

# Optimising reorder intervals and order-up-to levels in guaranteed service supply chains

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## Optimising reorder intervals and order-up-to levels in guaranteed service supply chains

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We consider the problem of determining the optimal reorder intervals  $R$  and order-up-to levels  $S$  in a multi-echelon supply chain system where all echelons are assumed to have fixed ordering costs and to operate with a  $(R, S)$  policy with stationary nested power-of-two reorder intervals. By using the guaranteed service approach to model the multi-echelon system facing a stochastic demand, we formulate the problem as a deterministic optimisation model in order to simultaneously determine the optimal  $R$  and  $S$  parameters as well as the guaranteed service times. The model is a non-linear integer programming (NLIP) problem with a non-convex and non-concave objective function including rational and square root terms. Then, we propose a sequential optimisation procedure (SOP) to obtain near-optimal solutions with reasonable computational time. The numerical study demonstrates that for a general acyclic multi-echelon system with randomly generated parameters, the SOP is able to obtain near-optimal solutions of about 0.46% optimality gap in average in a few seconds. Moreover, we propose an improved direct approach using a global optimiser, bounding the decision variables in the NLIP model and considering the SOP solution as an initial solution. Numerical examples illustrate that this reduces significantly the computational time.

**Keywords:** inventory control; multi-echelon system; guaranteed service model; power-of-two policies

### 1. Introduction

Many real-world supply chains are complex multi-echelon systems consisting of suppliers, manufacturers, wholesalers and retailers that have geographically dispersed facilities. One challenge these supply chains face is the efficient management of inventory when demand is uncertain, operating costs are important and customer service requirements are high. This requires specifying the inventory policy at different echelons so that to minimise the total cost of the whole multi-echelon system subject to customer service levels (Simchi-Levi and Zhao 2012). The guaranteed service model (GSM) which is among the relevant approaches that can be used in multi-echelon inventory systems has gained interest in recent years. In particular, this model enabled to realise important benefits in practice in general multi-echelon systems which combine distribution and assembly systems (see e.g. Billington et al. 2004; Farasyn et al. 2011).

In this paper, we build on the power-of-two (PO2) and the GSM research to find a reasonable solution to the problem of simultaneously optimising the reorder intervals and order-up-to levels for general multi-echelon systems facing stochastic demand. Finding an optimal policy for this problem would be extremely difficult. Indeed, the optimal policy is not known even for two-echelon distribution systems with deterministic demand (Snyder and Shen 2011). In order to deal with demand variations, we use the original assumptions of the GSM that are the guaranteed service time and the bounded demand assumptions. Besides, we assume that each stage of the supply chain operates with a periodic-review, order-up-to  $(R, S)$  policy with stationary nested PO2 reorder intervals.

This paper has several contributions. First, in order to use the GSM approach, we show how to set the demand bound functions associated with the supply chain stages. The demand bound function proposed here is the generalisation of the one of Graves and Willems (2000). Second, we propose a deterministic optimisation model for general multi-echelon systems to determine the optimal parameters  $R$  and  $S$  as well as the corresponding service times. This leads to a non-linear integer programming (NLIP) problem with a non-convex and non-concave objective function including rational and square root terms. Third, we propose a sequential optimisation procedure (SOP) to obtain near-optimal solutions with reasonable computational time for this problem. We measure the performance of this procedure on randomly generated instances pertaining to two supply chain structures. Fourth, by defining reasonable bounds for the decision variables of the NLIP model, we propose an improved direct (ID) approach.

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The paper is organised as follows. Section 2 reviews the related literature. In Section 3, we develop the NLIP. We then present the SOP in Section 4. Section 5 establishes the bounds for the decision variables of the NLIP and, hence, proposes the ID approach. Numerical analysis on the SOP and the ID approach for serial and general acyclic multi-echelon structures is summarised in Section 6. Finally, Section 7 draws some conclusions and suggests potential future research directions.

## 2. Related literature

Our paper is built upon two research streams which are the problem of dimensioning safety stocks in multi-echelon systems and the one of determining optimal reorder intervals.

The stochastic modelling (SM) and the GSM approach are the main approaches used for dimensioning safety stocks in multi-echelon systems. In the SM approach, each stage of the supply chain maintains safety stock to ensure its target service level. Even if the processing times of stages are deterministic; replenishment times between stages become stochastic due to the occasional stock-outs caused by demand uncertainty. The aim is to characterise these replenishment times properly (Graves and Willems 2003). Since the SM approach is computationally intractable for general multi-echelon systems, related research mostly focuses on two-echelon pure distribution systems (see e.g. Hwarng et al. 2005; Chu and Shen 2010). We refer the reader to Simchi-Levi and Zhao (2012) for more details on the SM approach.

The GSM approach aims at determining the optimal placement and amount of safety stocks in a multi-echelon system to ensure the overall target service level at the lowest cost. In this approach, it is assumed that each stage promises a guaranteed service to its downstream stages. For the purpose of satisfying the service time guarantee, demand is assumed to be bounded (see e.g. Graves and Willems 2000; Minner 2000; Sitompul et al. 2008) which enables to render the model deterministic and allows considering general multi-echelon systems. Early research on the GSM mainly focus on basic multi-echelon systems such as serial (Simpson 1958), distribution or assembly systems (e.g. Inderfurth 1991; Inderfurth and Minner 1998) which are extended to more general systems (e.g. Graves and Willems 2000; Humair and Willems 2006, 2011). Among others, an interesting extension of the GSM is the one of Bostert and Willems (2007) that studies stage-dependent reorder intervals which are considered as known input parameters of the model. We refer the reader to Eruguz et al. (2012) for a deep literature review pertaining to the GSM approach.

The problem of determining the appropriate  $R$  and  $S$  parameters has also attracted interest in the literature for single-stage inventory systems. Under stochastic demand, it is a common practice to optimise the policy parameters sequentially to obtain a near-optimal solution (see Silver, Pyke, and Peterson 1998). Silver and Robb (2008) provide some counter-intuitive results about how the best reorder interval changes as the values of various parameters are modified. Recently, Liu and Song (2012) develop efficient algorithms to compute the global optimal (GO) policy parameters. The cost formulation considered in such studies is based on the approximation of Hadley and Whitin (1963). However, this has several shortcomings for one or few weeks long reorder intervals. In order to remedy these shortcomings, Chiang (2006, 2007) introduces different dynamic programming models for both backorder and lost-sales cases in which holding and shortage costs are computed based on the ending inventory of each period that composes the reorder interval. For the lost-sales case, he assumes that lead time is less than or equal to the length of reorder intervals. Bijvank and Johansen (2012) extend this assumption and develop new models allowing constant lead times of any length.

