BIRGIT PEPIN, OLE KRISTIAN BERGEM & KIRSTI KLETTE

CHAPTER 3

Rethinking Algebra Teaching in the Light of 'Orchestration of Signs' – Exploring the "Equal Sign" in a Norwegian Mathematics Classroom

INTRODUCTION

Algebra continues to be the focus of reform efforts and research in mathematics education in many countries worldwide (e.g., Kieran, 1992 & 2006; National Council of Teachers of Mathematics (NCTM), 2000; Watson, 2009). There is a general concern (e.g., from policy makers, teachers and Higher Education councils) that students leave compulsory schooling with inadequate understandings of and preparation in algebra, and that they seem to be ill-prepared for future educational or professional opportunities (Moses & Cobb, 2001). Many universities (e.g., in the UK and the US) provide 'transition' or support courses for students who study mathematically demanding subjects in order to equip them with the relevant knowledge (see transition research, e.g., 'TransMaths' project at the University of Manchester[1]), also in algebra. In fact, there appears to be agreement that algebra reforms require a re-conceptualisation of algebra in school mathematics (e.g., NCTM, 2000).

A fundamental concept in algebra that has received considerable attention in mathematics education is that of equality and, connected to this, particularly the understanding of the equal sign (e.g., Kieran, 1981 & 2006; McNeil & Alibali, 2005). Knuth et al. (2006) claimed that 'the ubiquitous presence of the equal sign at all levels of mathematics highlights its importance' (p.298). It is generally acknowledged that the notion of 'equal' is complex and difficult for students to understand, and numerous studies have explored student understandings and use of equality and the equal sign (e.g. Alibali, 1999; Kieran, 1981).

Exploring *equivalence*, Gattegno (1974) stated that:

> We can see that *identity* is a very restrictive kind of relationship concerned with actual sameness, that *equality* points at an attribute which does not change, and that *equivalence* is concerned with a wider relationship where one agrees that for certain purposes it is possible to replace one item by another. Equivalence being the most comprehensive relationship, it will be the most flexible, and therefore the most useful. (p.83)

However, research into algebra tells us that the equal sign is not always interpreted by the learner, and presented by the mathematics teacher, in terms of equivalence.

In this chapter we explore the meanings and use of the equal sign in a 9th grade Norwegian mathematics classroom from a cultural semiotic perspective, in order to develop deeper understandings of algebraic learning and teaching in classroom/school environments.

UNDERSTANDINGS OF EQUALITY AND THE EQUAL SIGN

After decades of research to establish the importance of the equality concept (e.g., Baroody & Ginsberg, 1983; Falkner et al., 1999), the NCTM Standards (2000) reflected this importance by contending that "equality is an important algebraic concept that students must encounter and begin to understand in the lower grades" (p.94). Researchers have generally agreed on the important distinction between two understandings of equality: the 'operational'; and the 'relational' (e.g., Kieran, 1981; Filloy et al., 2003; Knuth et al., 2006). Kieran (1981) reported that:

> The equal sign is a 'do something signal', is a thread which seems to run through the interpretation of equality sentences throughout elementary school, high school, and even college. Early elementary school children … view the equal sign as a symbol which separates a problem and its answer. (p.324)

Another important finding of algebraic research has been that pupils do not develop a more nuanced understanding of the equal sign 'by themselves' or as a matter of 'natural' mathematical maturation (Kieran, 1981). Saenz-Ludlow and Walgamuth (1998) reported a year-long study in which pupils were taught to use different meanings of the equal sign, for example to use the verb 'to be' instead of the equal sign in the tasks. This had implications for their view of the sign and there was an apparent shift from 'find the answer' to 'is the same as'.

In terms of difficulties with the development of relational meaning (typically in transition from arithmetic to algebra), different approaches have been used, in particular with respect to the symmetric and relational use of the equal sign (e.g., Kieran, 1981; Theis, 2005). These differences are exacerbated by the fact that relational meaning has different interpretations (e.g., Malle, 1993; Cortes et al., 1990); Prediger (2010) summarized six different meanings for equality (p.81):

1. Operational meaning: operation equals answer (e.g. $24:6–3=1$ or $f'(x)=(3x^2)'=6x$)
2. Relational meaning:
   a. Symmetric arithmetic identity, e.g. '$5+7=7+5$' or '$19=10^2–9^2$'
   b. Formal equivalence describing equivalent terms, e.g. '$x^2+x–6=(x–2)(x+3)$, '$(a–b)(a+b)=a^2–b^2$'
   c. Conditional equations characterizing unknowns, e.g. 'solve $x^2=x+6$'
   d. Contextual identities in formulae, e.g. volume formula for cone: '$V=\frac{1}{3}·π·r^2·h$' or 'right angles with hypotenuse c and legs a, b satisfy $a^2+b^2=c^2$'
3. Specification, e.g. '$m:=\frac{1}{2}(a+b)$' or '$y=2x + 52$'

