we find the same scalar λ for any other hyperbolic pair (x,z) on x and hence also for any hyperbolic pair (z,x). Now connectedness of $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g})$ implies that the scalar for all hyperbolic pairs is the same. But then it is fixed for all other pairs of elements in $E \times E$. Since E generates $\mathfrak{g}$, we find for all $x, y \in \mathfrak{g}$ that $[x,y]_1 = \lambda[x,y]$.

◇

5.4. Conclusions

Combining the results of the previous two sections, we finally can characterize the Lie algebras under consideration by their extremal geometry.

THEOREM 5.4.1. *Let $\mathfrak{g}$ be a Lie algebra generated by its set E of extremal elements with respect to the extremal form g with trivial radical. If $\Gamma(\mathfrak{g})$ is nondegenerate and the natural embedding of the extremal geometry $\Gamma(\mathfrak{g})$ into $\mathbb{P}(\mathfrak{g})$ admits an absolute universal cover, then $\mathfrak{g}$ is uniquely determined (up to isomorphism) by $\Gamma(\mathfrak{g})$.*

PROOF. We combine the previous results. So let $\mathfrak{g}_1$ be a second Lie algebra with isomorphic extremal geometry $\Gamma(\mathfrak{g}_1) \cong \Gamma(\mathfrak{g})$. By 5.2.3, the projective embeddings of $\mathfrak{g}$ and $\mathfrak{g}_1$ are equivalent and therefore have the same Lie structure, as a consequence of 5.3.9. ◇

COROLLARY 5.4.2. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras as in 5.2.1 with extremal geometries $\Gamma(\mathfrak{g}_1) \cong \Gamma(\mathfrak{g}_2)$ isomorphic to a long root geometry of type* $\mathrm{BC}_{n,2}$, $\mathrm{D}_{n,2}$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, *or* $\mathrm{F}_{4,1}$, *where $n \geq 3$. Then $\mathfrak{g}_1 \cong \mathfrak{g}_2$.*

PROOF. As a consequence of the main result 4.4.6 in [KS01], the root filtration space of $\mathfrak{g}_1$ (and $\mathfrak{g}_2$) of the given types has an absolute universal cover. So Theorem 5.4.1 applies. $\diamond$

The following result about Chevalley Lie algebras can be obtained from Theorem 5.4.1, together with the results of [Rob12, Theorem 5.2.15], that cover the full A_n-case.

COROLLARY 5.4.3. *Suppose* $\mathfrak{g}$ *is a Lie algebra and* $\Gamma(\mathfrak{g})$ *is isomorphic to* $\Gamma(\mathfrak{ch})$ *for some Lie algebra* $\mathfrak{ch}$ *of Chevalley type* $X_n \neq \mathrm{C}_n$ *where* $n \geq 3$. *Then*

$$\mathfrak{g}/Rad(\mathfrak{g}) \cong \mathfrak{ch}/Rad(\mathfrak{ch}).$$

PROOF. The extremal geometry of a Lie algebra of Chevalley type $X_n \neq \mathrm{C}_n$ with $n \geq 3$ is a long root geometry of type $\mathrm{A}_{n,\{1,n\}}$, $\mathrm{BC}_{n,2}$, $\mathrm{D}_{n,2}$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, or $\mathrm{F}_{4,1}$ (see e.g. [CRS14]). So, we can apply [Rob12, Theorem 5.2.15] in case of a geometry of type $\mathrm{A}_{n,\{1,n\}}$ and 5.4.2 otherwise.

$\diamond$

REMARK 5.4.4. The extremal geometry of a Lie algebra of Chevalley type C_n does not have lines. So, we cannot apply the results of the present chapter. This case will be covered in the next chapter.

REMARK 5.4.5. Notice that in the simply laced case, i.e., X_n is A_n, D_n or E_n, the above result has also been obtained by Cuypers, Roberts and Shpectorov in [CRS14]. In this paper the authors reconstruct a Chevalley basis (or its image in a quotient) for the Lie algebra starting from the shadow space of an apartment of the building in the extremal geometry.

CHAPTER 6

A characterization of $\mathfrak{sp}$

In the previous chapter, we used the classification of non-degenerate root filtration spaces by Cohen and Ivanyos (2007) for the identification of a Lie algebra $\mathfrak{g}$ with $\mathcal{E}_{-1}(\mathfrak{g}) \neq \emptyset$ via its extremal geometry.

Now we consider the case of a Lie algebra $\mathfrak{g}$ with $\mathcal{E}_{-1}(\mathfrak{g}) = \emptyset$. As a consequence, also $\mathcal{E}_1(\mathfrak{g}) = \emptyset$ holds (see [i'p09] for a proof). A typical example of a simple Lie algebra with $\mathcal{E}_{\pm 1}(\mathfrak{g}) = \emptyset$ which is generated by its extremal elements is $\mathfrak{sp}_n(\mathbb{F})$ provided that $\operatorname{char}\mathbb{F} \neq 2$ (where $n \geq 2$ is even). In the following we will assume $\mathcal{E}_{-1}(\mathfrak{g}) = \mathcal{E}_1(\mathfrak{g}) = \emptyset$ and $\operatorname{char}\mathbb{F} \neq 2$. Our goal is to characterize $\mathfrak{sp}_n(\mathbb{F})$ under these assumptions.

As a result of Cuypers and in 't panhuis (see [i'p09]), the adjacency defined on points in $\mathcal{E}_0$-relation gives a polar graph on $\mathcal{E}(\mathfrak{g})$, i.e. $(\mathcal{E}, \mathcal{E}_0)$ is the collinearity graph of a polar space. Moreover, Cuypers and in 't panhuis proved that any triple of elements (x, y, z) with $x, y, z \in \mathcal{E}(\mathfrak{g})$ such that $\langle x, y\rangle \cong \mathfrak{sl}_2(\mathbb{F}) \cong \langle y, z\rangle$ generates a subalgebra of $\mathfrak{g}$ which can be of two possible types. The first case is that $\langle x, y, z\rangle$ is contained in some subalgebra generated by a symplectic triple (as defined in 2.1.5) and is isomorphic to $\mathfrak{sp}_3(\mathbb{F})$ (if the subalgebra is 6-dimensional) or its central quotient $\mathfrak{psp}_3(\mathbb{F})$ (if it is 5-dimensional). We define and consider these subalgebras in the next section of this chapter.
In the second case, the triple (x, y, z) generates a subalgebra isomorphic to $\mathfrak{su}_3(\mathbb{F})$, or its central quotient $\mathfrak{psu}_3(\mathbb{F})$ as shown in [i'p09]. In particular, we find that over a quadratic extension $\hat{\mathbb{F}}$ of $\mathbb{F}$, the three elements x, y, z generate a subalgebra isomorphic to $\mathfrak{sl}_3(\hat{\mathbb{F}})$ or its central quotient $\mathfrak{psl}_3(\hat{\mathbb{F}})$. But within $(\mathfrak{p})\mathfrak{sl}_3(\hat{\mathbb{F}})$, one finds pairs of strongly commuting extremal elements, i.e. $\mathcal{E}_{-1}((\mathfrak{p})\mathfrak{sl}_3(\hat{\mathbb{F}})) \neq \emptyset$. So we are back in the situation considered in the previous chapter.
Of course it still remains to find the isomorphism type of $\mathfrak{g}$ as the results of chapter 5 just provide a list of possible types, and to consider the situation *before* the quadratic field extension. But it is natural to first consider the

case where we only have subalgebras isomorphic (up to center) to $\mathfrak{sp}_3(\mathbb{F})$. We provide the following characterization.

MAIN THEOREM 6.0.6. *Let $\mathfrak{g}$ be a simple Lie algebra of finite even dimension over the field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ and generated by its set of extremal points $\mathcal{E}$ where $\mathcal{E}_{\pm 1}(\mathfrak{g}) = \emptyset$ and for any $(x, y), (y, z) \in \mathcal{E}_2(\mathfrak{g})$, the subspace $\langle x, y, z\rangle$ embeds into a subalgebra isomorphic to $\mathfrak{sp}_3(\mathbb{F})$ or $\mathfrak{psp}_3(\mathbb{F})$. Then $\mathfrak{g} \cong \mathfrak{sp}_n(\mathbb{F})$ for some (even) $n \geq 2$.*

6.1. The symplectic Lie algebra

We begin with a description of the symplectic Lie algebra in terms of symmetric tensors. Using this, we provide a description of the extremal elements of $\mathfrak{sp}_{2m}(\mathbb{F})$ being the *pure* symmetric tensors.

6.1.1. Symmetric tensors. Let (V, f) be a symplectic space as defined in chapter 1. According to the results of section 1.2, we can describe the symplectic Lie algebra $\mathfrak{sp}(V, f)$ in terms of tensors of the form $v \otimes f(v, \cdot)$, where $v \in V$ and $f(v, \cdot) \in V^*$, provided that f is nondegenerate.
To simplify the notation, we will denote in the following the dual vector $f(v, \cdot) \in V^*$ by ϕ_v.
Let (V, f) be a nondegenerate symplectic space with hyperbolic basis $\{v_i | i \in I\}$ for some index set I. Then, using the notation above, $\{\phi_{v_i} | i \in I\}$ forms an independent set of elements in V^*, which is moreover a basis for V^* if V has finite dimension. We will denote ϕ_{v_i} by ϕ_i if it is clear which vector v_i we refer to.
By $\big(\mathfrak{s}(V \otimes V^*), f\big)$ or, by abuse of notation just $\mathfrak{s}(V \otimes V^*)$, we denote the subspace of $V \otimes V^*$ generated by the tensors of the form $v \otimes \phi_v$, $v \in V$, so we consider the "symmetric" elements in $V \otimes V^*$. Then, the vector space $\mathfrak{s}(V \otimes V^*)$ is spanned by elements of the form

$$\begin{aligned} w_{ii} :=& v_i \otimes \phi_{v_i} \text{ and} \\ w_{ij} :=& (v_i + v_j) \otimes (\phi_{v_i} + \phi_{v_j}) \text{ for } i < j, \end{aligned}$$

where $i, j \in I$.
So an arbitrary element of $\mathfrak{s}(V \otimes V^*)$ is of the form

$$\sum_{i \leq j, i, j \in I} \lambda_{ij} w_{ij} \text{ for } \lambda_{ij} \in \mathbb{F}.$$

With the result of Proposition 2.4.2, we find that

$$\begin{aligned}[v \otimes \phi_v, w \otimes \phi_w] =&(v \otimes \phi_w)\phi_v(w) - (w \otimes \phi_v)\phi_w(v) \\ =&(v \otimes \phi_w) f(v,w) - (w \otimes \phi_v) f(w,v) \\ =&f(v,w)(v \otimes \phi_w + w \otimes \phi_v)\end{aligned}$$

is a Lie product on $\mathfrak{s}(V \otimes V^*)$.
(Note that since $(v+w) \otimes (\phi_v + \phi_w) = v \otimes \phi_v + v \otimes \phi_w + w \otimes \phi_v + w \otimes \phi_w \in \mathfrak{s}(V \otimes V^*)$, we can consider $v \otimes \phi_w + w \otimes \phi_v$ as an element of $\mathfrak{s}(V \otimes V^*)$.)
The pure tensors $v \otimes \phi_v$ are extremal in $\mathfrak{s}(V \otimes V^*)$ with respect to the extremal form g defined by $g(v \otimes \phi_v, w \otimes \phi_w) = f(v,w)^2$, see 2.4.4.
The results of 2.4.3 give

$$\varphi : \big(\mathfrak{s}(V \otimes V^*), f\big) \xrightarrow{\cong} \mathfrak{fsp}(V, f),$$

provided f is nondegenerate.