Concerning multi-echelon systems, most of research interested in determining optimal reorder intervals has considered the integer-ratio policies in which the reorder interval of each stage is an integer multiple of a base planning period (e.g. a day, a week or a month). The so-called PO2 policy (which is a subset of the integer-ratio policies) has received considerable attention. Under a PO2 policy, reorder intervals are PO2 multiples of the base planning period. PO2 policies have some practical and computational advantages (see Muckstadt and Roundy 1993). Besides, stationary PO2 policies have been shown effective for deterministic multi-echelon systems. Available algorithms generate solutions within 6% (see Roundy 1985a) or 2% (see Roundy 1985b) of optimality. Additionally, under stochastic demand, it is shown that the integer-ratio policies obtained by solving the deterministic counterpart of the problem can be an effective heuristic approach for serial (see e.g. Chen and Zheng 1998; Shang 2008) and distribution systems (see e.g. Chu and Shen 2010; Shang and Zhou 2011).

There are many policies within the class of PO2 policies that could be considered. Among others, stationary nested replenishment policies have been popular in the literature. Under a nested policy, every replenishment epoch of an upstream stage coincides with a shipment epoch towards its downstream stage. Stationary nested policies are proven to be optimal for deterministic serial (see Schwarz 1973) and assembly systems (see Muckstadt and Roundy 1993). However, they may be suboptimal for distribution and general multi-echelon structures (see Roundy 1985b). Nevertheless, many researchers assume nested policies for distribution and general multi-echelon systems because of

their significant practical importance and ease of computation (see e.g. Maxwell and Muckstadt 1985; Graves 1996; Yao and Wang 2006).

### 3. Optimisation model

This section presents the optimisation model we propose: Section 3.1 introduces our assumptions; Section 3.2 calculates the associated appropriate demand bounds; and Section 3.3 provides the mathematical programming formulation.

#### 3.1 Assumptions of the model

**A1.** We consider a general multi-echelon system. We assume that with each stage of the system, certain process is associated such as procurement of raw materials, production or transportation of items. Each stage is also a potential location to store the processed item. We model this system as a network where nodes represent stages and arcs denote the precedence relationship between stages. We denote the set of nodes by  $\mathbf{N}$  and the set of arcs by  $\mathbf{A}$ . We partition the set of nodes into three disjoint sets:  $\mathbf{N}_S$ ,  $\mathbf{N}_I$  and  $\mathbf{N}_D$  where  $\mathbf{N}_S$  is the set of nodes without predecessors, i.e. the set of supply nodes. Nodes in the set of demand nodes, i.e.  $\mathbf{N}_D$ , have no successors. The set of internal nodes, i.e.  $\mathbf{N}_I$ , is the set of nodes having at least one predecessor and one successor.

**A2.** We assume that external demand occurs only for nodes  $j \in \mathbf{N}_D$ . For each node  $j \in \mathbf{N}_D$  demand follows a stationary i.i.d. process with mean  $\mu_j$  and standard deviation  $\sigma_j$  per base planning period which can be a shift, a day, a week or a month. For internal and supply nodes, i.e. when node  $j \in \mathbf{N}_I \cup \mathbf{N}_S$ , we can compute the mean demand  $\mu_j$  per base planning period by:

$$\mu_j = \sum_{k:(j,k) \in \mathbf{A}} \theta_{jk} \mu_k$$

where the scalar  $\theta_{jk}$  represents the number of items at upstream node  $j$  required for downstream node  $k$ . We consider the case of no risk pooling. Hence, demand variability at stage  $j \in \mathbf{N}_I \cup \mathbf{N}_S$  is equal to the sum of demand variability associated with its successors. For  $j \in \mathbf{N}_I \cup \mathbf{N}_S$ , we compute the standard deviation  $\sigma_j$  per base planning period by:

$$\sigma_j = \sum_{k:(j,k) \in \mathbf{A}} \theta_{jk} \sigma_k$$

Furthermore, as in the original GSM, we assume that demand satisfied from stock is bounded for any long period  $\tau_j$  and for every stage  $j$ . The existence of demand bounds does not imply that arrival demand can never exceed the bounds. When arrival demand exceeds the upper bound, excess demand would be handled by some extraordinary measures such as subcontracting, overtime production, expediting and/or premium freight transportation. However, the impact of these extraordinary situations on the company is not studied in the GSM, neither in our model. We assume that bounds are defined by the company policy in such a way that the effect of excess demand is tolerated.

**A3.** At each stage, we assume a known and constant lead time  $L_j$  which corresponds to the duration of the process being realised at each stage, given that all necessary components are available. It also includes the waiting time and the transportation time to put the processed item into inventory.

**A4.** We assume that each stage  $j$  operates with a stationary  $(R_j, S_j)$  policy where  $R_j$  is the reorder interval and  $S_j$  is the order-up-to level. There is no time delay in ordering. We restrict attention to stationary nested PO2 policies. Thus, the reorder interval  $R_j$  can take the following values:  $\{1, 2, \dots, 2^{l_j}\}$  where  $l_j$  is a non-negative integer. Furthermore, since we consider nested policies, the reorder interval of a stage  $j \in \mathbf{N}_I \cup \mathbf{N}_D$  cannot be greater than the reorder interval(s) of its upstream stage(s). We note that the reorder epochs are offset to allow each stage to replenish from its immediate upstream stage(s) at the exact moment an order arrives at the upstream stage(s) and equidistant times of length  $R_j$  thereafter.

**A5.** As in the original GSM, we assume that each echelon  $j$  promises a guaranteed outbound service time  $s_j^{out}$  to its customers. That is we assume that demand occurred at time  $t$  within the demand bounds is fully satisfied with 100%

service at time  $t + s_j^{out}$ . Demand nodes should ensure the maximum service time  $s_j^{client}$  tolerated by the final customer. As in Graves and Willems (2000), we also assume that each echelon  $j$  proposes a unique guaranteed service time for its customers. Clearly, node  $j$  cannot start its process without receiving all inputs. Here, it is useful to introduce an additional variable which is called the inbound service time  $s_j^{in}$ . These service times define the time for node  $j$  to get all the inputs from node  $i : (i, j) \in \mathbf{A}$  to start the process. The inbound service time of a stage cannot be smaller than the maximum service time of its suppliers. Hence,  $s_j^{in} \geq s_i^{out}$  should be ensured for all arcs  $(i, j) \in \mathbf{A}$ . We note that  $s_j^{out}$  and  $s_j^{in}$  are decision variables for our optimisation problem. These decision variables serve to determine the safety stock level and the target order-up-to level  $S_j$  at each stage  $j$ . We assume that the outbound and the inbound service times are integer multiples of the base planning period likewise the review periods and lead times. For the sake of simplicity, we will further consider the base planning period as one unit of time and the decision variables as positive integers.