This last category was introduced by Cortes et al. (1990); in this, identities are not described but are provided as in definitions. Many researchers have emphasised the context specificity of meanings of the equal sign, and they relate it to the long-term shift in meaning across age, for example in transition from arithmetic to algebra (e.g., Cortes et al., 1990; Kieran, 1981). However, for the student (even the older student) the problem remains if there are changes of meanings within one problem, and this is seen as a main obstacle in the learning process. At the same time, and in the mathematical problem-solving situation where students are expected to distinguish and switch, it also represents an important characteristic and strength of learning algebra.

Broadly speaking, there are at least three issues in terms of difficulties about the meaning of the equal sign. First, and as mentioned above, learners (and often teachers) continue to use the equal sign to mean 'calculate', because this is familiar and meaningful for them. Second, the equal sign is often used differently within mathematics, also by mathematics teachers. Students have to develop an understanding of equivalence, as compared to equality, when using the sign. For example, equivalence can mean that expressions give the same equal values for a range of input values of the variables, or that expressions are transformations of the same form. Kieran and Sfard (1999) used a graphical function approach, and hence the students in their study had the opportunity to recognize that equivalent algebraic representations of functions could generate the same graphs, and hence represent the same relationship between variables. Equality is seen as the intersection of graphs (as compared to equivalence when graphs coincide) (Watson, 2010). Third, using the equal sign differently may imply different meanings for letters which may be used as variables, parameters or hidden values, for example.

THEORETICAL FRAMEWORK

Radford (2010) introduced a typology of forms of algebraic thinking that rests on a semiotic theoretical approach which, in turn, is based on current research in the field (e.g., Kieran 2006). For him, "signs lose the representational and ancillary status with which they are usually endowed in classical cognitive theories in order to become the *material* counterpart of thought" (p.2). In this semiotic perspective algebraic signs and formulas can be seen in a different light. Whereas, traditionally, letters and signs, including the equal sign, have been regarded as *the* semiotic system, Radford included words and gestures, amongst others, in this system. Without challenging previous research on symbolic algebra, Radford, in principle, claimed that there are many semiotic ways other than (and along with) the symbolic one to express algebraic ideas (in his case the 'unknown'). This leaves room for a large conceptual zone which Radford termed the "zone of emergence of algebraic thinking". He and other colleagues, such as Arzarello and his team, Nunez and Edwards, Roth and his colleagues, and others, have paid attention to the "embodied nature of mathematical cognition" (Radford 2010, p.4).

In this theoretical framework 'signs' encompass and include the traditional meanings of the 'sign'. Radford outlined, first, that signs are "considered in a broad sense, as something encompassing written as well as oral linguistic terms, mathematical symbols, gestures, etc." (p.3), and, second, that they are considered as parts of algebraic thinking (and not mere indicators of thinking).

> In more precise terms, within this semiotic-cultural perspective, thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts. (Radford, 2010, p.3)

Moreover, Radford (2010) coined the term of *objectification* and, according to him, the processes of *objectification* are the social processes through which the student develops deeper understandings of the 'culturally' built in logic and becomes confident with forms and actions related and involved in these processes. Moreover, constructs such as the 'formula' develop different meanings: for example, the formula becomes a *narrative* (in a condensed manner) which tells about students' mathematical experiences.

In terms of narrative Bruner's notion of "narrative construction of reality" (1991) focuses on the idea of narrative as a 'cultural product' with which the mind structures its sense of reality - narrative operates as an instrument of mind in the construction of reality. He identified (at least) nine features of narrative (Bruner, 1991: narrative diachronicity; particularity; intentional state entailment; hermeneutic composability; canonicity and breach; referentiality; normativeness and the centrality of trouble; context sensitivity and negotiability; narrative accrual). For this purpose, that is to use narrative as a perspective that can inform algebraic thinking in teaching, we draw on one of the key themes reflected in Bruner's analysis, canonicity and breach or, in other words 'troubles' and how to overcome them.

The combination of these theoretical stances provides the basics of our 'semiotic' analysis of teacher pedagogic practice in an algebra lesson. More particularly, we will explore the following questions:

- What signs are used in this lesson, and how is the equal sign used by this teacher?
- How can we interpret this semiotically, and 'collectively'?
- What may this mean for algebra teaching, and more particularly for the use of the equal sign in mathematics classrooms?