LEMMA 6.1.1. *Let $(\mathfrak{s}(V \otimes V^*), f)$ as defined before with f nondegenerate. The extremal elements in the Lie algebra $\mathfrak{s}(V \otimes V^*)$ are of rank at most 2.*

PROOF. Let x be an extremal element in the symplectic Lie algebra $\mathfrak{s}(V \otimes V^*)$ and consider its action on the natural module V (see also 2.4.3). Then for any other element y in $\mathfrak{s}(V \otimes V^*)$ we find that for all v the following holds:

$$[x,[x,y]](v) = 2g(x,y)x(v),$$

where g is the extremal form on $\mathfrak{s}(V \otimes V^*)$. If we apply this with y being the pure tensor $w \otimes \phi_w$, where $w \in V$, we find

$$\begin{aligned} 2g(x, w \otimes \phi_w)x(v) =&f(w,v)x^2(w) - 2f(w,x(v))x(w) + f(w,x^2(v))w \\ =&f(w,v)x^2(w) + 2f(x(w),v))x(w) + f(x^2(w),v)w. \end{aligned} \tag{6.1}$$

If $g(x, w \otimes \phi_w) \neq 0$, it follows from equation (6.1) that $x(v) \in \langle w, x(w), x^2(w)\rangle$ for all $v \in V$ and without restriction for $w \in V$, so in particular, also $x(v) \in \langle w', x(w'), x^2(w')\rangle$ for any $w' \in V$ with $g(x, w' \otimes \phi_{w'}) \neq 0$ and $w' \notin \langle w, x(w), x^2(w)\rangle$. So $x(v) \in \langle w, x(w), x^2(w)\rangle \cap \langle w', x(w'), x^2(w')\rangle$, and therefore rk $x \leq 2$.
If $g(x, w \otimes \phi_w) = 0$ or $g(x, w' \otimes \phi_{w'}) = 0$, we can always find new elements $u, u' \in V$ with $g(x, u \otimes \phi_u) \neq 0$ and $g(x, u' \otimes \phi_{u'}) \neq 0$, and come back to the previous case. The reason is as follows. Let W be the 2-space spanned by w and w'. Then $W_{\mathfrak{s}} := \big\langle r \otimes \phi_r | r \in W \big\rangle$ is a 3-space inside $\mathfrak{s}(V \otimes V^*)$. The points

$\langle r \otimes \phi_r \rangle$ with $r \in W$ form a quadric in this 3-space. Now $H := \ker g(x, \cdot)$ is a hyperplane in $\mathfrak{s}(V \otimes V^*)$, so it meets $W_\mathfrak{s}$ in a 2-space. Since $|\mathbb{F}| \geq 3$, we find at least two more points $\langle u \otimes \phi_u \rangle$ and $\langle u' \otimes \phi_{u'} \rangle$ on this quadric that are not in H, and therefore u and u' fulfill our requirements. $\diamond$

PROPOSITION 6.1.2. *Let $(\mathfrak{s}(V \otimes V^*), f)$ as defined before with f nondegenerate. Then the extremal elements in the Lie algebra $\mathfrak{s}(V \otimes V^*)$ are exactly the pure tensors, so the elements of the form $\lambda v \otimes \phi_v$, $v \in V$, $\lambda \in \mathbb{F}$.*

PROOF. As we already have seen in Section 2.4.3, pure tensors of $\mathfrak{s}(V \otimes V^*)$ are extremal and, as they generate the algebra, they define the extremal form g by the following

$$g(v \otimes \phi_v, w \otimes \phi_w) = f(v, w)^2.$$

Let x be an extremal element. Then by Lemma 6.1.1 x is of rank at most 2. If x is of rank 1, then clearly it is a pure tensor.
So, assume that the rank of x equals 2. Then we can find independent $v, w \in V$ and $\phi, \psi \in V^*$ with $x = v \otimes \phi + w \otimes \psi$.
Now for every $u, u' \in V$ we have $f(x(u), u') = -f(u, x(u'))$. Thus $f(\phi(u)v + \phi(u)w, u') = -f(u, \phi(u')v + \psi(u')w)$.
So, if $f(u, v) = f(u, w) = 0$, then $f(x(u), u') = 0$ for all $u' \in V$. So, $\phi, \psi \in \langle \phi_v, \phi_w \rangle$ and we can write x as

$$x = \alpha v \otimes \phi_v + \beta w \otimes \phi_w + \gamma(v \otimes \phi_w + w \otimes \phi_v)$$

for some $\alpha, \beta, \gamma \in \mathbb{F}$.
Since x is extremal, we have for every $u \in V$

$$2g(x, u\phi_u) \cdot x = [x, [x, u \otimes \phi_u]].$$

For any element u with $f(v, u) = 1$ and $f(w, u) = 0$ we have

$$\begin{aligned}[x, u \otimes \phi_u] &= [\alpha v \otimes \phi_v + \beta w \otimes \phi_w + \gamma(v \otimes \phi_w + w \otimes \phi_v), u \otimes \phi_u] \\ &= \alpha(v \otimes \phi_u + u \otimes \phi_v) + \gamma(u \otimes \phi_w + w \otimes \phi_u)\end{aligned}$$

and, as a straightforward computation reveals,

$$\begin{aligned}[x, [x, u \otimes \phi_u]] = 2\alpha^2 v \otimes \phi_v + 2\gamma^2 w \otimes \phi_w + 2\alpha\gamma(v \otimes \phi_w + w \otimes \phi_v) \\ + f(v, w)(\gamma^2 - \alpha\beta)(w \otimes \phi_u + u \otimes \phi_w),\end{aligned}$$

which has to be equal to

$$\kappa \cdot x = \kappa\alpha v \otimes \phi_v + \kappa\beta w \otimes \phi_w + \kappa\gamma(v \otimes \phi_w + w \otimes \phi_v),$$

where

$$\begin{aligned}\kappa &= 2g(x, u \otimes \phi_u)\\ &= 2\alpha f(v,u)^2 + \beta f(w,u)^2 + \gamma(f(v+w,u)^2 - f(v,u)^2 - f(w,u)^2) = 2\alpha.\end{aligned}$$

This implies that

$$(\gamma^2 - \alpha\beta)(2w \otimes \phi_w + f(v,w)(w \otimes \phi_u + u \otimes \phi_w)) = 0.$$

If $f(v,w) = 0$ we find $\gamma^2 - \alpha\beta = 0$. If $f(v,w) \neq 0$, then we take $u = f(v,w)^{-1}w$, and we find

$$(\gamma^2 - \alpha\beta)(4w \otimes \phi_w) = 0,$$

so, again

$$\gamma^2 - \alpha\beta = 0.$$

But this condition on α, β, γ implies that x is a rank 1 element, which is against our assumption.

Indeed, if α, β and γ are nonzero elements of $\mathbb{F}$ satisfying $\gamma^2 - \alpha\beta = 0$, then with $\delta = \gamma/\alpha$, we have

$$\begin{aligned}\alpha(v + \delta w) \otimes \phi_{v+\delta w} &= \alpha v \otimes \phi_v + \alpha\delta^2 w \otimes \phi_w + \alpha\delta(v \otimes \phi_w + w \otimes \phi_v)\\ &= \alpha v \otimes \phi_v + \beta w \otimes \phi_w + \gamma(v \otimes \phi_w + w \otimes \phi_v)\end{aligned}$$

is of rank 1. ◇

We consider an example.

6.1.2. Example: the 4-dimensional case. We consider the symplectic Lie algebra for a nondegenerate symplectic vector space (V, f) of dimension 4, with standard hyperbolic basis $\{e_1, e_2, e_3, e_4\}$, and identify it with the subalgebra $\mathfrak{s}(V \otimes V^*)$ of $\mathfrak{g}(V \otimes V^*)$ generated by the elements $v \otimes f(v, \cdot)$ as in 2.4.3.

Let $W \subset V$ be the 3-dimensional subspace spanned by e_1, e_2, and e_3, and denote by $\mathfrak{s}$ the subalgebra of $\mathfrak{s}(V \otimes V^*)$ spanned by the elements $v \otimes f(v, \cdot)$, where $v \in W$.

For $1 \leq i \leq 4$ denote by ϕ_i the element $f(e_i, \cdot) \in V^*$. Then notice that

$$f(e_1, e_2) = f(e_2, e_2) = f(e_3, e_2) = 0$$

$$\begin{aligned} &\Longrightarrow e_2 \perp \langle e_1, e_2, e_3 \rangle \\ &\Longrightarrow \langle e_2 \rangle = \mathrm{rad}(f|_W), \end{aligned}$$

so $e_2 \otimes \phi_2$ is nontrivial but in the center of $\mathfrak{s}$. Note that in the Lie subalgebra $\mathfrak{s}_0$ of $\mathfrak{g}(W \otimes W^*)$ generated by the elements $v \otimes f(v, \cdot)$ where $v \in W$, we have $e_2 \otimes \phi_2 = 0$, since $f(e_2, \cdot)$ is constantly zero on W. We have $Z(\mathfrak{s}) = \langle e_2 \otimes \phi_2 \rangle$ and $\mathfrak{s}_0 \cong \mathfrak{s}/Z(\mathfrak{s})$.
Let $v \in W$ and consider the elements

$$\begin{aligned} v_\lambda :=& v \otimes f(v, \cdot) - (v + \lambda e_2) \otimes f(v + \lambda e_2, \cdot) \\ =& v \otimes f(v, \cdot) - v \otimes f(v + \lambda e_2, \cdot) - \lambda e_2 \otimes f(v + \lambda e_2, \cdot) \\ =& v \otimes f(v, \cdot) - v \otimes f(v, \cdot) - \lambda v \otimes f(e_2, \cdot) - \lambda e_2 \otimes f(v, \cdot) - \lambda^2 e_2 \otimes f(e_2, \cdot) \\ =& - e_2 \otimes f(\lambda v, \cdot) - \lambda v \otimes f(e_2, \cdot) - \lambda^2 e_2 \otimes f(e_2, \cdot) \end{aligned}$$

for $\lambda \in \mathbb{F}$. Then for all $w \in W$ we have

$$\begin{aligned} &[v_\lambda, w \otimes f(w, \cdot)] \\ &\quad =[-e_2 \otimes f(\lambda v, \cdot) - \lambda v \otimes f(e_2, \cdot) - \lambda^2 e_2 \otimes f(e_2, \cdot), w \otimes f(w, \cdot)] \\ &\quad = - f(\lambda v, w)(e_2 \otimes f(w, \cdot) + f(w, \lambda v) w \otimes f(e_2, \cdot) \\ &\quad = f(\lambda v, w)(-e_2 \otimes f(w, \cdot) - w \otimes f(e_2, \cdot)) \\ &\quad = w_{f(\lambda v, w)} + f(\lambda v, w)^2 e_2 \otimes f(e_2, \cdot). \end{aligned}$$