**A6.** Two types of cost are considered in our model: the fixed ordering and the holding cost.

Let  $A_j$  be the fixed ordering cost and  $\chi$  be the number of base planning periods per year. The annual fixed ordering cost (AFOC) is calculated similarly to economic order quantity model:

$$\text{AFOC} = \sum_{j \in \mathbf{N}} \frac{A_j}{R_j} \chi$$

The annual holding cost formulation is based on the approximation of Hadley and Whitin (1963) which is the sum of the cycle stock and the safety stock costs.

In order to compute the cycle stock cost, we use the echelon stock approach. The on-hand stock evolutions for echelon stocks are always of the saw-tooth form no matter the network topology. Thus, it is easier to compute the average echelon stock compared to average on-hand stock. Besides, the two approaches yield the same cycle stock costs for the multi-echelon system with nested PO2 policies (Muckstadt and Roundy 1993). The echelon holding cost of stage  $j$  is denoted by  $h_j^e$ . The annual cycle stock cost (ACSC) of the system can be calculated by:

$$\text{ACSC} = \sum_{j \in \mathbf{N}} \frac{1}{2} u_j h_j^e R_j$$

where,  $h_j^e = h_j$  for all node  $j \in \mathbf{N}_S$  and  $h_j^e = h_j - \sum_{i:(i,j) \in \mathbf{A}} h_i$  for all node  $j \in \mathbf{N}_I \cup \mathbf{N}_D$ .

The annual safety stock cost (ASSC) of node  $j$  is the product of annual unit holding cost and the safety stock level  $SS_j$  of stage  $j$ :

$$\text{ASSC} = \sum_{j \in \mathbf{N}} h_j SS_j$$

In our mathematical model, the cost of pipeline stock is ignored since it depends only on input parameters and does not affect the optimisation. However, this is not to say that the pipeline stock is not a significant part of the inventory in a supply chain. Therefore, the annual cost function that we aim to minimise is the sum of the AFOC, the ACSC and the ASSC.

### 3.2 Demand bound functions

The existence of guaranteed service times implies that if stage  $j$  faces demand  $d_j(t)$  at time  $t$ , the demand within the demand bounds is fully satisfied with 100% service at time  $t + s_j^{out}$ . We assume that a replenishment is available to serve demand in its period of arrival. Let consider the replenishment mechanism at an internal or supply node  $j \in \mathbf{N}_I \cup \mathbf{N}_S$ . Without loss of generality, stage  $j$  places orders at times  $n \cdot R_j$  for  $n=0, 1, 2, \dots$ . Let  $t = v + n \cdot R_j$  where  $v \in \{1, 2, \dots, R_j\}$ . The stage  $j$  places an order for  $d_j(t)$  at time  $t + R_j - v$  and the order corresponding to this demand is received at time  $t + s_j^{in} + L_j + R_j - v$ . In the worst case,  $v = 1$  and the reception occurs at time  $t + s_j^{in} + L_j + R_j - 1$ . If a demand is served first and the replenishment corresponding to this demand occurs at a subsequent period, node  $j$  has to store the inventory that would satisfy the demand within the guaranteed service time. That is, if



$s_j^{in} + L_j + R_j - 1 > s_j^{out}$ , node  $j$  should have the amount of inventory to cover the demand over an interval of length  $\tau_j = s_j^{in} + L_j + R_j - 1 - s_j^{out}$ , that is called the net replenishment time of node  $j \in \mathbf{N}_I \cup \mathbf{N}_S$ .

We assume that the external demand occurs continuously over the base planning period. Thus, taking into account an additional increase of the net replenishment time by the base planning period, the net replenishment time  $\tau_j$  for a demand node  $j \in \mathbf{N}_D$  is equal to  $s_j^{in} + R_j + L_j - s_j^{out}$ . As in Graves and Willems (2000), one can set the demand bound function for demand nodes as follows:

$$D_j(\tau_j) = \tau_j \mu_j + z_j \sigma_j \sqrt{\tau_j} \quad \text{for } j \in \mathbf{N}_D$$

where  $z_j$  is the safety factor which relates to a non-stock-out probability in a node  $j$  during an arbitrary period. This function is mostly referred in the mono-echelon inventory theory to compute the reorder points or order-up-to levels (see Schneider 1981; Silver, Pyke, and Peterson 1998). In practice, the safety factor at different stages is chosen according to the company policy. In fact, the choice of  $z_j$  indicates how frequently the manager is willing to resort to extraordinary measures to cover demand variability at stage  $j$ . Besides, in some contexts, customer demand may be bounded due to capacity constraints (see Graves and Willems 2000).

We propose a more general demand bound function than Graves and Willems (2000) for internal and supply nodes to consider nested and stage-dependent reorder intervals. For these nodes, the maximum demand which can be observed during the net replenishment time depends on the reorder intervals of their immediate successors. The average size of an order placed by node  $k : (j, k) \in \mathbf{A}$  is  $R_k \theta_{jk} \mu_k$ . The number of orders placed by node  $k$  and observed by node  $j$  during the net replenishment time  $\tau_j$  can be calculated by the floor function  $n(\tau_j, R_k)$ :

$$n(\tau_j, R_k) = \left\lfloor \frac{\tau_j}{R_k} \right\rfloor \quad \text{for } (j, k) \in \mathbf{A}$$

The average demand requested by node  $k$  and observed by node  $j$  during the net replenishment time  $\tau_j$  is the product of the number of orders placed by node  $k$  during  $\tau_j$  and the average size of an order placed by node  $k$ . Since we consider the case of no risk pooling, the maximum demand  $D_{jk}(\tau_j, R_k)$  placed by node  $k$  and observed by node  $j$  during  $\tau_j$  can be calculated by:

$$D_{jk}(\tau_j, R_k) = n(\tau_j, R_k) R_k \theta_{jk} \mu_k + z_j \theta_{jk} \sigma_k \sqrt{n(\tau_j, R_k) R_k} \quad \text{for } (j, k) \in \mathbf{A}$$

To provide a guaranteed service time at stage  $j$ , the order-up-to level  $S_j$  should be equal to the demand upper bound during its net replenishment time:

$$S_j = D_j(\tau_j) \quad \text{for } j \in \mathbf{N}_D$$

$$S_j = \sum_{k:(j,k) \in \mathbf{A}} D_{jk}(\tau_j, R_k) \quad \text{for } j \in \mathbf{N}_I \cup \mathbf{N}_S$$

Thus, the safety stock level at stage  $j$ , i.e.  $SS_j$  becomes:

$$SS_j = z_j \sigma_j \sqrt{\tau_j} \quad \text{for } j \in \mathbf{N}_D$$

$$SS_j = z_j \sum_{k:(j,k) \in \mathbf{A}} \theta_{jk} \sigma_k \sqrt{n(\tau_j, R_k) R_k} \quad \text{for } j \in \mathbf{N}_I \cup \mathbf{N}_S$$

In the mathematical model, we will represent the safety stock function of internal or supply nodes without referring to the floor function. Let  $n_{jk}$  be the decision variables of the mathematical model representing the floor function value  $n(\tau_j, R_k)$ . The variables  $n_{jk}$  should verify the following constraints:

$$\tau_j - n_{jk}R_k \geq 0, \quad \forall (j, k) \in \mathbf{A}$$

$$\tau_j - n_{jk}R_k < R_k, \quad \forall (j, k) \in \mathbf{A}$$

$$n_{jk} \geq 0 \text{ and integer } \forall (j, k) \in \mathbf{A}$$

So considered, the term in the square root can be replaced by  $n_{jk}R_k$ . The safety stock level for a node  $j \in \mathbf{N}_I \cup \mathbf{N}_S$  is then equal to:

$$SS_j = z_j \sum_{k: (j,k) \in \mathbf{A}} \theta_{jk} \sigma_k \sqrt{n_{jk}R_k} \quad \text{for } j \in \mathbf{N}_I \cup \mathbf{N}_S$$

where  $n_{jk}$  verifies the constraints above.