THE STUDY

For this chapter we draw on data from the Learner's Perspective Study (LPS) (e.g Clarke et al., 2006) collected by the Norwegian LPS research team (see Klette, 2009; Bergem & Klette, 2010). The LPS aims to juxtapose observable classroom practices and the meanings attributed to those practices (e.g. by teachers and pupils). The LPS research design includes lesson sequences of about ten grade 8 lessons, using three video cameras and supplemented by the participants' accounts obtained in post-lesson Stimulated Recall interviews, and copies of classroom materials such

as textbooks and other curricular materials. For this study one teacher's data, and in particular the early algebra lesson, were chosen for 'semiotic investigation'. The subsequent nine lessons and teacher and student interviews were analysed in terms of teachers' general pedagogic practices and pupils' perceptions of their learning practices, in addition to the analysis of the textbook used.

The theoretical framework for the semiotic analysis was provided in the previous section. More practically, a procedure involving the analysis of themes similar to that described by Woods (1986) and by Burgess (1984) was adopted. This included, at one level, the identification of the different 'signs' used by the teacher and the meanings attributed to them and, more generally, it meant using our knowledge of Radford's semiotics and testing the hypotheses offered by the literature, and building explanations and theorisations grounded in the data.

At another level, we tried to maintain the coherence of the teacher's case in terms of her pedagogic practice and with a holistic view (and respondent validated by the participant teacher interview), anchored in the teacher's own interview and the observations. The cursory analysis of textbooks helped here in identifying issues for examination. In addition, we could also draw on pupil interviews. More generally, it was important to locate and understand teacher pedagogic practices and the mathematics classroom cultures in Norway, and it was useful to draw on knowledge gained from earlier and ongoing research (e.g. Pepin, 1999; Pepin, 2011) which has highlighted the complex nature of teachers' work and classroom environments.

THE FINDINGS

Contexts-Mathematics classroom environments

Learners of mathematics at lower secondary level work in particular environments. In Norway most pupils go to comprehensive schools until the age of 16 and are taught in mixed-ability groups. In the Norwegian LPS there appeared to be particular 'customary ways' that most teachers in used in their teaching. For example, most teachers asked their pupils to work on exercises from textbooks for a considerable amount of time so that the pupils could practise what had been explained and the teachers could monitor their understanding. In Norwegian classrooms every pupil is provided with a textbook by the school to be used in school and at home: pupils are said to be 'entitled' to a common curriculum. However, the textbook used in the classes observed for this study differentiates between 'Blå'; 'Gul' and 'Rød' (which mean blue; golden and red) exercises, indicating that there are likely to be three tiers in each group, reflecting the pupils' perceived achievement levels.

All of the Norwegian teachers who were observed during this study drew on textbooks for exercises and sometimes for classwork. This practice was confirmed by the results of an 'attitude' survey of 13 grade 6-10 classes (see Pepin, 2011), in which the pupils said that, for much of their time in mathematics lessons they work on exercises from the textbooks. In fact, this was one of the reasons why many pupils disliked mathematics ("there is too little variation in maths").

However, evidence from this study shows that, in practice, the teachers found it hard to differentiate and provide exercises so that every pupil could access the mathematics. The pupils came to the lessons with different mathematics backgrounds (depending what they had been taught in previous years). In the Stimulated Recall interview, one teacher described her pedagogic practices and explained her efforts "to keep the whole class together" and at the same time attend to the mathematical needs of individual pupils.

> It's not always that easy to explain to students who haven't been taught equations before and who might not be that interested in mathematics and always requires the practical side of it… you'll get a lot of different answers. There's not a clear answer from all of the class. You'll get a mix of answers. (S4 T1)

She explained that she had developed pedagogic strategies to help her pupils cope with the variety of understandings (e.g. use peer assessment).

> I don't know, I don't always have the general overview. I have to admit that. But I knew I had a group of students who knew how to check the answers that I had checked on, and that I had given some explanations about. They had solved some exercises already. I saw it was working, when I talked to them. So I knew I had some students who could do the teaching for me in a way … So I think the variation [of pedagogic strategies] is good. As a teaching method, to use students as teachers ... There is so much instruction from the blackboard in maths [lessons]. (S4 T1)

In terms of textbooks this school had chosen the above-mentioned textbook that differentiated between three levels of difficulties in terms of exercises: 'Blå'; 'Gul' and 'Rød', which in turn was perceived to provide opportunities, or not, for individual pupils to practise their developing understandings. In fact, this teacher did not mention the colours, but used the different level exercises for pupils to have more practice (at whichever level).

> …sometimes the exercises on the plan don't suffice [to have enough practice]. Sometimes you have to do more. (S4 T1)

However, the testing system seemed to have an effect on the teachers' practice, and they used the grades to evaluate pupil performance on the work plan. This, in turn, provided a dilemma for them in terms of their support for those 'weak' students who invested a lot of effort, for apparently little effect.