Moreover, for $w \in W$ and $\mu \in \mathbb{F}$ we find

$$\begin{aligned} [v_\lambda, w_\mu] =& [-e_2 \otimes f(\lambda v, \cdot) - \lambda v \otimes f(e_2, \cdot) - \lambda^2 e_2 \otimes f(e_2, \cdot), \\ & - e_2 \otimes f(\mu w, \cdot) - \mu w \otimes f(e_2, \cdot) - \mu^2 e_2 \otimes f(e_2, \cdot)] \\ =& f(\lambda v, \mu w) e_2 \otimes f(e_2, \cdot) - f(\mu w, \lambda v) e_2 \otimes f(e_2, \cdot) \\ =& 2 f(\lambda v, \mu w) e_2 \otimes f(e_2, \cdot). \end{aligned}$$

This implies that the elements v_λ, with $v \in W$ and $\lambda \in \mathbb{F}$ generate an ideal $\mathfrak{i}$ of $\mathfrak{s}$ which is, modulo the center $\langle e_2 \otimes f(e_2, \cdot) \rangle$, isomorphic to the natural 2-dimensional module for $\mathfrak{s}/\mathfrak{i} \simeq \mathfrak{sl}_2(\mathbb{F})$.
We notice that both $\mathfrak{s}$ and $\mathfrak{s}_0$ can be generated by a symplectic triple, i.e., a triple of elements x, y, z with (x, y) and (y, z) in $\mathcal{E}_2$, $z \notin \langle x, y \rangle$ and $[x, z] = 0$.

For example, we can choose the extremal elements

$$x := e_1 \otimes \phi_1,\ y := e_3 \otimes \phi_3,\ z := (e_1 - e_2) \otimes (\phi_1 - \phi_2)$$

generating both $\mathfrak{s}$ and $\mathfrak{s}_0$.
Indeed, we have

$$f(e_1, e_3) \neq 0 \text{ and } f(e_3, e_1 - e_2) = f(e_3, e_1) - f(e_3, e_2) = f(e_1, e_3) \neq 0$$

and

$$[x, z] = f(e_1, e_1 - e_2)\big(e_1 \otimes (\phi_1 - \phi_2) + (e_1 - e_2) \otimes \phi_1\big) = 0,$$

so $(x, z) \in E_0$ and $(x, y), (y, z) \in E_2$. Hence (x, y, z) is a symplectic triple, which is easily seen to generate $\mathfrak{s}$ or $\mathfrak{s}_0$.
The non-trivial pure tensors in $\mathfrak{s}$ and $\mathfrak{s}_0$ are extremal. They are scalar multiples of elements of the form

$$s := (\alpha e_1 + \beta e_2 + \gamma e_3) \otimes (\alpha \phi_1 + \beta \phi_2 + \gamma \phi_3),$$

with $\alpha, \beta, \gamma \in \mathbb{F}$.
Note that all pure tensors commuting with $x = e_1 \otimes \phi_1$ are scalar multiples of elements of the form

$$(\alpha e_1 + \beta e_2) \otimes (\alpha \phi_1 + \beta \phi_2) = \alpha^2 e_1 \otimes \phi_1 + \beta^2 e_2 \otimes \phi_2 + \alpha\beta(e_1 \otimes \phi_2 + e_2 \otimes \phi_1),$$

so they are in the 3-space U_x spanned by $\{e_1 \otimes \phi_1, e_2 \otimes \phi_2, e_1 \otimes \phi_2 + e_2 \otimes \phi_1\}$, and their coefficients $\alpha, \beta, \gamma \in \mathbb{F}$ w.r.t. this basis are described by the quadratic equation $\alpha\beta = \gamma^2$. Therefore the corresponding 1-spaces form a quadric inside U_x. The same holds for any other choice of extremal x not in the center $\langle e_2 \otimes \phi_2 \rangle$.

If we define an $\mathfrak{sl}_2$-line to be the set of 1-spaces generated by pure tensors inside an $\mathfrak{sl}_2$ which is generated by two such points, then these $\mathfrak{sl}_2$-lines induce the structure of a symplectic plane (as defined in 4.2.7) on the extremal points spanned by pure tensors.

Using the notation $A := e_1 \otimes \phi_1, B := e_2 \otimes \phi_2, C := e_3 \otimes \phi_3, D := e_1 \otimes \phi_2 + e_2 \otimes \phi_1, E := e_1 \otimes \phi_3 + e_3 \otimes \phi_1, F := e_2 \otimes \phi_3 + e_3 \otimes \phi_2$, we have the following multiplication table for $\mathfrak{s}$.

	A	B	C	D	E	F
A	0	0	E	0	$2A$	D
B	0	0	0	0	0	0
C	$-E$	0	0	$-F$	$-2C$	0
D	0	0	F	0	D	$2B$
E	$-2A$	0	$2C$	$-D$	0	F
F	$-D$	0	0	$-2B$	$-F$	0

Clearly B is in the center of $\mathfrak{s}$, and the space spanned by B, D and F is the ideal $\mathfrak{i}$.

The algebra $\mathfrak{s}_0$ is spanned by the elements A, C, D, E, F with the multiplication table

	A	C	D	E	F
A	0	E	0	$2A$	D
C	$-E$	0	$-F$	$-2C$	0
D	0	F	0	D	0
E	$-2A$	$2C$	$-D$	0	F
F	$-D$	0	0	$-F$	0

The generators A, C, D, E and F are linearly independent, we have $\dim(\mathfrak{s}_0) = 5$. Moreover, we see that $\mathfrak{s}_0$ is ismorphic to $\mathfrak{s}$ modulo its center, and it has a 2-dimensional ideal spanned by D and F.

Definition 6.1.3. By $\mathfrak{sp}_3(\mathbb{F})$, we denote a Lie algebra isomorphic to $\mathfrak{s}$ from the example above.

By $\mathfrak{psp}_3(\mathbb{F})$, we denote a Lie algebra isomorphic to $\mathfrak{s}_0$ from the example above. Notice that $\mathfrak{psp}_3(\mathbb{F})$ is isomorphic to $\mathfrak{sp}_3(\mathbb{F})$ modulo its center.

6.2. The geometry of $(\mathcal{E}, \mathfrak{sl}_2)$

Setting 6.2.1. In the following, $\mathfrak{g}$ denotes a Lie algebra over the field $\mathbb{F}$ of characteristic $\neq 2$, generated by its set of extremal elements $E(\mathfrak{g})$ and equipped with the extremal form g (as defined in 2.3.3). Let $\mathcal{E} := \mathcal{E}(\mathfrak{g})$ be the set of projective extremal points of the Lie algebra $\mathfrak{g}$ and $\{\mathcal{E}_i\}_{i=-2}^{2}$ denote the symmetric relations on $\mathcal{E}$ as defined in 5.1.1. As always in this chapter, we assume that $\mathcal{E}_{-1} = \mathcal{E}_1 = \emptyset$. Moreover, we assume the graph $(\mathcal{E}, \mathcal{E}_2)$ to be connected.

A triple of elements $x, y, z \in \mathcal{E}$ with $(x, y), (y, z) \in \mathcal{E}_2$ and $(x, z) \in \mathcal{E}_0$ is called a *symplectic triple* (cf. Definition 2.1.5).

PROPOSITION 6.2.2. *A symplectic triple (x, y, z) of elements of the Lie algebra $\mathfrak{g}$ generates either a subalgebra isomorphic to $\mathfrak{sp}_3(\mathbb{F})$, in which case it is of dimension* 6, *or to its quotient by its center, so isomorphic to $\mathfrak{psp}_3(\mathbb{F})$ of dimension* 5.
Under this isomorphism x, y and z are mapped onto pure tensors of $\mathfrak{sp}_3(\mathbb{F})$ or their cosets in the quotient $\mathfrak{psp}_3(\mathbb{F})$.

PROOF. Let (x, y, z) be a symplectic triple and $\mathfrak{s}$ the subalgebra generated by x, y and z.
Consider the six elements x, y, z, $[x, y]$, $[z, y]$ and $[x, [y, z]]$. Notice that after rescaling we can assume that $g(x, y) = g(z, y) = -1$ and $g(x, z) = 0$. The Premet identities and the relations from 2.1.4 imply that the subspace of $\mathfrak{g}$ spanned by these six elements is closed under multiplication. Moreover, the multiplication table of these six generators is completely determined by the values $g(x, y) = g(z, y) = -1$ and $g(x, z) = 0$ and the values $g(a, [b, c])$, where a, b, c are equal to x, y or z. But by associativity of g, we have $g(a, [b, c]) = 0$ for all choices of a, b, c.
So, $\mathfrak{s}$ has dimension at most 6 and hence is isomorphic to a quotient of $\mathfrak{sp}_3(\mathbb{F})$ (compare with the Lie algebra considered in Example 6.1.2). Moreover, this isomorphism can be chosen to map x, y and z onto pure tensors (modulo Z). $\diamond$

Since we consider in this chapter Lie algebras with the property $\mathcal{E}_{\pm 1} = \emptyset$, the extremal geometry as defined in 5.1.1, where we defined lines to be spanned by points in relation $\mathcal{E}_{-1}$, is no appropriate choice to characterize $\mathfrak{g}$. We have to proceed differently.
We assume moreover, that for any $(x, y), (y, z) \in \mathcal{E}_2(\mathfrak{g})$, the subalgebra $\langle x, y, z\rangle$ of $\mathfrak{g}$ embeds into a subalgebra isomorphic to $\mathfrak{sp}_3(\mathbb{F})$ or $\mathfrak{psp}_3(\mathbb{F})$.
For such a Lie algebra $\mathfrak{g}$, we consider the point line space $\Gamma(\mathfrak{g}) := (\mathcal{E}, \mathfrak{sl}_2\text{-lines})$ that corresponds naturally to the $\mathfrak{sl}_2$-graph $\Gamma_{\mathfrak{sl}_2}(\mathfrak{g}) := (\mathcal{E}, \sim_{\mathfrak{sl}_2})$ (as introduced in 2.5). So in $\Gamma(\mathfrak{g})$, denoted abbreviatory by Γ if it is clear what Lie algebra we refer to, the points are the extremal points and two points $x, y \in \mathcal{E}$ are on a line if and only if $g_x(y) = g_y(x) \neq 0$. This $\mathfrak{sl}_2$-line consists of all extremal points in the subalgebra $\langle x, y\rangle \cong \mathfrak{sl}_2$ (see 2.5.1). Note that if a pair (x, y) of

distinct points is not hyperbolic, it must be commuting since we assumed that $\mathcal{E}_1$ and $\mathcal{E}_{-1}$ are empty.