### 3.3 The mathematical programming formulation

The problem  $P0$  of finding the optimal PO2 reorder intervals and guaranteed service times in order to minimise the total annual cost of the multi-echelon system can be formulated as follows:

$$P0 : \text{Min} \sum_{j \in \mathbf{N}} \left( \frac{A_j}{R_j} \chi + \frac{1}{2} u_j h_j^e R_j \right) + \sum_{j \in \mathbf{N}_I \cup \mathbf{N}_S} \sum_{k: (j,k) \in \mathbf{A}} h_j z_j \theta_{jk} \sigma_k \sqrt{n_{jk}R_k} + \sum_{j \in \mathbf{N}_D} h_j z_j \sigma_j \sqrt{\tau_j} \quad (1)$$

$$R_j = 2^{l_j}, \quad \forall j \in \mathbf{N} \quad (2)$$

$$R_i \geq R_j, \quad \forall (i, j) \in \mathbf{A} \quad (3)$$

$$\tau_j = s_j^{in} + L_j + R_j - s_j^{out}, \quad \forall j \in \mathbf{N}_D \quad (4)$$

$$\tau_j = s_j^{in} + L_j + R_j - 1 - s_j^{out}, \quad \forall j \in \mathbf{N}_I \cup \mathbf{N}_S \quad (5)$$

$$\tau_j - n_{jk}R_k \geq 0, \quad \forall (j, k) \in \mathbf{A} \quad (6)$$

$$\tau_j - n_{jk}R_k < R_k, \quad \forall (j, k) \in \mathbf{A} \quad (7)$$

$$s_j^{in} \geq s_i^{out}, \quad \forall (i, j) \in \mathbf{A} \quad (8)$$

$$s_j^{out} \leq s_j^{client}, \quad \forall j \in \mathbf{N}_D \quad (9)$$

$$l_j, s_j^{in}, s_j^{out}, \tau_j \geq 0 \text{ and integer } \forall j \in \mathbf{N} \quad (10)$$

$$R_j \geq 1 \text{ and integer } \forall j \in \mathbf{N} \quad (11)$$

$$n_{jk} \geq 0 \text{ and integer } \forall (j, k) \in \mathbf{A} \quad (12)$$

The decision variables of this problem are: the reorder intervals ( $R_j$ ), the integer variables representing the PO2 values ( $l_j$ ), the net replenishment times ( $\tau_j$ ), the outbound service times ( $s_j^{out}$ ), the inbound service times ( $s_j^{in}$ ) and the number of orders placed by stage  $k$  to stage  $j$  ( $n_{jk}$ ) for each  $(j, k) \in \mathbf{A}$ .



The problem  $P0$  minimises the total cost function (1). Constraint (2) restricts the reorder intervals to PO2 solutions. Constraint (3) is necessary to ensure nestedness. Constraints (4) and (5) give the net replenishment times of nodes. The non-linear constraints (6) and (7) determine the number of orders placed by an internal node during the net replenishment time of its immediate supplier. Constraint (8) ensures that the outbound service time of a node's immediate supplier is not greater than its inbound service time. Constraint (9) ensures that the demand nodes satisfy their service guarantee. With Constraints (10)–(12), decision variables are forced to be positive integers.

The problem  $P0$  is a NLIP problem with a neither convex nor concave objective function on the feasible region (see Appendix A) including rational and square root terms.

After having solved the problem  $P0$ , the optimal order-up-to levels  $S_j^*$  of stages can be obtained by:

$$S_j^* = D_j(\tau_j^*) \quad \text{for } j \in \mathbf{N}_D$$

$$S_j^* = \sum_{k: (j,k) \in \mathbf{A}} D_{jk}(\tau_j^*, R_k^*) \quad \text{for } j \in \mathbf{N}_I \cup \mathbf{N}_S$$

where  $\tau_j^*$  and  $R_j^*$  are the optimal solutions of the problem  $P0$ .

#### 4. Sequential optimisation procedure

For large multi-echelon systems, the problem  $P0$  becomes computationally intractable with direct solution approaches because of the combinatorial nature of the problem and non-linear non-convex terms. We thus propose a SOP to obtain near-optimal solutions with reasonable computational time. Our method consists of two optimisation procedures. First, we determine the convenient reorder intervals using available optimisation models for nested PO2 policies with deterministic demand. Second, we obtain convenient order-up-to levels, guaranteed service times and safety stock placements using the results of the first procedure as input parameters.

The first optimisation procedure aims at determining a nested PO2 solution to the deterministic counterpart of this problem. Therefore, we first consider the problem  $P1$ :

$$P1 : \text{Min} \sum_{j \in \mathbf{N}} \frac{A_j}{R_j} \chi + \frac{1}{2} u_j h_j^e R_j \quad (13)$$

$$R_j = 2^{l_j}, \quad \forall j \in \mathbf{N} \quad (14)$$

$$R_i \geq R_j, \quad \forall (i, j) \in \mathbf{A} \quad (15)$$

$$l_j \geq 0 \quad \text{and integer} \quad \forall j \in \mathbf{N} \quad (16)$$

$$R_j \geq 1 \quad \text{and integer} \quad \forall j \in \mathbf{N} \quad (17)$$

The problem  $P1$  is studied in the literature for general acyclic multi-echelon systems. To find an optimal solution to this problem, one can use the polynomial time algorithm presented by Maxwell and Muckstadt (1985) and Muckstadt and Roundy (1993).