> And then I have… I have sort of categorised them in accordance to where they're at …, so that you can keep up with them. So I evaluate them according to their grades, the formal grade, and you have the running evaluation that takes place all the time. Then we have the formal tests, hand-ins and so on. Then I get a very…I think that…on their path to a summative assessment within a topic. (S4 T1)

One Norwegian teacher told us that she found it difficult to attend to individual pupils' problems and misconceptions in mathematics- there was too little time. However, she said that she had identified and used particular strategies to 'reach everyone'. One of these practices was the ½ hour teaching time when she had the whole class together. During this time she tried to "get students to reflect on their own", that is to step back as a teacher and encourage the pupils to think for themselves and initiate and encourage discussion in class. Often pupils who may generally have performed poorly in mathematics tests were encouraged to explain their answers at the blackboard; sometimes several pupils were taken to the board. This showed to the pupils that there are methods, "different ways of calculating".

In summary, the teachers found it difficult to attend to the needs of individual pupils. Although the textbook and individual 'work plans' supported individualistic pedagogic practices, the teachers mentioned that they had insufficient time and expertise to deal with every pupil's developing mathematical understandings.

Description of the lesson and identification/use of 'signs'

In the following we briefly outline and describe the early algebra lesson, identifying particular 'signs' (our bolding).

At the start of the lesson the teacher gave the pupils a text question:

Per and Kari have five **apples** jointly. Per has two apples. How many has Kari? Really simple, you know…yes, and then you laugh a little…But discuss in your groups, how can we calculate the answer and how would you put your calculations down in writing?

Photo 1

45

She explained that they would have to work in a particular way with these kinds of **text questions** ('read the text first', 'find the information', ...) . However, the pupils did not seem to be convinced that this was necessary:

that's easy ... five minus two equals three ... that's easy ... [S4, P3]

The teacher insisted and tried to build up a structured approach, to introduce the problem algebraically:

But, eh, how many apples Kari has is the **unknown**, what we are going to find, and that's the core of **equations.** When we work with equations we have something called the unknown, we are going to find an answer or something we don't know anything about. And the unknown…we name with a letter, usually it's **X**. ... If we are going to solve this as an equation…(writes on the board).

She used expressions such as 'known' and 'unknown', and wrote it on the board as an equation:

How many apples Kari has, so, the unknown (**writes on the board while reading out load what she writes**), how…many…apples…Kari has. If we are going to write this as an equation ... (writes on the board) two, that's Per's apples ... and then I write X instead and that equals five. Do you all understand that we can write it down in this way? When X is the unknown, the answer we are going to find and it represents how many apples Kari has.

Photo 2

Some pupils appeared to be puzzled and a whole conversation developed about 'knowns' and 'unknowns':

P1: But what is the known?
T: What is the known? Yes, what is that? What is known in this? Nina?
P2: How many apples Per has and how many they have jointly?
T: How many apples Per has, because that is her (**points at the board**), and how many they have jointly ... ? That is five, and that is what we know and ... the unknown.
P1: Don't we have to have a **letter** for that?
T: No, we don't, because that is not something that is unknown. ... You are thinking like in algebra when we had **As and Bs and Cs** and so on. But now we have only one unknown and that is what we are working with all the time, that we have one unknown.

Although the pupils expressed their 'uneasiness'- they were not convinced that this question necessitated algebraic thinking - the teacher continued:

T: But what we, now we are demonstrating how to solve it and we start with very simple exercises to show you how you solve more difficult equation exercises. That is why we make use of such simple **numbers**, so that everyone will understand.
P: Why do we have this **X** anyway?

The teacher then proceeded to explain equations and introduced the equal sign:

T: ... we name all the unknowns X…then this is Per's apples (**writes below the arithmetic calculation on the board**) and here is the apples they have jointly. Kari plus Per (…) is Kari (**writes below the X**), that we don't know. And that is the unknown. This sign here, what sign is this? Iselin?
P: It is an **equal sign**, so should be just as much on the right side as on the left side.
T: This is an equal sign, and we use this in other situations as well when we calculate, and we say that it should be just as much on the left side as on the right side of the equal sign (**writes 'equal sign' below the sign and underlines the wording twice**).

She also moved directly to the concept of equivalence and the scale:

T: Just as much on the right side as on the left side ...always just as much on one side as the other in **numerical value**. ... Numerical value? The value of the numbers here, is just as much as the value there. ... what one often does with equations to make it, to make you learn from the ground up, it is often a good idea to…to use a gauge, or this **scale**. We draw a scale so that you get a visual image of equations…Now we will draw a balance, we will not calculate any further before we demonstrate it with a scale (she **draws a balance on the blackboard**).