DEFINITION 6.2.3. On $\mathcal{E}$ we define the relation $\perp$ by:

$$x \perp y \Leftrightarrow (x, y) \in \mathcal{E}_0 \cup \mathcal{E}_{-2}.$$

The point-line space $\Gamma = (\mathcal{E}, \mathfrak{sl}_2\text{-lines})$ is called **nondegenerate** if it is connected and for any pair of elements $x, y \in \mathcal{E}$ with $x^\perp = y^\perp$, it follows $x = y$.

PROPOSITION 6.2.4. *Let $\Gamma(\mathfrak{g}) = (\mathcal{E}, \mathfrak{sl}_2\text{-lines})$ for a Lie algebra $\mathfrak{g}$ fulfilling the assumptions of 6.2.1. Then every pair of points $x, y \in \mathcal{E}$ is on at most one $\mathfrak{sl}_2$-line, and two intersecting $\mathfrak{sl}_2$-lines generate inside the geometry Γ a subspace isomorphic to a symplectic plane.*

PROOF. The first statement is clear by definition of the $\mathfrak{sl}_2$-lines: they are exactly the lines between hyperbolic pairs of elements $x, y \in \mathcal{E}$, i.e. $g_x(y) \neq 0$, (and two extremal elements cannot generate two different $\mathfrak{sl}_2$-subalgebras). We already considered the extremal elements on $\mathfrak{sl}_2$-lines in 5.3.6.
For the second property, consider these two intersecting $\mathfrak{sl}_2$-lines l and m. There exists an intersection point $z \in \mathcal{E}$ with $z \in l \cap m$ and we can find points $x \in l$ and $y \in m$ sucht that (x, z, y) is a symplectic triple.
We have seen in 6.2.2 that a symplectic triple in a Lie algebra $\mathfrak{g}$ generates a $\mathfrak{sp}_3(\mathbb{F})$ subalgebra or its central quotient $\mathfrak{psp}_3(\mathbb{F})$. Moreover, the elements x, y and z are mapped to pure tensors (or their cosets). As the pure tensors not in the center of $\mathfrak{sp}_3(\mathbb{F})$ form a subspace isomorphic to a symplectic plane, see example 6.1.2, we find that x, y and z generate a subspace of Γ isomorphic to a symplectic plane. $\diamond$

LEMMA 6.2.5. *If $\operatorname{rad}(g) = \{0\}$, then $\Gamma(\mathfrak{g}) = (\mathcal{E}, \mathfrak{sl}_2\text{-lines})$ is nondegenerate.*

PROOF. By assumption Γ is connected. Now consider two elements $x, y \in \mathcal{E}$ with $x^\perp = y^\perp$. Then $\mathfrak{g} = \langle x^\perp, z\rangle$ for any element $z \in \mathcal{E}_2(x)$. Indeed, each element z' in $\mathcal{E}$ is either in $x^\perp$ or in $\langle x, z\rangle$ or generates together with x and z a subalgebra isomorphic to $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ which is generated by x, z and some $w \in \mathcal{E}_0(x) \cap \mathcal{E}_2(z)$, see 6.2.4.
Let x_0, y_0 and z_0 be nonzero extremal elements in x, y, and z, respectively. Then we can find a $\lambda \in \mathbb{F}$ such that $g(\lambda x_0, z_0) = g(y_0, z_0)$. It follows

$$g(\lambda x_0 - y_0, z_0) = 0.$$

Together with

$$g(\lambda x_0 - y, v_0) = 0$$

for each extremal element v_0 in an extremal point $v \in x^\perp = y^\perp$, it follows $\lambda x_0 - y_0 \in \mathrm{rad}(g) = \{0\}$, so $x_0 = y_0$ and $x = y$. ◇

Geometries in which any two intersecting lines generate a subspace isomorphic to a symplectic plane have been studied by Cuypers in [Cuy94]. Using the main result of [Cuy94], as stated in 4.2.8, we obtain the following.

THEOREM 6.2.6. *The connected partial linear space* $\Gamma = (\mathcal{E}, \mathfrak{sl}_2$*-lines) as defined above is isomorphic to the geometry* $HSp(V, f)$ *of hyperbolic lines of a symplectic space* (V, f) *over* $\mathbb{F}$ *as defined in 1.2.2. This isomorphism is denoted by*

$$\varphi : (\mathcal{E}, \mathfrak{sl}_2\text{-}lines) \xrightarrow{\cong} HSp(V, f).$$

The form f *is nondegenerate if* $\mathrm{rad}(g) = \{0\}$.

PROOF. If Γ contains a single line, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ and Γ is isomorphic to $HSp(2, \mathbb{F})$.
So, assume that Γ contains at least two lines. By assumption, $(\mathcal{E}, \mathfrak{sl}_2$-lines) is connected. By 6.2.4, we moreover know that $(\mathcal{E}, \mathfrak{sl}_2$-lines) is a partial linear space, and any pair of intersecting lines is contained in a symplectic plane. Our assumption $|\mathbb{F}| \geq 3$ guarantees that we have a line with more than 3 points in $(\mathcal{E}, \mathfrak{sl}_2$-lines). This gives us all conditions for 4.2.8 and we conclude that $(\mathcal{E}, \mathfrak{sl}_2$-lines) forms a geometry of hyperbolic lines in a symplectic geometry.
As each symplectic plane can be coordinatized by $\mathbb{F}$, we obtain that Γ is isomorphic to $HSp(V, f)$ for some symplectic space over $\mathbb{F}$. ◇

Concretely, the geometric structure of $HSp(V, f)$ translates to the following: Let $\varphi(x) = p, \varphi(y) = q$ be distinct points, then

$$\begin{aligned} &(p, q) \text{ are on a hyperbolic line in } (V, f) \\ &\quad \Leftrightarrow (x, y) \text{ is a hyperbolic pair in } (\mathcal{E}, \mathfrak{sl}_2\text{-lines}) \\ &\quad \Leftrightarrow x \not\perp y. \end{aligned}$$

So the hyperbolic lines in $HSp(V, f)$ are the lines obtained from the $\mathfrak{sl}_2$-lines in $(\mathcal{E}, \mathfrak{sl}_2$-lines). The second type of lines in the symplectic space (V, f), the singular lines, correspond to commuting extremal points. Indeed, for two distinct points $\varphi(x) = p, \varphi(y) = q$, we have

$$\begin{aligned} &(p, q) \text{ are on a singular line in } (V, f) \\ &\Leftrightarrow x \not\sim_{\mathfrak{sl}_2} y \\ &\Leftrightarrow (x, y) \in \mathcal{E}_0 \\ &\Leftrightarrow x \perp y. \end{aligned}$$

For later use, we need a name for the equivalent of the singular lines in the symplectic geometry for elements in $\mathcal{E}$.

Definition 6.2.7. Suppose Γ is nondegenerate and $x, y \in \mathcal{E}$ with $x \perp y$. Then the **polar line** through x and y is the set $(\{x, y\}^\perp)^\perp$.

Remark 6.2.8. With the previous construction and the result of Theorem 6.2.6, we find, in case Γ is nondegenerate, that

$$\mathbb{P}(V) \cong (\mathcal{E}, \{\mathfrak{sl}_2\text{-lines}\} \cup \{\text{polar lines}\}).$$

6.3. Veroneseans

In this section we introduce Veroneseans, following the definitions of Schillewaert and Van Maldeghem in [SVM13].

Definition 6.3.1. Let W be a vector space of dimension $m(2m+1)$ over the field $\mathbb{F}$. The **quadric Veronesean of index** $l = 2m-1$, denoted by $\mathcal{V}_l$, is the set of points in $\mathbb{P}(W)$ with projective coordinates y_{ij}, $i, j = 0, 1, \ldots, l$ and $i \leq j$, such that the corresponding symmetric matrix $(y)_{ij} \in M := \{(m)_{ij}, i, j = 0, 1, \ldots, l \mid m_{ij} = m_{ji} \text{ if } i > j\}$ is of rank 1.

The above definition implies that in the vector space $M_{sym}^{2m\times 2m}(\mathbb{F})$ of all symmetric matrices over the field $\mathbb{F}$ the set of projective points spanned by rank 1 matrices is a quadric Veronesean of index $l = 2m-1$.
A second example is the set of 1-spaces spanned by rank 1 symplectic matrices inside the space $M_{sp}^{2m\times 2m}(\mathbb{F})$ of all $2m \times 2m$ symplectic matrices.
We recall that a matrix M is called symplectic if and only if $M^t F = -FM$, where (with I_m the $m \times m$-identity matrix)

$$F = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Note that there is a linear isomorphism between the space of symmetric matrices $M_{sym}^{2m\times 2m}$ and of symplectic matrices $M_{sp}^{2m\times 2m}(\mathbb{F})$ via the bijection

$$\begin{aligned}\tau : M_{sym}^{2m\times 2m} &\to M_{sp}^{2m\times 2m}\\ M &\mapsto -FM\end{aligned}$$

Indeed, since $F^t = -F = F^{-1}$ we find for a symmetric matrix M that

$$(-FM)^t F = -M^t F^t F = -M = -F^t FM = F(FM) = -F(-FM)$$

and hence $-FM$ is symplectic. Vice versa, if M is symplectic then FM, the image under the inverse map, is symmetric. Indeed,

$$(FM)^t = M^t F^t = -M^t F = -(-FM) = FM.$$

As we can identify $\mathfrak{s}(V \otimes V^*)$, for finite dimensional V, with the space of symplectic matrices, the pure tensors corresponding to the rank 1 matrices, we find that the extremal points, which by 6.1.2 are generated by pure tensors, form a quadric Veronesean in $\mathfrak{s}(V \otimes V^*)$.