Let  $R_j^{seq}$  be the reorder interval of stage  $j$  obtained by solving the problem  $P1$ . By considering reorder intervals as input parameters, reorder interval of stage  $j$ ,  $R_j^{seq}$  can be aggregated into its lead time  $L_j$ . Hence, lead times of stage  $j$  may be replaced by  $L_j'$  where:

$$L_j' = L_j + R_j^{seq} \quad \text{for } j \in \mathbf{N}_D$$

$$L_j' = L_j + R_j^{seq} - 1 \quad \text{for } j \in \mathbf{N}_I \cup \mathbf{N}_S$$

Therefore, the problem  $P0$  can be reduced to problem  $P2$ :

$$P2 : \text{Min} \sum_{j \in \mathbf{N}} c_j(s_j^{in}, s_j^{out}) \quad (18)$$

$$s_j^{in} + L'_j - s_j^{out} \geq 0, \quad \forall j \in \mathbf{N} \quad (19)$$

$$s_j^{in} \geq s_i^{out}, \quad \forall (i, j) \in \mathbf{A} \quad (20)$$

$$s_j^{out} \leq s_j^{client}, \quad \forall j \in \mathbf{N}_D \quad (21)$$

$$s_j^{in}, s_j^{out} \geq 0 \quad \text{and integer} \quad \forall j \in \mathbf{N} \quad (22)$$

Where;

$$c_j(s_j^{in}, s_j^{out}) = h_j z_j \sigma_j \sqrt{s_j^{in} + L'_j - s_j^{out}} \quad \text{for } j \in \mathbf{N}_D$$

$$c_j(s_j^{in}, s_j^{out}) = \sum_{k: (j,k) \in \mathbf{A}} h_j z_j \theta_{jk} \sigma_k \sqrt{\left[ \frac{s_j^{in} + L'_j - s_j^{out}}{R_k^{seq}} \right] R_k^{seq}} \quad \text{for } j \in \mathbf{N}_I \cup \mathbf{N}_S$$

The second procedure aims at finding an optimal solution to the problem  $P2$ . The problem  $P2$  is a GSM with a non-continuous objective function. The cost function of stages only depends on its own service times and is increasing in  $s_j^{in}$  and decreasing in  $s_j^{out}$ . Therefore, considering the multi-echelon system structure, generic solution techniques developed in Graves and Willems (2000), Humair and Willems (2006, 2011) can be used to solve this problem to optimality. In fact, for these techniques, there are no structural limitations on  $c_j(s_j^{in}, s_j^{out})$  as long as the cost functions of a stage  $j$  depend on only  $s_j^{in}$  and  $s_j^{out}$ .

By solving the problem  $P2$ , we obtain the best service times for the multi-echelon system given reorder intervals  $R_j^{seq}$ . As presented in Section 3.3, we can deduce the safety stock and order-up-to levels corresponding to this solution. A feasible solution for the problem  $P0$  is then obtained by combining the solutions found for problems  $P1$  and  $P2$ .

## 5. Improved direct approach

A direct approach to obtain a GO solution for the problem  $P0$  is to solve it by using a global optimiser such as BARON with 0% optimality margin. BARON provides global optima for this problem if finite lower and upper bounds on the decision variables are properly specified. When the default decision variable bounds are too large, this approach requires significant computational time (cf. Section 6). Otherwise, if these bounds are too tight, global optima may not be obtained. We improve this default direct (DD) approach: first, we establish the solution obtained by the SOP as an initial solution. Second, we develop appropriate decision variable bounds using the solution obtained by the SOP.

In what follows, we will show how to establish the decision variable bounds in order to develop an ID approach. By solving the problem  $P1$ , we obtain for each node  $j$  the reorder interval  $R_j^{seq}$  that optimises the convex part of the cost function including the annual fixed ordering cost and the annual cycle stock cost. If the optimal reorder interval of a supply node  $j \in \mathbf{N}_S$  is greater than  $R_j^{seq}$  found by the sequential optimal solution, the cost of the convex part increases. However, this also increases the net replenishment time of node  $j \in \mathbf{N}_S$  and, hence, its annual safety stock cost. Therefore, it is not beneficial for a supply node to set a reorder interval greater than its  $R_j^{seq}$ . Besides, since we only consider nested policies, the reorder interval of a non-supply node must be smaller than or equal to the maximum reorder interval of the supply nodes. Hence, we can establish the upper bounds for all reorder intervals by:

$$R_j \leq \max\{R_j^{seq} \mid j: j \in \mathbf{N}_S\} \quad \text{for } j \in \mathbf{N} \quad (23)$$

Similarly, the upper bound for the integer decision variable  $l_j$  becomes:

$$l_j \leq \max\{l_j^{seq} | j: j \in \mathbf{N}_S\} \quad \text{for } j \in \mathbf{N} \quad (24)$$

Since an upper bound can be defined for reorder intervals, we can deduce upper bounds for service times as well. We can define the maximum replenishment time for nodes by:

$$M_j = L_j - 1 + \max\{R_j^{seq} | j: j \in \mathbf{N}_S\} \quad \text{for } j \in \mathbf{N}_S$$

$$M_j = L_j - 1 + \max\{R_j^{seq} | j: j \in \mathbf{N}_S\} + \max\{M_i | i: (i, j) \in \mathbf{A}\} \quad \text{for } j \in \mathbf{N}_I$$

$$M_j = L_j + \max\{R_j^{seq} | j: j \in \mathbf{N}_S\} + \max\{M_i | i: (i, j) \in \mathbf{A}\} \quad \text{for } j \in \mathbf{N}_D$$

The total cost increases when the inbound service times or the net replenishment times increase and when the outbound service times decrease. Hence, as in the original GSM, there always exists an optimal solution for the problem  $P0$  such that all the inbound service times of the supply nodes are equal to 0 and the inbound service time of each non-supply node is equal to the maximum service time of its upstream nodes (see Lesnaia 2004). Therefore, we can establish upper bounds for the inbound and outbound service times as follows:

$$s_j^{out} \leq M_j \quad \text{for } j \in \mathbf{N} \quad (25)$$

$$s_j^{in} \leq \max\{M_i | i: (i, j) \in \mathbf{A}\} \quad \text{for } j \in \mathbf{N}_I \cup \mathbf{N}_D \quad (26)$$

$$s_j^{in} \leq 0 \quad \text{for } j \in \mathbf{N}_S \quad (27)$$

Hence, upper bounds for  $\tau_j$  and  $n_{jk}$  become:

$$\tau_j \leq M_j \quad \text{for } j \in \mathbf{N} \quad (28)$$

$$n_{jk} \leq M_j \quad \text{for } (j, k) \in \mathbf{A} \quad (29)$$

Besides, the natural lower bounds for these decision variables are given by Constraints (10)–(12). The ID approach is then obtained by setting the bounds (23)–(29) to the decision variables and by considering the sequential optimal solution as an initial solution.

## 6. Numerical analysis

In this section, computational experiments are carried to test the relevancy of the SOP and the direct approaches. Randomly generated five-echelon serial and five-echelon general acyclic multi-echelon systems are used to show different results. While Section 6.1 presents the data generation procedure for both systems, Section 6.2 provides results on the GO reorder intervals based on illustrative examples. Section 6.3 discusses the optimality gap of the SOP and identifies conditions under which the SOP performs relatively bad. Finally, Section 6.4 compares the performance of the SOP and the direct approaches in terms of computational time.