... There is just as much **weight** on both sides, then we can say that the **midline here is the sign of equation seeing as it is in balance**. Here we have a **question mark** (**writes on one of the scales**) and that is Kari's apples, and then we have two apples which is…

... And they have the same weight, or there is just as much on both sides. It will always be so when it is an equation.

Photo 3

She subsequently used the scale to show equivalence:

> When we solve an equation we always want the X on the left side of the scale or the equal sign. It should always be on the left side alone. We want it all alone to find out what it equals.

> If I take away those two apples… What happens to the scale if I remove two apples? ... The other side will go straight down. (**makes handsign**) ... The other side will go straight down, because? What do I have to do so that the other side doesn't go straight down? What do I have to do with those five apples here? How many do I have to remove? Two. Do you get it? If I remove two here now…I remove the two apples…remove two apples…like…what do we call it when the scales are on the same horizontal line? Balance. .. What do we have left on the scales now? What do we have left on the left side when we have done this? ...

Photo 4

Furthermore, she formalised the process:

> And then we have found out what X is…Now I am going to **show you on paper**, no, not on paper, how we write it out. ... X plus two equals five (writes on the board). This is what we started with and on the scales we did X plus two and then minus two apples, to remove them so we were left with only X. And then I had to remember to subtract on the right side as well…Are you following me?...I am allowed to subtract or add on the left side, but then I have to do the same at the right side. And two minus two, how much is that? Iselin?

49

```
P:   Nought
T:   Is nought and I am left with X. And five minus two
     is? ...
P:   Three.
T:   Three. Now I have solved the equation. Then I have to
     write it in text because it is a text exercise. Kari
     has three apples.
```


Photo 5

We have described and followed this early part of the lesson in great detail, in order to see how the teacher introduced and developed the concepts, and used the different signs for her purpose. The lesson proceeded to solve another similar problem. The teacher then asked one pupil to come to the board to explain her answer and she then provided comments to the whole class about the solution:

When we solve an equation Iselin wants to get X alone. First she finds out what X is, you have to define what the unknown is in the task, and that is how old Olav is. Then she has used the information already given and writes it as an equation, because this is her equation, X plus seven equals 16. Then she starts to solve the equation…and she says she removes, why do you do this Iselin? Seven minus seven?.. Always when we solve an equation it is our job to get X by itself on the left side. And when we talk about the left side …it is the left side of the equation sign we talk about. We always want to get the X by itself like Iselin has tried to do here, remove the seven on one side and then we are allowed to remove seven on the other side, and then we have to do that in order

for it to be correct, otherwise it wouldn't be balance, like Rino said, in our scales. ...When we have text exercises it is important that we write our answer in text. Another tip we will try to follow up on is to write the equal sign below one another throughout our calculations. It will bring order to our equations, in order to see what we are doing. That's a tip. Now we don't have time to go through any more. It wasn't much we did, but we had just half an hour today. We'll continue with equations on Wednesday…

In the following section we discuss the different 'signs' the teacher used in this particular algebra lesson (see bolded words), and the different sources of meanings she assigned to the equal sign.

DISCUSSION OF FINDINGS

One can identify different sources of meaning of the equal sign in this Norwegian mathematics classroom. Looking at the lesson description above, the teacher's use of the equal sign has implications for its meaning in the process of teaching. At a first level we could identify at least three ways that are outlined by the literature (e.g., Kieran, 1981): Firstly, the equal sign is seen as a 'do something' signal and separates the problem and its answers. The teacher mentioned that the equal sign is also used "when we calculate" and "in other situations".

This is an equal sign, and we use this in other situations as well when we calculate, and we say that it should be just as much on the left side as on the right side of the equal sign (writes 'equal sign' below the sign and underlines the wording twice)

Second, the equal sign signifies that 'both sides yield the same value'. The teacher explicitly mentioned this when explaining the equation.

...always just as much on one side as the other in **numerical value**. ...Numerical value? The value of the numbers here, is just as much as the value there. ..

Third, the equal sign signifies equivalence relations when linked to when balance model.

... There is just as much **weight** on both sides, then we can say that the **midline here is the sign of equation seeing as it is in balance**.

Moreover, and looking through a slightly different lens, the equal sign can be regarded as sign or symbol within a semiotic system that is important for mathematical activity (e.g., Steinbring 2005) and where the sign is seen as part of a mathematical sign language. Steinbring (2005) drew on Otte's notion of *metaphor* (Otte 1984) to investigate the meaning of 'equation'. For him, algebra is a mathematical sign language, and he asked what the particularities of this language are, and in which ways meanings are attributed to the 'words' and 'phrases'. For this he used and

analysed the 'equation'- equivalence- balance situation. In terms of results, he defined three levels of the relations between the 'object' and 'sign/symbol':

1. algebraic signs and symbols (ASS) serve as *names* for objects and as descriptors of reality;
2. ASS describe *relations* and *structures* within the context;
3. there is a *reciprocal action* between ASS, and structures and relations (Steinbring 2005, p.101).