Definition 6.3.2. An **oval** C in a projective plane π is a set of points of π where no three of them are collinear and for every point $x \in C$, there is a unique line L through x intersecting C in only x. The line L is called the **tangent** line at x to C.

Notice that for each 2-dimensional subspace U of V the points in $\mathbb{P}(\mathfrak{s}(V \otimes V^*))$ spanned by pure tensors $u \otimes \phi_u$ with $u \in U$ form an oval. Indeed, if $U = \langle u, v\rangle$, then $\langle w \otimes \phi_w \mid w \in U\rangle$ is a 3-dimensional subspace $U_{\mathfrak{s}}$ of $\mathfrak{s}(V \otimes V^*)$. As we have seen in the previous section, the pure tensors in $U_{\mathfrak{s}}$ are all scalar multiples of $\alpha u \otimes \phi_u + \beta v \otimes \phi_v + \gamma(u \otimes \phi_v + v \otimes \phi_u)$ where $\gamma^2 - \alpha\beta = 0$. The pure tensors in $\langle u \otimes \phi_u, v \otimes \phi_v\rangle$ are only the scalar multiples of $u \otimes \phi_u$ and $v \otimes \phi_v$. But then the only 2-space of $U_{\mathfrak{s}}$ containing only pure tensors from $\langle u \otimes \phi_u\rangle$ is $\langle u \otimes \phi_u, u \otimes \phi_v + v \otimes \phi_u\rangle$.

The set of all ovals obtained from such 3-spaces $U_{\mathfrak{s}}$ induces the structure of $\mathbb{P}(V)$ on the quadric Veronesean of 1-spaces spanned by pure tensors in $\mathfrak{s}(V \otimes V^*)$.

The following theorem, proven by Schillewaert and Van Maldeghem in [SVM13], provides a characterization of the quadric Veronesean by this projective structure.

THEOREM 6.3.3 ([SVM13, Thm. 2.3]). *Let W be a vector space of dimension d over a field $\mathbb{F}$ of order at least three, and X be a spanning point set of $\mathbb{P}(W)$ and $\mathbb{K}$, and suppose*

- **(V1*):** *for any pair of points $x, y \in X$, there is a unique plane denoted by $\langle x|y\rangle$ such that $\langle x|y\rangle \cap X$ is an oval, denoted by $X(\langle x|y\rangle)$.*
- **(V2*):** *the set X endowed with all subsets $X(\langle x|y\rangle)$ has the structure of the point-line-geometry of a projective space $\mathbb{P}(V)$ for some vector space V of dimension $n \geq 3$, or of any projective plane Π (and we put $n = 2$ in this case).*
- **(V3*) :** *$d \geq \frac{1}{2}n(n+1)$.*

Then $d = \frac{1}{2}n(n+1)$ and X is the point set of a quadric Veronesean of index $n-1$.

With the notations of the previous Theorem, we define the injective map

$$\mathcal{V} : \mathbb{P}(V) \to \mathbb{P}(W)$$

mapping points to points and lines to ovals, such that for any two points $x \neq y$ in $\mathbb{P}(V)$, we have $\mathcal{V}(\langle x, y\rangle) = \langle x|y\rangle$. We call this map the **Veronesean embedding** of $\mathbb{P}(V)$ if property (V1*) is fulfilled for the image of $\mathcal{V}$ (note that (V2*) holds automatically by construction). If moreover (V3*) holds, $\mathcal{V}$ is unique (up to isomorphism) and we call it the **universal Veronesean embedding**.

The above implies that the map

$$\begin{aligned} \mathbb{P}(V) &\to \mathbb{P}(\mathfrak{s}(V \otimes V^*)) \\ \langle v\rangle &\mapsto \langle v \otimes \phi_v\rangle \end{aligned}$$

is a universal Veronesean embedding.

6.4. The uniqueness of the Lie product on the Veronesean

Recall our original situation as introduced in section 6.2. We have an (unidentified) Lie algebra $\mathfrak{g}$ generated by its set $\mathcal{E}$ of extremal points with $\mathcal{E}_{\pm 1} = \emptyset$ and extremal form g with trivial radical. Moreover, we assume that every three elements x, y and z in $\mathcal{E}$ with (x, y) and (y, z) in $\mathcal{E}_2$ generate a Lie subalgebra isomorphic to $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$. Then the geometry $\Gamma(\mathfrak{g}) = (\mathcal{E}(\mathfrak{g}), \mathfrak{sl}_2) \cong HSp(V, f)$, where (V, f) is a nondegenerate symplectic space. This isomorphism is denoted by ϕ.

The inverse of this isomorphism, ϕ^{-1} provides us with a Veronesean embedding of each of the $\mathfrak{sl}_2$-lines into a 3-dimensional subspace of $\mathfrak{g}$.

The goal of this section is to show that, up to a scalar, the Lie product $[\cdot,\cdot]$ of $\mathfrak{g}$ is the unique Lie product on the vector space $\mathfrak{g}$, whose $\mathfrak{sl}_2$-geometry coincides with that of $\mathfrak{g}$.

The following Lemma gives a translation of the result of 5.3.7 to the situation in this chapter. There, we used the lines coming from the relation $\mathcal{E}_{-1}$; now we use the $\mathfrak{sl}_2$-relation.

LEMMA 6.4.1. *Let (x, y) be a hyperbolic pair generating a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then there is a $\lambda \in \mathbb{F}^*$ such that for all $v, w \in \mathfrak{h}$ we have $[v, w]_1 = \lambda[v, w]$.*

PROOF. Without loss of generality we can assume $g(x, y) = 1$. The subalgebra $\mathfrak{h}$ is isomorphic to $\mathfrak{sl}_2$. Its extremal points are the 1-spaces spanned by elements $ax + by + c[x, y]$ satisfying the equation $ab = c^2$ (see the proof of Proposition 5.3.6).

In $\mathfrak{g}_1$, the extremal elements in the subalgebra generated by x, y are the 1-spaces spanned by elements $y + \lambda[x, y]_1 + \lambda^2 g^1(x, y)x$. Since any two points in $\Gamma(\mathfrak{g})$ are on at most one line, these extremal elements generate points which are extremal points in $\mathfrak{h}$. So, all these extremal elements are in the subspace $\mathfrak{h}$. In particular, we find that $[x, y]_1$ is in $\mathfrak{h}$ and hence can be expressed as

$$[x, y]_1 = \alpha x + \beta y + \gamma[x, y]$$

for some fixed α, β and γ in $\mathbb{F}$. But that implies that

$$y + \lambda[x, y]_1 + \lambda^2 g^1(x, y)x = (\lambda^2 g^1(x, y) + \lambda\beta)x + (1 + \lambda\alpha)y + \lambda^2\gamma^2[x, y]$$

satisfies the equation

$$(1 + \lambda\alpha)(\lambda^2 g^1(x, y) + \lambda\beta) = \lambda^2\gamma^2.$$

As in the proof of 5.3.7 we deduce that, restricted to $\mathfrak{h}$, the Lie product $[\cdot,\cdot]_1$ is a scalar multiple of $[\cdot,\cdot]$. $\diamond$

LEMMA 6.4.2. *Let (x, y, z) be a symplectic triple in $\mathcal{E}$ generating a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ isomorphic to $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$.*

Let $[\cdot,\cdot]_1$ denote a Lie product defined on the vector space $\mathfrak{s}$, such that the $\mathfrak{sl}_2$-geometries of $[\cdot,\cdot]_1$ coincides with the symplectic plane of Γ generated by x, y

and z. Then there is a scalar $\lambda \in \mathbb{F}^*$ such that for any two elements $v, w \in \mathfrak{s}$ we have $[v,w]_1 = \lambda[v,w]$.

PROOF. Let $\mathcal{S}$ be the set of extremal points in the symplectic plane generated by x, y and z. For any subset $\mathcal{T}$ of $\mathcal{S}$ we denote by $E_{\mathcal{T}}$ the set of extremal elements whose span is in $\mathcal{T}$.
To prove the lemma, it suffices to show that there exists a $\lambda \in \mathbb{F}$ such that for all $v, w \in E_{\mathcal{S}}$ we have $[v,w]_1 = \lambda[v,w]$.
As two points in $\mathcal{S}$ commute if and only if they are not collinear, we find for all $v, w \in S$ that $[v,w] = 0 \Leftrightarrow [v,w]_1 = 0$.
Let L be any line of the symplectic plane on $\mathcal{S}$. Then by Lemma 6.4.1 there is an $\lambda_L \in \mathbb{F}$ with $[v,w] = \lambda_L[v,w]_1$ for all $v, w \in E_L$. Suppose L, M are two lines in the symplectic plane on $\mathcal{S}$. We will prove that $\lambda_L = \lambda_M$.
Let p be a point on L but not on M and let q, r, s be three distinct points on M collinear with p, such that $s \in L$. Denote the line through p and q by Q and through p and r by R. By t we denote the unique point on M not collinear to p. Let p_1, q_1, r_1 and s_1 be extremal elements in p, q, r and s, respectively, such that $0 \neq q_1 + r_1 + s_1 = t_1 \in t$. Then

$$\begin{aligned} 0 &= [p_1, t_1] \\ &= [p_1, q_1 + r_1 + s_1] \\ &= [p_1, q_1] + [p_1, r_1] + [p_1, s_1] \end{aligned}$$

and, moreover

$$\begin{aligned} 0 &= [p_1, t_1]_1 \\ &= [p_1, q_1 + r_1 + s_1]_1 \\ &= [p_1, q_1]_1 + [p_1, r_1]_1 + [p_1, s_1]_1 \\ &= \lambda_Q[p_1, q_1] + \lambda_R[p_1, r_1] + \lambda_L[p_1, s_1]. \end{aligned}$$

This implies that

$$(\lambda_L - \lambda_Q)[p_1, q_1] + (\lambda_L - \lambda_R)[p_1, r_1] = 0.$$

If $[p_1, q_1]$ and $[p_1, r_1]$ are linearly independent, we find $\lambda_L = \lambda_Q = \lambda_R$. If $[p_1, q_1]$ and $[p_1, r_1]$ are linearly dependent, then $\lambda_Q = \lambda_R$, as then $[p_1, [p_1, q_1]]_1 = \lambda_Q[p_1, [p_1, q_1]]$ but also $[p_1, [p_1, q_1]]_1 = \lambda_R[p_1, [p_1, q_1]]$.
With a similar argument, but permuted L, Q and R, we find $\lambda_L = \lambda_Q = \lambda_R$.