### 6.1 Data generation

For numerical analysis, we consider a five-echelon serial (Figure 1) and a five-echelon general acyclic system (Figure 2). Common parameters for both structures are as follows. The length of the base planning period is a business day and there are 260 business days in a year. The safety factors for all stages are the same and equal to 1.645 (which correspond to a 95% service level). The maximum service times at demand nodes are set to 0. The parameter  $\theta_{jk}$  is set equal

to 1 for all stages  $(j, k) \in \mathbf{A}$ . Besides, we consider different groups as in Bossert and Willems (2007) based on the ordering cost ratios (i.e. the ratio  $A_j/h_j$  for stage  $j$ ). For the serial five-echelon structure, we define three profiles for each group (see Table 1). For each echelon of the five-echelon general acyclic system, we specify intervals in which the ordering cost ratios are generated randomly if the ratio is not set to 0 (see Table 2).

For the five-echelon serial supply chain system, mean and standard deviation of daily demand at the demand node (Stage 5) are respectively  $\mu=150$  and  $\sigma=45$ . The annual unit holding cost for Stage 1 is generated randomly in  $U[0, 20]$ . Then, the annual unit holding cost of other stages is obtained by adding a random number in  $U[0, 20]$  to the annual unit holding cost of its upstream stage. The lead time value of each stage is an integer, generated randomly in  $U[1, 20]$ . Following these rules, 15 instances that comprise the lead time and the annual holding cost data are obtained. The 15 instances (Table 3) permuted with the 21 ordering cost profiles (Table 1) generate the 315 problem instances considered.

The considered five-echelon general acyclic system corresponds to the real-world supply chain presented in Willems (2008). For this system, we use data provided by Willems (2008) that includes the lead times (the average values are considered), the stage costs (holding cost rate is set to 10%) and mean and standard deviation of demand at demand nodes. For the ordering costs, we generate 15 instances for each of the 21 ordering cost profiles using intervals reported in Table 2. Hence, we obtain 315 problem instances.

The SOP and direct approaches are coded in GAMS 23.7 on a VAIO computer with Intel Core i3–2310 M processor (2.10 GHz) and 4 GB RAM. BARON (version 9.3.1) is used for the computational experiments. For all problem instances, the GO solutions are obtained by the ID approach using the global optimiser BARON. The sequential optimal (SO) solutions are the feasible solutions obtained from the SOP.

## 6.2 Results

Results presented in this section concern reorder intervals associated with the SO and GO solutions over the problem instances considered.

We observe that when the SO and GO solutions are not the same, the GO solution may lead to smaller reorder intervals than the SO solution. In this case, the sum of total annual fixed ordering and annual cycle stock costs ( $AFOC_j + ACSC_j$  of stages  $j \in \mathbf{N}$ ) increases in comparison with the SO solution. Besides, this may also increase the annual safety stock cost  $ASSC_i$  at the upstream stage(s)  $i : (i, j) \in \mathbf{A}$  since stage  $j$  will order more frequently during the net replenishment time of stage  $i$ . On the other hand, this leads to a potential reduction of the annual safety stock cost at stage  $j$  or at the one(s) of its downstream stage(s). The additive effect of these deviations may reduce the total cost. To illustrate this result, we provide in Table 4 an example case representing the SO and GO solutions pertaining to a five-echelon serial structure. For this example, the total cost of the SO and GO solutions is respectively \$90,100 and \$89,208, which represents a relative gap of 1.00%.

Another interesting result is the reduction of the total cost when the reorder interval of a non-supply stage  $j$  is increased over its SO reorder interval. This action may increase the sum  $AFOC_j + ACSC_j$  for stage  $j$ . However, it may decrease  $ASSC_i$  at the upstream stage(s)  $i : (i, j) \in \mathbf{A}$ . This is due to a better order coordination between customer–supplier stages. Besides, this may also reduce the safety stock cost at other stage(s) sharing a same supplier with stage  $j$  since each supplier quotes a unique service time for all of its customers. To illustrate this result, we provide in Table 5 the GO and SO solutions for a general acyclic problem instance. For this example, the total SO and GO costs are respectively \$3180,765 and \$3141,906 and this represents a relative gap of 1.24%.

Concerning the reorder intervals obtained for different groups of ordering cost ratio profiles, ending, uniform and increasing groups lead to the same reorder interval among all stages since we only consider nested policies. In this case, decreasing the reorder intervals of all stages together may improve the SO solution. In the general acyclic structure, this may imply high cost deviations and may significantly reduce the total cost. For decreasing and random groups, the total cost of the system may be reduced by increasing or decreasing the reorder interval of the SO solution for one or several stages. For the starting group, the SO solution may be improved by decreasing the reorder interval at supply stages. Similarly, for the middle group, a better solution than the SO solution may be found by decreasing the reorder intervals at the first three upstream echelons. However, for this group, the improvement of the total cost function is restrictive and it usually implies small cost deviations.

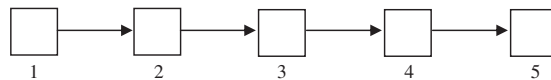


Figure 1. Five-echelon serial supply chain system used for the numerical analysis.

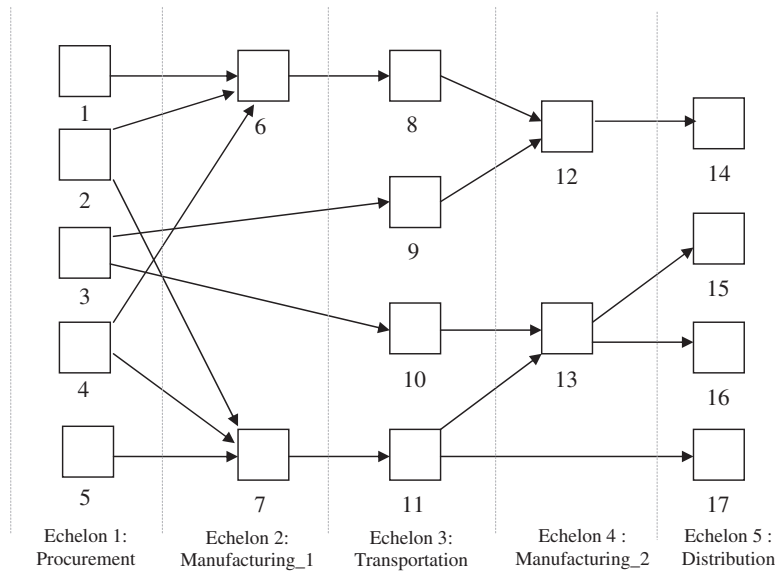


Figure 2. Five-echelon general acyclic system used for the numerical analysis.

Table 1. Ordering cost ratios used for the serial structure.