Here it is important to identify and draw out the mathematical concept, in order not to confuse the sign with the concept (e.g., Duval, 1993). Duval claimed that mathematical signs do not represent empirical things, but embody relations:

> There is an important gap between mathematical knowledge and knowledge in other sciences ... we do not have any perceptive or instrumental access to mathematical objects, even the most elementary ... we cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. This conflicting requirement makes the specific core of mathematical knowledge. (Duval, 2000, p.61)

For the equal sign and its use in this lesson, questions can be asked about when it is used as a 'sign', and when as a balancing 'tool', and when it signifies the process of balancing. The most noteworthy lesson episode here is when the teacher draws the equal sign above the centre of the scale (see photo 3), signifying the process of balancing the 'left' and 'right' hand side of the scale, which at this moment is 'in balance'.

Furthermore, and at a third level we take the semiotic perspective of Radford where 'signs' encompass and include linguistic terms, mathematical symbols and gestures as constituent parts of mathematical thinking.

In Radford's terms it can be said that the teacher uses a number of signs, and we can identify the following in this lesson sequence (see bolded words in last section):

- Words/expressions: 'apples'; 'unknown'; 'known'
- Gestures: underlining the equal sign; pulling the hand down signifying when the scale is pulled down on one side, gets 'out of balance'
- Algebraic signs and symbols: 'X'; 'A'
- Other signs: '?', underline
- Representations: scale/ balance
- Text: text/ question as relations
- Names: Kari; Per
- Numbers: 5 (and at times linked to names)

Radford's claim was that these signs become unique "by their mode of signifying" and help students to develop their "zone of emergence of algebraic thinking" (Radford, 2010).

Focussing on the equal sign, it is evident that this is used in several different semiotic ways, sometimes expressing 'doing', sometimes 'balancing', or 'unbalancing', for example. First, it is worth noticing which kind of text question the teacher has chosen for the topic of 'equations' and the use of the equal sign. Radford claimed that the mathematical problem at hand plays a crucial role. This is evident when pupils question the value of the test question with respect to the mathematical concepts they are meant to learn. Clearly, students did not see much sense in solving this problem algebraically: "it is easy ... why do we have this X anyway ... ?". It is also likely that pupils attach meanings in contexts, and when these contexts change, they have difficulties in establishing meanings for the symbol. "When do/can I use the scale model, when do I have to use the formal algebraic way of solving the problem?" In fact, several authors have investigated the affordances and constraints, and the usefulness of models, such as the balancing model, for particular problem situations (e.g., Vlassis, 2002). For example, Vlassis claims that the balance model can provide students with an 'operative' mental image containing principles to be applied, but it cannot overcome all obstacles linked to processes of abstraction in algebraic balancing (linked to negative numbers for example).

Second, we can identify different embodied and semiotic resources that are used to look at the problem in analytic ways. In the first stage the equal sign is used in the equation 'X+2=5' to tell the story of Per and Kari, and how many apples they each have. At the same time it is exploited for the explanation of the concept of 'unknown' and 'known' (photo 2) and the equal sign is written out in words and underlined twice - a strong gesture to emphasise the importance of the sign. In the second stage it is worked into the 'picture' of the balance, separating left and right with the labels 'left = right' above the scale, and explaining the process of balancing, whilst at the same time the equation (X+2=5) remains part of the 'story' (see photo 3). Here another gesture supports the meaning of the process: a sign of a hand pulling down as if the scale is being pulled down. In the third stage the equal sign is 'woven' into a more complicated balancing process (but principally in the same way as before) where apples and names are added (photo 4). At a fourth stage the sign is built into an algebraic equation which is meant to formalise the process of 'balancing': 'X+2-2=5-2' and, finally, linking a number to X ('X=3'). This then concludes the 'story' with a final answer written/elucidated in text (photo 5).

CONCLUSIONS AND IMPLICATIONS FOR THEORY AND PRACTICE

From the above, and considering the various uses of the equal sign and its meanings as reflected in teacher pedagogic practice, gestures and use of resources, we can develop a deeper understanding of teacher pedagogic practice, in particular with respect to the concept of 'balancing'. We propose the concept of *orchestration of signs* in the algebra classroom, to explain teacher pedagogic practice in terms of the use of signs and the meanings attached to them.