This shows that for all lines L' on p inside $\mathcal{S}$ we have $\lambda_{L'} = \lambda_L$. But by connectedness of the symplectic plane, we find this to be true for any line L' in $\mathcal{S}$. $\diamond$

PROPOSITION 6.4.3. *Let* $[\cdot,\cdot]_1$ *denote a Lie product defined on the vector space* $\mathfrak{g}$*, such that the* $\mathfrak{sl}_2$*-geometries of* $[\cdot,\cdot]_1$ *coincides with* Γ*. Then there is a scalar* $\lambda \in \mathbb{F}^*$ *such that for any two elements* $v, w \in \mathfrak{g}$ *we have* $[v,w]_1 = \lambda[v,w]$*.*

PROOF. Let L be a line of Γ. By Lemma 6.4.1 there is a $\lambda \in \mathbb{F}^*$ with $[\cdot,\cdot]_1 = \lambda[\cdot,\cdot]$ restricted to the subalgebra generated by L.
The above Lemma 6.4.2 implies that for any line M intersecting L we have that $[\cdot,\cdot]_1 = \lambda[\cdot,\cdot]$ restricted to the subalgebra generated by M. But then connectedness of Γ implies that $[\cdot,\cdot]_1 = \lambda[\cdot,\cdot]$ restricted to the subalgebra generated by any line N, which clearly implies the proposition. $\diamond$

PROPOSITION 6.4.4. *Let* $\Gamma(\mathfrak{g}) \cong HSp(V,f)$ *with* (V,f) *a nondegenerate symplectic space. Suppose* $\mathfrak{g}$ *has vector space-dimension* $m(2m+1)$*. If the projective embedding of* $\mathfrak{g}$ *into* $\mathbb{P}(\mathfrak{g})$ *induces a universal Veronesean embedding of* $\mathbb{P}(V) \cong (\mathcal{E}, \{\mathfrak{sl}_2\text{-lines}\} \cup \{\text{polar lines}\})$ *into* $\mathbb{P}(\mathfrak{g})$*, then* $\mathfrak{g} \cong \mathfrak{sp}_{2m}(\mathbb{F})$*.*

PROOF. We extend the isomorphism $(\mathcal{E}, \mathfrak{sl}_2\text{-lines}) = \Gamma(\mathfrak{g}) \cong \Gamma(\mathfrak{sp}_{2m}(\mathbb{F}))$ uniquely to $\mathbb{P}(V) \cong (\mathcal{E}, \mathfrak{sl}_2\text{-lines} \cup \text{polar lines}) = \mathbb{P}(\Gamma(\mathfrak{g}))$. This, together with the uniqueness (up to isomorphism) of the universal Veronesean embedding of $\mathbb{P}(V)$ into the projective space of $\mathfrak{sp}_{2m}(\mathbb{F})$, allows us to identify the underlying vector spaces of $\mathfrak{g}$ and $\mathfrak{sp}_{2m}(\mathbb{F})$ as well as the Veronesean embeddings of $\Gamma(\mathfrak{g})$ and $\Gamma(\mathfrak{sp}_{2m}(\mathbb{F}))$. So w.l.o.g., we assume the equality of the vector spaces, the sets of extremal elements $\mathcal{E}(\mathfrak{g}) = \mathcal{E}(\mathfrak{sp}_{2m}(\mathbb{F}))$ and their relations $\mathcal{E}_i(\mathfrak{g}) = \mathcal{E}_i(\mathfrak{sp}_{2m}(\mathbb{F}))$ for $i \in \{-2, \dots, 2\}$.
Now we can apply Proposition 6.4.3 and find that up to a scalar multiple the two Lie products of $\mathfrak{g}$ and $\mathfrak{sp}_{2m}(\mathbb{F})$ are the same and hence these Lie algebras are isomorphic. $\diamond$

6.5. The Veronesean embedding

Before we begin with the last steps of the identification of the Lie algebra $\mathfrak{g}$, we subsume the previous results. We started with an unknown simple Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$ with and $\operatorname{char}\mathbb{F} \neq 2$, spanned by its extremal points $\mathcal{E}$ and with $\mathcal{E}_{\pm 1} = \emptyset$ and the radical of the extremal form g trivial. For any pairs of extremal points $(x,y), (y,z) \in \mathcal{E}_2(\mathfrak{g})$, the span $\langle x, y, z\rangle$ embeds into a subalgebra isomorphic to $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$. The partial linear space $(\mathcal{E}, \mathfrak{sl}_2\text{-lines})$ is isomorphic to $HSp(V, f)$, for some nondegenerate symplectic space (V, f) of dimension $2m$ and if $\mathcal{E}(\mathfrak{g})$ forms a quadric Versonesan in $\mathbb{P}(\mathfrak{g})$, then $\mathfrak{g}$ is uniquely identified to be isomorphic to a symplectic Lie algebra $\mathfrak{sp}_{2m}(\mathbb{F})$, where $2m = \dim V$.

So it is left to prove that the embedding of $\mathcal{E}(\mathfrak{g})$ into $\mathbb{P}(\mathfrak{g})$ is indeed a universal Veronesean embedding. Therefore, we consider the geometry on $HSp(V, f)$ as the geometry of hyperbolic and singular lines, isomorphic to the geometry of $\mathfrak{sl}_2$-lines and polar lines between the elements of $\mathcal{E}(\mathfrak{g})$. In the following, we will denote by L the set of $\mathfrak{sl}_2$-lines and by S the set of polar lines (as defined in 6.2.7). Now $L \cup S$ induces the structure of a projective space on $\mathcal{E}(\mathfrak{g})$ isomorphic to $\mathbb{P}(V)$. We prove the properties (V1*), (V2*) and (V3*) of Theorem 6.3.3 in the following propositions.

PROPOSITION 6.5.1 (V1*). *Any two points $x, y \in \mathcal{E}(\mathfrak{g})$ lie in a unique plane π of $\mathbb{P}(\mathfrak{g})$, where $\mathcal{E}(\mathfrak{g}) \cap \pi$ forms a quadric, and we denote π by $\langle x|y\rangle$ and $\mathcal{E} \cap \pi$ by $\mathcal{E}\langle x|y\rangle$.*

PROOF. In general, we have to distinguish two cases for $x, y \in \mathcal{E}(\mathfrak{g})$, namely either $(x, y) \in \mathcal{E}_2$ or $(x, y) \in \mathcal{E}_0$. Let us first consider $(x, y) \in \mathcal{E}_2$. Since the subalgebra $\mathfrak{sl}_2$ spanned by x and y is as a vector space 3-dimensional, it defines a unique plane $\langle x|y\rangle$. Finally in 5.3.6 we have seen that the extremal points in a Lie algebra generated by a hyperbolic pair form a quadric, so the same holds for $\mathcal{E}\langle x|y\rangle$.

If $(x, y) \in \mathcal{E}_0$, we have a bit more to do. Let l be the singular line on x, y. We claim that the linear span of l is a 3-dimensional subspace of $\mathfrak{g}$ meeting $\mathcal{E}$ just in l. Moreover, the points on l form a quadric in this 3-space. So, $\langle l\rangle$ will be the required plane $\langle x|y\rangle$.

Let z be a point in $\mathcal{E}$ such that $(x, z), (y, z) \in \mathcal{E}_2$, so (x, z, y) is a symplectic triple. Then z is collinear with all but one point, say a, on the polar line l. Clearly $l \setminus \{a\} \subseteq \langle x, y, z\rangle$. As we see in the symplectic plane generated by x, y and z, the points of $l \setminus \{a\}$ are all contained in a subspace of $\mathfrak{g}$ of dimension 3 if $\langle x, y, z\rangle \simeq \mathfrak{sp}_3(\mathbb{F})$ and of dimension 2 if $\langle x, y, z\rangle \simeq \mathfrak{psp}_3(\mathbb{F})$. In the first case they are all but one of the points of a quadric (the missing point being the center of $\langle x, y, z\rangle$) and in the second case all but one of the points of the 2-space.

Now consider a second point z' with (x, y, z') is another symplectic triple but this time z' collinear with a, but not with some $a' \neq a$ in l. As above, we find that all points of $l \setminus \{a'\}$ are contained in a subspace of $\mathfrak{g}$ of dimension 3 or 2. Moreover, in the first case they are all but one of the points of a quadric (the missing point being the center of $\langle x, y, z'\rangle$) and in the second case all but one of the points of the 2-space.

If $l \setminus \{a\}$ generates a 2-space, then this 2-space is contained in $\langle x, y, z'\rangle$, and we find at least three extremal points in it that are not commuting with z'. But this implies that also $l \setminus \{a'\}$ generates a 2-space and $l \setminus \{a\}$ and $l \setminus \{a'\}$ generate the same 2-space. In particular, a is contained in this 2-space. But since $a \in \langle x, y, z\rangle \cong (\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ and the center of $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ is trivial, it follows $[a, z] \neq 0$, contradicting that a is not collinear to z.

Hence $l \setminus \{a\}$ (and $l \setminus \{a'\}$) generates a 3-dimensional subspace.

Let c be the center of $\langle x, y, z\rangle$. Then every element $u \in \mathcal{E}$ that commutes with a also commutes with at least three points of some singular line on a that are contained in $\langle x, y, z\rangle$. As c is in the span of these points, we find $[u, c] = 0$.

Let c_1 be a nonzero element of c and a_1 be a nonzero element of a and fix $\lambda, \mu \in \mathbb{F}$, not both 0, such that $g(z', \lambda c_1 + \mu a_1) = 0$. As also $g(u, \lambda c_1 + \mu a_1) = 0$ for all $u \in \mathcal{E}$ with $a \perp u$, we find that $g(v, \lambda c_1 + \mu a_1) = 0$ for all $v \in \langle u^\perp, z'\rangle = \mathfrak{g}$. This implies that $\lambda c_1 + \mu a_1$ is in the radical of g and hence 0. But then $a = c$, so l is a quadric in l and we have proven the proposition. $\diamond$

Proposition 6.5.2 (V2*). *The point-line space $(\mathcal{E}(\mathfrak{g}), L \cup S)$ has the structure of the point-line-geometry of a projective space $\mathbb{P}(V)$, with V vector space over the field $\mathbb{F}$ with $|\mathbb{F}| \geq 3$.*

Proof. This follows immediately from 6.2.6. $\diamond$

Since in the following the dimension $2m$ of the vector space V with $\Gamma(\mathfrak{g}) \cong HSp(V, f)$ is of some importance, we denote the corresponding Lie algebra by $\mathfrak{g}_{2m}$ instead of $\mathfrak{g}$. Hereby, $\mathfrak{g}_{2m}$ still fulfills the same conditions as $\mathfrak{g}$ before.