Ordering Cost Profiles	Stages				
	1	2	3	4	5
Starting	4	0	0	0	0
	25	0	0	0	0
	80	0	0	0	0
Middle	0	0	4	0	0
	0	0	25	0	0
	0	0	80	0	0
Ending	0	0	0	0	4
	0	0	0	0	25
	0	0	0	0	80
Uniform	2	2	2	2	2
	25	25	25	25	25
	50	50	50	50	50
Increasing	0	2	2	2	25
	0	2	4	25	80
	1	2	16	25	50
Decreasing	25	2	2	2	0
	80	25	4	2	0
	50	25	16	2	1
Random	11	8	9	0	2
	13	25	0	4	12
	60	10	8	24	11

We note that for the considered general acyclic problem instances, the GO reorder intervals usually tend to be the same among all stages. This stems from the benefit obtained due to the order coordination. This benefit is significant since the considered general acyclic system represents high demand variability at demand nodes. However, the SOP does not consider demand variability to compute the SO reorder intervals.

### 6.3 Optimality gap

Table 6 summarises the performance of the SOP for both structures and for different groups of ordering cost profiles in terms of optimality gap. Optimality gap is computed by  $(Obj_{SO}-Obj_{GO})/Obj_{GO}$  where  $Obj_{SO}$  denotes the SO objective value and  $Obj_{GO}$  the GO objective value. The average (Avr.) and the maximum (Max.) gaps are reported for each

Table 2. Ordering cost ratio intervals used for the general acyclic structure.

Ordering cost profiles	Echelons				
	1	2	3	4	5
Starting	U[0, 5]	0	0	0	0
	U[15, 25]	0	0	0	0
	U[65, 80]	0	0	0	0
Middle	0	0	U[0, 5]	0	0
	0	0	U[15, 25]	0	0
	0	0	U[65, 80]	0	0
Ending	0	0	0	0	U[0, 5]
	0	0	0	0	U[15, 25]
	0	0	0	0	U[65, 80]
Uniform	U[5, 10]	U[5, 10]	U[5, 10]	U[5, 10]	U[5, 10]
	U[20, 25]	U[20, 25]	U[20, 25]	U[20, 25]	U[20, 25]
	U[45, 50]	U[45, 50]	U[45, 50]	U[45, 50]	U[45, 50]
Increasing	U[0, 5]	U[5, 10]	U[10, 15]	U[15, 20]	U[20, 25]
	U[0, 10]	U[10, 20]	U[20, 30]	U[30, 40]	U[40, 50]
	U[0, 15]	U[15, 30]	U[30, 45]	U[45, 60]	U[60, 75]
Decreasing	U[20, 25]	U[15, 20]	U[10, 15]	U[5, 10]	U[0, 5]
	U[40, 50]	U[30, 40]	U[20, 30]	U[10, 20]	U[0, 10]
	U[60, 75]	U[45, 60]	U[30, 45]	U[15, 30]	U[0, 15]
Random	U[0, 15]	U[0, 15]	U[0, 15]	U[0, 15]	U[0, 15]
	U[0, 30]	U[0, 30]	U[0, 30]	U[0, 30]	U[0, 30]
	U[0, 50]	U[0, 50]	U[0, 50]	U[0, 50]	U[0, 50]

Table 3. Lead time and annual holding cost data of the serial problem instances.

Instance	Stage $j$	$L_j$	$h_j$ (\$)	Instance	Stage $j$	$L_j$	$h_j$ (\$)	Instance	Stage $j$	$L_j$	$h_j$ (\$)
1	1	10	15.9	6	1	16	15.6	11	1	17	5.3
	2	14	32.3		2	8	22.7		2	15	16.5
	3	2	36.4		3	1	25.1		3	6	23.0
	4	8	55.8		4	15	30.4		4	6	37.1
	5	11	61.1		5	16	43.7		5	6	49.3
2	1	16	13.2	7	1	20	9.3	12	1	8	12.9
	2	5	28.4		2	13	12.3		2	11	20.5
	3	12	36.9		3	15	29.5		3	19	31.7
	4	20	53.4		4	4	42.4		4	1	47.8
	5	13	58.5		5	20	56.7		5	12	49.1
3	1	15	10.8	8	1	10	14.6	13	1	5	8.9
	2	6	15.6		2	9	19.6		2	11	12.9
	3	1	18.7		3	9	29.8		3	4	13.0
	4	12	20.8		4	8	36.4		4	3	21.9
	5	9	24.0		5	8	39.7		5	10	35.3
4	1	4	14.8	9	1	3	15.3	14	1	16	7.0
	2	18	31.6		2	3	21.4		2	14	19.9
	3	18	35.7		3	12	24.5		3	19	28.4
	4	1	43.2		4	2	36.9		4	11	36.9
	5	9	59.6		5	2	43.8		5	13	47.8
5	1	20	10.1	10	1	13	17.9	15	1	12	10.2
	2	19	13.9		2	19	33.2		2	13	26.0
	3	5	25.7		3	8	45.1		3	4	35.5
	4	6	26.0		4	11	50.8		4	5	53.5
	5	16	31.9		5	8	68.6		5	14	69.0

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Table 4. The SO and GO solutions for the serial problem obtained by permuting instance 14 with the second decreasing ordering cost profile.

Stage $j$	The SO solution				The GO solution			
	$R_j$ (days)	$s_j^{out}$ (days)	ASSC $_j$ (\$)	AFOC $_j$ +ACSC $_j$ (\$)	$R_j$ (days)	$s_j^{out}$ (days)	ASSC $_j$ (\$)	AFOC $_j$ +ACSC $_j$ (\$)
1	16	0	2061	17,451	16	0	2524	17,451
2	16	22	–	23,508	8	14	–	23,850
3	8	45	–	8829	8	37	–	8829
4	4	59	–	7326	4	51	–	7326
5	1	0	30,109	817	1	0	28,411	817
Total			32,169	57,931			30,935	58,273

Table 5. The SO and GO solutions for the general acyclic problem pertaining to the first decreasing ordering cost profile.

Stage $j$	The SO solution				The GO solution			
	$R_j$ (days)	$s_j^{out}$ (days)	ASSC $_j$ (\$)	AFOC $_j$ +ACSC $_j$ (\$)	$R_j$ (days)	$s_j^{out}$ (days)	ASSC $_j$ (\$)	AFOC $_j$ +ACSC $_j$ (\$)
1	16	45	–	152,224	16	45	–	152,224
2	16	38	–	163,693	16	38	–	163,693
3	16	31	–	329,261	16	31	–	329,261
4	16	38	69,373	109,106	16	38	69,373	109,106
5	16	26	–	188,098	16	26	–	188,098
6	16	57	–	103,575	16	57	–	103,575
7	16	48	–	95,483	16	48	–	95,483
8	16	59	–	45,912	16	59	–	45,912
9	16	48	–	32,549	16	48	–	32,549
10	16	48	–	36,358	16	48	–	36,358
11	16	58	–	54,822	16	50	–	54,822
12	16	69	–	46,135	16	69	–	46,135
13	16	80	–	38,300	16	60	–	38,300
14	16	0	832,814	35,425	16	0	832,814	35,425
15	16	0	352,945	37,653	16	0	316,074	37,653
16	4	0	296,591	5146	16	0	282,947	12,858
17	8	0	129,588	25,713	16	0	129,588	29,657
Total			1681,312	1499,453			1630,798	1511,109

ordering cost profile that comprises 45 problem instances. The last column (titled ‘Occ.’) reports the number of occurrences where the SO solution is different from the GO solution.