Leaning on the work of Trouche (2003; 2004), who developed this perspective to explain teacher pedagogic practice in technology-rich environments,

> *...an instrumental orchestration* is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a -in this case computerised learning environment in a given mathematical task situation, in order to guide students' instrumental genesis (Trouche, 2004). (p. 214/15, Drijvers et al., 2010)

The metaphor of orchestration relates to teacher pedagogic practice, the didactical performance to the musical performance, where the interplay between the conductor (teacher) and the musicians (students) describes and is likened to the learning situation (in our case) in the mathematics classroom. We are aware that the metaphor has its limitations (see Drijvers et al., 2010), and moreover we do not adhere strictly to the construct as it was used by Trouche (2004) for the technology-rich classroom and the use of tools and artefacts. However, in terms of using *signs* (and in particular the different meanings of the equal sign) *orchestration* is a helpful construct.

If we see the sign at the level of an artefact, we can identify three different levels:

- As a primary artefact: the equal sign written on the board means 'is equal to';
- As a secondary artefact: the modes of action attached to the equal sign, e.g. 'calculate';
- As a tertiary artefact: as simulating 'balancing' and with the representation of the scale; hand sign signifying 'off-balance'.

This relates to Wartofsky's (1983) distinctions between three levels of artefacts. Building a system from and with these, and *orchestrating the signs* at these different levels in order for mathematical learning to emerge, can be said to be one of the main goals of teacher pedagogic practice.

However, how the teacher organises this is an individual 'enterprise' where every teacher uses his/her skills to weave their stories using the signs (and tools) in complex ways. In our case, the teacher told the story of Kari and Per, and the unequal distribution of apples between them. There are also more subtle, and unexpected, 'trouble' elements in the lesson; for example, why should one use equations to solve this 'easy' question?; the 'known' and the 'unknown'; etc. Seen in this way, the equal sign becomes more than a sign in the formula; the sign is part of the story that is told by the formula and it has a narrative character. The dilemma for students here is that if the context changes, the story changes, and this may shift meanings too- at least for students who have problems with algebraic understanding.

In conclusion, by bringing together a semiotic and 'instrumental' approach, we have been able to analyse algebra teaching in a different way. This has highlighted the complex relationships between the signs in particular the equal sign, as artefacts, and the meanings attached to and uses of signs by the teacher. We have claimed that *orchestration of signs* may be a useful construct to describe these processes. Furthermore, we have compared the sign in the 'context' of the formula, and in

teacher *orchestration*, to a narrative told and given meaning by the teacher. This emphasises the highly individualised nature of teaching and of mathematics teachers' pedagogic practices; this individualism may not be a characteristic wished for by policy makers in their endeavour to standardise teaching.

In terms of the implications of these findings for teacher education, it can be argued that teacher educators may consider using frameworks such as 'orchestration of signs' in order to raise teacher awareness of the potential threats to mathematical learning when students move from one class to another, from one *orchestration* to another. However, the *orchestration of signs* highlights not only the results of the immediate activities, but the different shapes that these can take depending on the meanings attached to the signs. Trouche (2003) stressed the necessity for 'didactic management' of a system of what we would interpret as 'signs'. What signs should be proposed to learners, and how should teachers be guided in their *orchestration of signs*? What kinds of learner activities should be provided (in order to develop understandings of the different signs), and for what kinds of mathematical knowledge? Considering these questions, the following issues need to be addressed:

- New signs suggest new meanings in new environments which may require new sets of mathematical problems;
- It is important/essential to understand the constraints and potential of signs;
- It is important to understand and manage the orchestration process, and how the signs 'work together'.

In this respect the concept of *orchestration of signs* links issues of mathematics teacher pedagogic practices to teacher knowledge and pupil learning. Thus, and related to teacher education, we argue that the 'sign' can be regarded as a 'new' pedagogic resource to build competent teacher practice around its use; and that the *orchestration of signs* can be viewed as a creative pedagogic resource to develop an awareness of, or to notice, what constitutes important instructional moments.

Considering the theoretical implications, it is argued that the concept of *orchestration of signs* provides an analytic tool to investigate mathematics lessons in more detail by paying attention to the meanings of signs, and the role the different (algebraic) signs play in pedagogic practice whilst, at the same time, overseeing the 'whole', thus realising that, in the *orchestration of signs,* 'the whole equals more than the sum of its parts'.

NOTE

[1]   http://www.education.manchester.ac.uk/research/centres/lta/LTAResearch/transmaths/into-he/`

REFERENCES

Alibali, M. W. (1999). How children change their minds: strategy change can be gradual or abrupt. *Developmental Psychology*, *35*, 127–145.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the 'equals' sign. *The Elementary School Journal, 84*(2), 198–212.