Proposition 6.5.3 (V3*). *Let $\mathfrak{g}_{2m}$ be a Lie algebra generated by its extremal elements corresponding to the points in $\mathcal{E} := \mathcal{E}(\mathfrak{g}_{2m})$ and suppose that $\Gamma(\mathfrak{g}_{2m}) = (\mathcal{E}, \mathfrak{sl}_2\text{-lines}) \cong HSp(V, f)$ for some nondegenerate symplectic space (V, f) of dimension $2m$. Then $\dim \mathfrak{g}_{2m} \geq m(2m + 1)$.*

Proof. We prove this by induction on m.

If $m = 1$, then $\Gamma(\mathfrak{g}_2) = (\mathcal{E}, \mathfrak{sl}_2)$ is a line and $\dim \mathfrak{g}_2 = \dim \mathfrak{sl}_2 = 3 = 1(2 \cdot 1 + 1)$.

Now suppose the statement is true for some $m \in \mathbb{N}$. Then consider $\mathfrak{g}_{2(m+1)}$ with $\Gamma(\mathfrak{g}_{2(m+1)}) = (\mathcal{E}, \mathfrak{sl}_2) \cong HSp(V, f)$ with (V, f) a nondegenerate symplectic space of dimension $2(m+1)$. We fix an $\mathfrak{sl}_2$-line $\langle x, y\rangle$ in $\Gamma(\mathfrak{g}_{2(m+1)})$ and consider $\mathfrak{g}_0 = \langle z \in \mathcal{E} | [z, x] = [z, y] = 0\rangle$. Its geometry is isomorphic to $HSp(V', f')$, where (V', f') is a nondegenerate symplectic space of dimension $2m$, so by induction $\mathfrak{g}_0$ has dimension $m(2m+1)$.

Now consider $\mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0)$, where $\mathfrak{g}_x := \langle z \in \mathcal{E} | [z, x] = 0\rangle$. Each singular line s on x spans a 3-space in $\mathfrak{g}_x$ which meets $\mathfrak{g}_0$ in at most one point, so s maps to a space of dimension at most 1 in $\mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0)$. Now assume that $s \subseteq \langle \mathfrak{g}_0 + x\rangle$. Then the set $C_s(y)$ of elements in s commuting with y contains $\mathfrak{g}_0 \cap s$, which is at least 2-dimensional. But inside the $\mathfrak{sp}_3(\mathbb{F})$-subalgebra spanned by x, y and s we see that $C_s(y)$ is a 1-space, a contradiction. So it follows that indeed s is mapped to a 1-dimensional subspace in $\mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0)$.

Let $\mathfrak{s} := \mathfrak{sp}_3(\mathbb{F})$ as in Example 6.1.2 be a Lie algebra generated by a symplectic triple, such that x spans the center of $\mathfrak{s}$. Then the intersection of the geometries of $\mathfrak{s}$ and $\mathfrak{g}_0$ is a hyperbolic line l (so a 3-space in $\mathfrak{g}_0$) and $\mathfrak{s}$ is mapped to a subspace of dimension at most $6 - (3+1) = 2$ in $\mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0)$. We prove that this subspace is indeed of dimension 2. We use that $\mathfrak{s} \cong N : \mathfrak{sl}_2$, where $N \cong \mathbb{F}^{1+2}$ is an ideal isomorphic to a non-split extension of the natural module for $\mathfrak{sl}_2$ by a 1-dimensional center. Note that the elements of $\mathfrak{s}$ that are in $\mathfrak{sl}_2$ commute with y, as stated before. So assume there is an $n \in N$ that commutes with y. Clearly n is not in the center of N. But the action of $\mathfrak{sl}_2$ on $N/\langle x\rangle$ is the action on the natural module, so the images of n under this action will generate the full ideal N and commute with y. This implies $[x, y] = 0$, a contradiction. So $\mathfrak{s}$ maps to a 2-dimensional subspace in $\mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0)$.

Note that the geometry of the space spanned by singular lines l on x together with all possible subspaces $\mathfrak{s}$ as above on x is isomorphic to $HSp(V', f')$.

As follows from the above, this space is naturally embedded into $\mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0)$, which therefore has dimension $2m$ (by 4.2.8) and is isomorphic to the natural module for $\mathfrak{g}_0$.

A similar construction of the spaces $\mathfrak{g}_y$ and $\mathfrak{g}_y/\langle y\rangle + \mathfrak{g}_0$ leads to similar conclusions.

So $\mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0)$ and $\mathfrak{g}_y/(\langle y\rangle + \mathfrak{g}_0)$ are both $2m$-dimensional and by the above construction natural modules for $\mathfrak{g}_0$. These natural modules are irreducible, and we deduce

$$\begin{aligned}\dim \mathfrak{g}_{2(m+1)} \geq & \dim\langle x,y\rangle + \dim \mathfrak{g}_x/(\langle x\rangle + \mathfrak{g}_0) + \dim \mathfrak{g}_y/(\langle y\rangle + \mathfrak{g}_0) + \dim \mathfrak{g}_0 \\ =& 3 + 2m + 2m + m(2m+1) \\ =& 2m^2 + 5m + 3 \\ =& (m+1)(2(m+1)+1).\end{aligned}$$

◇

The consequence of 6.5.1, 6.5.2 and 6.5.3 is the following, again using the notation $\mathfrak{g}_{2m}$ for the Lie algebra $\mathfrak{g}$ with $\Gamma(\mathfrak{g}) \cong HSp(V,f)$, where V of dimension $2m$:

Corollary 6.5.4. *$\mathcal{E}(\mathfrak{g}_{2m})$ is a quadric Veronesean of index $2m-1$ in $\mathbb{P}(\mathfrak{g}_{2m})$.*

We can finally identify our Lie algebra.

Theorem 6.5.5. *Let $\mathfrak{g}$ be a Lie algebra with $\Gamma(\mathfrak{g}) \cong HSp(V,f)$ for some nondegenerate symplectic space (V,f) of dimension $2m$. Then $\mathfrak{g} \cong \mathfrak{sp}_{2m}(\mathbb{F})$.*

Proof. We have seen in Corollary 6.5.4 that the conditions of 6.3.3 are fulfilled for $\Gamma(\mathfrak{g})$, so $\mathcal{E}(\mathfrak{g})$ is a quadric Veronesean of index $2m-1$. Now application of Proposition 6.4.4 finishes the proof.

◇

Now, the Main Theorem 6.0.6 is the direct consequence of Theorem 6.5.5 and Theorem 6.2.6.

Our main result of this chapter, Theorem 6.0.6, characterizes Lie algebras $\mathfrak{g}$ generated by the set of their extremal elements $\mathcal{E}$ with $\mathcal{E}_{\pm}(\mathfrak{g}) = \emptyset$ under the additional condition that $\mathfrak{g}$ is simple and of finite dimension. However, the geometric results of Cuypers [Cuy94] do not have any restrictions. This suggests that one should be able to remove both the condition of $\mathfrak{g}$ being simple and of finite dimension.
Indeed, if the radical of the form g is nontrivial, then the geometry allows us to find a complement of the radical which is then a simple symplectic Lie algebra. Moreover, it seems possible to use the methods as in the proof of Proposition 6.5.3 to show that, up to the center of $\mathfrak{g}$, the radical of g is just a direct sum of natural modules for this complement.

Also the restriction on the finiteness of the dimension of $\mathfrak{g}$ might be removed. The only restriction in our present proof is the analogue of Theorem 6.3.3. But, again, the geometry $\Gamma(\mathfrak{g})$ can be of help. Indeed, as an infinite dimensional $\mathfrak{g}$ has a basis consisting of extremal elements, every element in $\mathfrak{g}$ is a finite sum of extremal elements and therefore inside a subalgebra $\mathfrak{g}'$ generated by a subset $\mathcal{E}'$ of $\mathcal{E}$ whose geometry is a subspace of $\Gamma(\mathfrak{g})$, which can be chosen to be isomorphic to $HSp(V', f')$ for some nondegenerate finite dimensional symplectic space (V', f'). So, $\mathfrak{g}'$ is isomorphic to $\mathfrak{sp}_{2m}(\mathbb{F})$ for some finite m. In this way we are able to construct a local system of subalgebras for $\mathfrak{g}$ whose members are all simple finite dimensional symplectic Lie algebras. Now methods as used in [Hal95] and [BS02] should identify $\mathfrak{g}$ as a symplectic Lie algebra.

APPENDIX A

Extremal forms on Cartan subalgebras

Here, we give the concrete values of the extremal form on the Cartan subalgebra of the Chevalley algebras considered in chapter 2. For the G_2-case, we give the table of the full extremal form. This is an application of the rules stated in 3.4.4. We use the result to obtain the radicals of the extremal form (that are just nontrivial in a few distinct characteristics) in 3.4.6.
Note that all tables are symmetric since the extremal form is symmetric.

TABLE 1. A_n
The fundamental roots of a type A_n root system are
$e_0 - e_1, e_1 - e_2, \ldots, e_{n-1} - e_n$.

g	$h_{e_0-e_1}$	$h_{e_1-e_2}$	$h_{e_2-e_3}$	$\ldots$	$h_{e_{n-1}-e_n}$
$h_{e_1-e_2}$	2	-1	0	$\ldots$	0
$h_{e_2-e_3}$	-1	2	-1	$\ldots$	0
$h_{e_3-e_4}$	0	-1	2	$\ldots$	0
$\ldots$					
$h_{e_{n-1}-e_n}$	0	$\ldots$	0	-1	2

TABLE 2. B_n
The fundamental roots for a system of type B_n are given by
$e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n$.

g	$h_{e_1-e_2}$	$h_{e_2-e_3}$	$h_{e_3-e_4}$	$\ldots$	$h_{e_{n-2}-e_{n-1}}$	$h_{e_{n-1}-e_n}$	h_{e_n}
$h_{e_1-e_2}$	2	-1	0		$\ldots$	0	0
$h_{e_2-e_3}$	-1	2	-1		$\ldots$	0	0
$\ldots$							
$h_{e_{n-1}-e_n}$	0	$\ldots$	0		-1	2	-2
h_{e_n}	0	$\ldots$	0		0	-2	4

TABLE 3. C_n

The fundamental roots for a system of type C_n are given by $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n$.

g	$h_{e_1-e_2}$	$h_{e_2-e_3}$	$h_{e_3-e_4}$	$\ldots$	$h_{e_{n-1}-e_n}$	h_{2e_n}
$h_{e_1-e_2}$	8	-4	0	$\ldots$	0	0
$h_{e_2-e_3}$	-4	8	-4	$\ldots$	0	0
$h_{e_3-e_4}$	0	-4	8	$\ldots$	0	0
$\ldots$						
$h_{e_{n-1}-e_n}$	0	$\ldots$	0	-4	8	-4
h_{2e_n}	0	$\ldots$	0	0	-4	2

TABLE 4. D_n

The fundamental roots for a system of type D_n are given by $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n$.