For the serial structure, the SO solution is usually equal to the GO solution. Decreasing and random profiles reveal a relatively inferior performance. The SOP is able to obtain near-optimal solutions of about 0.01% optimality gap in average with an observed worst case of 1.23%.

In the general acyclic structure, similarly to the serial one, the gap is important for decreasing and random groups. Besides, the gap of ending and increasing groups is higher compared with the serial structure. However, the SO solutions are still near-optimal with 0.46% optimality gap in average and with an observed worst case of 4.87% for the considered structure.

The demand variability considered in the general acyclic system lays between 0.50 and 1.05. In order to investigate the impact of demand variability on the SOP performance, we perform a second set of experiments for the serial structure. When we increase the coefficient of variation from 0.30 to 1 for this system, the performance of SO solutions deteriorates. Particularly, the performance of the starting, uniform, decreasing and random profiles gets worst. The observed worst case still belongs to the random profile with 3.41% optimality gap. However, the SO solutions still represent an average optimality gap of 0.15% for all groups of ordering cost profiles.

#### 6.4 Computational time

For serial problem instances, computational time is less than 1, 2 and 6 s using respectively the SOP, the ID and DD approaches. Therefore, the computational times of these approaches are very short and similar for five-echelon serial structure. However, the differences become significant for the general acyclic structure.

Table 6. Optimality gap results for the considered systems.

Ordering cost profiles	Serial system			General acyclic system		
	Avr. Gap (%)	Max. Gap (%)	Occ.	Avr. Gap (%)	Max. Gap (%)	Occ.
Starting	0.00	0.00	0	0.29	1.27	31
Middle	0.00	0.00	0	0.01	0.53	2
Ending	0.00	0.00	0	0.60	3.63	17
Uniform	0.00	0.21	1	0.39	2.58	13
Increasing	0.00	0.00	0	0.41	2.22	15
Decreasing	0.05	1.00	5	1.07	4.87	27
Random	0.03	1.23	1	0.45	4.61	14

Table 7. Running times for the SOP and ID approaches in the general acyclic structure.

Ordering Cost Profiles	SOP			ID Approach		
	Avr. Time (secs)	Min. Time (secs)	Max. Time (secs)	Avr. Time (secs)	Min. Time (secs)	Max. Time (secs)
Starting	7.57	3.37	12.43	18684.17	161.60	69699.39
Middle	6.90	4.36	12.18	439.69	102.68	1085.88
Ending	4.84	2.27	8.96	122.90	2.61	489.28
Uniform	3.77	2.62	8.36	214.18	13.90	1472.73
Increasing	2.99	1.91	4.15	105.53	7.41	753.98
Decreasing	5.25	2.85	11.59	853.43	26.60	4600.47
Random	4.75	1.81	9.33	243.90	9.27	3041.4

We notice that the DD approach requires significant computational time when the complexity of the supply chain network increases. With the ID approach, the computational time may be significantly reduced. For instance, for general acyclic problem instances belonging to the first starting profile, the DD approach cannot converge within 18,000 s whereas the ID approach provides global optima in 2785 s on average. In this case, the average gap between the best feasible solution obtained by the DD approach and the global optima is about 15.31%. Therefore, the ID approach clearly dominates the DD approach.

Besides, we notice that SOP requires significantly shorter computational time than the ID approach for all profiles. The SOP provides near-optimal solutions within 13 s whereas the ID approach requires 2951 s in average to provide global optima. Table 7 reports the running times of the SOP and the ID approaches for all groups of ordering cost profiles.

We observe that for the starting group, computational time of the ID approach is surprisingly long (see Table 7). Using decision variable bounds presented in Section 5, decision variable bounds of non-supply nodes remain too large for the starting group and this prevents a fast convergence of BARON to global optima. This also explains the relatively long computational times for middle and decreasing groups. A converse effect is observed in ending, increasing and uniform groups.

## 7. Conclusion and future research directions

In this paper, we have presented a NLIP model that determines nested PO2 reorder intervals and order-up-to levels in a multi-echelon inventory system. The GSM approach is used to model the multi-echelon system facing stochastic demand. Our computational studies demonstrate that the performance of the solution procedure may deteriorate when demand variability and the complexity of the supply chain network increase. However, for a five-echelon general multi-echelon system with 17 stages and 18 arcs facing high demand variability, the SOP provides near-optimal solutions of about 0.46% optimality gap in average within 13 s. Besides, we also propose an ID approach to reduce the computational time when the problem is solved to global optimality using a global optimiser. For problem instances for which the global optimiser cannot converge within 18,000 s, the ID approach provides global optima in 2785 s on average.

Some additional relevant issues remain for future consideration. The first one is the performance evaluation of the SOP for more complex and larger multi-echelon systems. For those systems, a faster global optimisation method must be developed in order to realise this analysis. The second issue is the extension of the model to consider non-nested

policies which would be more relevant for general supply chain structures. By considering a non-nested policy, a better solution in terms of total supply chain cost may be obtained. The third issue concerns the relaxation of the model so that stages are allowed to have arbitrary integer reorder intervals. This extension would enable to estimate the cost of the PO2 restrictions. All these extensions represent challenging future research directions for general multi-echelon systems facing stochastic demand.

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## Appendix A: Proof

In order to show that the objective function (1) is neither convex nor concave on the feasible region of the problem  $P_0$ , we consider the Hessian matrix of function (1) corresponding to the variables  $R_j$  where  $j \in \mathbf{N}_S$  and to  $\tau_k$  where  $k \in \mathbf{N}_D$  is:

$$\mathbf{H} = \begin{pmatrix} \frac{2A_j}{R_j^3} & 0 \\ 0 & -\frac{h_k z_k \sigma_k}{4(\tau_k)^{3/2}} \end{pmatrix}$$

We assume that the input parameters for node  $j$  and  $k$  are strictly positive. The determinant of matrix  $\mathbf{H}$  is then strictly negative when  $R_j > 0$  and  $\tau_k > 0$ . Therefore, the function (1) is neither convex nor concave with respect to  $R_j$  and  $\tau_k$  on the feasible region of the problem  $P_0$ . Hence, the function (4) is neither convex nor concave with respect to all of its variables on the feasible region of the problem  $P_0$ .