Bergem O. K., Klette, K. (2010). Mathematical tasks as catalysts for student talk: Analysing discourse in a Norwegian mathematics classroom. In Y Shimizu, B. Kaur, R. Huang, and D. Clark (Eds.), *Mathematical tasks in classrooms around the world* (pp. 35–62). Sense Publishers.

Burgess, R. G. (1984). *In the field: An introduction to field research*. London, UK: Allan & Unwin.

Cortes, A., Vergnaud, G., & Kavafian, N. (1990). From arithmetic to algebra: negotiating a jump in the learning process. In G. Brooker, P. Coob, & T. de Mendicuti (Eds), *Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education*(Vol. 2, pp. 27–34). Oaxtepex, Mexico: PME.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75,* 213–234.

Duval, R. (2000). Basic issues for research in mathematics education. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 55–69). Hiroshima, Japan: Nishiki Print Co. Ltd.

Duval, R. (1993). Registres de representation semiotique et fonctionnement cognitive de la pensee. In *Annales de Didactique et des Sciences cognitives, 5*, 37–65.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching children mathematics, 6*, 56–60.

Filloy, E., Rojano, T., Solares, A. (2003). Two meanings of the 'equal' sign and senses of comparison and substitution methods. In N. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 223–230). Honolulu, HI: CRDG.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317–326.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of Research on mathematics teaching and learning* (pp. 390–419). New York, NY: MacMillan.

Kieran, C. (2006). Research on the learning and the teaching of algebra: a broadening of sources of meaning. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: past, present, future* (pp. 23–49). Rotterdam: The Netherlands: Sense Publication.

Klette K. (2009). Challenges in Strategies for Complexity Reduction in Video Studies. Experiences from the PISA + Study: A video study of teaching and learning in Norway. In T. Janik & T. Seidel (Eds.), *The power of video studies in investigating teaching and learning in the classroom* (pp. 61–83). Waxmann Publishing.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education. 36*(4), 297–312.

Malle, G. (1993). *Didaktische Probleme der Elementaren Algebra*. Wiesbaden: Vieweg.

McNeil, N. M., & Alibali, M. W. (2005). Knowledge changes as a function of mathematics experience: all contexts are not created equal. *Journal of Cognition and Development*, *6*, 285–306.

Moses, R., & Cobb, C. (2001). *Radical equations: math literacy and civil rights*. Boston, MA: Beacon Press.

National Council of Teachers of Mathematics- NCTM. (2000). *NCTM Standards 2000: Principles and standards for school mathematics*. Reston. Retrieved from http://standards.nctm.org/document/index.htm

Otte, M. (1984). *Was ist Mathematik? Occasional Paper 43 des*. IDM, Bielefeld: Universitaet Bielefeld.

Pepin, B. (2011) Pupils' attitudes towards mathematics: a comparative study of Norwegian and English secondary students. *ZDM—The International Journal on Mathematics Education*, *43*(4), 535–546.

Pepin, B. (1999). The influence of national cultural traditions on pedagogy: classroom practices in England, France and Germany. In J. Leach & B. Moon (Eds.), *Learners and pedagogy* (pp. 124–139). London: Sage Publications.

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: the case of meanings of the equal sign. *Journal of Mathematics Teacher Education, 13*, 73–93.

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12*(1), 1–19.

Steinbring, H. (2005). Do mathematical symbols serve to describe or construct 'reality'? In M. H. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign- grounding mathematics education* (pp. 91–104). New York, NY: Springer.

Theis, L. (2005). L'apprentissage du signe =: un obstacle cognitif important. *For the learning of Mathematics, 25*(3), 7–12.

Trouche, L. (2003). From artefact to instrument: mathematics teaching mediated by symbolic calculators. *Interacting with Computers, 15*, 783–800.

Trouche, L. (2004). Managing complexity of human/machine interactions in computerised learning environments: guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*, 281–307.

Vlassis, J. (2002) The balance model: hindrance or support for the solving of linear equations with one unknown, *Educational Studies in Mathematics, 49*, 341–359.

Wartofsky, M. (1983). From genetic epistemology to historical epistemology: Kant, Marx and Piaget. In L. S. Liben (Ed.), *Piaget and the foundations of knowledges.* Hilseide, NJ: Lawrence Erlbaum.

Watson, A. (2009) Algebraic reasoning. In T. Nunez, P. Bryant & A. Watson (Eds.), *Key Understandings in mathematics learning.* London: Nuffield Foundation.

Woods, P. (1996). *Inside schools: Ethnography in educational research.* London: Routledge & Kegan Paul.

AFFILIATIONS

Birgit Pepin
Høgskolen i Sør-Trøndelag
Norway

Ole Kristian Bergem
Kirsti Klette
University of Oslo
Norway