g	$h_{e_1-e_2}$	$h_{e_2-e_3}$	$h_{e_3-e_4}$	$\ldots$	$h_{e_{n-2}-e_{n-1}}$	$h_{e_{n-1}-e_n}$	$h_{e_{n-1}+e_n}$
$h_{e_1-e_2}$	2	-1	0	$\ldots$	0	0	0
$h_{e_2-e_3}$	-1	2	-1	$\ldots$	0	0	0
$h_{e_3-e_4}$	0	-1	2	$\ldots$	0	0	
$\ldots$							
$h_{e_{n-2}-e_{n-1}}$	0	$\ldots$	0	-1	2	-1	-1
$h_{e_{n-1}-e_n}$	0	$\ldots$	0	0	-1	2	0
$h_{e_{n-1}+e_n}$	0	$\ldots$	0	0	-1	0	2

TABLE 5. E_8
The fundamental roots for a system of type E_8 are given by
$e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, e_6 - e_7, e_6 + e_7$
and $-\frac{1}{2}\sum_{i=1}^{8} e_i =: e_m$.

g	$h_{e_1-e_2}$	$h_{e_2-e_3}$	$h_{e_3-e_4}$	$h_{e_4-e_5}$	$h_{e_5-e_6}$	$h_{e_6-e_7}$	$h_{e_6+e_7}$	h_{e_m}
$h_{e_1-e_2}$	2	-1	0	0	0	0	0	0
$h_{e_2-e_3}$	-1	2	-1	0	0	0	0	0
$h_{e_3-e_4}$	0	-1	2	-1	0	0	0	0
$h_{e_4-e_5}$	0	0	-1	2	-1	0	0	0
$h_{e_5-e_6}$	0	0	0	-1	2	-1	-1	0
$h_{e_6-e_7}$	0	0	0	0	-1	2	0	0
$h_{e_6+e_7}$	0	0	0	0	-1	0	2	-1
h_{e_m}	0	0	0	0	0	0	-1	2

TABLE 6. E_7
The fundamental roots for a system of type E_7 are given by
$e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, e_6 - e_7, e_6 + e_7$
and $-\frac{1}{2}\sum_{i=1}^{8} e_i =: e_m$.

g	$h_{e_2-e_3}$	$h_{e_3-e_4}$	$h_{e_4-e_5}$	$h_{e_5-e_6}$	$h_{e_6-e_7}$	$h_{e_6+e_7}$	h_{e_m}
$h_{e_2-e_3}$	2	-1	0	0	0	0	0
$h_{e_3-e_4}$	-1	2	-1	0	0	0	0
$h_{e_4-e_5}$	0	-1	2	-1	0	0	0
$h_{e_5-e_6}$	0	0	-1	2	-1	-1	0
$h_{e_6-e_7}$	0	0	0	-1	2	0	0
$h_{e_6+e_7}$	0	0	0	-1	0	2	-1
h_{e_m}	0	0	0	0	0	-1	2

TABLE 7. E_6

The fundamental roots for a system of type E_6 are given by $e_3 - e_4, e_4 - e_5, e_5 - e_6, e_6 - e_7, e_6 + e_7$ and $-\frac{1}{2}\sum_{i=1}^{8} e_i =: e_m$.

g	$h_{e_3-e_4}$	$h_{e_4-e_5}$	$h_{e_5-e_6}$	$h_{e_6-e_7}$	$h_{e_6+e_7}$	h_{e_m}
$h_{e_3-e_4}$	2	-1	0	0	0	0
$h_{e_4-e_5}$	-1	2	-1	0	0	0
$h_{e_5-e_6}$	0	-1	2	-1	-1	0
$h_{e_6-e_7}$	0	0	-1	2	0	0
$h_{e_6+e_7}$	0	0	-1	0	2	-1
h_{e_m}	0	0	0	0	-1	2

TABLE 8. F_4

The fundamental roots for a system of type F_4 are given by $e_1 - e_2, e_2 - e_3, e_3, \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$.

g	$h_{e_1-e_2}$	$h_{e_2-e_3}$	h_{e_3}	$h_{\frac{1}{2}(-e_1-e_2-e_3+e_4)}$
$h_{e_3-e_4}$	2	-1	0	0
$h_{e_4-e_5}$	-1	2	-1	0
h_{e_3}	0	-2	4	-2
$h_{\frac{1}{2}(-e_1-e_2-e_3+e_4)}$	0	0	-2	4

TABLE 9. G_2

Here, we give the extremal form of the full Lie algebra (see the table on the next page). Note that the G_2-case has already been considered in Example 3.2.4, and all structure constants can be found in chapter 3, Table 3. We obtain that indeed in $\operatorname{char} \mathbb{F} = 3$, many entries of the following table vanish (see 3.4.6).

g	h_α	h_β	x_α	x_β	$x_{\alpha+\beta}$	$x_{2\alpha+\beta}$	$x_{3\alpha+\beta}$	$x_{3\alpha+2\beta}$	$x_{-\alpha}$	$x_{-\beta}$	$x_{-\alpha-\beta}$	$x_{-2\alpha-\beta}$	$x_{-3\alpha-\beta}$	$x_{-3\alpha-2\beta}$
h_α	6	-3	0	0	0	0	0	0	0	0	0	0	0	0
h_β	-3	2	0	0	0	0	0	0	0	0	0	0	0	0
x_α	0	0	0	0	0	0	0	0	3	0	0	0	0	0
x_β	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$x_{\alpha+\beta}$	0	0	0	0	0	0	0	0	0	0	3	0	0	0
$x_{2\alpha+\beta}$	0	0	0	0	0	0	0	0	0	0	0	3	0	0
$x_{3\alpha+\beta}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$x_{3\alpha+2\beta}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$x_{-\alpha}$	0	0	3	0	0	0	0	0	0	0	0	0	0	0
$x_{-\beta}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$x_{-\alpha-\beta}$	0	0	0	0	3	0	0	0	0	0	0	0	0	0
$x_{-2\alpha-\beta}$	0	0	0	0	0	3	0	0	0	0	0	0	0	0
$x_{-3\alpha-\beta}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$x_{-3\alpha-2\beta}$	0	0	0	0	0	0	0	1	0	0	0	0	0	0

The extremal form on G_2

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Summary:
A Geometric Approach to Classical Lie Algebras

The second half of the 20th century has been very successful for many areas of mathematics. Also in algebra, various striking results have been obtained, including the classifications of the finite simple groups and of the modular simple Lie algebras in characteristic at least 5.

The classification of all finite simple groups states that a finite simple group is either cyclic or alternating, a group of Lie type, or one of 26 sporadic examples. The classification of finite-dimensional modular simple Lie algebras is complete for algebraically closed fields of characteristic greater than or equal to 5. It implies that a simple modular Lie algebra in characteristic at least 5 is either classical, of Cartan type or Melikian.

The groups of Lie type and the classical (modular) Lie algebra are strongly related with each other and both form a central part in the conclusions of above mentioned classification results. They can be connected by Tits' unifying geometric concept of buildings and their related geometries.

Within the theory of finite simple groups, the interaction of groups and geometries has been very fruitful. The geometric method in finite group theory, as started by the pioneering work of Fischer, has been one of the key ingredients in the theory of finite simple groups. This successful interaction is a model for the relations between Lie algebras and geometries that we explore in this thesis.

An element x of a Lie algebra $\mathfrak{g}$ is called *extremal* if the image of $\mathfrak{g}$ under the square of left multiplication by x is contained in the 1-dimensional subspace generated by x. We notice that the long root elements of the classical Lie algebras are extremal. However, extremal elements also occur in other classes of Lie algebras. Indeed, Premet showed that, if the characteristic of the underlying field is at least 5, each simple Lie algebra $\mathfrak{g}$ contains an extremal element. Moreover, over algebraically closed fields these simple Lie algebras

are, up to a single exception, generated by their extremal elements as follows from work of Cohen and others. The two collections of simple Lie algebras, the classical algebras and the Cartan type algebras (including the Melikian algebras), are distinguished by the dimension of the image of $\mathfrak{g}$ under the square of left multiplication by the extremal element x: If this dimension is 0, then x is called a *sandwich* and L is of Cartan type. Otherwise, the dimension is 1, and $\mathfrak{g}$ is classical. The construction of geometries related to Lie algebras was introduced by A. Cohen and G. Ivanyos; they obtained a natural way to associate a geometry to a Lie algebra generated by extremal elements that are no sandwiches. The resulting geometric structure is a *root filtration space*, that is (under some mild restrictions) the shadow space of a spherical building. This construction was inspired by the geometric methods used in finite simple group theory.

The resulting geometries have been classified, which raises the natural question: can the Lie algebra be recovered from the building in a canonical way? If one can develop the reverse construction, recovering $\mathfrak{g}$ from the building Δ in a canonical way, then this implies that $\mathfrak{g}$ is, in fact, a known classical Lie algebra.

In this thesis, we take this path from the geometries to the Lie algebras, concentrating on classical modular Lie algebras. Given a specific geometry related to a building, we study to what extent a Lie algebra whose associated geometry is related to that building is unique.

In his Ph.D. thesis, K. Roberts already obtained this result for the A_n-case. We extend this result to the general case of classical Lie algebras. We show under some weak assumptions on the underlying field that a simple Lie algebra that is generated by extremal elements that are not sandwiches and whose associated geometry is related to a spherical building of rank at least 3 is a classical Lie algebra.

Curriculum Vitae

Yael Fleischmann was born on March 25, 1987 in Frankfurt am Main, Germany. Before she finished her pre-university education with honours in 2004, she was a junior student at the Johannes Gutenberg-Universität in Mainz, attended lectures in mathematics and successfully participated repeatedly in the German young researchers competition *Jugend forscht* with topics on graph theory and combinatorics. In 2004, she started studying mathematics and computer science in Mainz and completed her Vordiplom in 2005. From 2007 to 2008, she was also enrolled at the medical school of the university. During her studies in Mainz, she worked and taught as a student assistant at the mathematical institute. In 2010, she graduated within the group of algebraic geometry and topology. The topic of her Diploma thesis was on the cohomology of orbifolds, with the title *"Questions of rationality in Ruan's conjecture of crepant resolutions"*. From April to May 2010, she was invited for a research stay at the Université de Poitiers, France, before she started her Ph.D. project on Lie algebras and Geometry at the Technische Universiteit Eindhoven, the Netherlands, in September 2010. During her time in Eindhoven, she enthusiastically continued teaching students, was a member of PromoVE, the consultative Ph.D. body of the university and spent research stays at the Justus-Liebig-Universität Gießen and the Mathematisches Forschungsinstitut Oberwolfach, Germany. The results of her Ph.D. research project are presented in this thesis. Since 2015, she works at the field of risk and finance at d-fine GmbH, Germany.